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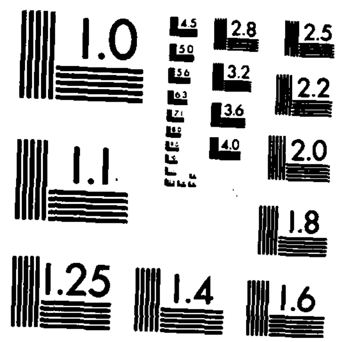

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER ARO 21096.2-MA	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) The Visibility of Stationary and Moving Targets in the Plane Subject to a Poisson Field of Shadowing Elements		5. TYPE OF REPORT & PERIOD COVERED Technical Report 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) S. Zacks, M. Yadin		8. CONTRACT OR GRANT NUMBER(s) DAAG29-83-K-0176
9. PERFORMING ORGANIZATION NAME AND ADDRESS New York, St U-Binghamton		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS N/A
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office Post Office Box 12211 Research Triangle Park, NC 27709		12. REPORT DATE Feb 84 13. NUMBER OF PAGES 33
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

The view, opinions, and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Visibility probability, stationary and moving targets; measure of proportional visibility; Poisson random shadowing process.

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ARO 21096.2-MA

Center for Statistics, Quality Control and Design

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THE VISIBILITY OF STATIONARY AND MOVING TARGETS IN THE PLANE SUBJECT TO A POISSON FIELD OF SHADOWING ELEMENTS

by

S. ZACKS and M. YADIN

Technical Report No. 2

February 15, 1984

Prepared Under Contract DAAG29-83-K-0176 with the
U. S. Army Research Office

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THE VISIBILITY OF STATIONARY AND MOVING TARGETS IN THE PLANE SUBJECT TO A POISSON FIELD OF SHADOWING ELEMENTS*

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ABSTRACT

A methodology for an analytical derivation of visibility probabilities of n stationary target points in the plane is developed for the case when shadows are cast by a Poisson random field of obscuring elements. In addition formulae for the moments of a measure of the total proportional visibility along a star-shaped curve are given. The general methodology is illustrated by some examples of visibility along the circumference of half a circle, when the obscuring elements are centered within a concentric annular region. The distribution of the measure of proportional visibility is approximated by a mixed-beta distribution.

Key Words: *Visibility probability, stationary and moving targets; measure of proportional visibility; Poisson random shadowing process.*

*Partially supported by Contract DAAGZ983K0176 with the U. S. Army Research Office.

0. Introduction

The present paper is the first one in a series of articles dealing with problems of visibility of targets through random fields of obscuring elements (trees, bushes, clouds, etc.) and similar problems of detection and hitting of targets. The problems dealt with are of stochastic nature. The exact number, location and dimensions of the obscuring elements are unknown. A probability model is formulated concerning these variables. Given such a probability model it is required to determine probabilities of certain events and distributions of certain random variables. To illustrate some of the problems that can be solved by the methodology developed, consider the following examples:

Example 1: Visibility of Stationary Targets

An observer is placed at a given location in a forest, in order to detect specified targets (vehicles, animals, etc.). Due to the random location of the trees it is important to determine the probabilities that individual targets are observed and the distribution of the number of targets observed. For this purpose one has to determine the probabilities that any specified r points out of n, $1 \leq r \leq n$, are simultaneously visible. The locations of the targets may be specified or random.

Example 2: Visibility of Moving Targets

A target is moving along a specified path C. An observer is located at a point $\underset{\sim}{0}$. Trees or other obscuring elements are distributed between $\underset{\sim}{0}$ and C. It is often required to determine the distribution function of the total visible portion of C.

Example 3: Detecting and Hitting Moving Targets

A hunter is located at a point $\underset{\sim}{0}$ in the forest and a target is moving along a path C, which is partially obscured by the random elements. In order to detect a target the hunter should observe it uninterruptedly for t_0 units of time. It is interesting to determine the probability of detecting a target at a specified location or anywhere on C. In addition, after detecting a target the hunter could attempt to hit it. Each such attempt requires t units of time in which the target is visible. A hitting attempt may be successful with probability p. If an attempt fails the hunter can attempt again, as long as the target is visivle. Once the target disappears, it has to be detected again. The total number of possible trials, N, depends on the length of the visible portions of C. N is a random variable, and it is interesting to determine the distribution of N, and the probability of hitting a target.

Example 4: Energy Penetration Through Random Fields

A light beam from the origin $\underset{\sim}{0}$ is focused in a specified direction. If there are no obscuring elements, a given sector of the plane is in the light. On the other hand, a portion of the sector may be in the shadow of random elements. In such a case part of the energy in a beam does not penetrate the field. The objective is to determine the distribution function of the amount of penetrating energy.

In the above examples we restricted attention to visibility problems in the plane. These problems can be readily generalized to visibility in three dimensional spaces, where the obscuring elements are crowns of trees, clouds, sattelites, stars, etc. Problems of similar nature can be found in various areas of applied science and technology and in military applications.

The literature dealing with these kinds of shadowing problems contains a small number of papers. Chernoff and Daly [2] discussed the problem of determining the distribution of the length of shadowed and of visible segments on a straight line in the plane. Likhterov and Gurin [6] studied the probability of detecting a moving target on a straight line. The shadowing problem can be reduced to a coverage problem. This reduction is achieved by specifying the shadows cast by random elements on a target curve. In a previous paper, Yadin and Zacks [12] applied methods of coverage probabilities to study visibility problems on a circle. The literature on coverage problems is very rich (see Eckler [3]). The papers of Ailam [1], Greenberg [5], Robbins [8,9] and Siegel [10,11] provide methods which are relevant to the study of shadowing problems. In particular the methods developed in the present paper and those of [12] are based on the theory given by Robbins [9, 10]. The geometry of the problem may be complicated, and a reduction to a coverage problem may be inconvenient. In such cases it is often easier to study the geometry of the shadowing elements rather than that of the shadows. For this reason, the present paper deals with the shadowing problems directly.

The fact that the number of papers on shadowing problems published is small does not reflect a lack of interest in these problems in various areas of application. One can find a large number of technical reports which treat shadowing problems by simulations or by fitting simple statistical models to empirical data. Such "solutions" do not shed light on the intrinsic structure of the stochastic phenomenon. The methodology developed in the present paper analyzes the stochastic structure and provides methods for numerical approximations which are more accurate and require less computing time than simulation procedures. For developing the general methodology we construct in Section 1 a probability model of random shadowing elements. Sections 2 and 3 deal with visibility probabilities of single and several stationary targets (Example 1). Section 4 deals with a visibility measure for moving targets (Example 2). The methodology for solving problems discussed in Example 3 and 4 will be presented in forthcoming papers. The methods of the present paper are extended in [13] to solve three dimensional visibility problems.

1. Random Fields of Shadowing Objects

In general, a shadowing object is characterized by parameters of location, shape and size. In the present paper we restrict attention to disks in the plane. An extension to spheres in three dimensional spaces is given in [13]. A disk in the plane is characterized by a vector (ρ, θ, y), where (ρ, θ) are the polar coordinates of its center and y is its diameter. The specification of disks location by polar coordinates is a matter of convenience and could be replaced in some application by Cartesian coordinates. Let $S = \{(\rho, \theta, y);\ 0 \le \rho < \infty,\ -\pi \le \theta \le \pi,\ 0 \le a \le y \le b\}$ denote the sample space of all possible shadowing disks. Let $\mathcal{B}$ denote the Borel σ field on S. A set B in $\mathcal{B}$ represents geometrical conditions on the location and the size of disks. We consider a countable collection of disks. Let N{B}, for any B ε $\mathcal{B}$, denote the number of disks having coordinates in B, i.e., satisfying geometrical conditions specified by B. $N\{\cdot\}$ is a σ-finite measure on $\mathcal{B}$. We consider stochastic structures in which $N\{\cdot\}$ are random measures. In these cases, for each $B \varepsilon \mathcal{B}$, N{B} is a random variable. In the case that N{B} has, for each $B \varepsilon \mathcal{B}$, a Poisson distribution, the spatial stochastic process is called a Poisson random field of shadowing disks. In a Poission random field, the conditional distribution of $(N\{B_1\}, \ldots, N\{B_m\})$, given $N\{S\} = n$, is multinomial, for every finite partition $\{B_1, \ldots, B_m\}$ of S. Accordingly, if the number $N\{S\}$ in a Poisson random field of disks is known, we speak of multinomial fields. In the present study we discuss Poisson random fields.

The Poisson random field model is obviously an idealization of natural phenomena. For example, on a planar cut of a forest, trunks of trees are represented by random disks rather than by their actual random shape. Additional parameters could represent other shapes of shadowing objects, like ellipses, triangles, etc. Furthermore, the Poisson random field models assumes that, given $N\{S\} = n$, the centers of random disks are independently distributed in the plane. Accordingly, with certain (usually negligible) probabilities disks may overlap. Clearly, in practical applications the fit of the Poisson model to the empirical data should be verified. In Poisson random fields $N\{B\}$ has a Poisson distribution with mean

$$\text{(1.1)} \qquad \nu\{B\} = \lambda \iiint_B dG(y|\rho,\theta)H(d\rho,d\theta),$$

where $\lambda H(d\rho,d\theta)$ is the expected number of centers located in the rectangle $[\rho, \rho+d\rho) \times [\theta, \theta+d\theta)$. $G(y|\rho,\theta)$ is the conditional c.d.f. of the diameter Y, given the location of the center (ρ,θ). If Y is independent of (ρ,θ), and if the centers are uniformly distributed, i.e., $H(d\rho,d\theta) = \rho\, d\rho\, d\theta$, we say that the random field is <u>standard</u>. In this case

$$\text{(1.2)} \qquad \nu\{B\} = \lambda \int_a^b H\{B(y)\}dG(y),$$

where $H\{B(y)\}$ is the area of the subset of B containing disks with a given diameter y. Standard Poisson and multinomial fields are not the only interesting ones. In many applications, scattering of centers of disks follow other distributions than uniform. In particular, clusters of disks, each one following a different bivariate normal distribution are of interest in various applications (ecological problems, etc.).

2. Visibility of Individual Targets

A target is represented in the present paper by a point $\underset{\sim}{P}$ in the plane whose location is specified by the polar coordinates (r,s). The observer is located at the origin $\underset{\sim}{0}$. The point $\underset{\sim}{P}$ is visible from $\underset{\sim}{0}$ if the line segment $\overline{\underset{\sim}{OP}}$ is not intersected by any random disk. A random disk interacts the line segment $\overline{\underset{\sim}{OP}}$ if its coordinates belong to the set

$$(2.1) \qquad B(r,s) = \{\rho,\theta,y); \ (\rho,\theta) \in B(r,s,y), \ a \le y \le b\},$$

where $B(r,s,y)$ is the set of points having distances from $\overline{\underset{\sim}{OP}}$ smaller than y/2 (see Figure 1). A disk which intersects $\overline{\underset{\sim}{OP}}$ is said to cast shadow on $\underset{\sim}{P}$. A natural requirement for shadowing processes is that the source of light $\underset{\sim}{0}$ is uncovered. We therefore introduce the structural condition

$$(2.2) \qquad C_o = \{(\rho,\theta,y); \ (\rho,\theta) \in C_o(y), \ a<y\le b\},$$

where

$$(2.3) \qquad C_o(y) = \{(\rho,\theta); \frac{y}{2}<\rho<\infty, \ -\pi < \theta < \pi \}.$$

Figure 1 provides a graphical illustration of the set $B(r,s,y) \cap C_o(y)$. A point $\underset{\sim}{P}$ = (r,s) is visible if $N\{B(r,s) \cap C_o\} = 0$. The probability that $\underset{\sim}{P}$ is visible is, therefore,

$$(2.4) \qquad \psi_1(r,s) = \exp\{-\nu\{B(r,s) \cap C_o\}\}.$$

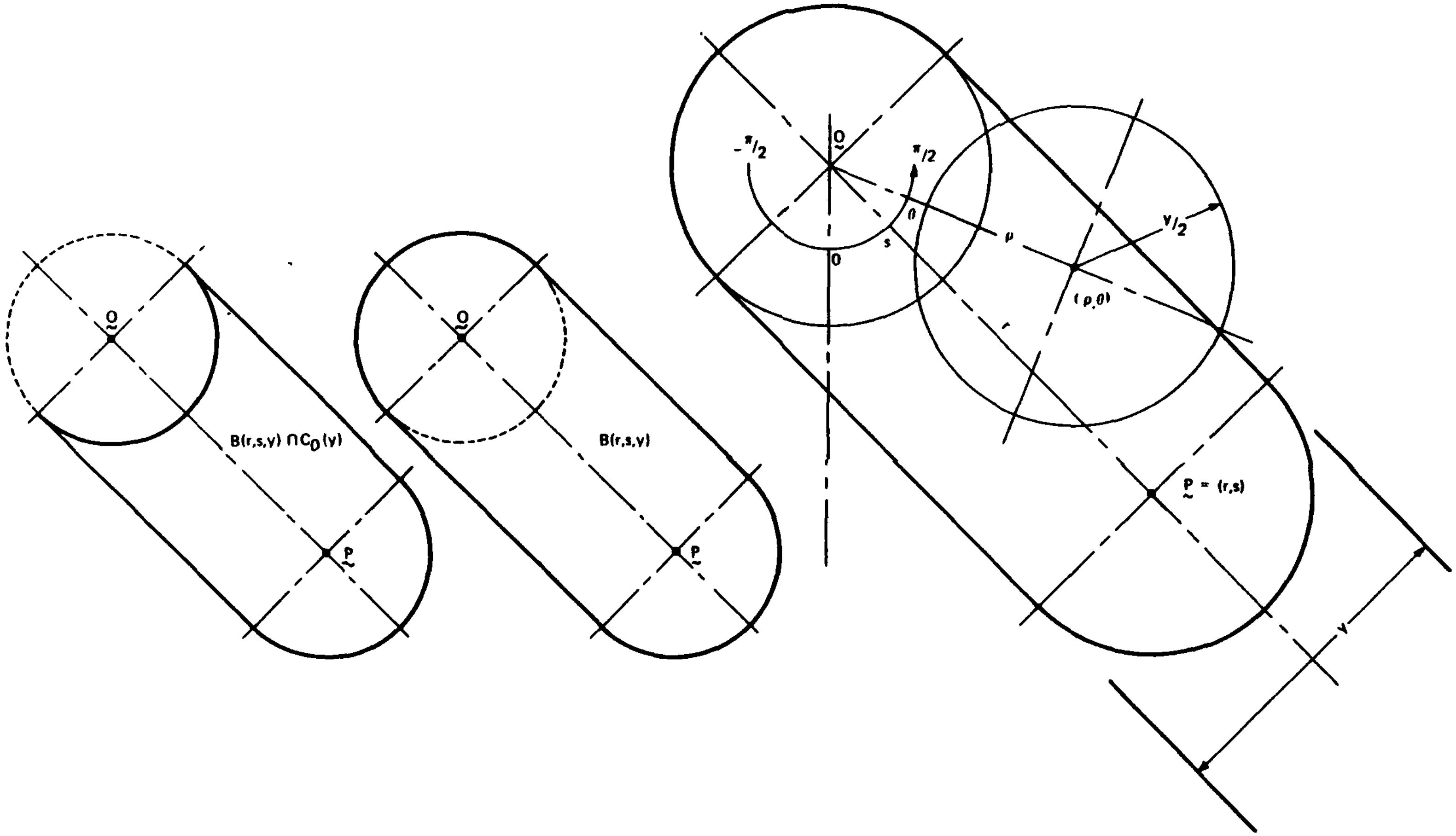

Fig. 1. The Geometry Of Disks Casting Shadows On A Point

In the standard case, the area of $B(r,s,y) \cap C_o(y)$ is yr. Hence, $\nu\{B(r,s) \cap C_o\} = \lambda r \xi$, where $\xi = \int_a^b y dG(y)$ is the expected diameter of a shadowing disk. In this case, the probability that $\underset{\sim}{P}$ is visible is $\psi_1(r,s) = \exp(-\lambda r \xi)$ irrespectively of the orientation parameters. A <u>line of sight</u>, L_s, is the set of all points, in direction s, which are visible from $\underset{\sim}{0}$. The length of L_s, $||L_s||$, has a c.d.f.

(2.5) $$Pr\{||L_s|| \leq r\} = 1 - \psi_1(r,s).$$

In the standard case, the distribution of $||L_s||$ is negative exponential with parameter $\rho = \lambda\xi$ (see also Feller [4,p.10]). One can generalize the above results to cases where the target $\underset{\sim}{P}$ is randomly located. For For example, in the standard case, if $F_R(r)$ is the c.d.f. of the distance of $\underset{\sim}{P}$ from $\underset{\sim}{O}$, the probability that P is visible is

(2.6) $$\psi_1 = \int_0^\infty e^{-r\rho} dF_R(r).$$

3. Simultaneous Visibility of Several Targets

Let $\underset{\sim}{P}_1, \ldots, \underset{\sim}{P}_n$, $n \geq 2$, be arbitrary points in the plane and let $\underset{\sim}{0}$ designate the origin. The polar coordinates of $\underset{\sim}{P}_j$ are (r_j, s_j), $j = 1, \ldots, n$. These n points are simultaneously visible if and only if no one of the line segments $\overline{OP}_j$, $j=1, \ldots, n$, is intersected by a random disk. Let $\underset{\sim}{r} = (r_1, \ldots, r_n)$, $\underset{\sim}{s} = (s_1, \ldots, s_n)$ and consider the set $B(\underset{\sim}{r}, \underset{\sim}{s}, y)$ of points having distances not exceeding $y/2$ from any one of the line segments $\overline{\underset{\sim}{O}\underset{\sim}{P}_j}$, $j = 1, \ldots, n$ (see Figure 2). Let

$$(3.1) \qquad B_n(\underset{\sim}{r}, \underset{\sim}{s}) = \{(\rho, \theta, y);\ (\rho, \theta) \in B_n(\underset{\sim}{r}, \underset{\sim}{s}, y),\ a < y \leq b\}.$$

The expected number of disks which cast shadows on one or more of the specified points is $\nu\{B_n(\underset{\sim}{r}, \underset{\sim}{s}) \cap C_o\}$. Accordingly, the probability that $\underset{\sim}{P}, \ldots, \underset{\sim}{P}_n$ are simultaneously visible is

$$(3.2) \qquad \psi_n(\underset{\sim}{r}, \underset{\sim}{s}) = \exp\{-\nu\{B_n(\underset{\sim}{r}, \underset{\sim}{s}) \cap C_o\}\}$$

We provide general formulae for the determination of $\nu\{B_n(\underset{\sim}{r}, \underset{\sim}{s}) \cap C_o\}$ for the case where the n points $\underset{\sim}{P}_j$, $j = 1, \ldots, n$, satisfy the condition

$$(3.3) \qquad -\pi \leq s_1 < s_2 < \ldots < s_n \leq \pi$$

Condition (3.3) means that no two points are on the same ray. In addition we require that all disks satisfy some condition C, where $C \subset C_o$, and $\nu\{C\} < \infty$. Disks which belong to the set C may

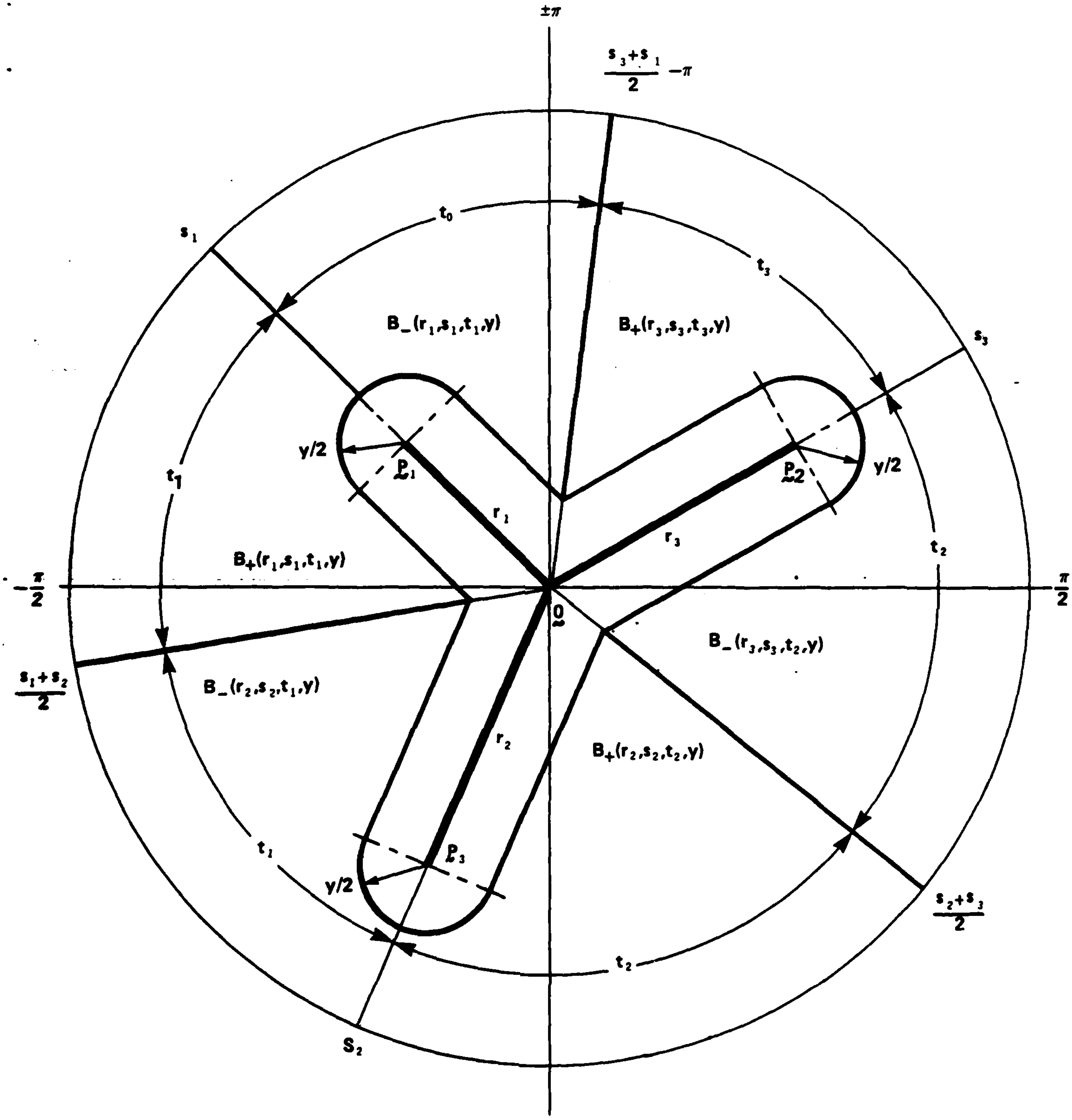

Fig.2 The Geometry of Disks Casting Shadows On n(n=3) Points

or may not cast shadows on the specified points. In order to compute the expected number of disks which cast shadows on at least one point we subtract from the total expected number of disks, $\nu\{C\}$, the expected number of disks which do not cast shadows on any one of the n specified points. For this purpose we define 2n disjoint sets $B_+(r_i,s_i,t_i)\cap C$ and $B_-(r_i,s_i,t_{i-1})\cap C$, $i=1, \ldots, n$, where

$$t_o = t_n = \pi - \frac{s_n-s_1}{2},$$

(3.4) and

$$t_i = \frac{s_{i+1}-s_i}{2}, \quad i = 1, \ldots, n-1.$$

$B_+(r_i,s_i,t_i)\cap C$ is the set of disks having orientation coordinate $\theta\varepsilon[s_i, s_i+t_i)$ which do not intersect the line segment $\overline{OP}_i$. The t_i values were chosen as bisectors of two adjacent directions in order to avoid the possibility that a disk belonging to $B_+(r_i, s_i,t_i)\cap C$ will intersect any other line segment $\overline{OP}_j$, $j\neq i$. The sets $B_-(r_i, s_i, t_{i-1})\cap C$ have similar interpretation, for disks in C with orientation coordinates $\theta\varepsilon[s_i-t_{i-1}, s_i)$. The sets $B_\pm(r,s,t)$ of disks parameters (ρ,θ,y) are described in terms of sets of center ponits (ρ,θ) of disks having a given diameter y. These sets are denoted by $B_\pm(r,s,t,y)$. In Figure 2 we illustrate a partition of the (ρ,θ) plane in terms of such sets for a given y. Accordingly

(3.5) $B_\pm(r,s,t)=\{(\rho,\theta,y);\ (\rho,\theta)\varepsilon B_\pm(r,s,t,y),\ a<y\leq b\}$.

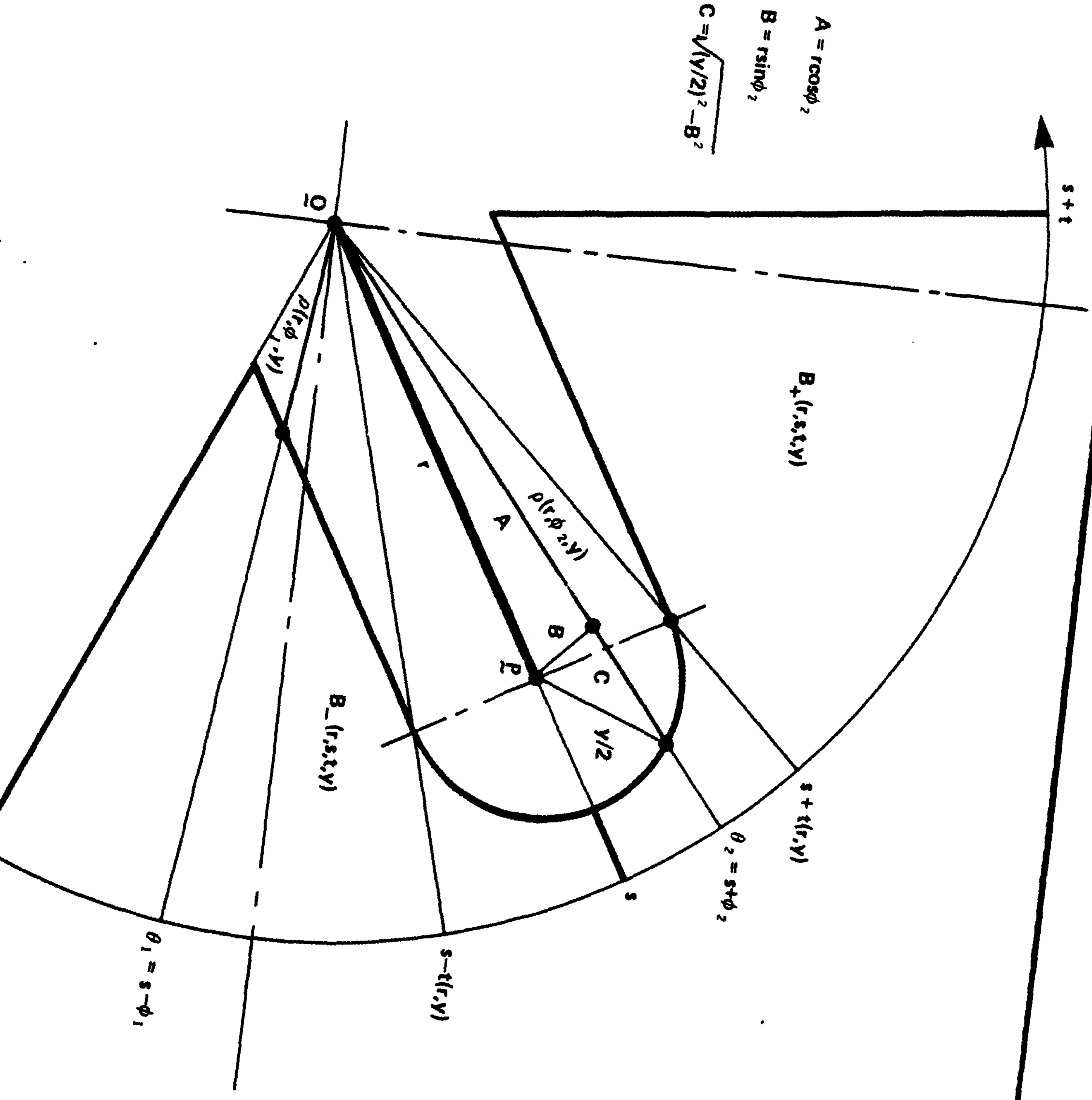

Fig.3.The Geometry of Sets of Centers (ρ,θ) of Disks of Diametery

Figure 3 provides geometrical details for the definition of $B_\pm(r,s,t,y)$. For a formal definition, let $t(r,y) = \min(t, \tan^{-1}(y/2r))$ and let

$$(3.6)\qquad \rho(r,\phi,y) = \begin{cases} r + y/2 & , \text{ if } \phi = 0 \\ r[\cos\phi + ((y/2r)^2 - \sin^2)^{1/2}], & 0<\phi<t(r,y) \\ y/(2\sin\phi) & , t(r,y) \le \phi \le t. \end{cases}$$

Then,

$$(3.7)\qquad B_+(r,s,t,y) = \{(\rho,\theta);\ \rho(r,\theta-s,y)<\rho<\infty \text{ and } 0 <\theta-s\le t\}.$$

Similarly,

$$(3.8)\qquad B_-(r,s,t,y) = \{(\rho,\theta);\ \rho(r,s-\theta,y) < \rho<\infty, \text{ and } 0\le s-\theta <t\}.$$

The expected number of disks in $B_\pm(r,s,t) \cap C_0$ is denoted by $\lambda K_\pm(r,s,t)$. We formally define $K_\pm(r,s,t) = 0$ for all $t\le 0$. Generally, for $t>0$,

$$(3.9)\qquad K_\pm(r,s,t) = \iiint_{B_\pm(r,s,t)\cap C} dG(y|\rho,\theta)H(d\rho,d\theta).$$

The above K-functions can be used to express the probabilities $\psi_n(\underset{\sim}{r},\underset{\sim}{s})$ more explicitly. The expected number of disks in C, which cast shadows on one or more of the n specified points, is

$$(3.10)\qquad \nu\{B_n(\underset{\sim}{r},\underset{\sim}{s})\cap C\} = \nu\{C\} - \lambda \sum_{i=1}^{n} [K_-(r_i,s_i,t_{i-1}) + K_+(r_i,s_i,t_i)].$$

Substitution of (3.10) in (3.2) yields more explicit formula for $\psi_n(\underset{\sim}{r},\underset{\sim}{s})$.

4. Visibility of a Moving Target

Consider a target moving at a constant speed along a curve C in the plane. We are interested in determining the distribution function of the proportion of time in which the target is not shadowed by random obscurring elements while moving between points $\underset{\sim}{P}_L$ and $\underset{\sim}{P}_u$ on C. This proportion of time is equivalent to the proportion of the length of the visible segments of C between $\underset{\sim}{P}_L$ and $\underset{\sim}{P}_u$. We assume that C is a star-shaped curve. Such a curve intersects any ray from the origin at most once. The curve C is specified by a function $r(\theta)$, $s_L \le \theta \le s_u$, which yields the distance of C from $\underset{\sim}{0}$ in direction θ. We further assume that $r(s)$ is piecewise continuously differentiable. Let s_L and s_u be the orientation coordinates of $\underset{\sim}{P}_L$ and $\underset{\sim}{P}_u$, respectively. The length of C, between these two points is

$$(4.1) \qquad L = \int_{s_L}^{s_u} \ell(s)ds \quad ,$$

where

$$(4.2) \qquad \ell(s) = [r^2(s) + (\frac{d}{ds} r(s))^2]^{1/2}$$

The proportion of the visible segments of C, between $\underset{\sim}{P}_L$ and $\underset{\sim}{P}_u$, is

$$(4.3) \qquad V = \frac{1}{L} \int_{s_L}^{s_u} I(s)\ell(s)ds \quad ,$$

where $I(s) = 1$ if the point $(r(s),s)$ on C is visible, and $I(s) = 0$ otherwise. V is a random variable assuming values in [0 1]. The distribution of V is a mixture of a two-point distribution concentrated on {0,1} and a distribution $F_V^*(v)$ on (0,1). Under some general conditions the distribution concentrated on (0,1) is absolutely continuous. It sufficies, for example, that $G(y|\rho,\theta)$ is absolutely continuous, or that $H(d\rho, d\theta) = h(\rho,\theta)d\rho d\theta$, $h(\rho,\theta) \geq 0$. Let $p_o = \Pr\{V=0\}$ and $p_1 = \Pr\{V=1\}$. The c.d.f V of can be generally represented by the formula.

$$(4.4) \qquad F_V(v) = \begin{cases} 0 & , v < 0 \\ p_o + (1-p_o-p_1)\, F_V^*(v) & , 0 \leq v < 1 \\ 1 & , 1 \leq v . \end{cases}$$

The probability that the section of C, between $\underset{\sim}{P}_L$ and $\underset{\sim}{P}_u$, is completely visible is given by

$$(4.5) \qquad p_1 = \exp\{-[\nu\{C\} - \lambda[K_+(r(s_u), s_u, t) + K_-(r(s_L), s_L, t)]]\}$$

where $t = \pi - (s_u - s_L)/2$ and C is the set of shadowing disks. $\lambda K_+(r(s_v), s_u, t)$ is the expected number of random disks having orientation coordinate $\theta \epsilon (s_L, s_v + t)$, which do not cast shadows on the segment of C under consideration. Similarly, $\lambda K_-(r(s_L), s_L, t)$ is the expected number of random disks with orientation coordinate $\theta \epsilon (s_L - t, s_L)$, which do not cast shadows on the intersecting segment of C. Explicit expressions for the probability of complete invisibility p_o and of the c.d.f. $F_V(v)$ are not available. An approximation to $F_V(v)$ by a beta distribution will be discussed in Section 4.2. A discrete approximation will be studied elsewhere.

4.1 Moments of V

The n-th moment of V is, according to (4.3),

$$(4.6)\qquad \mu_n = \frac{1}{L^n} E\{(\int_{s_L}^{s_u} \ell(s)I(s)ds)^n\}$$
$$= \frac{1}{L^n}\int_{s_L}^{s_u} \cdots \int_{s_L}^{s_u} E\{\prod_{i=1}^{n} I(s_i)\} \prod_{i=1}^{n} \ell(s_i)ds_i$$
$$= \frac{n!}{L^n}\int \underset{A}{\cdots}\int E\{\prod_{i=1}^{n} I(s_i)\} \prod_{i=1}^{n} \ell(s_i)ds_i .$$
$$= \frac{n!}{L^n}\int \underset{A}{\cdots}\int p_n(s_1, \ldots, s_n) \prod_{i=1}^{n} \ell(s_i)ds_i ,$$

where $A = \{s_L \le s_1 < \ldots < s_n \le s_u\}$ and

$$(4.7)\qquad p_n(s_1, \ldots, s_n) = \psi_n(\underset{\sim}{r}, \underset{\sim}{s}) = E\{\prod_{i=1}^{n} I(s_i)\}.$$

Applying the K-functions, which are defined in (3.7), one can express the probability of simultaneous visibility $p_n(s_1, \ldots, s_n)$ in the form

$$(4.8)\qquad p_n(s_1, \ldots, s_n) = \exp\{-\nu\{C\}\}.$$
$$.\exp\{\lambda \sum_{i=1}^{n} [K_-(r(s_i), s_i, t_{i-1}) + K_+(r(s_i), s_i, t_i)]\}$$
$$= \exp\{-\nu\{C\}\}A_o(s_i, s_n) \prod_{i=1}^{n-1} A(s_i, s_{i+1}),$$

where

(4.9) $$A_o(x,y)=\exp\{\lambda[K_-(r(x),x,\pi-\frac{y-x}{2})+K_+(r(y),y,\pi-\frac{y-x}{2})]\}$$

for $-\pi \le x \le y \le \pi$, and

(4.10) $$A(x,y)=\exp\{\lambda[K_+(r(x),x,\frac{y-x}{2}) + K_-(r(y),y,\frac{y-x}{2})]\}, \quad -\pi \le x \le y \le \pi.$$

Substitution of (4.8) and (4.6) yields a recursive formula for the moments of V. Let $\ell^*(s) = \ell(s)/L$ and define recursively,

(4.11)
$$Q_1(x,y) \equiv A(x,y),$$
$$Q_{i+1}(x,y) = (i+1)\int_x^y \ell^*(z)A(z,y)Q_i(x,z)dz \quad , i=1,2,\ldots$$

Finally, the moments of V are

$$\mu_1=\exp\{-\nu\{C\}\}\int_{s_L}^{s_u} \ell^*(x)A_o(x,x)dx \quad ,$$

(4.12) and for $n\geq 2$

$$\mu_n = n\exp\{-\nu\{C\}\}\int_{s_L}^{s_u}\ell^*(x)\int_x^{s_u}\ell^*(y)A_o(x,y)Q_{n-1}(x,y)dy\,dx.$$

Notice that if $s_u - s_L < \pi$ then $A_o(x,y) = A_1(x)A_2(y)$ and expression (4.12) assumes a simpler form. Formulae (4.11) and (4.12) are convenient for the recursive determination of the moments. Finally we remark that the sequence $\{\mu_n; n\geq 1\}$ is monotonically decreasing and $\lim_{n\to\infty}\mu_n = p_1$. Indeed, according to the dominated convergence theorem,

(4.13)
$$\lim_{n\to\infty}\mu_n = \lim_{n\to\infty}\left[(1-p_o-p_1)\int_o^1 u^n dF_V^*(u)+p_1\right]$$
$$= (1-p_o-p_1)\int_o^1 (\lim_{n\to\infty} u^n)dF_V^*(u)+p_1$$
$$= p_1.$$

4.2 A Beta-Mixture Approximation to the CDF of N.

In order to approximate the c.d.f. $F_V(v)$ we consider a mixture of a two-point distribution and a beta-distribution. This approximation is given by

$$\tilde{F}_V(v)=\begin{cases} 0 & , v < 0 \\ \tilde{p}_o + (1-\tilde{p}_o-p_1)\ I_V(\alpha,\beta) & , 0 \le v < 1 \\ 1 & , 1 \le v \end{cases} \qquad (4.14)$$

where $I_x(a,b) = \frac{1}{B(a,b)} \int_o^x u^{a-1}(1-u)^{b-1}du$ is the incomplete beta function ratio, $0<a,\ b<\infty$. The parameters $\tilde{p}_o$, α and β are determined by equating the first three moments of V to those of $\tilde{F}_V(V)$. The moment equations yield the formulae

$$\begin{aligned} \tilde{p}_o &= 1-p_1 - (\mu_1-p_1)(\alpha+\beta)/\alpha , \\ \alpha &= (2(\mu_2^*)^2-\mu_3^*(\mu_1^*+\mu_2^*))/D \\ \beta &= (\mu_1^*-\mu_2^*)(\mu_2^*-\mu_3^*)/D, \end{aligned} \qquad (4.15)$$

where $\mu_n^* = \mu_n-p_1$ and $D = \mu_1^*\ \mu_3^* - (\mu_2^*)^2$.

This approximation was previously applied by Yadin and Zacks [12] in a related shadowing problem. In Section 6 we provide a numerical example which illustrates this approximation in a special case.

5. The Evaluation of the K-Functions Under a Special Field Structure

The evaluation of the K-functions depends on the particular structure of the field of random disks. In the present section we consider a class of field structures, which is prevalent in many applications, and which yields relative simple formulae. This class is characterized by the condition that centers of disks are scattered within a region S which is situated between $\underset{\sim}{0}$ and the observed points $\underset{\sim}{P}, \ldots, \underset{\sim}{P}_n$, or the curve C.

The regions in which centers of disks are scattered are specified by two star-shaped curves U and W between the origin and the target curve (or points). These curves are specified by two star-shaped functions $u(\theta), w(\theta), \theta \varepsilon [\theta_L, \theta_u]$, respectively, and the region S is given by

$$(5.1) \qquad S = \{(\rho,\theta);\ u(\theta) \le \rho \le w(\theta),\ \theta_L \le \theta \le \theta_u\} .$$

The requirement that disks having centers in the region S and diameter in [a,b], would not cover the origin or intersect target points implies the following conditions on $u(\theta)$ and $w(\theta)$:

$$(5.2) \qquad \frac{b}{2} \le u(\theta) \quad , \ \theta \varepsilon [\theta_L, \theta_u] \quad ;$$

and

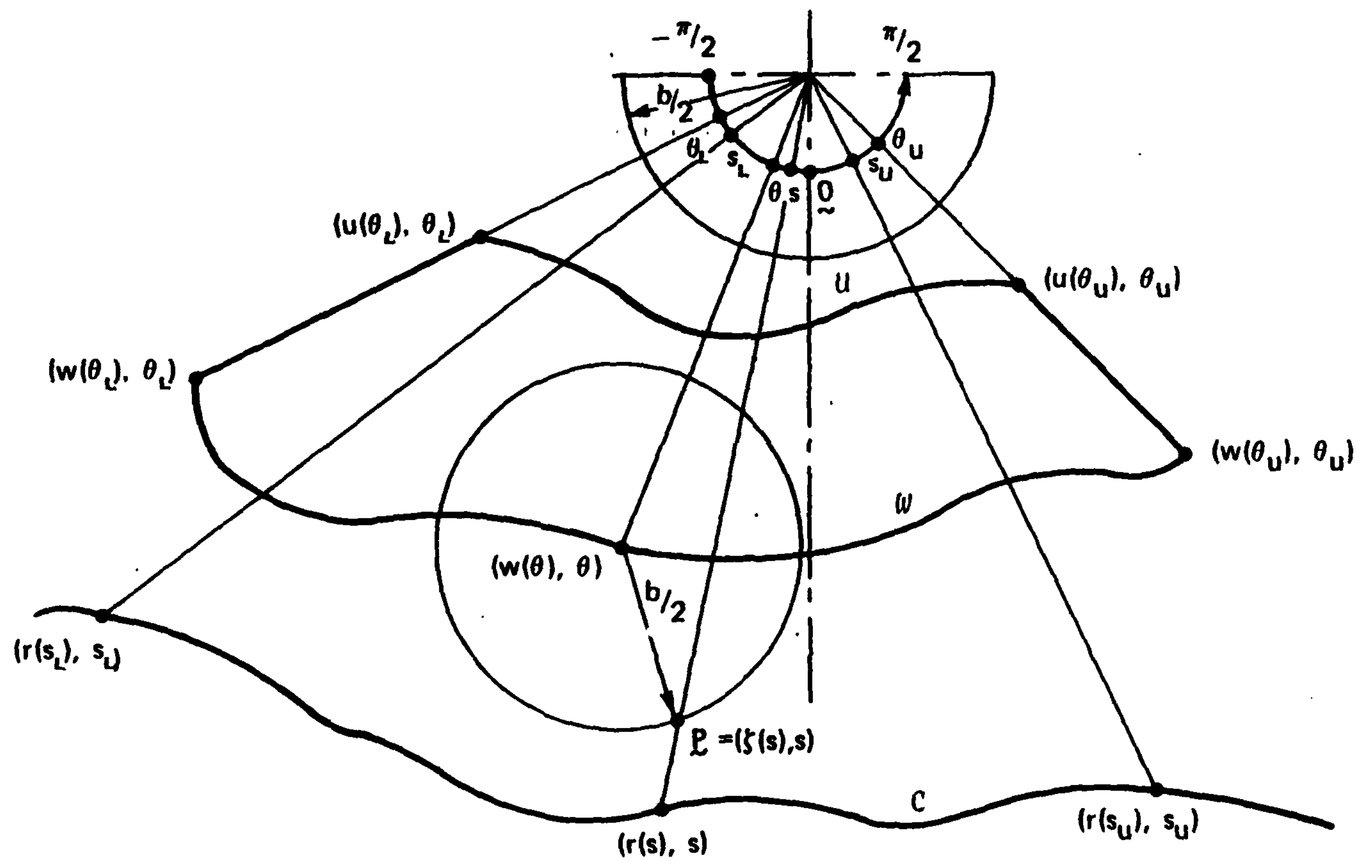

Fig. 4. The Geometry Of Disks Under A Special Field Structure

$$(5.3) \qquad w(\theta)\cos(\theta-s) + [(\tfrac{b}{2})^2 - w^2(\theta)\sin^2(\theta-s)]^{\frac{1}{2}} < r(s), \quad \text{for all } \theta\varepsilon[\theta_L,\theta_u],$$

where $(r(s),s)$ is any target point. Condition (5.3) insures that the distance $r(s)$ between $\underset{\sim}{0}$ and $(r(s),s)$ is greater than the distance $\xi(s)$ between $\underset{\sim}{0}$ and $\underset{\sim}{P} = (\xi(s),s)$, where $\underset{\sim}{P}$ is a point on a circumference of a disk of the maximal diameter, b, whose center $(w(\theta),\theta)$ is on $\mathcal{W}$ (see Figure 4). The structural condition imposed on the field of random disk is $C = S \times [a,b]$. In the following development we assume, for the sake of simplifying notation that the centers of disks are distributed within S according to an absolutely continuous measure, i.e.,

$$(5.4) \qquad H(d\rho,d\theta) = \begin{cases} h(\rho,\theta)d\rho d\theta & , \text{ if } (\rho,\theta)\varepsilon S \\ 0 & , \text{ otherwise.} \end{cases}$$

A disk centered at (ρ,θ) does not intersect a ray from $\underset{\sim}{0}$ having orientation s, if its diameter y is smaller than $2\rho\sin|\theta-s|$, or if $|\theta-s| > \pi/2$. Let $G^*(\rho,\theta,s)$ denote the probability of this event, i.e.

$$(5.5) \qquad G^*(\rho,\theta,s) = \begin{cases} G(2\rho\sin|\theta-s|\,|\rho,\theta) & , \text{ if } |\theta-s| < \sin^{-1}(\frac{b}{2\rho}) \\ 1 & , \text{ otherwise} \end{cases}$$

Let

$$(5.6) \qquad k(s,\theta) = \int_{u(\theta)}^{w(\theta)} G^*(\rho,\theta,s)\, h(\rho,\theta)d\rho \, .$$

Then,

$$K_{+}(r,s,t) = \int_{s}^{\min(s+t,\theta_u)} k(s,\theta)d\theta \quad ,$$

(5.7) and

$$K_{-}(r,s,t) = \int_{\max(s-t,\theta_L)}^{s} k(s,\theta)d\theta.$$

Notice that, due to conditions C, $K_{\pm}(r,s,t)$ is independent of r.

6. An Example

In the present example we develop all the previous formulae explicitly and provide numerical illustrations for the following simple case.

(1) The Poisson field is standard, with intensity λ;

(ii) The diameters of disks are uniformly distributed in $[a,b]$, $0 \le a < b$;

(iii) The target curve C is an arc on a circle of radius r, specified by

$C = \{(\rho,\theta);\ \rho = r\ ,\ s_L \le \theta \le s_u\}$.

(iv) The region S is an annular region bounded between two concentric arcs, U and W of radii u and w, respectively, which satisfy the condition

$\frac{b}{2} < u < w < r - b/2$.

That is

$$S = \{(\rho,\theta);\ \ u \le \rho \le w;\ \ \theta_L \le \theta \le \theta_u\}$$

and

$$[s_L, s_u] \subset [\theta_L, \theta_u] \subset [-\frac{\pi}{2}, \frac{\pi}{2}].$$

The expected number of disks in S obtains the simple expression

$$\text{(6.1)} \qquad \nu\{C\} = \lambda \int_{\theta_L}^{\theta_u}\int_{u}^{w} \rho d\rho d\theta = \frac{\lambda}{2}(w^2 - u^2)(\theta_u - \theta_L).$$

To develop the K-functions we substitute in (5.6) the function

$$G^*(\rho,s,\theta) = \begin{cases} 0 & \text{, if } |\theta-s| \leq \sin^{-1}(\frac{a}{2\rho}) \\ \frac{2\rho\sin|\theta-s|-a}{b-a} & \text{, if } \sin^{-1}(\frac{a}{2\rho}) < |\theta-s| < \sin^{-1}(\frac{b}{2\rho}) \\ 1 & \text{, if } \sin^{-1}(\frac{b}{2\rho}) \leq |\theta-s|. \end{cases} \tag{6.2}$$

Thus one obtains

$$k(s,\theta) = k_w(|s-\theta|) - k_u(|s-\theta|) \quad , \tag{6.3}$$

where, for $t \geq 0$

(6.4) $k_v(t) =$

$$= \begin{cases} 0 & \text{, if } t \sin^{-1}(\frac{a}{2v}) \\ [\frac{2}{3} v^3 \sin t - \frac{a}{2} v^2 + \frac{a^3}{24\sin^2 t}]/(b-a) & \text{, if } \sin^{-1}(\frac{a}{2v}) \leq t \leq \sin^{-1}(\frac{b}{2v}) \\ \frac{1}{2}[v^2 - \frac{a^2+ab+b^2}{12\sin^2 t}] & \text{, if } \sin^{-1}(\frac{b}{2v}) \leq t < \frac{\pi}{2} \\ \frac{1}{2}[v^2 - (a^2+ab+b^2)/12] & \text{, if } \pi/2 \leq t. \end{cases}$$

For positive t values not exceeding $\min(\theta_u - s, s - \theta_L)$, the K-functions satisfy

$$(6.5)\quad K_+(r,s,t) = K_-(r,s,t) = K^*(t) = K_w^*(t) - K_u^*(t),$$

where

$$(6.6)\qquad K_v(t) = \int_0^t k_v(\tau)\,d\tau$$

$$= \begin{cases} 0 & , \text{ if } t < \sin^{-1}(\frac{a}{2v}) \\ K_v^{(1)}(t) & , \sin^{-1}(\frac{a}{2}) \le t < \sin^{-1}(\frac{b}{2v}) \\ K_v^{(2)}(t) & , \sin^{-1}(\frac{b}{2v}) \le t < \pi/2 \\ K_v^{(3)}(t) & , \pi/2 \le t \end{cases},$$

in which

$$(6.7)\qquad K_v^{(1)}(t) = \frac{1}{b-a}\{\frac{v^2}{3}[4v^2-a^2)^{1/2} - 2v\cos(t)] - \frac{a}{2}v^2\,[t-\sin^{-1}(\frac{a}{2v})] + \frac{a^2}{24}[4v^2-a^2)^{1/2} - a\,\mathrm{cotan}(t)]\},$$

$$(6.8)\qquad K_v^{(2)}(t) = K_v^{(1)}(\sin^{-1}(\frac{b}{2v})) + \frac{v^2}{2}(t-\sin^{-1}(\frac{b}{2v})) - \frac{a^2+ab+b^2}{24}(\frac{1}{b}(4v^2-b^2)^{1/2} - \mathrm{cotan}(t))\ ;$$

and

(6.9) $$K_V^{(3)}(t) = K_V^{(2)}(\frac{\pi}{2}) + (t-\frac{\pi}{2})(\frac{v^2}{2} - \frac{a^2+ab+b^2}{24}).$$

If $\underset{\sim}{P}_1, \ldots, \underset{\sim}{P}_n$ are points on C such that $s_L \leq s_1 < s_2 < \ldots < s_n \leq s_u$ then, the probability of simultaneous visibility is given by the formula

(6.10) $$p_n(s_1, \ldots, s_n) = \exp\{-\frac{\lambda}{2}(w^2-u^2)(\theta_u-\theta_L)\}$$
$$\cdot\exp\{\lambda[K^*(s_1-\theta_L) + K^*(\theta_u-s_n) + 2\sum_{i=1}^{n-1} K^*(\frac{s_{i+1}-s_i}{2})]\}.$$

Furthermore, the probability that C is completely visible is

(6.11) $$p_1 = \exp\{-\frac{\lambda}{2}(w^2-u^2)(\theta_u-\theta_L)\}.$$
$$\exp\{\lambda[K^*(s_L-\theta_L) + K^*(\theta_u-s_u)]$$

In Table 6.1 we provide numerical values of the probabilities $p_n(s_1,\ldots, s_n)$ and p_1 for the following case. The region S is located in an annular region, with $\theta_L = -\pi/2$, $\theta_u = \pi/2$, u - .5, w = .75. The distributions of Y is uniform on the interval (.1, .3). The intensity of the Poisson field is $\lambda = 1(2)9$. n target points are placed on a half-circle with radius r = 1, and orientation parameters, $s_i^{(n)} = (i - \frac{n+1}{2})\frac{\pi}{2}$, $i=1,\ldots,n$ and $n = 1, \ldots, 4$. In this special case, the functions $p_n(s_1, \ldots, s_n)$ is given by the formula

$$p_n(s_1, \ldots, s_n) = \exp\{-.15625\lambda\pi\}\exp\{\lambda 2nK^*(\frac{\pi}{2n})\} \quad (6.12)$$

λ	$p_1(s_1^{(1)})$	$p_2(s_1^{(2)},s_2^{(2)})$	$p_3(s_1^{(3)},\ldots,s_3^{(3)})$	$p_4(s_1^{(4)},\ldots,s_4^{(4)})$
1	.95096	.90432	.85998	.81780
3	.85998	.73956	.63600	.54695
5	.77770	.60481	.47036	.36580
7	.70329	.49462	.34786	.24465
9	.63600	.40450	.25726	.16362

Table 6.1. Probabilities of Simultaneous Visibility of n Points Uniformly Placed on a Half Circle; $u = .5$, $w = .75$, $r = .1$, $b = .3$.

For the computation of moments of V, we consider a target curve C, on the half-circle of radius $r = 1$ with $s_L = -\pi/3$ and $s_u = \pi/3$. In this case, $\ell^*(z) = \frac{3}{2\pi}$ for $-\frac{\pi}{3} \le z \le \pi/3$. $A_o(x,y) = A_1(x)A_2(y)$,

where $A_1(x) = \exp\{\lambda K^*(x + \frac{\pi}{2})\}$, and $A_2(y) = \exp\{\lambda K^*(\frac{\pi}{2} - y)\}$,

$-\frac{\pi}{3} \le x < y \le \pi/3$. Similarly, $A(x,y) = \exp\{2\lambda K^*(\frac{y-x}{2})\}$.

$-\frac{\pi}{3} \le x \le y \le \pi/3$. Since $A_o(x,y)$ is factored to the product of

$A_1(x)A_2(y)$, the recursive integration needed for the moments obtains the following form.

Let $\tilde{Q}_o(x) = A_1(x)$

and for $j \geq 1$, let

(6.13) $$Q_j(x) = \frac{3j}{2\pi}\int_{s_L}^{x} \tilde{Q}_{j-1}(y)A(y,x)dy \quad .$$

The moments of V are determined by the formula

(6.14) $$\mu_n = \exp\{-.15625\lambda\pi\} \frac{3n}{2\pi}\int_{s_L}^{s} \tilde{Q}_{n-1}(y)A(y,s_u)dy.$$

The probability that C is completely visible is given by

(6.15) $$p_1 = \exp\{-.15625\lambda\pi\}\exp\{2\lambda K^*(\pi/6)\}.$$

The integrations in (6.13) and (6.14) were performed numerically. In Table 6.2 we present the values of μ_n for n = 1(1)10 and the value of $\mu_\infty = p_1$ for the same parameters as in Table 6.1.

μ	1	2	3	4	5	6	7	8	9	10	∞
1	.951	.912	.880	.854	.832	.814	.799	.787	.776	.767	.686
3	.860	.758	.682	.623	.577	.541	.511	.487	.467	.451	.322
5	.778	.631	.529	.456	.402	.360	.328	.303	.282	.266	.152
7	.703	.526	.412	.335	.280	.241	.212	.189	.171	.157	.071
9	.636	.438	.321	.246	.196	.162	.137	.118	.104	.094	.033

Table 6.2 Moments of V for $\lambda=1(2)9$ and $s_L = -\pi/3$, $S_u = \pi/3$, $\theta_L - \pi/2$, $\theta_u = \pi/2$, $u = .5$, $w = .75$, $r = 1.0$, $a = .1$, $b = .3$

In Table 6.3 we present the moments of V and the corresponding mements of the mixed-beta approximation, for a slightly different case, in which C is an arc on a circle of radius $r = 1$, between $s_L = -\pi/18$ to $s_u = -s_L$.

Intensity	Order of Moments										
λ	1	2	3	4	5	6	7	8	9	10	∞
1.	0.951	0.934	0.926	0.921	0.918	0.917	0.917	0.919	0.921	0.925	.901
	0.951	0.934	0.926	0.921	0.917	0.915	0.913	0.912	0.910	0.910	
3.	0.860	0.816	0.794	0.781	0.773	0.768	0.765	0.763	0.763	0.765	.730
	0.860	0.816	0.794	0.780	0.772	0.765	0.761	0.757	0.754	0.752	
5.	0.778	0.713	0.681	0.662	0.650	0.643	0.638	0.634	0.633	0.633	.592
	0.778	0.713	0.681	0.662	0.649	0.640	0.634	0.629	0.625	0.621	
7.	0.703	0.623	0.584	0.562	0.548	0.538	0.532	0.527	0.525	0.524	.480
	0.703	0.623	0.584	0.562	0.547	0.536	0.528	0.522	0.518	0.514	
9.	0.636	0.545	0.502	0.477	0.461	0.451	0.444	0.439	0.435	0.434	.389
	0.636	0.545	0.502	0.477	0.460	0.449	0.440	0.434	0.429	0.425	

Table 6.3 Moments of V (upper) and the corresponding moments of the mixed-beta approximation (lower) for the case of $s_L = - /18$, $s_u = -s_L$, $u = .5$, $w = 75$, $r = 1.0$, $a = .1$, $b = .3$

Notice that μ_1 is the same in tables 6.2 and 6.3, for all λ values. The moments of the mixed-beta approximation are generally close to those of V. this shows an adequate approximation. The values of μ_n, $n \geq 1$ should be monotonically decreasing. The slight deviation for high order moments and small λ is due to numerical errors in the integration. In Table 6.4 we present the parameters of the mixed-beta approximating distribution corresponding to Table 6.3.

Intensity λ	σ	$\tilde{p}_o$	p_1	α	β
1	.1733	.0020	.9005	1.1161	1.0398
3	.2765	.0152	.7302	1.1405	1.0967
5	.3290	.0326	.5921	1.0956	1.1194
7	.3590	.0536	.4801	1.0450	1.1386
9	.3754	.0768	.3893	0.9937	1.0967

Table 6.4 Parameters of the Mixed-Beta Distributions Corresponding to Table 6.3. σ Denotes the Standard Deviation.

REFERENCES

1. Ailam, G. (1966). Moments of Coverage and Coverage Space, J. Appl. Prob. 3: 550-555.

2. Chernoff, H. and J. F. Daly (1957) The Distribution of Shadows, Jour. of Mathematics and Mechanics, 6: 567-584.

3. Eckler, A. R. (1969) A Survey of Coverage Problems Associated with Point and Area Targets, Technometrics, 11:561-589

4. Feller, W. (1966). An Introduction to Probability Theory and Its Applications, Vol. II, 2nd ed. John Wiley, NY

5. Greenberg, I. (1980). The Moments of Coverage of a Linear Set, J. Appl. Prob.,17:865-868.

6. Likhterov, Ya. M. and Gurin, L.S. (1966) The Probability of Interval Overlap by a System of Random Intervals, Engineering Cybernetics, 4:45-55 (in Russian).

7. Robbins, H. E. (1944) On the Measure of Random Set, Annals of Math. Statist., 15:70-74.

8. Robbins, H. E. (1945) On the Measure of Random Set, II. Annals of Math. Statist., 16:342-347.

9. Siegel, A. F. (1978) Random Space Filling and Moments of Coverage in Geometric Probability, J. Appl. Prob., 15:340-355.

10. Siegel, A. F. (1978) Random Arcs on the Circle, J. Appl. Prob. 15:774-789

11. Solomon, H. (1978) Geometric Probability, SIAM, Philadelphia.

12. Yadin, M. and S. Zacks (1982) Random Coverage of a Circle with Applications to a Shadowing Problem, J. App. Prob. 19:562-577.

13. Yadin, M. and S. Zacks (1984) The Distribution of Measures of Visibility on Line Segments in Three Dimensional Spaces Under Poisson Shadowing Processes, Submitted for Publication.

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