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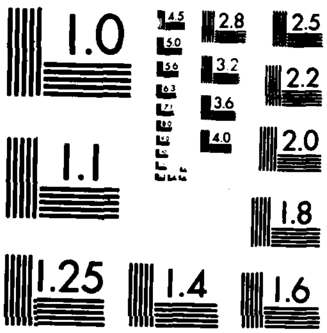

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Research Report CCS 477

A LIMIT THEOREM ON CHARACTERISTIC FUNCTIONS VIA AN EXTREMAL PRINCIPLE

by

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December 1983

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Abstract

This [illegible]

~~We~~ proves a classical limit theorem on characteristic functions by using duality between a pair of optimization problems, one of which is an infinite dimensional minimization involving the relative entropy functional.

KEY WORDS: Characteristic Functions, Optimization in infinite dimensional spaces, Duality, Relative Entropy.

Introduction

Modern optimization theory has been employed successfully in many diverse fields such as Economics, Physics, Statistics, Biology and Engineering. This paper is a small step toward demonstrating the use of optimization theory as proof mechanism in Probability.

The result from Probability Theory in question here is a limit theorem on characteristic functions. Let X be a random variable with distribution F_X, support $[x_L, x_R]$ and characteristic function $\psi(t) = Ee^{itX}$

Then

$$x_R = \lim_{y\to\infty} \frac{1}{y} \log \psi(-iy)$$

$$x_L = -\lim_{y\to\infty} \frac{1}{y} \log \psi(iy) .$$

The result is given in Lucas' classical book "Characteristic Functions" [3]; first a weaker result, concerning only analytic characteristic functions, is proved in Chapter 7. The full statement is given in Chapter 11, as part of Theorem 11.1.2. It is derived from the result in Ch. 7 via a chain of lemmas on boundary characteristic functions.

Here we prove the above limit theorem by using duality relations between two extremum problems. One of these problems is an infinite-dimensional convex program involving the minimization of the relative entropy functional, which is of dundamental importance in Statistical Information Theory, Thermodynamics and Communication Theory.

The plan of the paper is as follows: Section 1 gives a formal statement of the limit theorem (Theorem A). Section 2 gives the duality theorem (Theorem B), which is in fact an adaptation of a result in the authors' paper [1]. In Section 3 we prove Theorem A via Theorem B.

1. A limit theorem on characteristic functions

Let X be a random variable, and $F_X(x)$ its distribution function. Let $\psi(t)$ denote the characteristic function of F_X, i.e.

$$\psi(t) = \int_{-\infty}^{\infty} e^{itx}\, dF_X(x) .$$

The left extremity of F_X is the number x_L with the property

$$\forall\varepsilon > 0 : F_X(x_L-\varepsilon) = 0 , \; F_X(x_L+\varepsilon) > 0 .$$

and the right extremity of F is the number x_R with the property

$$\forall\varepsilon > 0 : F_X(x_R-\varepsilon) < 1 \quad F_X(x_R) = 1$$

The interval $[x_L, x_R]$ is the support of F_X. Clearly

$$\psi(t) = \int_{x_L}^{x_R} e^{itx}\, dF_X(x) .$$

F_X is bounded to the left if $x_L > -\infty$ and bounded to the right if $x_R < \infty$.

Theorem A. If F_X be bounded to the right, then its right extremity is given by

$$x_R = \lim_{y\to\infty} \frac{1}{y} \log \psi(-iy) \tag{1}$$

If F is bounded from the left, then its left extremity is given by

$$x_L = -\lim_{y\to\infty} \frac{1}{y} \log \psi(iy) . \tag{2}$$

2. A duality theorem on relative entropies

Let D be the class of generalized densities $f = \frac{dF}{dt}$ (Radon-Nikodym derivatives) of distribution functions F, on a given probability space, with support $[x_L, x_R]$. In particular $F_X \in D$ and $f_X = \frac{dF_X}{dt}$

is the corresponding density. The relative entropy (divergence, Kullback-Leibler distance) between $F \in D$ and F_X is given by the quantity

$$I(f;f_X) = \int_{x_L}^{x_R} f(t)\log\left[\frac{f(t)}{f_X(t)}\right]dt .$$

It is well known that $I(\cdot;f_X)$ is a nonnegative convex functional and is equal to zero if and only if $f = f_X$ (a.e. with respect to dt) see [2].

A special case of a problem studied in [1,Ch.3] is the infinite-dimensional convex program

$$\text{(E)} \qquad \inf_{f \in D}\{I(f,f_X) : \int_{x_L}^{x_R} g(t)f(t)dt \geq a\}$$

where $g(t)$ is a given summable function. It was shown in [1] that a dual problem is given by

$$\text{(H)} \qquad \sup_{y \geq 0}\{ay - \log \int_{x_L}^{x_R} e^{yg(t)} f_X(t)dt\} .$$

Moreover, from Th. 1 in [1] the following duality relations hold between (E) and (H).

Theorem B. If (E) is feasible then inf(E) is attained, sup(H) is finite and

$$\min(E) = \sup(H) .$$

3. Proof of Theorem A via Theorem B

First note that the trivial inequality

$$\forall y > 0 : Ee^{yX} \leq e^{yx_R}$$

implies

$$\lim_{y\to\infty} \frac{1}{y} \log E\, e^{yX} \leq x_R$$

i.e.

$$(3) \qquad \lim_{y\to\infty} \frac{1}{y} \log \psi(-iy) \leq x_R \ .$$

Consider now the problem

$$(E_\varepsilon) \qquad \inf_{f\in D}\{I(f;f_X): \int_{x_L}^{x_R} tf(t)dt \geq x_R - \varepsilon\}$$

for some fixed $\varepsilon > 0$. This is a special case of problem (E) with $g(t) = t$, $a = x_R - \varepsilon$. The dual is

$$\sup_{y\geq 0}\{y(x_R-\varepsilon) - \log \int_{x_L}^{x_R} e^{yt} f_X(t)dt\}$$

i.e.

$$(D_\varepsilon) \qquad \sup_{y\geq 0}\{y(x_R-\varepsilon) - \log \psi(-iy)\}$$

Problem (E_ε) is clearly feasible for every $\varepsilon > 0$, and we infer from Theorem B:

$$\infty > \sup(D_\varepsilon) \geq \lim_{y\to\infty}\{y(x_R-\varepsilon) - \log \psi(-iy)\} =$$

$$= \lim_{y\to\infty} y[x_R-\varepsilon - \frac{1}{y} \log \psi(-iy)]$$

Now, for the limit to be finite, it is necessary that:

$$(4) \qquad x_R - \varepsilon \leq \lim_{y\to\infty} \frac{1}{y} \log \psi(-iy) \qquad \forall\, \varepsilon > 0 \ .$$

Combining (3) and (4) we obtain equation (1).

To prove equation (2) we note that the inequality

$$(5) \qquad -\lim_{y\to\infty} \frac{1}{y} \log \psi(iy) \geq x_L$$

is trivial, while the inequality

$$(6) \qquad -\lim_{y\to\infty} \frac{1}{y} \log \psi(iy) \leq x_L + \varepsilon \qquad \forall\, \varepsilon > 0$$

follows by applying Theorem B (in the above manner) to the dual pair:

$$\inf\{I(f,f_x) : \int_{x_L}^{x_R} tf(t)dt \geq x_L - \varepsilon\}$$

$$\sup_{y\geq 0}\{-y(x_L+\varepsilon) - \log \psi(iy)\} .$$

Now, (5) and (6) indeed imply (2) , and the proof of Theorem A is thereby completed.

References

[1] Ben-Tal, A., "The entropic penalty approach to stochastic programming", Math. of Operations Research (to appear).

[2] Kullback, S., Information Theory and Statistics, Wiley, New York, 1959.

[3] Lucas , E., Characteristic Functions (Second Edition), Griffin, London, 1970.

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