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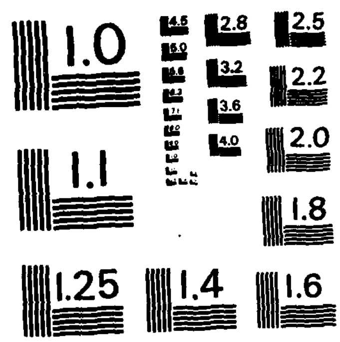

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THE FLORIDA STATE UNIVERSITY

COLLEGE OF EDUCATION

DEVELOPING AND MEASURING AN UNDERSTANDING OF THE CONCEPT OF THE LIMIT OF A SEQUENCE

by

TUIREN A. BRATINA

A Dissertation submitted to the
Department of Curriculum and Instruction
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Approved:

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DEVELOPING AND MEASURING AN UNDERSTANDING
OF THE CONCEPT OF THE LIMIT
OF A SEQUENCE

(Publication No.)

Tuiren A. Bratina, Ph.D.
The Florida State University, 1983

Major Professor: Herbert Wills, Ph.D.

This study concerned limits of sequences. Since limits are such an important mathematical concept for students to 'understand,' the major purposes of this study were to:

1. Develop a meaning of 'the understanding of the limit of a sequence' based upon students' behavior.
2. Construct an instrument for measuring the understanding described in 1.

An additional purpose was to:

3. Investigate subskills related to understanding the limit concept.

A good test for measuring the understanding in "1" would prove useful in helping teachers at various levels to answer the question, "Do my students understand limits?" as opposed to just finding limits.

Naturally, such an endeavor would require some thought on what indeed it means to understand limits.

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Prior to this study such a definition of understanding limits appeared to be lacking.

Thus, behavioral objectives were established by identifying the main features of limits and gaining a consensus from well-qualified professionals whose work involves an intimate knowledge of limits.

Test development involved constructing an initial version of the limits instrument, and then performing many revisions so that certain standards of measurement theory were satisfied. The final version of the instrument was administered to 263 subjects who had studied limits. The results for this 53 item test were reliability, alpha = 0.817; mean, 35.9 (67.7%); and standard deviation, 6.99 (13.2%). Validity checks were made on the instrument by comparing performance on this instrument and other related measures.

This study also involved identifying specific subskills related to understanding limits. This is noteworthy in that a variety of illustrious professors shared their views with regard to these subskills. Linear relationships were found between scores received on the limits instrument and scores on five subskills test.

Finally, specific information gleaned from the analyses performed in this study would directly benefit classroom teachers. Students did poorly on absolute value, distance, inequality, and segments or intervals. They do not have a good formal level of understanding limits, although they did fine at seemingly lower levels of understanding. Repeating decimals caused students confusion. Also some specific misconceptions of which teachers should be aware, surfaced during this study.

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Although the amount of space permitted here precludes naming all of the participants who were so actively involved in test administration, I would like to express my appreciation to the following individuals: Dr. Charles Aplin, Mr. Archie Hatcher, Mr. Dale Hiedeman, Mrs. Mary Lois King, Dr. Peter Knepell, Dr. Ralph McWilliams, Maj. Eugene Paulk, Mrs. Carolyn Riggs, Mr. Eugene Sadler, and Mrs. Sarah Wagner. These individuals greatly facilitated data collection procedures. Special recognition is also owed to Mr. Ross Butler, Mrs. Linda Fisher, Mrs. Marty Hopkins, and Mr. Fred Woodward for repeatedly furnishing their skills to critique many of the instruments used in this study.

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Special thanks go to my friend, Ernie Mac Donald, for his constant encouragement for the duration of my graduate studies. Finally, credit for providing the background which enabled me to attain this degree goes to my parents.

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CHAPTER I

INTRODUCTION

Background to the Study

This study investigates understanding of limits. The concept of limit is one of the fundamental ideas in mathematics. The limit concept is a broad one. Thus, to establish a focus for this study, the researcher concentrated on limits of sequences.

It should not be difficult to convince the reader of the importance of limits. The limit concept is the foundation upon which differential and integral calculus are built. The all important question of whether an infinite series converges or diverges rests with limits of sequences of partial sums. to p. 2

Since the topic of limits is such an important one, it is essential that students understand it. It is also desirable to be able to evaluate whether a student understands the concept of the limit of a sequence. In order to evaluate understanding, it is necessary to identify what is meant by "understanding of the limit concept."

There were considerable differences among educational researchers as to their meaning of "understanding the concept of the limit of a sequence." This was evident from the instruments they used to measure the understanding of the limit concept. These instruments varied greatly in the reviewed studies. Variation in the meaning of "understanding limits" was also noted in correspondence from nationally recognized experts in the mathematics and mathematics education disciplines. Several experts correctly indicated that decisions related to "understanding limits" depend upon what is meant by "understanding limits."

So, it is easy to say, "Limits are important. Students need to understand them." But it is a trying task to establish a meaning for "understanding the limit concept." Even after this feat has been accomplished, it is still necessary to construct an instrument to measure this understanding. The direction of this particular research will now be discussed.

The entire study was related to characterizing and measuring understanding of the concept of the limit of a sequence. The population of concern consisted of those persons who had been exposed to a unit covering this topic. The major purposes were:

1. To develop a meaning of 'the understanding of the limit of a sequence' based upon students' behavior; and

2. To measure the meaning of 'the understanding of the limit of a sequence.' This was to be done by → to P 3

constructing an instrument that is valid, reliable, objective, and suitable for measuring this understanding.

The related topic was:

3. To investigate specific subskills that might be related to understanding the concept of the limit of a sequence.

Prior to this study, there was not an operational definition for the understanding of the limit of a sequence. Also, no test was found which adequately measured the understanding of the limit of a sequence. This study was designed to fill these voids.

Review of the Literature

Understanding Concepts

The topic of concept learning is very expansive and a detailed discussion of it is not the intention of the research proposed here. However, a few preliminary remarks must be considered. Many definitions of "concept" have been published. Without being critical of other definitions, this researcher is most impressed with a definition proposed by Klausmeier, Ghatala, and Frayer (1972). They state that a concept is

> ordered information about the properties of one or more things--objects, events, or processes--that enables any particular thing or class of things to be differentiated from, and also related to other things or classes of things (p. 3).

Since this study intended to determine individual understanding of the limit of a sequence, it was pertinent to investigate the literature for information concerning measuring the understanding of a concept.

Summative evaluation is one means by which to evaluate this understanding. Begle and Wilson (1970) discuss the associated difficulty that is encountered:

> Ideally, mathematics educators should be able to take any task which students are to learn and to develop criteria by which teachers can determine when the student has learned it. There are not very many nontrivial examples of such criteria in the mathematics education literature. Usually, the criteria are what some norm group has done on a measure, or more typically, coverage of mathematical material in class without assessment of student proficiency (p. 370).

They suggest that a model of mathematics achievement facilitates the task of describing or specifying the pupil performance criteria which determine student proficiency. "The word 'model' is used in the sense of providing an organizational framework; it represents a categorization system with some stated rules and relationships for using the system" (Begle and Wilson, 1970, p. 372). They present several models that have been used to measure student proficiency in mathematics.

One key component of the models Begle and Wilson discuss is the use of categories of cognitive behavior. These are "levels of cognitive behavior assumed to be associated with the outcome or its measure" (1970, p. 372). Cognitive levels represent categories of expected behaviors. "The primary value of describing cognitive levels is to insure a more balanced coverage within a content area" (1970, p. 372). Without using this dimension in the model, Begle and Wilson express the opinion that mathematics educators, teachers, and test constructors overemphasize certain objectives and tend to ignore others (p. 372). The same authors also state the need for these levels to be defined in such a manner as to be comprehensible across the disciplines of mathematics, education, and psychology.

Bloom receives much of the credit for the use of cognitive levels in evaluating educational outcomes (Begle, 1979, p. 14; Begle and Wilson, 1970, p. 371; Romberg and Wilson, 1968, p. 490). Bloom's taxonomy contains six major classes: knowledge, comprehension, application, analysis, synthesis, and evaluation (Bloom, 1956). Definitions of these classes and their subclasses are contained in Bloom's _Taxonomy of Educational Objectives, Handbook 1: Cognitive Domain_. Modifications of Bloom's taxonomy have been adopted in some mathematics testing projects, such as the examples presented by Begle (1979) and Begle and Wilson (1970).

The mathematics achievement tests of the National Longitudinal Study of Mathematical Abilities, NLSMA, were constructed using a model with four levels of cognitive behavior: computation, comprehension, application, and analysis (Begle, 1979, p. 15; Begle and Wilson, 1970, p. 374). Five models, similar in their use of cognitive levels, are cited in Begle and Wilson. The influence of Bloom can be clearly observed by the widespread use of cognitive levels in these models.

This researcher reviewed many models which have been used to describe what it means to understand a concept and was not convinced that any one model already in existence applies to the meaning of the understanding of the concept of the limit of a sequence. The model which was used for this study incorporated the following levels of expected behaviors:

> Intuitive level: This level of understanding requires the student to demonstrate the ability to comprehend situations which involve or imply the contexts from which the concept of limit of a sequence will develop. Vocabulary usage is non-rigorous, corresponding to pre-formal experiences with limits.
>
> Identification level: This level of understanding requires the student to be able to classify sequences in terms of their convergence or divergence, using only elementary methods (inspection or elementary arithmetic or algebraic properties). In the case of

convergence, the student can specify the limit.

Production level: This level of understanding requires the student to be able to produce an example of a sequence which meets certain prescribed conditions, if such an example is possible; and to be able to state that no such example will satisfy the conditions if it is not possible. The prescribed conditions will be of a nature that can be easily understood by the student.

Comprehension level: This level of understanding requires that the student demonstrate knowledge of the general principles which characterize the convergence/divergence of a sequence.

Formal level: This level of understanding requires that the student be able to communicate a precise definition of the limit of a sequence and demonstrate a knowledge of the relationships among the component parts of the definition.

It should be noted that this model is not intended to be hierarchical. Although this author does not rule out the possibility of an ordering, further research would be required to determine if the levels form a hierarchy.

Testing the Understanding of Limits

Many tests have been given on limits. It appears that these tests involve a preponderance of computations of limits or proving that particular limits exist. Another observation by this investigator is that in practice the scoring of items which attempt to identify greater understanding of limits tends to be subjective. For example, a thorough computer search of the available literature about limits revealed no cases in which partial credit problems were accompanied by explicit directions for the judges to use to award the partial credit. A substantial effort was made in locating and evaluating existing tests that are concerned specifically with the limit concept. The limits tests found were carefully reviewed and evaluated with regard to several basic principles of measurement theory. Thus, prior to addressing the quality of these tests, various aspects of measurement theory used in assessing tests will be discussed.

This study concerns itself strictly with summative tests. Even though there are other models, the most frequently encountered one used in testing in various areas of mathematics and mathematics education is the summative model. Nunnally gives the major features of summative tests:

1) the relationship between the sum of item scores and the attribute being measured is approximately linear

and 2) the test result is a linear combination of items (1978, pp. 83-84).

An instrument which uses the simple sum of item scores is an example of a summative test. Most achievement and attitude tests in mathematics are summative. The topic of item weighting in scoring summative tests has been given considerable attention by experts in the field of measurement. Nunnally (1978) questions indiscriminant differential weighting and actually recommends the unweighted summation of item scores in nearly all cases (pp. 296-297).

"An appraisal instrument that measures what it claims to measure is valid" (Van Dalen, 1979, p. 135). Reliability concerns the stability of the measurements over a variety of conditions (Nunnally, 1978, p. 191). Reliabilities can be computed, and it would be helpful to note that Nunnally (1978) places 0.70 as the lowest estimate that should be a standard of reliability for research purposes (p. 245). Some people confuse the terms validity and reliability, so a quick aside will serve to point out the meaning of the two terms. Consider several situations which might occur during an official weigh-in for a heavyweight division wrestling match. Three separate recordings of each competitor's weight are made.

Case 1. One wrestler's weights are listed as 161, 178, and 189.

Case 2. The defending champion weighed in before his heavyweight match. The scale read 120, 119, and 121 for each of the three times he stood on the scale.

Case 3. The challenger for the heavyweight title steps up and weighs in. The weights recorded are 184, 184, and 185.

Applying the ideas of validity and reliability to cases 1, 2, and 3; the reader probably noticed that:

Case 1. The scale was not very reliable as was apparent from the wild fluctuations in the readings.

Case 2. Although the scale is very reliable in that approximately the same number is being recorded each time, something is wrong with the validity of the instrument. Weights approximating 120 pounds are not valid measures for a defending heavyweight wrestler.

Case 3. Not only are the readings reliable, but the reasonableness of a heavyweight contestant weighing approximately 184 pounds causes no concerns regarding a lack of validity for this instrument. Notice, there is no guarantee on validity here--simply support for the claim that the scale is a valid instrument. More situations would be used to investigate the validity to a greater extent. For example, comparing weights with another scale would be a good method of checking validity here.

Van Dalen's main concern for obtaining an objective test was that the same score be received regardless of who scores the test. Examiners should not have to make subjective judgments when scoring the tests. Campbell and Stanley (1963, p. 9) emphasize that objectivity is threatened when "autonomous changes in the measuring instrument" account for a difference in observations. An example of this would be changes in the mood of the scorer when giving out partial credits. If there are not strict guidelines for awarding the partial credit, the scorer would probably grade differently according to whether he or she was happy or sad; energetic or tired; etc. This researcher has had positive experiences being on grading teams with the purpose of scoring items which were awarded partial credit. Rigorous guidelines had been written telling how to grade the item, practice solutions were graded and compared to those of other scorers, and further grader training was given as the need indicated. So this researcher is not opposed to partial credit items, however, the subjectivity that occurs does present problems. When conducting research it is necessary to avoid any situations in which experimenter bias might influence the results. In this setting, that would indicate that experimenter grading is not good practice. In fact, Van Dalen (1979, pp. 296-297) specifically warns against this practice (p. 141).

Suitability was the last of the four terms mentioned by Van Dalen (1979). He says that an instrument must also be suitable for one's purposes, e.g., the proper degree of precision, reasonable cost, ease of administration, etc. (pp. 141-142). Most readers can identify with the ideas here because of so many experiences giving tests and grading them. It is important to realize that suitability may vary according to context. For example, it might not be a problem giving half-hour interviews to five or six people, but for thirty people this would become too time consuming.

Now, some limits tests are presented below. They shall be discussed with the ideas of validity, reliability, objectivity, and suitability in mind.

Shelton (1965) compared inductive and deductive methods of teaching limits, and Lackner (1969, 1972) and Stock (1971) performed replication studies using some of Shelton's materials. The criterion test on limits of functions was used in all three studies, as well as a prerequisite test. Some commendable steps were taken in the development of his test: He did report reliability data. He appears to have used a scheme to insure adequate coverage of the concept--content is the dimension by which he categorizes his items.

Shelton's instrument also had some problems: Split-half reliabilities, which Shelton reports, are not the most desirable reliability estimates. More importantly, his estimates did not reach an acceptable level. Recall that Nunnally used 0.70 as a "cutoff" for the reliability of instruments used for research (Nunnally, 1978, p. 245). The objectivity of the test is threatened because the tests were all experimenter graded. Since four items were scored by partial credit, this is not desirable, as was mentioned earlier in this section. Shelton's instrument was one of the instruments reviewed that used different weights for some items. Since Shelton did not provide any justification for doing this, it is not condoned by this researcher. As for validity, this researcher noticed considerable interference from such extraneous factors as algebraic skills and manipulations. Three specific items required testees to be able to

1) factor the difference of cubes,
2) eliminate radicals in the numerator

or 3) divide polynomials.

Thus, a student might very well miss an item for lack of mastery at an algebraic manipulation. How could we conclude that this student does not understand the concept of limits from several such items?

Coon (1972) investigated the intuitive concept of limit possessed by pre-calculus college students. She modified, and deleted some of, Taback's (1970) tasks which were designed to measure the child's intuitive understanding of the concept of limit. Taback's tasks examined:

1) the functional rule of correspondence,

2) neighborhood of a point,

3) convergence,

and 4) limit points (Taback, 1970).

Coon's instrument was comprised of six tasks which were administered in an interview setting. A subjective five-point Likert type rating, ranging from "clear evidence of not understanding" to "clear evidence of understanding" was employed (p. 32). A conversion of the ratings to a numerical score was devised by Coon (pp. 52-55). The material in the test is too lengthy to include in this report; however, the general situation of one task, Halfway Rabbit, is presented below.

> A rabbit hops halfway from one end A of a line segment AB toward the other end B, then hops halfway again from where he is toward point B. The rabbit continues to hop, following the same pattern of hopping halfway from wherever he is toward point B. The questioning relates to the convergence of the hops to the limit point B (p. 119).

No significant relationships were found between scores on this test and any of the following: mathematics scores on the ACT entrance test, achievement in calculus, or possession of a high school calculus background. Coon (1972) cites some limitations of the instrument as being the subjective rating system (p. 103), the suitability of the instrument with respect to administrative details (p. 103), and the confinement of the content to sequences which are monotonic (p. 102). Her suggestions for further research include the invention of a similar test which utilizes shortened tasks so that students could focus directly on convergence/divergence (p. 103), and that this test be in a written format to enable the administration of the instrument to more than one person at a time (p. 104).

Pavlick (1968) constructed a test "designed to measure student learning of the limit concept" (p. 26). He used a well-organized plan for test development, including using responses from test items of a preliminary version of the test for distractors of the same items presented in a multiple-choice format in the final version of the test (p. 33). The use of a pilot study to revise the test is also a commendable practice. Several reliability estimates were provided (p. 29)--all of which were respectable, i.e., ranging from 0.75 to 0.85 which easily meet the standards proposed by Nunnally (1978, p. 245). The test items corresponded to items which might be used to investigate the identification and formal levels of the model used in this study. The absence of items involving the other levels was a concern of this researcher. The grading of the proofs on the test was subjective. The scores for the proofs were wisely not combined with the scores from the objective section, since, as the author notes, no justification was provided for the differential weighting of the items for the two sections (p. 42).

Macey (1970) developed a test which was "designed to measure the student's understanding of the definition of the limit of a sequence and its application in proving limits of sequences" (p. 21). The majority of items would be associated with the identification and formal levels used in this study, but there were also some excellent items which Macey included "to assist in determining whether a student had developed a proper conception of the limit of a sequence" (p. 22). Two proofs are contained in the second part of the test. The subjectivity of the scoring on the proofs and the weighting of each item in part two of the test, ten times the weighting of each item in part one, are concerns of this researcher. Finally, Macey notes that "only sequences which had limits were considered" (p. 60), which was reflected in the fact that only one test item involved a divergent sequence.

In an effort to summarize this section, the reader will recall that validity, reliability, objectivity, and suitability of instruments were explained. Since no formal critiques of the instruments were available, this researcher presented her own critiques of those instruments by applying some of the standards for validity, reliability, objectivity, and suitability. There was no instrument reviewed by this investigator which satisfactorily met all of the criteria which are required to measure the understanding of the limit concept. That is, none of the instruments contained items covering each of the five cognitive levels; intuition, identification, production, comprehension, and formal; mentioned earlier in this report. However, critiques of the instruments presented in this section were useful for developing the instrument in this study.

Test Construction

Many steps are involved in constructing an evaluative instrument. The purpose of this section is to discuss some of those steps.

Although this section presents specific ideas involved in test construction that are frequently mentioned in the literature, the plan for test construction should also involve some practical steps. For example, a part of the initial process of test construction is administering rough drafts of the test to a few persons for the purpose of discovering possible shortcomings. Based on this, revisions are made. Then the improved test is administered to larger groups for such purposes as validity and reliability checks.

Van Dalen (1979, p. 151) mentions some of the tasks that are required for constructing an instrument. Identifying the population of concern, defining the precise property that is to be measured, analyzing factors that make up the property, constructing items corresponding to each of these factors, and developing a good format are some of the tasks he lists in the initial stages of test construction.

Van Dalen's approach is consistent with the methodology employed by Krathwohl and Payne (1970) for test construction. They insist that goals, in terms of expected student behaviors, should be formulated during the early stages of test development, and these goals should be used as guides to the total process of instrument construction (pp. 20-21). They suggest that different levels of specificity of these expected student behaviors, which they call "behavioral objectives," are important and useful. Statements of specific behaviors that describe performance capabilities for students successfully completing an instructional unit need not be as specific and detailed as objectives from which test items are to be created and/or chosen (p. 21). For example, a globally described objective might be, "the student can classify instances and noninstances of a concept for familiar examples." With respect to limits, an objective might be to have the student identify instances and noninstances related to sequences. In this regard, the student would identify those sequences which have limits, as well as those which do not. Specifically, the objective would include information related to observable student performance, prevalent conditions, and acceptable level of performance. Perhaps in more understandable terms a useful behavioral objective addresses the characteristics identified above as the student:

1) does what

2) with what

and 3) how well.

Once the preliminary test form has been administered to a sample of subjects, many more steps are required. The data are subjected to statistical techniques which identify weak items (Van Dalen, 1979, p. 151). Initial validity and reliability data are also obtained from this preliminary draft of the test. Other additional steps that might be required at this stage of development include revising directions that were not clear, standardizing scoring procedures where judgments of graders differed, and improving the test format. The revised test can then be administered to another sample.

Evaluation of instrument validity, reliability, suitability, and objectivity are essential aspects of test construction. The validation process depends upon the use of the test. For an instrument such as the one being developed in this study, content and construct validity are of prime importance (Cronbach, 1971, p. 463). Content validity deals with the adequacy with which a specified domain of content is sampled (Nunnally, 1978, p. 91). Construct validity is a property that is hypothesized to explain some aspect of human behavior (Van Dalen, 1979, p. 137)--in the case of this study "understanding the concept of the limit of a sequence."

To maximize the content validity of a test, a representative collection of items must be included and "sensible" methods of test construction must be used (Nunnally, 1978, p. 92). The use of qualified judges is a common practice for obtaining general approval and suggestions for revisions (APA, 1974, p. 45; Nunnally, 1978, p. 259; Cronbach, 1971, p. 446; and Van Dalen, 1979, p. 136). Cronbach explicitly states that one method of improving content validity is to translate educational objectives into item specifications (p. 458). Begle and Wilson (1970) encourage the use of a model in preparing measuring instruments (p. 403), and discuss improved content coverage when cognitive levels are a dimension of that model (p. 372). Perhaps it was best presented by Nunnally (1978), who said

> Inevitably content validity rests mainly on appeals to reason regarding the adequacy with which important content has been sampled and on the adequacy with which the content has been cast in the form of test items (p. 93).

Content validity is also supportive of construct validity in that it provides circumstantial evidence to construct validity (Nunnally, 1978, p. 110).

A question useful in determining construct validity is: "Does the test measure the attribute it is said to measure?" (Cronbach, 1971, p. 446). One way to obtain evidence of this is to compare scores on an instrument with measures of behavior in certain other situations (p. 446). If such a test is available, the useful procedure would be to compare the scores on the newly constructed instrument with the scores on the test that already exists. The basic idea is to use circumstantial evidence to validate the test (Nunnally, 1978, p. 109). Evidence of construct validity is based on an accumulation of research results rather than on a single study (APA, 1974, p. 30; Cronbach, 1971, p. 465; Nunnally, 1978, p. 99; Van Dalen, 1979, p. 138). Campbell (1960) discusses the matter more fully:

> No a priori defining criterion is available as a perfect measure or defining operation against which to check the fallible test. Instead, the validation seeks out some independent way of getting at "the same" trait. Thus he may obtain specially designed ratings for the purpose. This independent measure has no status as _the_ criterion for the trait, nor is it given higher status for validity than is

> the test. Both are regarded as fallible measures, often with known imperfections, such as halo effects for the ratings and response sets for the test (pp. 547-548).

So, although the imperfections of the different measures are admitted, comparisons can still be made. Regarding any such comparisons, Cronbach (1971) emphasizes that it is the strength of the result that is of importance in these studies, more than statistical significance (p. 465).

The reliability of an instrument must also be evaluated. It is worth noting here that there is a relationship between reliability and validity--"reliability being a necessary but not sufficient condition for any type of validity" (Nunnally, 1978, p. 237). So, if an instrument is not reliable, then it is not valid.

Recall that reliability is concerned with the stability of measurements over a variety of conditions (Nunnally, 1978, p. 191). "Estimates of reliability based on the average correlation among items within a test are said to concern the 'internal consistency'" (Nunnally, 1978, p. 229). Special formulas such as the coefficient alpha are available for this and should be used on every new instrument.

The measurement error which reduces reliability can be minimized by including specific steps in test construction. Writing items clearly, making directions specific and easily understood, and adhering to the prescribed conditions for test administration are a few such actions (Nunnally, 1978, p. 242). Also, using item analyses information is helpful for constructing a reliable test. One interesting correlation that is frequently used in an item analysis is the corrected item-total correlation. A corrected item-total correlation is a correlation of performance on an item with the total test score with that item removed. Low correlations here serve as a flag for identifying weak items. The use of corrected item-total correlations in test construction is one of the best ways of insuring that the resulting test is reliable (Nunnally, 1978, pp. 279-287).

Modern day computers are helpful in obtaining these correlations. The Statistical Package for the Social Sciences (SPSS) includes software that provides such information as coefficient alpha, inter-item correlations, corrected item-total correlations, and item frequencies. This package has been a real asset to test constructors.

One comment that warrants mentioning concerns the suitability of the test. Although the format, directions, length, etc., are important when considering suitability a more practical question arises. What good is a single test that will measure the understanding of the concept of the limit of a sequence? It could only be administered infrequently so that academic security would not be sacrificed. So, to this test developer, suitability requires a set of item specifications that would lead to an equivalent alternate form of the test. The specifications, contained in Appendix A, should prove useful in the construction of just such an alternate form.

To insure that the test was objective, only items which are dichotomously scored were used. That is, each item was judged to be either right or wrong--no partial credit. This contributes positively to higher test reliability by eliminating any possible fluctuations and/or disagreements in grading. It also results in greater ease of scoring. Furthermore, subject matter can be more widely sampled when an objective test is given (Nunnally, 1978, p. 260).

In summary, the careful construction of a useful evaluative instrument involves many steps. During the initial stages the attribute to be measured must be operationally defined. From this, specific test items eventually evolve. The rough draft of the test should be administered to small groups to detect weaknesses of the instrument. A revision addresses such weaknesses. Larger samples of people are used on the revised instrument and statistical techniques are employed to spot poor items. Validation and reliability data are also obtained from these larger groups.

CHAPTER II

METHODS

Procedures Used

The procedures used in this study were designed to fulfill the purposes that were stated at the beginning of this report (p. 2). That is, these methods were designed to:

1. establish a definition of "understanding limits of sequences" by identifying the main features of this concept and gaining a consensus from a group of well-qualified professionals.
2. actually construct an instrument which measured the understanding as developed in part 1.
3. identify certain subskills for understanding limits of sequences and investigate the relationships of these subskills with understanding limits.

Understanding of Limits

Some observations that came from reviewing the literature were that:

A. There were no attempts to couch a definition of understanding limits as is found for understanding some other concepts.

B. There were no attempts to specify observable behaviors that would provide evidence of such understanding.

C. There is a lack of reported criteria to which the limits tests were subjected in determining the validity of the limits tests.

Since no definition for "understanding limits of sequences" was found, and since no attempts at specifying observable behaviors were noted, this became one of the major purposes for this study. Because the use of observable behaviors is widely accepted, it was felt beneficial to couch the definition of "understanding limits of sequences" in terms of behavioral objectives. Contributions to this definition came from many sources. First, a model was developed and modified by combining the experiences of this researcher with the ideas presented in many models found in the literature. The personal experiences of this researcher in teaching limits of sequences had been enhanced by a case study of a high school student studying this topic. In documenting over ten hours of one-on-one study sessions, this researcher could more easily focus on those behaviors which were pertinent to learning limits. Also considered in formulating the objectives were ideas from various textbooks covering the topic of limits, observations of student performance in a class that was studying limits, and professional experiences of persons familiar with limits. The next step in accomplishing phase one of the study was to present the limits objectives to a panel of experts for an evaluation of comprehensiveness and communicability. Since literature concerning other limits tests did not provide evidence that detailed evaluations of the tests were being conducted, this study tried to avoid that. The materials evaluated by the panel of experts are found in Appendix B.

It has already been mentioned that the use of qualified judges is a common practice helpful in maximizing validity (p. 20). This researcher was fortunate to be able to benefit from the guidance provided by several experts in the mathematics and mathematics education disciplines. These experts were key figures for helping to develop a definition of understanding using behavioral objectives. Dr. Lawrence A. Couvillon, The Florida State University, provided invaluable assistance to this researcher in formulating the objectives which comprise the meaning of this understanding. The panel of experts then evaluated each objective to determine if it was appropriate for understanding limits of sequences. For example, one behavioral objective presented to the judges was:

> The student can determine the limit of certain convergent sequences by using only inspection.

Judges were presented with the statement, this objective is appropriate for understanding limits of sequences at the identification level, to which each judge responded to one of five choices corresponding to

1. Strongly disagree
2. Disagree
3. No opinion
4. Agree
5. Strongly agree.

Judges responding "strongly disagree" or "disagree" were asked to express their opinions why the objective was not appropriate for this level of understanding.

Accompanying this objective was one of the following item specifications:

(General Description)	The student can specify the limit of a bounded monotonic sequence.
(Stimulus attributes)	1. The stimulus will be: "In the blank provided, write the limit(s) of each sequence that appears below. If the particular sequence does not have a limit, write DNE. (a_n represents the value of the n^{th} term of a sequence, where n stands for a natural number). This statement will appear before a block of items (i.e., not repeated above each item).

2. The general expression for the n^{th} term of a sequence which converges to a non-zero number will be given.

3. The expression will be written as:

   $a_n =$.

(Response attribute) A blank line will follow the example.

(Sample item) In the blank provided, write the limit(s) of each sequence that appears below. If the particular sequence does not have a limit write DNE. (a_n represents the value of the n^{th} term of a sequence, where n stands for a natural number).

$a_n = 98 + \frac{3}{n^2}$ ______

Panel members were asked to react to three more statements:

1. The item specifications would be clearly understood by an item writer.
2. The item specifications are appropriate for this objective.
3. The sample item is consistent with the item specifications.

The panel used the same five ratings as previously mentioned to indicate their level of agreement with each of the three statements. In addition to the fixed responses to which the panel could respond, the panel members were encouraged to write any comments about any of the material in the package they received. (See Appendix B.) Members of the panel were Drs. Janice L. Flake, Dwight B. Goodner, Wolfgang H. Heil, Robert Kalin, and Dewitt L. Sumners, of The Florida State University and Dr. Paul J. Fairbanks of the United States Air Force Academy. Use of the term "panel" is reserved for these six experts throughout the remainder of this written report. The verbal communication with the individuals on the panel was also beneficial to the accomplishment of the first task.

<u>Constructing the Instrument</u>

Some problems that have been observed with instruments that were designed to measure understanding of limits were:

A. There was subjective grading on some of the tests.

B. One test was not suitable if it was to be given to many students because it took a long time to administer.

C. Some tests did not appear to be valid, and may have been measuring other achievement such as mastery of algebraic skills.

D. Some tests did not report good reliabilities.

In order to address the second aspect of this study, a specific plan of test construction was followed. For purposes of objectivity it was decided to use only those items which could be dichotomously scored. This provided for consistency in scoring, because it eliminated all subjective decisions. For purposes of suitability, it was decided that the instrument be a paper-and-pencil test which could be completed in one class period. Since most class periods are at least 50 minutes long, and since it usually takes about a minute to pass out and collect tests, the test duration was designated as 48 minutes.

Item specifications, including sample items, were written for each item.

It is a very involved process to consider many individuals' responses to items and their remarks concerning the instrument. However, efforts spent on this aspect of the study seemed to alleviate problems later. For example, initially many students were writing "∞" in the blanks that were provided when the directions were stated to write "NONE" in those blanks for the examples for which the limit did not exist. A change in the test directions was helpful in alleviating this problem. Another thing that was checked was whether each distractor of a multiple choice item was selected by someone who took the preliminary tests.

Thus, based upon preliminary administrations of early versions of the test, revisions were made to improve the directions and items included in the test. Also, remarks from the members of the panel contributed to revisions of the test.

Two types of validity, content and construct, were addressed in association with the development of this instrument. The content validity of the instrument was insured by having the panel members evaluate the objectives and specifications for the test. The experts' evaluations were also helpful in establishing the construct validity of the test.

However, since empirical evidence is the key by which to assess construct validity, the following research hypotheses were addressed

RH 2A: Teacher's rankings of high school students' understanding of limits of sequences will be positively correlated with the rankings obtained from the measurements of the instrument designed in this study.

RH2B: Student scores on part one of Macey's (1970) test will be positively correlated with student scores on the instrument constructed in this study.

RH 2C: Student scores obtained on Coon's (1972) instrument will be positively correlated with student scores on the intuitive subtest of the instrument constructed in this study.

The reader is reminded of the discussion by Campbell (1960) that acknowledges that although there is no perfect measure against which to compare the newly constructed instrument, specially designed ratings for the purpose should be obtained (pp. 547-548). In other words, this researcher admits that each of the measures just mentioned in the research hypotheses is fallible, however, each measure is aimed toward "getting at" the understanding of limits of sequences.

It would also be logical for students who have studied this topic in greater depth (e.g., students having taken real analysis, advanced calculus, and other advanced mathematics courses which include a rigorous study of limits of sequences) to do well on any valid instrument designed to measure the understanding of the concept of the limit of a sequence. With this in mind, the scores of such students on this limits of sequences test will be reported.

Prerequisite Skills

The computer search for literature yielded no information on subskills related to the understanding of limits of sequences. Thus, the need for discovering these skills surfaced.

The investigation of specific subskills which might be related to the understanding of limits began by identifying a variety of topics considered prerequisite for this understanding. The selection of these skills was made in several ways. First, the original panel of experts provided some unsolicited remarks which turned out to be very helpful in identifying prerequisite skills. Secondly, certain skills automatically seemed to be required when working with limits of sequences. A third source for determining prerequisite skills for this topic came from personal or postal correspondence with nationally recognized experts who have specifically worked with limits of sequences, either as teachers of the topic or as authors of books which include the study of limits. Most of these experts also offered specific suggestions which were useful in revising the directions and items that were designed to test understanding of these prerequisites.

It might interest the reader that the nationally known experts came from various parts of the country. This was intentional. The determination of prerequisite skills should not be dependent upon the philosophy or methodology that may be associated with a particular region.

The individuals listed in Table 1 provided their expertise in this research.

It was very fortunate to have such highly qualified professionals participating in this study. In fact, extensive comments were received by two of the most prolific writers on the topic of limits--Dr. Wilson M. Zaring and Dr. E. J. McShane. Professor Zaring has directed many National Science Foundation Institutes in mathematics and is currently the Director of Graduate Studies at the University of Illinois at Urbana-Champaign. Professor

Table I

Experts Participating in This Study

ROBERT G. BARTLE	Professor of Mathematical Sciences	University of Illinois
MARVIN L. BITTINGER	Professor of Mathematical Sciences	University of Purdue / Indiana University
LEXTON BUCHANAN	Head, Department of Mathematics	Fulton County Schools, Georgia
C. Y. CHAN	Professor of Mathematics and Computer Sciences	Florida State University
ARTHUR F. COXFORD	Professor of Mathematics Education	University of Michigan
DWIGHT B. GOODNER	Professor, Emeritus, of Mathematics and Computer Sciences	Florida State University
MATTHEW J. HASSETT	Associate Professor of Mathematics	Arizona State University
WOLFGANG H. HEIL	Associate Professor of Mathematics and Computer Sciences	Florida State University
E. J. McSHANE	Professor, Emeritus, of Mathematics	University of Virginia
ALBERT E. MEDER	Professor, Emeritus, of Mathematics	Rutgers University
DENNIS SENTILLES	Professor of Mathematics	University of Missouri
H. CARL WHITMAN	Associate Professor of Mathematics	Florida A & M University
ALICE G. WOODBY	Professor, Emeritus, of Mathematics	Ohio State University
WILSON M. ZARING	Professor of Mathematics	University of Illinois

E. J. McShane has served as head of the mathematics department at the University of Virginia and as president of the Mathematical Association of America. Professor McShane is well known for his work on limits.

Dr. A. E. Meder not only responded to a letter this researcher sent to him, but commented in great detail on more than one occasion. Dr. Meder was the executive director of the Commission on Mathematics of the College Entrance Examination Board for three years. Noteworthy is the fact that his directorship occurred during the extensive effort at reform of the school mathematics curriculum, immediately following "Sputnik."

It was also good fortune that several textbook authors participated in this study. Authors Robert G. Bartle, Marvin L. Bittinger, and Lexton Buchanan corresponded with this researcher about this study. Their books cover limits and have been used at Florida State University.

For the benefit of the reader, relevant correspondence that was received from these experts is included in Appendix C. For example, George Polya wrote a thoughtful and charming letter declining my request, for health reasons. Such letters were not included. It was not the intention of this researcher to quantify the responses received, but to react to the comments on a case-by-case basis. This was closely monitored by Dr. Herbert Wills of The Florida State University. As mentioned previously, the efforts of these reviewers was certainly beneficial to this research. In addition to the above, Drs. Goodner, Heil, and Whitman were interviewed by this researcher. They evaluated whether specific items designed to test understanding of the prerequisite skills did, in fact, test what was claimed to be tested.

Even though the correspondence was treated on a case-by-case basis, this researcher would like to share some of her impressions of the correspondence she received with the reader. Experts were overwhelmingly in support of including the topics of absolute value (as it is related to distance, inequalities, and segments or intervals), sequences, and algebraic generalizations as prerequisite for understanding limits of sequences. Experts were also clearly in favor of students understanding counterexamples before they begin to study limits of sequences.

The majority of responses considered conditional sentences, quantified statements, and denials to be important for the understanding of limits. Dr. Bartle may have expressed some of the concerns of the others when he wrote "nothing too fancy" for these topics. If one looks at the items used to test the understanding of these topics, it is evident that the understanding was tested at a very basic level. Many experts, in commenting directly about the items covering conditional sentences, quantified statements, and denials, indicated that they were good items. This would appear to be reasonable because the conditional sentences very closely resembled the sentences used in the widely accepted definition of limits. Dr. Hassett's comment about these topics deserves mention here. It was his view that these topics are necessary only for the formal definition and that many bright students can find limits, produce examples, and develop a good feel for limits without those skills. Since this study included the formal level of understanding of limits, Dr. Hassett's responses were viewed as "votes" for these topics being prerequisite skills.

Dr. Hassett was joined by Dr. Bittinger and Dr. Sentilles in wanting to specify which topics were important for which level of understanding of limits of sequences. Dr. Bittinger used the example that absolute value would be irrelevant to the understanding of limits of sequences on the intuitive level, but would be essential to understanding limits of sequences on the formal level. Actually, this researcher felt that the reactions of Dr. Bittinger, Dr. Hassett, and Dr. Sentilles were votes of confidence for establishing cognitive levels of understanding as had been done earlier in this study.

The issue of how much formal logic is necessary was addressed by Dr. McShane. His concern rested mainly on the symbolic notation employed for the algebraic generalizations test. He said that "except for the symbolic quantifiers," the items in the prerequisite skills tests "should be answerable by any one studying sequences." (From this researcher's experience, the use of such symbolism as "$\forall x$" can be quickly and easily taught to precollege students, and, in fact, should be taught.) Dr. McShane specifically mentioned "examples such as 'For each real number x, there exists some real number y such that $x + y = x$.' are easily intelligible."

Some isolated comments were also helpful in revising items testing the understanding of certain topics. For example, Dr. Heil suggested that some items on the absolute value/distance/inequalities/segments or intervals test could be correctly answered without really understanding the ideas involved. The items were revised accordingly. Dr. Sentilles and Dr. Woodby suggested using different directions on the test covering conditional sentences. The directions were improved using their suggestions. The domain was not specified for the algebraic generalizations items, as Dr. McShane noted in his letter. The set of real numbers was declared to be the domain in the revised test. There were also numerous instances where "good" was written by the reviewer in the margin beside a specific item or at the top of a page. All of these remarks were helpful for the final versions of the prerequisite tests.

The items used to test these prerequisites are found in Appendix D. The length and format were conducive to grouping the items into seven subtests. The topics were: 1) conditional sentences, 2) denials, 3) absolute value/distance/inqualities/segments or intervals, 4) sequences, 5) quantification, 6) general algebraic knowledge, and 7) counterexamples. At the encouragement of Dr. Goodner and Dr. Whitman, the last group of items was expanded to the set of items found in the appendix.

Furthermore, a practical consideration, namely time required to complete the tests, entered into combining topics into three individual instruments. This was done so that students could complete one instrument in one class period. The final three tests were those involving:

A. conditional sentences

B. denials and absolute value,
distance, inequalities,
segments or intervals

C. sequences, quantification,
general algebraic knowledge,
and counterexamples.

This arrangement was based upon the time required to finish each set of items. Several persons took the subtests for time, as well as for other helpful suggestions, and the arrangement of the subtests was intended for an administration time of less than fifty minutes--one class period. For administrations of items covering more than one topic, the order of subtests was rearranged for different students so that practice effects would not confound the results.

This researcher sought to address the third task of the study by finding correlations between the scores on the subtests with scores received by the same students on the limits of sequences test. To put this in a practical vein, it may turn out, for example, that students' "understanding of limits" scores do not have a high correlation with students' "counterexamples" scores. It would appear that students scoring high on "understanding limits" and low on a tested prerequisite skill would be evidence that that specific tested skill was, in fact, not a prerequisite. On the other hand, a high correlation between the "limits" scores and the scores for a particular suspected prerequisite skill would indicate that a linear relationship between the understanding of limits and the understanding of that tested skill does, in fact, exist. Granted, high correlations do not imply that causal relationships are present, but couple a high correlation with the opinions of persons with expertise in teaching limits of sequences, and a strong case can be made for declaring understanding of a topic as prerequisite for understanding limits of sequences. Chapter IV contains a more detailed discussion of performance on the subskills tests and the limits tests. Further research would be suggested to insure that a prerequisite relationship was truly the case.

Subjects

The subjects who were tested during various phases of the construction of the instrument measuring understanding of the limit concept were all persons who had been exposed to a unit covering the topic. Students from the following schools participated:

The Florida State University
The FSU Developmental Research School
Tallahassee's Godby High School
Tallahassee's Lincoln High School
Tallahassee Community College
The United States Air Force Academy.

The initial version of the instrument designed to measure the understanding of the limit concept was administered to 23 subjects. The educational backgrounds of these 23 people included high school, college, and graduate level schooling. Mathematics education, mathematics, meteorology, computer science, physics, and engineering were the majors of the college subjects. The responses of one of the public high school students were omitted since the student only attempted one item on the entire test.

A preliminary limits test, which included a revision of the initial items and directions that had been given to the earlier groups was then administered to 56 subjects. There were 53 high school students. Forty-six of these were attending public schools. Three college graduates with degrees in mathematics education, computer science, or engineering were among the 56 test takers.

The final version of the test measuring understanding of the limit concept was administered to 263 subjects. Those tested had a broad range of educational backgrounds. This was intentional. The study was concerned with understanding limits of sequences. It was not restricted to that understanding for a certain subgroup of persons who had studied limits, say Florida high school higher mathematics students. Thus, it was desirable to include persons with varied backgrounds. For statistical pursuits randomization of subjects is desirable. However, it was not feasible to attain a random sample of persons throughout the world who have studied limits of sequences. The subjects participating in this study certainly seemed representative of the population of concern. A discussion of this fact appears in the last chapter. High school students numbered 45; junior college and college students, 208; and those who had received college degrees or had studied beyond college totalled 10. The academic majors of the latter group were mathematics education, mathematics, and computer science.

Seventy-seven new subjects participated in the phase of the study that was geared toward investigating prerequisite skills. Data from two of these subjects were discounted because it was apparent that these two persons were unable to fully participate in this research as it was designed. These subjects were Florida and Georgia high school students. Subjects were first administered the limits of sequences instrument designed in this study. At a later date, each subject took at least one of the instruments measuring understanding of a suspected prerequisite skill. The numbers of pairs of scores obtained for the limits of sequences test and for each prerequisite test were as follows: conditional sentences, 59; denials, 58; absolute/distance/inequalities/segments or intervals, 55; sequences, 56; quantification, 56; general algebraic knowledge, 47; and counterexamples, 58. Although the subskills involving sequences, quantification, general algebraic knowledge, and counterexamples were administered at the same time, subjects receiving these tests do not all have recorded scores for these four subskills. One reason for this was that some students did not know what the symbol "$\forall$" meant (as reported to this researcher by one classroom teacher administering the tests). Consequently, they skipped the general algebraic knowledge instrument. The second reason for the omission of some of the four subskills scores was because these tests were left blank (or virtually so). Three persons appeared to run out of time on their last subskills test. One other student only completed the counterexamples instrument in this battery, and this researcher suspects that the time element problem may have arisen because this student devoted too much energy toward counterexamples (as evidenced by a fine 58 correctly answered items out of a possible 64, i.e., 91%). Similarly, there were a few subjects who appeared not to have time to complete both of the instruments measuring understanding of denials and absolute value/distance/ inequalities/segments or intervals. Three subjects appeared to experience time shortages.

Methods of Gathering Data

In the spring of 1981 each member of the panel of experts was given a listing of objectives and corresponding item specifications for understanding the concept of the limit of sequence. Appendix B contains a copy of the materials that were given to each panel member. As previously mentioned, the individuals provided their opinions concerning specific questions about the objectives and item specifications by checking one of the following responses: strongly agree, agree, no opinion, disagree, or strongly disagree. If an expert marked disagree or strongly disagree, that individual was also asked to write comments regarding this response. Comments were also encouraged regardless of the response checked, and such comments were freely given. In addition to the written responses gathered using the materials in Appendix B, panel members were also very helpful in providing assistance as the need arose. For example, the final version of the test contained directions which dictated that the testee write "NONE" in the blank provided for any example of a sequence which had no finite limit. These directions evolved from the early administrations in which the directions were to indicate for specific examples that the limit did not exist. Panel members were helpful in insuring that wording was correct.

The initial version of the limits test was administered to 23 subjects in the spring of 1982. The test was administered to small groups. The small group settings allowed for subjects' comments concerning test directions and duration as well as their responses to the test items.

The information gathered from the panel of experts and the results of small group testings was useful for making revisions. A preliminary version of the test instrument, reflecting these revisions, was now administered to 56 subjects. The reliability estimate, alpha, was approximately 0.87. The preliminary version contained

71 items. Frequency data and item-total correlations were helpful tools for revising this version of the instrument.

The final version of the instrument, along with specific research questions was administered to 263 subjects from May through August, 1982. The test appears in Appendix E, and the results of this administration are discussed in the next chapter.

Other information was collected for purposes of investigating tasks two and three of this study. To investigate the second area concerning the instrument measuring understanding of the limit of a sequence, three types of data were also collected:

a. teacher rankings of students' understanding of the limit of a sequence.

b. subjects' performance on an already existing test (Macey, 1970) covering limits of sequences.

c. subjects' performance on an already existing instrument (Coon, 1972) designed to measure intuitive understanding of the limit concept.

The teacher who provided the rankings was the teacher for a group of high school students who took this researcher's limits of sequences test. The teacher's own opinion of what it means to understand the limit of a sequence was used to rank the students. The data in b. and c. were obtained from subjects who had already taken the limits test designed in this study. The test in b. was a paper-and-pencil test, while the data in c. were collected by interviewing each subject individually. In order to minimize differences introduced by using different interviewers, this researcher conducted all of these interviews. The data in a., b., and c. were used to obtain measures of association with the limits test constructed in this study. The results are discussed in the next chapter.

To investigate part three of the study, involving prerequisite skills for understanding the limit concept, correlational data were used. Seven sets of data, containing a limits of sequences score paired with a particular subskills score, were used to find measures of association between each of the suspected prerequisites and limits. Seven confidence intervals were computed and scattergrams plotted to study the existence of any relationships of these subskills with understanding limits.

Description of Data-gathering Instruments Used

Appendix B contains the materials that were completed by the panel of experts. The opinions and comments voiced by the experts were used to perfect the objectives and specifications that had been established.

Other tools of data collection were: 1) the instrument constructed in this study, 2) teacher rankings, 3) part one of a limits of sequences test by Macey (1970), 4) a test measuring the intuitive understanding of limits by Coon (1972), and 5) a test of specific subskills that were investigated for this study.

The instrument designed in this study is the subject of this entire report. This instrument appears in Appendix E.

The rankings of student understanding of the limit of a sequence were provided by an instructor who had just completed teaching that topic. The students, whose understanding was being ranked, were members of an advanced high school mathematics class taught by that teacher. The basis for making the rankings was the teacher's own opinion of what it means to understand the limit of a sequence. Rankings were made prior to any instruments involved in this study being administered to those students.

Data were obtained from an already existing paper-and-pencil limits of sequences instrument constructed by Macey (1970). Fourteen objective questions (from part one of Macey's test) were administered to a subset of the population taking the instrument designed for this study. Items were dichotomously scored and total scores were obtained by summing the number of correct responses for each subject. Subjects spent no more than 15 minutes answering all of the items. Although the data were collected as a measure of association with the instrument designed in this study, the following information was obtained during the administration of Macey's fourteen items: reliability, alpha = 0.667; mean, 9.12 (65.1%);

and standard deviation, 2.58 (18.4%).

Interviews were conducted by this researcher using an instrument measuring the intuitive understanding of limits (Coon, 1972). The interviewers were persons who had already taken the test designed in this study. Six situations were presented to each subject and their responses to questions involving limits were scored using a five-point scale: clear evidence of understanding, some evidence of understanding, evidence lacking, uncertain evidence of understanding, and clear evidence of not understanding. A discussion and examples of the scoring of responses is provided by Coon (pp. 40-55). Numerical scores were assigned to each situation and those scores were summed to obtain a total score. The highest attainable score was 21.0. The interviews lasted approximately 30 minutes and were tape-recorded. The mean and standard deviation for the scores obtained by these subjects were 15.55 (74.0%) and 2.23 (10.6%) respectively.

Finally, a set of data was collected for the purpose of investigating the subskills that might have been required for understanding the limit of a sequence. Appendix D contains the tests that were used for this purpose. Although the focus of attention is not intended to be these prerequisite tests, a few remarks concerning each of these instruments would be helpful here.

There are 58 choices to be made for the conditional sentences instrument. The determination of whether a particular numeral, when placed in each occurrence of a "box," makes a resulting statement true, is the decision that the testee must make. The boxes appear in the antecedent and/or consequent for an assortment of "if, then" statements.

On the denials test, the examinee had to determine if a statement was a negation of a given statement. Fourteen original statements appeared, some of which were true and some of which were false. The student had to decide if three related statements were, in fact, denials of the original statement. The number of correct responses possible for the denials instrument was 42.

The instrument used as a measure of understanding of absolute value/distance/inequalities/segments or intervals involved 14 items for which statements had to be converted from one form to an equivalent statement in a different form. This was to be accomplished by the student placing the correct numeral(s) in the boxes and/or circles for each item.

The test on sequences was aimed at being able to determine a particular term of a given sequence. Four different sequences were used with seven items directed toward each of these sequences. (The reader should take note that item "b" was scored as four responses and item "c" counted as one response.) For the sequences test, 28 was the highest attainable score.

There were 32 true/false items on the test covering quantified statements. The intent of this instrument was to discover whether students recognized the truth or falsity of statements employing "for each" and "there exists" phraseology.

Students determined whether items concerning real numbers were true or false in the instrument designed for general algebriac knowledge. Since textbooks differ on whether zero is a natural number, two items (2e and 2f) were not scored. Hence, the total possible number of correct responses was 24.

The instrument designed to measure the ability to recognize counterexamples involved 16 different statements. Examinees were to determine whether each of four numerals was a counterexample. The counterexamples instrument, therefore, contained a total of 64 responses.

The means, standard deviations, and reliabilities of the administration of these prerequisite skills instruments are presented in Table 2. Since the total correct varies for each instrument the means and standard deviations are presented as percentages of the total score for that specific instrument. Reliability was determined using coefficient alpha.

Table 2

Data from Prerequisite Instruments

Instrument	Mean (percent)	Standard Deviation (percent)	Coefficient Alpha
Conditional Sentences	67	9	0.71
Denials	72	14	0.83
Absolute Value/ Distance/ Inequalities/ Segments or Intervals	58	29	0.87
Sequences	89	16	0.91
Quantification	67	14	0.76
General Algebraic Knowledge	76	13	0.67
Counterexamples	83	14	0.92

Notice how well the students performed on the sequences and counterexamples. It was good to see that 89 percent and 83 percent were the respective means for these two tests.

However, the mean for the instrument aimed at understanding absolute value/distance/inequalities/segments or intervals was very poor. The average for this instrument was only 58 percent. What could be the reason for such low scores? Is the mathematical language awkward compared to what is used in "everyday" vocabulary? The results on this test surprised this researcher, because she thought the items were quite simple. Although this study concentrates on limits, the topics of absolute value, distance inequalities, segments or intervals are such important ones in mathematics, that this area needs to be researched. High school higher level mathematics students should understand these concepts. What will it take to make certain they learn these?

The mean and standard deviation for the limits of sequences instrument for the 75 new subjects used to investigate subskills were 33.16 (62.6%) and 8.32 (15.7%) respectively.

CHAPTER III

RESULTS

The Limits of Sequences Instrument

The final version of the instrument measuring understanding of limits of sequences was administered to 263 subjects who had been exposed to this topic. The results obtained for this 53 item test were reliability, alpha = 0.817; mean, 35.9 (67.7%); and standard deviation, 6.99 (13.2%).

The 53 items are listed in Table 3. Other pertinent data are included in this table. The cognitive level under which each item falls was determined in the initial stages of this research to be intuitive, identification, production, comprehension, or formal. The ease index is the probability of correct responses for a given item. The corrected item-total correlation is a measure of association between the item score and the total score with that item removed.

A discussion of these items is held in the next chapter. Several items also appeared with the 53 items. These were not included in the item analysis for the instrument designed for this study, however, other information was obtained from these items, and will be

Table 3

Item Analysis

Item Number	Cognitive Level	Ease Index	Corrected Item-Total Correlations
1A	Identification	0.81	0.28
1B	Identification	0.80	0.31
1C	Identification	0.81	0.41
1D	Identification	0.61	0.40
1E	Identification	0.63	0.31
1F	Identification	0.64	0.35
1G	Identification	0.82	0.38
1H	Identification	0.34	0.36
1I	Identification	0.74	0.20
2A	Identification	0.38	0.36
2B	Identification	0.88	0.39
2D	Identification	0.76	0.37
2E	Identification	0.82	0.29
3A	Intuitive	0.67	0.33
3B	Intuitive	0.97	0.30
3C	Formal	0.47	-0.02
3D	Comprehension	0.76	0.35
3E	Comprehension	0.67	0.35
3F	Comprehension	0.54	0.29
3G	Comprehension	0.78	0.37
3H	Comprehension	0.51	0.17
3I	Comprehension	0.54	0.29
3J	Comprehension	0.75	0.28
4Aa	Intuitive	0.83	0.26
4Ab	Intuitive	0.59	0.37
4Ac	Intuitive	0.46	0.28
4Ad	Intuitive	0.55	0.15

Item Number	Cognitive Level	Ease Index	Corrected Item-Total Correlations
4Ba	Intuitive	0.79	0.32
4Bb	Intuitive	0.84	0.34
4Bc	Intuitive	0.82	0.24
4Bd	Intuitive	0.82	0.21
4Be	Intuitive	0.81	0.44
4C1	Intuitive	0.97	0.09
4C2	Intuitive	0.45	0.14
4C4	Intuitive	0.99	0.07
4C7	Intuitive	0.96	-0.02
4C19	Intuitive	0.88	0.00
5A	Comprehension	0.83	0.23
5B	Comprehension	0.83	0.27
5C	Comprehension	0.29	0.12
5D	Comprehension	0.74	0.20
5E	Formal	0.51	0.11
5F	Formal	0.70	0.35
5G	Formal	0.55	0.04
5I1	Formal	0.50	0.23
5I2	Formal	0.31	0.03
6A	Production	0.55	0.40
6B	Production	0.40	0.44
6C	Production	0.18	0.46
6D	Production	0.55	0.28
6E	Production	0.61	-0.02
6F	Production	0.51	0.06
6G	Production	0.89	0.33

discussed momentarily.

It should also be noted that Item 5H is not included in this instrument. As a result of students failing to correctly provide a definition of the limit of a sequence in earlier stages of this research, a battery of items were selected. They were directly aimed at understanding the definition of the limit of a sequence, but unfortunately an error occurred in 5H. Thus, it was omitted from any analyses.

A quick look at the results by cognitive levels will provide useful information. Table 4 includes the number of items designed to test that level, the percentage of those questions which were correctly answered, the correlation of the score for that level with the overall test score with those items removed, and the reliability (coefficient alpha) for that level.

Table 4

Analysis of the Cognitive Levels

Cognitive Level	Number of Items	Percentage Correct	Correlation with Total with that Level Removed	Reliability
Intuitive	16	95	0.52	0.60
Identification	13	73	0.47	0.77
Production	7	53	0.46	0.53
Comprehension	11	66	0.63	0.50
Formal	6	51	0.26	0.15

The data obtained to test RH2A, that there is no significant difference in teacher's rankings of high school students' understanding of limits of sequences and the rankings obtained from the measurements of the instrument designed in this study, yielded a Spearman Correlation Coefficient of 0.7759. The probability of obtaining a correlation greater than 0.4959 is 0.01 for 23 pairs of ranked data (as computed using Olds, 1938, Table V, p. 148). The scattergram that was generated by the data here was suggestive of the existence of a linear relationship, as can be seen in Figure 1.

The data gathered to evaluate RH2B, that scores on part one of Macey's (1970) test will be positively correlated with scores on the limits test designed in this study, yielded a Pearson Product Correlation Coefficient of 0.590. Forty-three pairs of scores were used to determine this coefficient. The 97.5% confidence interval computed from this was ($0.353 \leq \rho \leq 0.756$) (Minium, 1978, pp. 356-7). It was reasonable to accept the straight line as the curve of best fit from viewing a scattergram produced from the pairs of data collected here, Figure 2.

In order to investigate RH2C, that scores obtained on Coon's (1972) instrument would be positively correlated with student scores on the intuitive subtest of the instrument developed in this study, 30 pairs of scores were used to obtain a Pearson Product Correlation Coefficient

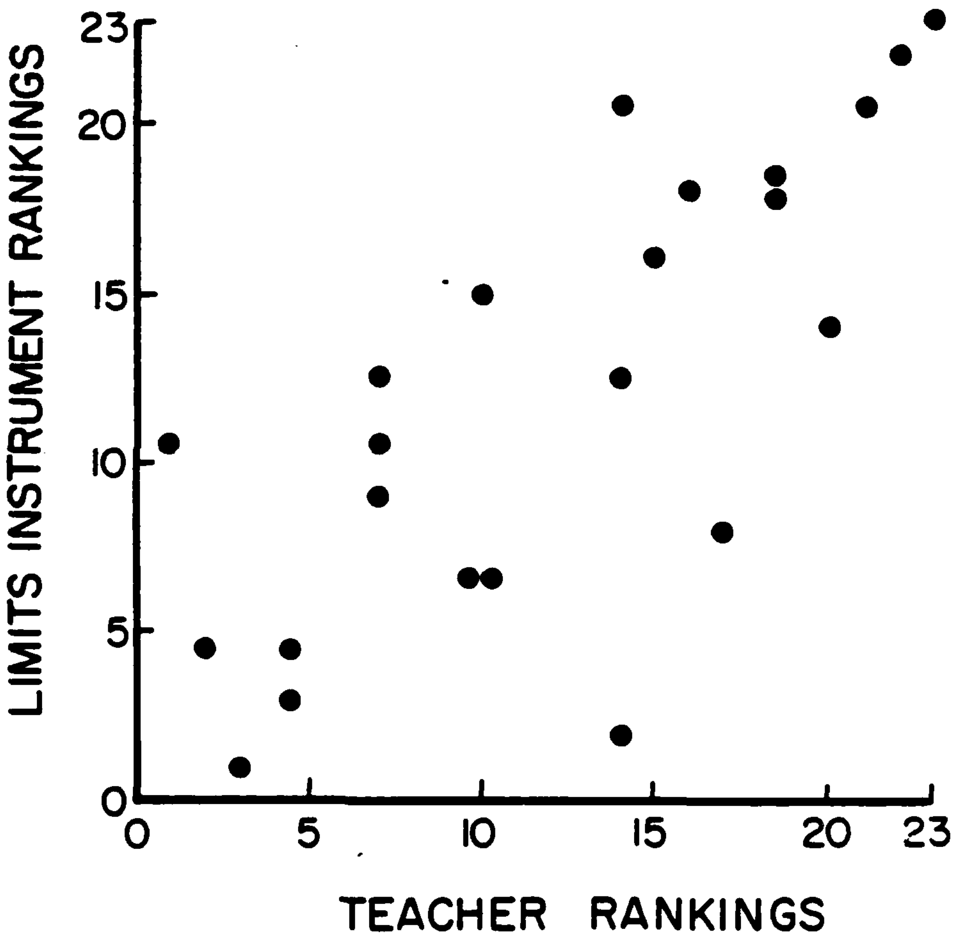

Figure 1. Scattergram of rankings by teacher vs. instrument

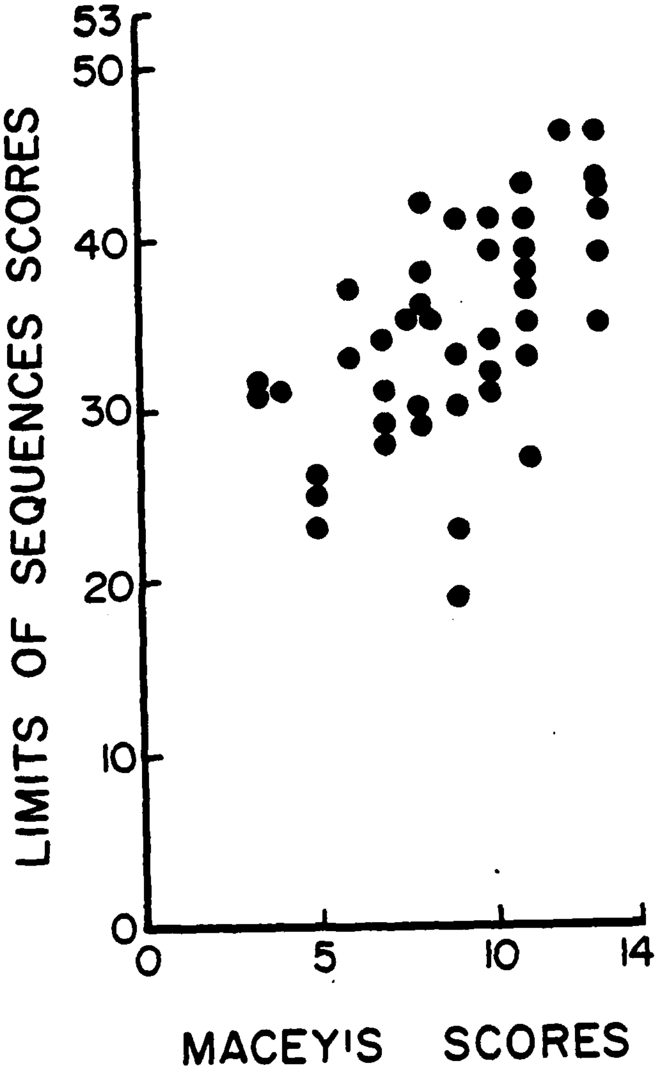

Figure 2. Scattergram of scores on Macey's test vs. scores on the limits of sequences test in this study

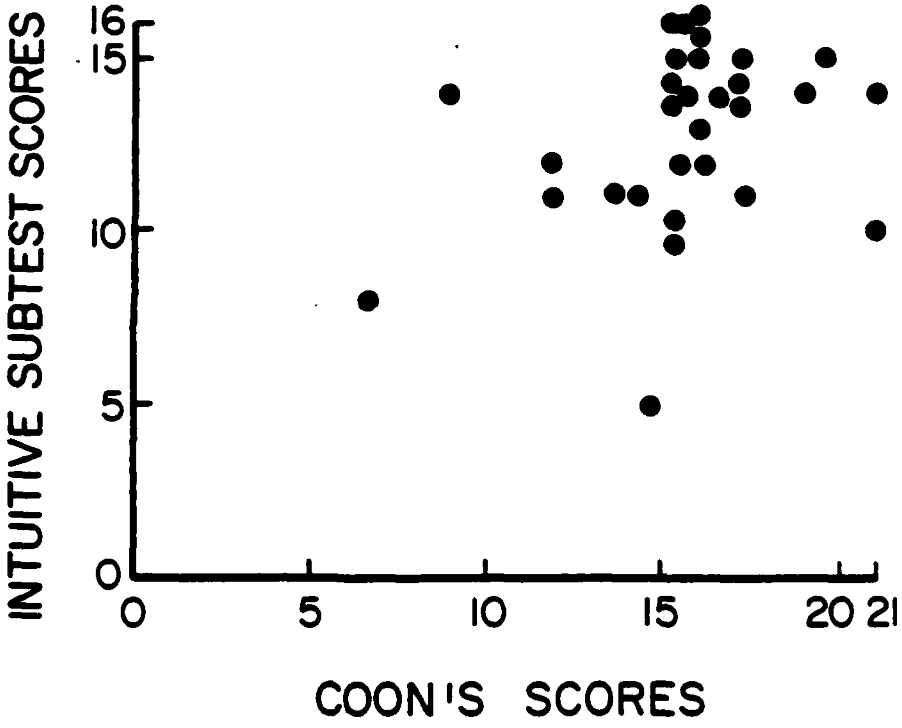

Figure 3. Scattergram of scores on Coon's test vs. scores on the intuitive subtest in this study

of 0.307. The 97.5% confidence interval associated with this has a lower limit of $\rho = -0.592$ and an upper limit of 0.601 (as computed from Minium, E. W., 1978, pp. 356-7). The presence of any particular relationship between the variables was not apparent from a scattergram produced using the pairs of data here, Figure 3.

Scores for students who had studied advanced math courses such as real analysis and advanced calculus are reported separately, as well as being included in the overall item analysis. The scores for the seven individuals who had studied limits in greater depth than most were: 41, 42, 42, 45, 51, 51, and 52.

Items Accompanying Limits Test

Twenty-seven items accompanied the 53 items that were specifically testing understanding of limits of sequences. These 27 items were useful for checking students' ideas about equality and notation. The item numbers and the associated probability of correct responses (ease index) appear in Table 5.

Here are a few interesting notes that concern these items.

1. Item 1J was intended to test whether students could find the limit of a sequence whose terms were constant. The intent of Item 2A was the same. The difference in the items was in the way the sequence was presented, i.e.,

Table 5

Results for Items Accompanying
Limits of Sequences Items

Item Number	Ease Index	Item Number	Ease Index
1J	0.90	4C17	0.89
2C	0.59	4C18	0.88
4C3	0.33	4C20	0.90
4C5	0.92	4C21	0.89
4C6	0.92	4C22	0.40
4C8	0.95	4C23	0.88
4C9	0.42	4C24	0.94
4C10	0.42	4C25	0.90
4C11	0.96	4C26	0.95
4C12	0.91	4C27	0.75
4C13	0.92	4C28	0.79
4C14	0.51	4C29	0.95
4C15	0.98	5J	0.75
4C16	0.90		

Item 1J: For each natural number n, each term of the sequence is 19.

Item 2A:

$-1\frac{1}{2}, -1\frac{1}{2}, -1\frac{1}{2}, \ldots$ in which each term of the sequence is $-1\frac{1}{2}$.

Students had little difficulty with Item 1J as is evidenced by the 0.90 ease index. Basically, the persons responded in the same manner in which they answered Item 2A. A variation in the wording of the two problems caused no problems here.

2. Items 2C, 4C3, 4C5, 4C8-18 and 4C20-29 were designed to investigate understanding of equality. Repeating decimals and equivalent expressions were used to test this. Almost 60 percent of the students were able to find the limit of the sequence in Item 2C, but when they were asked this question using a different format, i.e., Item 4C2, less than 45 percent could correctly answer. The same basic question, relating to 0.999... = 1, reappears in Items 4C9, 4C14, and 4C22 with success rates of 0.42, 0.51, and 0.40, respectively. The people were relatively consistent in their responses as indicated by obtained inter-item correlations among each pair of these four items being greater than 0.69. The same cannot be said of Item 2C. Obtained inter-item correlations with each of the four previously mentioned items were all less then 0.23. Just what is going on in students' minds here? From the frequency of responses on Item 4C3, it seems that most students, approximately 67 percent, think

that 0.999... is less than 1. Certainly, more research would be welcome in this area.

3. The other questions pertaining to decimals were answered quite satisfactorily with the exceptions of 4C10 and 4C7. Compare 4C10 and 4C7.

   4C7: $1/3 = 0.333...$

   4C10: $2(1/3) = 2(0.333...)$

   Both are true/false items. The ease index for 4C7 was 0.96 while the ease index for 4C10 was only 0.42. What could the students be thinking? Could this involve test-taking strategies? Would the order of these items have made a difference? Perhaps the number of incorrect responses to 4C10 is related to the fact that it comes immediately after 4C9 which also had an ease index of only 0.42. More research would be beneficial not only to the mathematics community, but also to testing experts. Before ending the observations concerning student responses to decimals, let it be mentioned that it seems disconcerting that less than 75 percent of the students knew that 0.667 was not equal to 2/3.

4. Item 5J resulted because of difficulties this test constructor experienced in trying to direct students to state that the limit of a sequence does not exist when the terms of the sequence

are not bounded. Approximately 21 percent of those questioned, stated that for the situation for which $\lim_{n \to \infty} f(n) = \infty$, the function f does have a limit. Some students from preliminary work had voiced the opinion that they had learned this from presentations in textbooks and from instructors. This researcher is not making a judgment concerning this matter, but merely making an observation that persons teaching limits should be aware of this when presenting this topic.

Prerequisite Skills

Finally, results concerning the third area of the study, i.e., possible relationships of certain subskills with the understanding of limits of sequences, are reported.

Correlation coefficients obtained from the seven sets of pairs of scores were used to compute confidence intervals for the population correlations. Since subjects were taking so many tests, adjustments were made to guard against inflated correlations caused by multiple testings. That is, using $\alpha = 0.01$ and performing the calculations from Minium, E. W., 1978, pp. 356-7; the confidence intervals for ρ listed in Table 6 were obtained. These interval estimates are measures of association of the particular subskill with the understanding of limits of sequences.

Table 6

Interval Estimates of ρ

	Correlation Coefficient	Lower Limit	Upper Limit
Conditional Sentences	0.39	0.07	0.64
Denials	0.51	0.22	0.72
Absolute Value/ Distance/ Inequalities/ Segments or Intervals	0.61	0.34	0.79
Sequences	0.34	0.00	0.61
Quantification	0.44	0.12	0.68
General Algebraic Knowledge	0.65	0.37	0.82
Counterexamples	0.20	-0.14	0.50

Scattergrams corresponding to the pairs of scores, i.e., particular subskill score and limits of sequences score, were also plotted (Figures 4 through 10 below). The outlier that appears on the scattergram in Figure 10, (7,38), is interesting. This researcher believes that the student who only answered 7 out of 64 correctly on the counterexamples instrument was, in fact, determining which numerals "worked" for each example!

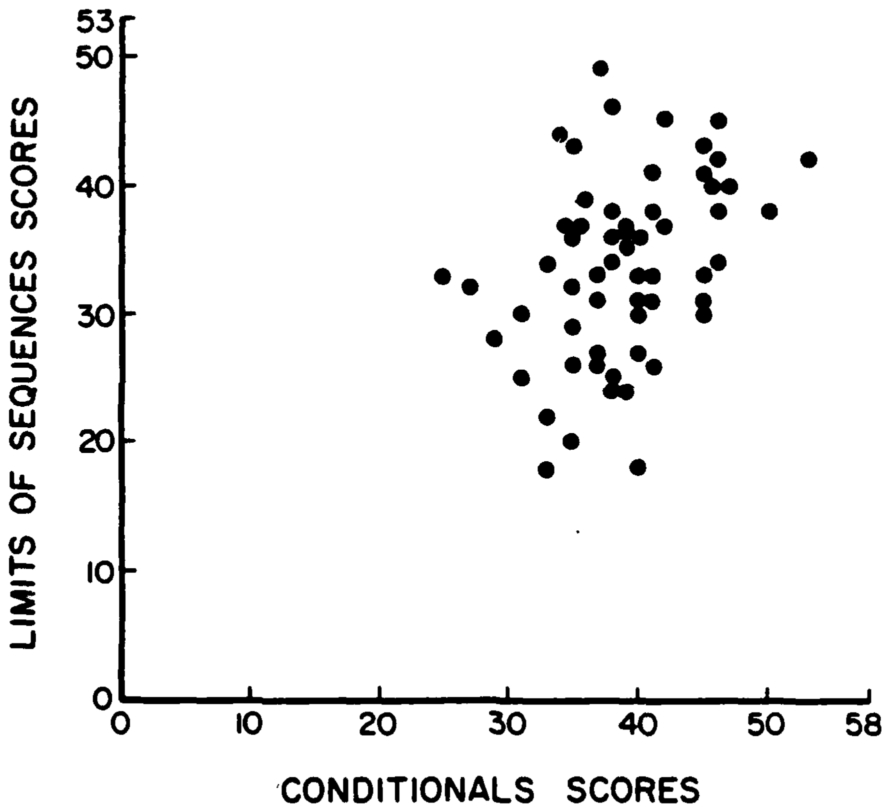

Figure 4. Scattergram of scores on conditionals vs. limits of sequences

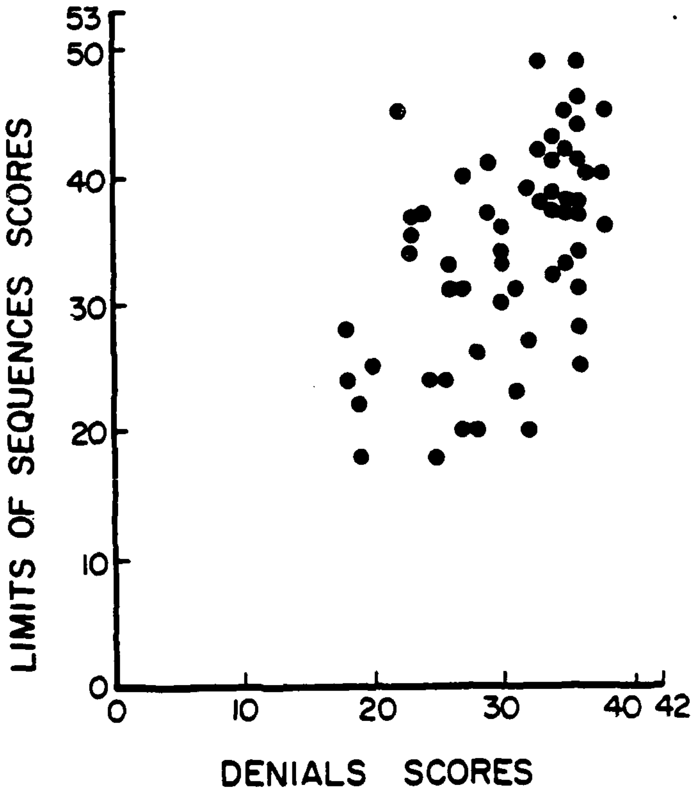

Figure 5. Scattergram of scores on denials vs. limits of sequences

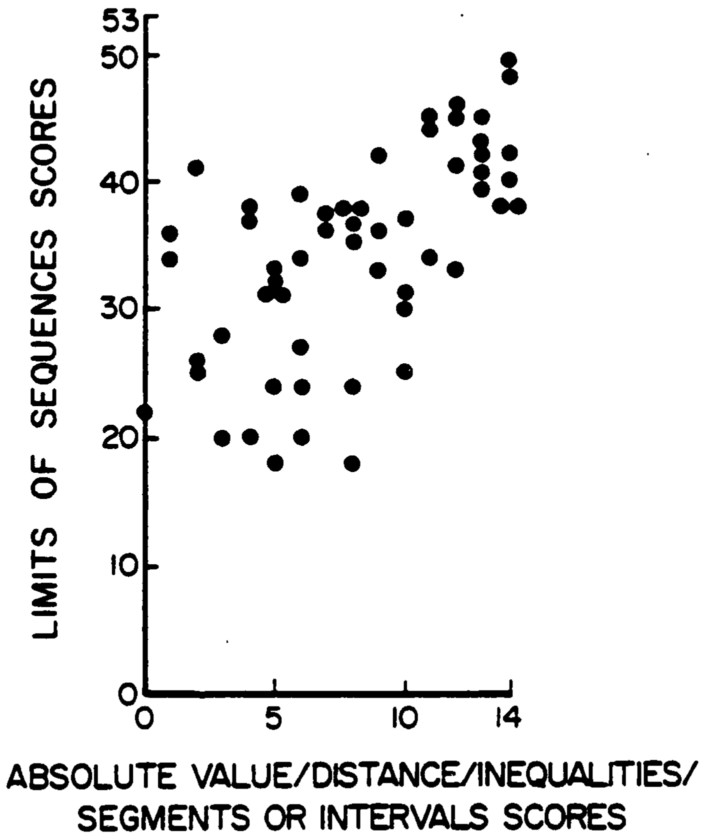

Figure 6. Scattergram of scores on absolute value/distance/inequalities/segments or intervals vs. limits of sequences

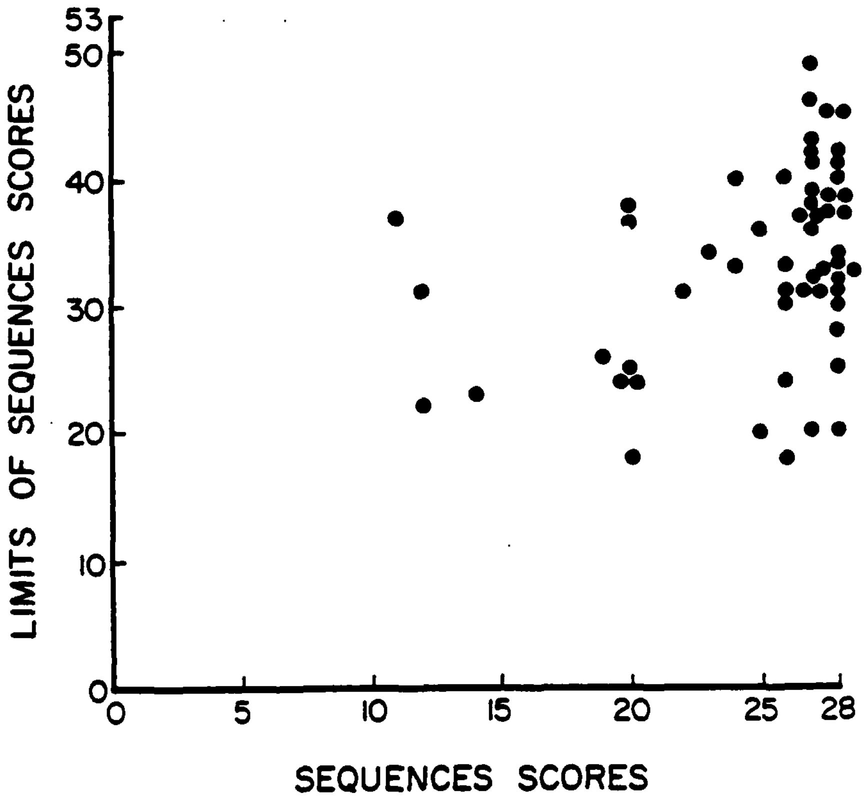

Figure 7. Scattergram of scores on sequences vs. limits of sequences

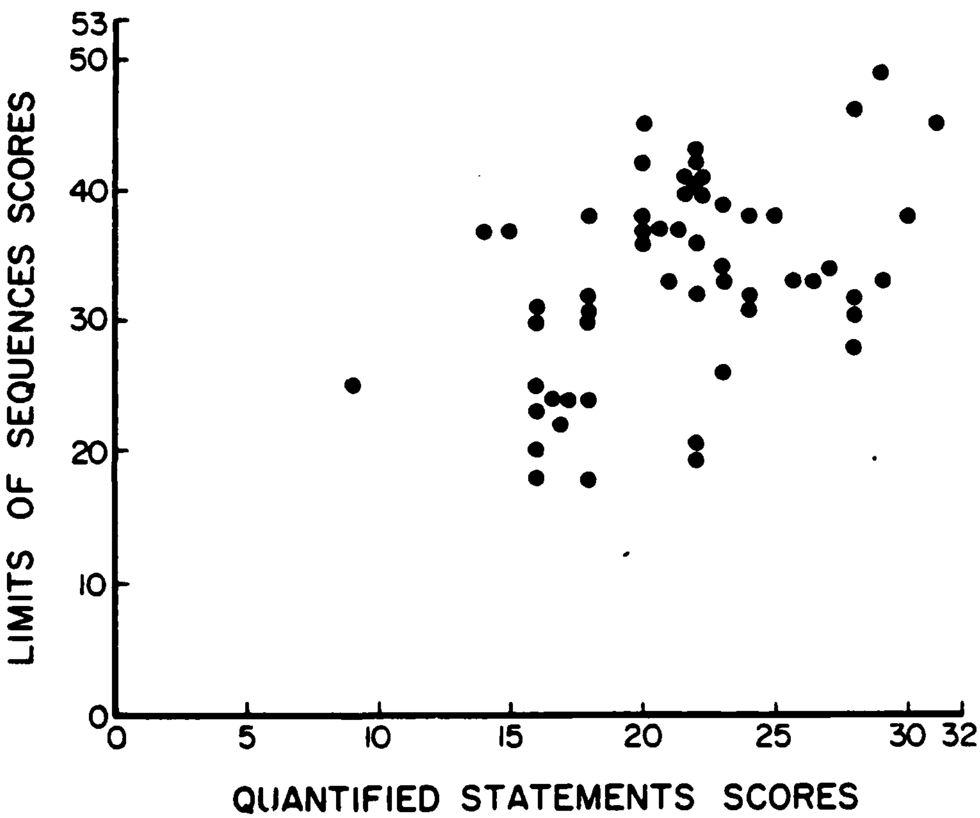

Figure 8. Scattergram of scores on quantified statements vs. limits of sequences

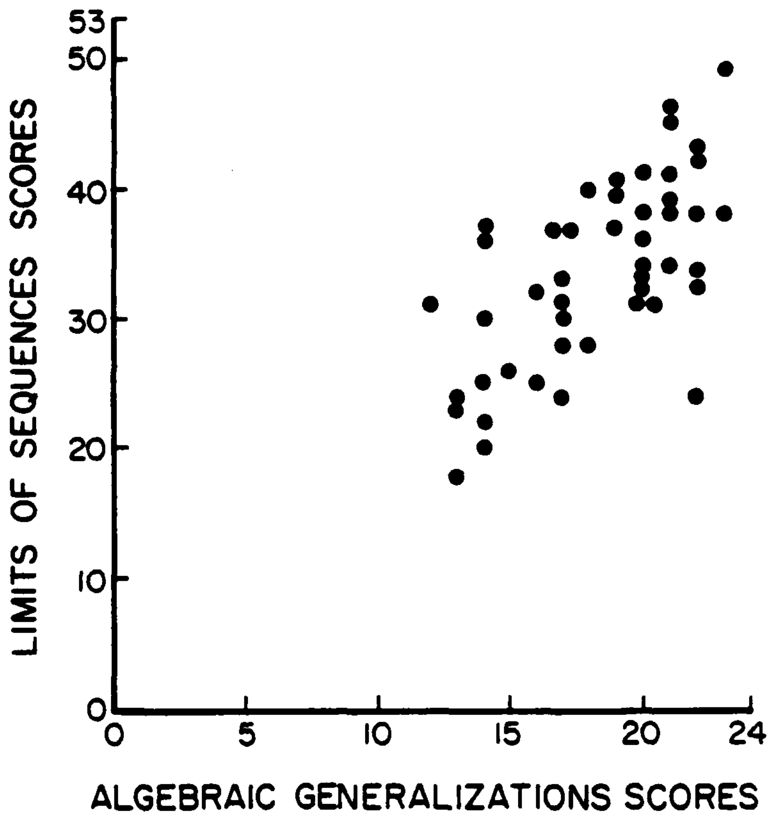

Figure 9. Scattergram of scores on algebraic generalizations vs. limits of sequences

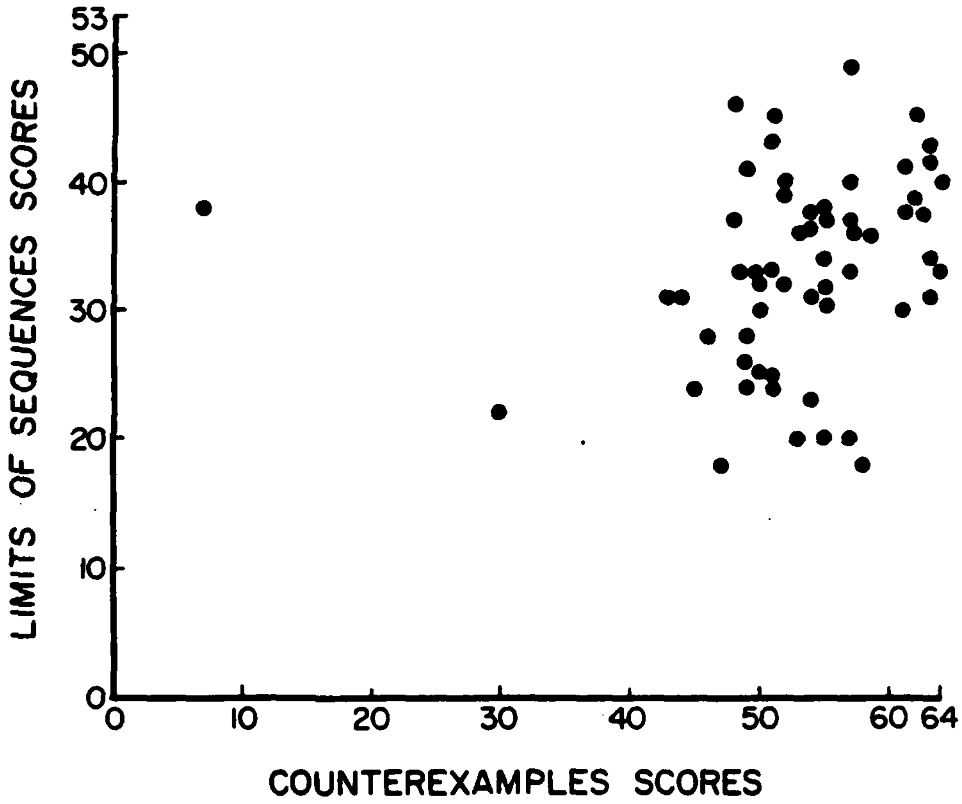

Figure 10. Scattergram of scores on counterexamples vs. limits of sequences

CHAPTER IV

DISCUSSION

Limits of Sequences Instrument

The methods used to develop the objectives for understanding the limit of a sequence and the instrument designed to measure this understanding were sound. Certain information that was gathered during the administration of the final version of this instrument is supportive of the instrument's validity and reliability being good. Other information gleaned from the data provides positive guidance for further perfecting this instrument.

The reliability, as computed by coefficient alpha = 0.817, is certainly acceptable and would undoubtedly be increased with a few changes to the instrument. The possibilities for changes will be discussed momentarily.

The validity is confirmed by the use of competent judges in determining content, and by administering preliminary versions of the instruments and revising the instrument according to the problems that arose. More concrete evidence of the validity is obvious when the instrument is associated with other methods of determining the understanding of the limit of a sequence.

When the results of the administration of this test to 23 high school students were paired with their teacher's independent rankings of these student's understanding of limits, the degree of association was very high. Their teacher's opinion was very consistent with the order established by the scores on the limits instrument, as evidenced by an obtained Spearman Correlation Coefficient of 0.7759.

Scores received on part one of Macey's already existing instrument designed to test understanding of this same topic also lend support to the claim that the instrument developed in this study is valid. The true correlation for 43 pairs of scores on these two limits tests was found to be between 0.353 and 0.756. The strength of the association between these two instruments is reflective of the intent of each instrument to measure the same thing.

The scores on this researcher's limits test that were reported for individuals who had studied more advanced mathematics courses lend credibility to the claim of this instrument's validity. Although it was not possible to obtain enough subjects to apply inferential statistics to the data, the scores are descriptive. Each of these persons scored considerably higher than the mean for this test. The scores ranged from approximately three-fourths of a standard deviation above the mean to more than two standard deviations above the mean. It seems quite logical that the performance of this select group would be high on any valid instrument measuring understanding of limits.

It is not so clear what the results of the correlation obtained from 30 scores obtained on Coon's instrument paired with scores obtained on the intuitive subtest of this researcher's instrument mean. The 97.5% confidence interval, $(-0.592 \leq \rho \leq 0.601)$, does not permit the conclusion that the two instruments are truly positively correlated--nor does it rule out that possibility. The use of more subjects would have increased the power of the statistical test corresponding to this interval, and the results would have been more conclusive. The fact that the strength of this association is not strong, may boil down to the intent underlying Coon's instrument compared to this researcher's intuitive subtest. Tasks in Coon's instrument "were not designed to measure knowledge gained in formal course work in mathematics but instead were intended to measure intuitive mathematical knowledge gained 'just by being alive in today's society'" (Coon, 1972, pp. 38-9). The intuitive cognitive level described by this researcher undoubtedly encompasses more complex situations. For example, one would probably not consider understanding of repeating decimals as "intuitive mathematical knowledge gained 'just by being alive in today's society.'"

The reliabilities reported for each of the subtests (see Table 3: Analysis of Cognitive Levels) give a strong indication of the area which requires the most work. The items aimed toward the formal cognitive level need to be improved and questions should be added. Rather than discuss each item separately, this researcher would attack the changes as a bigger project, i.e., what objectives and corresponding items are involved in understanding limits of sequences at the formal level. The problem here may also be complicated by a population that does not formally understand the concept of the limit of a sequence. This researcher recalls having no correct responses when asking students for the definition of the limit of a sequence in the preliminary stages of this research. More than 70 persons who had studied limits of sequences were unable to produce a correct definition. For that reason, items such as 5I1 and 5I2 were inserted into the test. More items like these would reflect whether the student really knows a definition--more so even than just writing a definition. Some students have been known to memorize definitions with no understanding of the meaning behind the definition whatsoever. So this subtest needs more work, the results of which may not even be applicable to the population of concern in this study. The population may be those persons who have studied limits of sequences in much greater depth than a normal first exposure course would provide.

The reliability of the identification level items was very good. These seemed to be the easiest items to construct, and also, students most frequently encounter these type of items when studying limits. Regardless of the instructional methodology used in a given course, students are repeatedly required to find the limit of a particular sequence, if that limit, in fact, exists.

The other subtests had reliabilities that were good. The group of items comprising each of these subtests would not require major changes. It may interest the reader to note that the easiest subtests, if measured by the percentage of correct responses, were those involving the intuitive and identification cognitive levels. On the other hand, students had trouble with the production level. Fifty-three percent correct was the average score for the production level items. This researcher proposes that this could be remedied by reversing the types of questions a teacher asks. For example, instead of asking

"What is the limit of the sequence f defined by:

For each n, $f(n) = 1/n$,"

the teacher should ask the students for a variety of examples of sequences with limit zero. Research using different instructional techniques should investigate this. Also, to be considered is the possibility that the proper notation required to answer these questions is interfering with the students obtaining the correct solution. Some evidence of that was noticed in their responses. The reader might recall that this researcher did not claim any particular hierarchy for these levels, although one might possibly be present.

Statistics from responses to individual items can be informative and sometimes suggest changes for a particular item. A discussion of some of the individual items is worthwhile here.

Item 3I provides us with some especially interesting information or conjectures. 3I was an item onto which this researcher accidentally stumbled during the one-on-one case study that was mentioned earlier. Note that only 54 percent of the examinees correctly responded that "b" was the correct choice for this problem:

Suppose that we have a sequence $\{a_n\}$ such that for each natural number n

$$a_n = \begin{cases} n^4 & \text{for } n \leq 1{,}000{,}000{,}000{,}000 \\ \dfrac{1}{n^4} & \text{for } n > 1{,}000{,}000{,}000{,}000. \end{cases}$$

$\underset{n\to\infty}{\text{Limit}}\ a_n$

a) is $1{,}000{,}000{,}000{,}000^4$.

b) is 0.

c) is not a real number because the terms of the sequence are not bounded.

One might suspect that notation was the primary problem causing students to do poorly on this item. However, after observing the case study student "plugging in large numbers for n" for similar problems, and rounding those results to get his answer, this same phenomenon was noticed in the classroom. The use of calculators may even be aiding this misconception. The idea of "what happens to the terms as n gets large" is being pursued by the students, but what does "as n gets large" mean? Research should be performed in this area. Similar examples, like $\lim_{n\to\infty} |987^{987} - n + 0.1| \div (987^{987} - n + 0.1)$, would strike at the same idea. Also, if a calculator is to be used while covering the topic of limits, how can it be used effectively? Would it be best to "punch in one big number" into the calculator, or should students try several cases for n and explain what is happening?

The items that were the most difficult, as measured by the percentage of persons correctly responding to them, were items 1H, 5I2, 5C, and 6C. Item 1H involves finding the limit of the sequence, $f(n) = \sqrt[n]{271.32}$. This researcher's experience has been that many people think this limit is zero. The difficulty in finding the limit of this sequence may not be in understanding the concept of limit, but actually in understanding the specific function being dealt with here. This researcher also wonders what the responses might have been had the function been presented as "$f(n) = (271.32)^{1/n}$." Item 5I2 was answered correctly by less than a third of the subjects taking this test. This researcher suspects a lack of a formal understanding of the concept of limit accounts for this. Improvement and expansion of the subtest covering the formal cognitive level would undoubtedly verify this.

Item 5C was only answered correctly by 29 percent of the subjects. A pedagogically popular approach to limits of sequences is one in which limits are described as the "number to which the terms get closer as one goes farther out in the sequence." So it should not be surprising that most of these subjects believe that "For each sequence S, if the limit of S is 22.7, then the 875th term is closer to 22.7 than the 874th term is." It is interesting to note that the responses to this item were not related to the responses to Item 1E, which required the student to find the limit of a sequence, $f(n) = \frac{2 + (-1)^n}{n}$. Most subjects correctly identified this limit. Since the inter-item correlation was 0.00 for these two items, subjects were apparently treating the two items as two totally distinct questions. Similarly there was no apparent relationship of either of these items with Item 5G, which embraces the same question of whether the terms of a sequence get "closer and closer to the limit" for each successive term of a convergent sequence. An interesting group of items that might be useful here would be the items presented in 4B, but in which the situation is different. Let the Point M_n "jump a fraction of the distance back toward the point X, say, every fifth jump." Would the students do as well on the true/false items related to this question? This would be good for additional research.

Less than 18 percent of the respondents could produce a sequence "for which the first five terms are negative, but the limit of that sequence is 7." It is not clear to this researcher whether the dearth of correct responses stems from ineptness with mathematics notation, or the lack of familiarity with the notion that a finite number of terms has no effect on the limit of a sequence.

The easiest items on the test were Items 3B, 4C1, 4C4, and 4C7. Each of these items was correctly answered by more than 96 percent of the subjects. Subjects responded that the greater the number of sides of a polygon circumscribed about a circle, the better the area of the polygon will approximate the area of that circle (Item 3B). A picture accompanied this item. Would as many people do as well on this type of question, if a picture were not present? It would require additional research to answer that question. That type of research is beyond the scope of this study, but it falls right in line with what one panel member had suggested about comparing many sample items for one objective. The panel member suggested that

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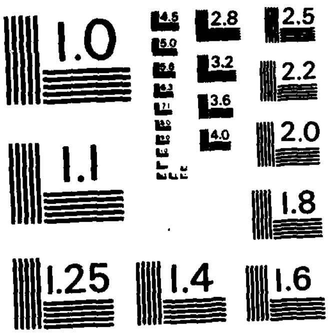

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

a researcher could construct many items for each objective and compare the performance of the same students on these items. This would be a good idea for further research. In Item 4C1, subjects correctly acknowledged that 0.999... $= 0.9 + 0.09 + 0.009 + 0.0009 + \cdots$. Students had no problems recognizing that $0.999\ldots > 1$ was false and that $1/3 = 0.333\ldots$ was true, as indicated by their responses to Items 4C4 and 4C7, respectively. However, as previously discussed, there appear to be several discrepancies in students' understanding of repeating decimals.

Since corrected item-total correlations can serve as a flag for items which need altering, Items 3C, 4C7, 4C19, and 6E deserve a closer look. The wording of Item 3C should be changed to improve this item. Items 4C7 and 4C19 have low correlations with the total score with those items removed. Since there are already three repeating decimals on the test, these two items can be deleted. Item 6E also was flagged. One subject vehemently objected to this item when told he answered it wrong. Consider the sequence, $f(n) = 3 + \frac{1}{n}$. Each successive term is closer to 2, but the limit of this sequence is clearly <u>not</u> 2. The student claimed it was a "trick question." This researcher has encountered independent discussions relating to a number which each successive term gets closer to, must be the limit of that sequence. However, this item was changed to one which asks for an example which fits this description. Appendix F contains an instrument which reflects the above mentioned changes to the final version used in this study. Item 5H was also corrected. These changes should result in an improved instrument that this researcher would suggest for use.

In summary, this researcher was happy to report that the instrument measuring understanding of limits of sequence was reliable and valid. This claim was supported by the acceptable reliability coefficient, 0.817, that was obtained. Also, research hypotheses 2A and 2B were supportive of the validity of the instrument designed in this study. That is, there was a high degree of association between both teacher rankings of students' performance and student scores on Macey's (1970) already existing written test with scores obtained on this researcher's instrument. The data collected for RH 3, related to the association between performance on an oral intuitive test and the intuitive subtest designed in this study, do not offer ammunition to the claim of the validity of this instrument, but certainly the data do not discredit that claim either. It was also good to observe that persons who had taken advanced mathematics scored high on the instrument measuring understanding of limits of sequences.

Prerequisite Skills

The question of whether, in fact, a particular suspected prerequisite skill is truly prerequisite for understanding limits of sequences is now addressed in greater detail.

Scattergrams, Figures 4 through 10, will be useful for discussing different combinations of performance on a subskill test and the limits of sequences test. Imagine each scattergram as being divided into four sections corresponding to:

1. low subskills score and low limits of sequences score
2. high subskills score and low limits of sequences score
3. low subskills score and high limits of sequences score

and 4. high subskills score and high limits of sequences score.

Table 7 should help the reader visualize the four possibilities for performance on the two tests.

The easiest case to explain is the situation in which students score low on a suspected prerequisite test and high on the limits of sequences test. This corresponds to the top left block in Table 7. This case, as Dr. A. E. Meder agrees, is very good evidence that the skill is <u>not</u> a prerequisite. When viewing the scattergrams, these points would be found in the upper left-hand corner of the graphs. Notice that there is no real trend like that occurring in Figures 4 through 10.

Table 7

Performance on Subskills and Limits of Sequences Tests

High Limits Score and Low Subskills Score	High Limits Score and High Subskills Score
Low Limits Score and Low Subskills Score	Low Limits Score and High Subskills Score

A person might be tempted to conclude that a high score on a particular "prerequisite" subskill coupled with a high score on understanding limits would validate that prerequisite. For example, a person who does well in long division must also do well with multiplication facts. However, upon closer inspection one recognizes the weaknesses of this reasoning. One can perform quite successfully at spelling familiar three-letter words as well as score high on long division. Are we to conclude that spelling familiar three-letter words is a prerequisite for long division?

What about the case of high subskills scores coupled with low limits scores? A more familiar example might be to discuss long division again. Students may score high on subtraction facts, but low on long division problems. Subtraction might well be a prerequisite skill, and a lack of understanding of multiplication could really be the culprit here. More investigative work would be required to find out if subtraction was truly a prerequisite skill.

Some of the students are scoring low on both of the mathematics tests they are taking. Low performance on an alleged prerequisite skill coupled with low performance on understanding limits does not disqualify that subskill as a prerequisite. These are the scores that are found at the bottom left-hand quarter of the scattergrams.

Thus, this researcher considered the four combinations of performance when viewing the scattergrams. The reader is reminded that confidence intervals for ρ were obtained, Table 6, and that nationally recognized experts offered their opinions about the suspected prerequisites. Again, even though linear relationships between variables do not always imply causation, this information coupled with the opinions of experts does provide stronger ground for making such a claim. With that in mind, the seven confidence intervals are considered.

Only two of the confidence intervals displayed in Table 6 contain $\rho = 0$. It can be stated with at least 90% confidence that five of the population correlations are positive. The interval estimate ρ for the measure of association between scores on the counterexamples test and scores on the limits of sequences test ($-0.14 \le \rho \le 0.50$), clearly included $\rho = 0$. These results do not provide sufficient evidence to conclude that a linear relationship exists. Furthermore, the scattergram, Figure 10, does nct illustrate an alternative pattern of how the two variables might be related. So although the understanding of counterexamples is basic in mathematics, this investigator concludes it is not essential for understanding limits of sequences.

The confidence interval determined from the correlation coefficient computed from the limits scores paired with the sequence scores ($0.00 \le \rho \le 0.61$) also included $\rho = 0$. This researcher is reluctant to deny, however, that any relationship exists between understanding sequences and understanding limits of sequences. A quick view of the scattergram (Figure 7) makes it apparent that the ceiling of 28 possible correct responses was achieved by several subjects. This investigator believes this low ceiling also shrouded any relationship that might exist between understanding sequences and understanding their limits.

Positive linear relationships were revealed for each of the other five cases. As previously mentioned, the ultimate statement of how understanding one of these sub-skills relates with understanding limits of sequences cannot be concluded from the scattergram and confidence interval, but the possibility of a causal relationship cannot be ruled out either.

Limitations

Because of practical considerations, certain adjustments were made in this study. The use of a random sample was not achievable for the population of concern (p. 2) here. Since it was not feasible to randomly select subjects from the entire group of persons who have studied limits of sequences, it was necessary to locate groups of those subjects who had studied limits at various levels and who were willing to participate in the study. While randomness is lost, some positive dimensions were realized by using the subjects who participated in this study.

1. The willingness of the participants does offer a positive dimension in that the element of seriousness in taking any of these instruments was desirable.
2. The backgrounds of the subjects were so varied that they would seem to be representative of the population of all persons who have studied limits of sequences.

3. Subjects were from different geographical areas, different educational levels, and had been exposed to different instructional presentations of the topic of limits. Since the intent of this study was not aimed at understanding limits according to a particular set of variables that might have been operating, the diversity of the subjects is considered to be an asset.

Suggestions for Further Research

Some of the results of this study suggest additional areas of investigation. A list of these suggestions follows.

1. Earlier it was mentioned that there was no attempt to place the levels of understanding limits of sequences according to their level of difficulty. Such information would be useful to classroom teachers by providing them with practical guidance on whether to immediately begin teaching a formal definition of limits, or to build from more informal experiences. While investigating this, determine what population can correctly master a formal definition of limit.

2. Since students performed so poorly at the production level of understanding limits of sequences, this is an area of concern for classroom teachers. The mean score for this part of the limits test was only 53 percent

correct. What should be done to improve students' level of understanding here? Investigate what instructional techniques will help students to produce examples of sequences which satisfy certain requirements relative to convergence or divergence. Include proper written communication here as well as oral.

3. One panel member suggested that it may be beneficial to construct more items and compare the performance of the same students on these items. In following this fine suggestion, a researcher could also investigate whether pictures help make an item easier, which may have been the reason so many people answered Item 3B correctly on this test. Clearly, this would necessitate additional items of each type which would automatically contribute to the suggestion.

4. Absolute value, distance, inequalities, segments or intervals are important topics in mathematics. It was very disconcerting to see such poor performance on the test aimed at understanding of these topics. This area needs attention. Thus, it is essential to investigate students' understanding of absolute value, distance, inequality, and segments or intervals more extensively. What methods are helpful for

teaching such important mathematical concepts? Is "everyday" vocabulary usage interfering with learning these concepts?

5. Investigate students' understanding of decimals. Particularly baffling to students is the repeating decimal 0.999... . Why are students inconsistent in their responses to items on decimals? Is it related to their understanding of decimals, or does it involve test-taking strategies? It may just be their lack of grasp of the ellipsis notation. It would prove interesting to know.
6. The responses to Item 31 were interesting. Research should be done to determine if students think that the limit of a sequence is the same as evaluating a sequence for a specific large number; say 1,000,000. Does the use of calculators or computers help students to understand limits? Indeed, can such devices hinder an understanding of limits? These answers are important in determining if calculators can be used effectively in teaching limits.
7. The items from the instrument developed in this study provided some very useful information to the classroom teacher. Teachers should be careful about their mathematical language in the classroom;

because if they say, "the limit is the number the terms get closer and closer to as you go farther out in the sequence," the students will apparently believe that is always true. The use of sequences which are not monotonic would be interesting here. Students in this study appear to think that each term of a sequence is closer to its limit, if the limit exists. More work should be done to learn how to correct this misconception in students' thinking.

8. Linear relationships were found to exist for five of seven subskills that were investigated with respect to understanding limits of sequences. That, however, does not guarantee causality. Suggested research along these lines would be to determine any causal relationships that might be present between understanding limits of sequences and some of the subskills investigated in this study.

9. This study focused directly on limits of sequences. No effort was made to investigate limits of real functions. A recommended activity would be to extend the understanding of limits of sequences to include limits of functions. Correspondingly, expand the instrument so that it measures not only the understanding of limits of sequences,

but the understanding of all limits.

Also, use procedures similar to the ones followed in this research to determine what it means to understand the continuity of functions, and to develop an instrument to measure that understanding. Since continuity is such an important topic in mathematics, an instrument measuring its understanding would be a giant feat. This researcher can already envision using the same five cognitive levels to approach this task.

Practical Implications

There are many benefits that can be derived from this study. Before this study there was not an operational definition for the meaning of "understanding" limits of sequences. Now classroom teachers have an operational definition for understanding this concept. Plus there is an instrument which will measure this understanding.

Not only is the instrument itself helpful, but some of the results in this study are suggestive of particular areas of awareness for classroom teachers. Students were especially weak on the productive level and formal level items. It will require a concerted effort to overcome the weaknesses here. Care must be taken to insure that students are not simply memorizing definitions. Notation is frequently a problem for students and there is no reason that the notation frequently used with limits would be an exception. Teachers should beware!

There are a few very specific details about limits to which teachers should pay strict attention.

1. Students think, "the limit is the number the terms get closer to each time you go farther out in the sequence." This misconception might be easily remedied if teachers use enough examples of convergent sequences which do not behave this way.

2. Students interpret "$\lim_{n\to\infty} f(n) = \infty$" differently. Thus, if teachers ask,

   "Does the '$\lim_{n\to\infty} n^2$' exist?"

   then students may respond,

   "Yes"

   while thinking

   "$\lim_{n\to\infty} n^2 = \infty$."

   It is important to know what the students are thinking when these problems appear.

3. Students may just believe that the limit of the sequence is the value of the term of the sequence for a very large number; say 1,000,000. There is reason to make that conjecture based on the performance on Item 3I of the limits test. The way the students use calculators may not help

this situation, and may even be hindering it. Teachers should expose the students to several sequences for which the terms are nowhere close to the limit for an unusually large number of initial terms.

4. Teachers should make a special effort to include coverage of decimals in their presentation of limits. 0.999... appears to be particularly confusing to students. This was apparent not only on the item analyses, but also during observations of student performance in the classroom. It may just be a misunderstanding of the notation, but the misunderstanding needs to be corrected.

There are a couple of comments that should be addressed to mathematics teachers even if they do not cover limits.

Students in this study performed poorly on absolute value, distance, inequalities, and segments or intervals. Teachers should have students take the necessary steps to master these important concepts. What those necessary steps are may be difficult to determine. This was an area that had been suggested for further research. This deficiency must be corrected.

Some students are not being exposed to standard mathematical notation. This was discovered when the meaning of the symbol "$\forall$" was not understood by many students in one high school higher mathematics class. This was not a problem on the limits of sequences instrument because the use of this and other mathematical symbols had been held to a minimum. This was to guard against any interference in determining the understanding of limits. However, since there were so many true/false items on the algebraic generalizations test, and since the wording got so long; it seemed much more efficient for the reader to read "$\forall$". Seventh and eighth graders have handled such symbols as "$\forall$" comfortably, so it would not be a difficult thing to teach. It would also not be time-consuming. Teachers should cover this in the classroom.

Finally, a practical implication for classroom teachers is that similar procedures to those used in this study can be used for all of the important mathematical topics that are taught. If a topic is important, then the student should understand it. But what does it mean to understand it? Once the meaning of "understanding of that topic" has been determined, can a good instrument be constructed that will measure that understanding? Granted that this is a huge task, but since teachers repeatedly cover the same important topics, it would be great to incorporate some of these procedures into the mathematics curriculum. It would also be beneficial to share meanings of "understanding a particular concept" and instruments measuring "understanding" with fellow teachers.

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APPENDIX A

ITEM SPECIFICATIONS FOR LIMITS OF SEQUENCES INSTRUMENT

INTUITIVE LEVEL

This level of understanding requires the student to demonstrate the ability to comprehend situations which involve or imply the contexts from which the concept of limit of a sequence will develop. Vocabulary usage is nonrigorous, corresponding to pre-formal experiences with limits.

Objective: The student can demonstrate a knowledge of convergence in situations involving "a limit object" of an implied sequence.

Item specifications:

(General description)	The student can demonstrate a knowledge of how successive approximations can be used in measurement.
(Stimulus attributes)	1. A drawing of four circles with equal radii will accompany the item. Four regular polygons (from left to right, a triangle, a square, a pentagon, and a sexigon) will be circumscribed about the circle.
	2. The stimulus will refer to the picture and describe the method of taking successively more circumscribed regular polygons and finding their areas.
	3. The stimulus will indicate that the correct statement concerning the circumscribed polygons is desired.

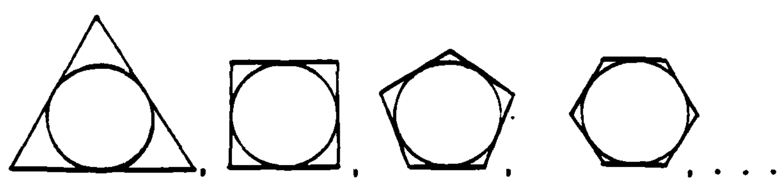
(Response attributes, The response attributes will indicate that:

a) the best approximation will result from averaging the areas of the polygons.

b) at some finite stage of this process, the area of the polygon and the area of the circle will be the same.

c) the more the number of sides for the polygon, the better the approximation will be.

(Sample item--3B)



A process of circumscribing regular polygons about a circle is shown in an attempt to approximate the area of that circle. The number of sides for each successive polygon increases by one. Which of the statements below is correct?

a) Averaging the areas of the polygons will provide the best approximation.

b) At some finite stage in the process, the area of the polygon will equal the area of the circle.

c) The greater the number of sides of the polygon, the better the approximation will be.

Item specifications:

(General description)	The student can make judgments related to the limit point and neighborhoods about the limit point of a "convergent" sequence.
(Stimulus attributes)	1. A situation will be described in which a sequence of points are identified on a line segment so that the points converge to a limit point. The limit point will not be a member of the sequence. 2. The stimulus will contain fewer than 70 words. 3. The item will be accompanied by a picture with appropriate points labelled. 4. Five specific statements about the situation will be preceded by directions telling the student to indicate whether each statement is true or false.

5. The statements will be that:

a) At some finite stage in the sequence, the sequence point and the limit point are the same.

b) For each successive point in the sequence, the distance between that point and the limit point decreases.

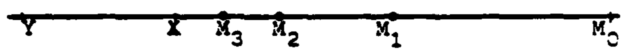
c) For each successive point in the sequence, the distance between that point and another point, which is not the limit point, decreases.

d) After a certain point in the sequence is reached, <u>all</u> of the points in that sequence will be inside a small circle placed around the limit point.

e) After a certain point in the sequence is reached, <u>some</u> point in that sequence will be inside a circle, no matter how small, which is placed around a point other than the limit point.

(Response attribute) The student will respond to each of the five statements by circling T for True or F for False.

(Sample items--4Ba, 4Bb, 4Bc, 4Bd, 4Be)



Point M_1 is the midpoint of the line segment with endpoints X and M_0. Shown above are the first three midpoints M_1, M_2, and M_3 of a sequence of midpoints which are obtained by the following continuing process:

For each natural number n, Point M_n is the midpoint of the segment with endpoints X and M_{n-1}.

Are the following statements, concerning this process, true or false? (Circle "T" for true or "F" for false for each statement.)

T F a) For some natural number n, Point M_n and Point X are the same.

T F b) For each successive natural number n, the distance between Point M_n and Point X is decreasing.

T F c) For each successive natural number n, the distance between Point M_n and Point Y is decreasing.

T F d) Regardless of how small a circle is placed around Point X, after a certain midpoint in the sequence is reached, all of the following midpoints in the sequence will be inside of that circle.

T F e) Regardless of how small a circle is placed around Point Y, there will be some midpoint in the sequence which will be inside of that circle.

<u>Item specifications</u>:

(General description)	The student can made correct judgments about repeating decimals.
(Stimulus attributes)	1. The student will be given five true/false statements, each involving a repeating decimal.
	2. The statements will be about thirds or ninths and will be statements of equality or inequality.
(Response attribute)	The student will respond to each of the five statements by circling T for True or F for False.

(Sample items--4C1, 4C2, 4C4, 4C7, 4C19)

T F 1) 0.999... = 0.9 + 0.09 + 0.009 + 0.0009 + ...

T F 2) 0.999... = 1

T F 4) 0.999... = 1

T F 7) 1/3 = 0.333...

T F 19) 2 9 = 0.222...

Objective: The student can make appropriate judgments about situations involving divergence.

Item specifications:

(General description) The student can make appropriate judgments about expressions which involve adding infinitely many terms and for which the sequence of partial sums diverges.

(Stimulus attributes)

1. Two expressions which involve adding infinitely many terms and for which the sequence of partial sums diverges will be given.

2. The two expressions will be:

   $a + (-a) + a + (-a) + a + (-a) + a + \ldots$

   and

   $a + (-a) + a + (-a) + a + (-a) + \ldots$

   for a particular value of a.

3. Four statements about these two expressions will be made. They will be that, for a particular value of a (listed above)

   a) the expression listed first has the value a.

   b) the expression listed second has the value 0

c) the expression listed first has no real value.

d) each of the expressions have the values 0 and a.

4. Directions will be given telling the students to indicate the validity of each of these statements.

(Response attribute) The student will indicate the validity of each statement by circling T for True or F for False for each of the four statements.

(Sample items--4Aa,4Ab,4Ac,4Ad)

Answer each "true/false" item that appears below.

M. 1 + (-1) + 1 + (-1) + 1 + (-1) + 1 + . . .

N. 1 + (-1) + 1 + (-1) + 1 + (-1) + . . .

Circle "T" for true or "F" for false for each statement concerning the two expressions above.

T F a) The expression in M has the value 1.

T F b) The expression in N has the value 0.

T F c) The expression in M has no real value.

T F d) The expression in M and N each have the values 0 and 1.

Item specifications:

(General description) Given a process which produces a sequence of polygons of unbounded area, the student recognizes that the area of the figures grows arbitrarily large.

(Stimulus attributes)

1. The stimulus will contain:
   a) first a general description of the situation (sequential process).
   b) a drawing depicting the situation.
   c) a question asking which of several responses is true.

2. The stimulus will use 75 words or less to describe the construction of the sequence of polygons.

(Response attributes)

1. The responses will be no more than 25 words long.

2. There will be three responses as follows:
   a) An incorrect statement

that the perimeter of the fifth polygon is the average of the perimeters of the fourth and sixth polygons.

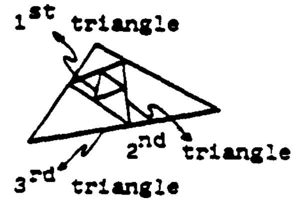
b) A statement that the area of any of the polygons can (cannot) exceed the area of a circle with a specific large radius.

c) Based on the choice in b, a statement that the length of any one of the sides of the polygons in the sequence cannot (can) exceed a specific linear measure.

3. Directions will be given to circle the one correct answer.

(Sample item--3A)

A sequence of triangles is constructed. The sides of each new triangle contain the vertices of the preceding triangle. The first three triangles in this sequence are pictured at the right. Which of the following statements about this sequence is true?



a) The perimeter of the fifth triangle is the average of the perimeters of the fourth and sixth triangles.

b) The process suggested could produce a triangle which has an area greater than that of a circle with a radius of 1,000,000 miles.

c) The process suggested cannot produce a triangle which has a side longer than 1,000,000 miles.

IDENTIFICATION LEVEL

This level of understanding requires the student to be able to classify sequences in terms of their convergence or divergence, using only elementary methods (inspection or elementary arithmetic or algebraic properties). In the case of convergence, the student can specify the limit.

Unless otherwise indicated, all items in this section will appear using the following format:

(Stimulus attributes)	1.	The stimulus will be: "In the blank provided, write the limit of each sequence that appears below. If the particular <u>sequence does not have a finite limit, write</u> "<u>NONE</u>." At the beginning of a new line, "For each natural number n, each term of the sequence is:," will appear.
	2.	These statements will appear before a block of items (i.e., <u>not</u> repeated above each item).
(Response attribute)	1.	A blank line will follow the example.

Objective 1: The student can determine the limit of certain convergent sequences by only using inspection.

<u>Item specifications</u>:

(General description)	The student can specify the limit of a bounded monotonic sequence.
(Stimulus attribute)	The sequence will converge to a non-zero number.
(Sample item--1A)	$98 + \frac{3}{n^2}$ ________________

<u>Item specifications</u>:

(General description) The student can specify the limit of a convergent sequence whose limit is nonmonotonically approached by consecutive values of the terms of the sequence.

(Sample item--1E) $\frac{2 + (-1)^n}{n}$ ____________

Item specifications:

(General description)	The student can specify that the limit of a constant function is that constant.
(Stimulus attributes)	1. This item will appear with a group of items. This group will be preceded by the directions, "Write the limit of each sequence described below. If the sequence has no finite limit, write 'NONE.'"
	2. The constant will be a negative number.
	3. The sequence will be described by presenting the first three terms of the sequence, separated by commas; followed by an ellipsis; followed by an explanation of how each term is determined.
(Response attribute)	The student's response will be indicated below the description of the sequence in the space marked "Ans. __________."

(Sample item--2A)

$-1\frac{1}{2}, -1\frac{1}{2}, -1\frac{1}{2}, \ldots$ in which each term of the sequence is $-1\frac{1}{2}$.

Ans. __________

Objective 2: The student can classify examples of divergent sequences as sequences for which a finite limit does not exist.

Item specifications:

(General description) The student can classify an example of an unbounded sequence as one which does not have a finite limit.

(Stimulus attribute) 1. A sequence which diverges to $-\infty$ will be given.

(Sample item--1B) $9 - n$ ______________

Item specifications:

(General description)	The student can classify an example of a sequence which has more than one cluster point as one which does not have a finite limit.
(Stimulus attribute)	1. An example of a sequence with two cluster points will be used.
(Sample item--1C)	$(-1)^n$ ____________

Item specifications:

(General description)	The student can classify an example of a sequence whose consecutive terms are increasing by a very small amount, as one for which there is no finite limit.
(Stimulus attributes)	1. This item will appear with a group of items. This group will be preceded by the directions, "Write the limit of each sequence described below. If the sequence has no finite limit, write 'NONE.'"
	2. The sequence will be described by presenting the first seven terms of the sequence, separated by commas; followed by an ellipsis; followed by the general expression for each term; followed by another ellipsis; followed by the words "for each natural number n."
	3. Each term will be .0001 larger than its predecessor.

4. The first term will be the sum of a natural number and .9997.

(Response attribute) The student's response will be indicated below the description of the sequence in the space marked "Ans. __________."

(Sample item--2B) 1.9997, 1.9998, 1.9999, 2.0000, 2.0001, 2.0002, 2.0003, . . . , 1.9997 + (n/10000), . . . for each natural number n.

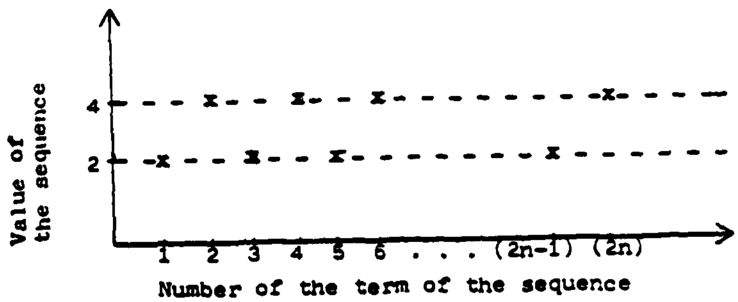
Ans. __________

Item specifications:

(General description)	The student can classify an example of a sequence which has more than one cluster point as one which does not have a finite limit.
(Stimulus attributes)	1. This item will appear with a group of items. This group will be preceded by the directions, "Write the limit of each sequence described below. If the sequence has no finite limit, write 'NONE.'"
	2. This sequence will be presented using a two-dimensional graph on which labels will appear that depict a sequence whose odd-numbered terms take on one value and whose even numbered terms take on a different value.
	3. The (2n-1)th and the (2n)th terms will appear on the graph.
(Response attribute)	The student's response will be indicated below the description

of the sequence in the space
marked "Ans. __________."

(Sample item--2D)



For each natural number n each term of the sequence appears on the graph above.

Ans. __________

<u>Item specifications</u>:

(General description) The student can classify a sequence which contains both unbounded and bounded subsequences as a sequence which does not have a finite limit.

(Stimulus attributes)

1. This item will appear with a group of items. This group will be preceded by the directions, "Write the limit of each seqeuence described below. If the <u>sequence has no finite limit</u>, <u>write</u> '<u>NONE</u>.'"

2. The subsequence formed by the odd-numbered terms will converge to zero.

3. The subsequence formed by the even-numbered terms will not be bounded.

(Response attribute) The student's response will be indicated below the description of the sequence in the space marked "Ans. __________."

(Sample item--2E) $1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, 7, \frac{1}{8}, \ldots,$

$\{\frac{1}{n}\}^{[(-1)^n]}$, . . . for each natural number n.

Ans. ________

Objective 3: The student can classify a sequence expressed as a ratio of two polynomials in terms of its convergence or divergence. In the case of convergence, the student can specify the limit.

<u>Item specifications</u>:

(General description) The student can classify a sequence expressed as the ratio of higher order polynomials, with the degree of the numerator exceeding that of the denominator, as one for which a finite limit does not exist.

(Stimulus attributes)

1. The example will be a sequence in which the degree of the polynomial in the numerator exceeds that of the denominator.
2. The degree of each polynomial will be greater than four.
3. The coefficients of the terms whose power is greater than four will be different from one and not share a common factor.

(Sample item--none on this test since item 3E was of the same nature)

$$\frac{4n^7 + 5}{3n^5 + 4n - 1}$$

<u>Item specifications</u>:

(General description)	The student can specify the limit of a sequence expressed as the ratio of two lower order polynomials with the degree of the denominator being greater than or equal to that of the numerator.
(Stimulus attributes)	1. The example will be a sequence in which the degree of the polynomial in the denominator is greater than or equal to that of the denominator. (If "greater than" is chosen for this example, then "equal to" should be chosen for the sample in the following item specification.) 2. Coefficients for the terms will be different from one and not share a common factor. 3. The degree of each polynomial will be less than or equal to two.

(Sample item--1G) $\frac{3n}{4n - 7}$

Item specifications:

(General description)	The student can specify the limit of a sequence expressed as the ratio of higher order polynomials with the degree of the denominator being greater than or equal to that of the numerator.
(Stimulus attributes)	1. The example will be a sequence in which the degree of the polynomial in the denominator is greater than or equal to that of the numerator. 2. The degree of each polynomial will be greater than four. 3. The coefficients of the terms whose power is greater than four will be different from one and not share a common factor.
(Sample item--1F)	$\frac{7n^9 - 12n^8 + 15n - 1}{2n^{10} + n^3 + 2}$

Objective 4: The student can classify sequences involving a constant raised to the nth power or the nth root of a constant, for all natural numbers n; and sequences which are formed by taking the square roots of the values of an algebraic function.

Item specifications:

(General description)	The student can classify an example of sequences of the form c^n, for all positive constants c, in terms of its convergence or divergence. In the case of convergence, the student can specify the limit.
(Sample item--1D)	$(0.7739)^n$

Item specification:

(General description) The student can classify a sequence which is formed by taking the square root of the values of an algebraic function.

(Sample item--1I)

$$\sqrt{\frac{3n - 1}{2}}$$

Item specifications:

(General description)	The student can classify sequences of the form $\sqrt[n]{c}$, for all positive numbers c, in terms of its convergence or divergence. In the case of convergence, the student can specify the limit.
(Stimulus attribute)	Negative exponents will not be used.
(Sample item--1H)	$\sqrt[n]{271.32}$

PRODUCTION LEVEL

This level of understanding requires the student to be able to produce an example of a sequence which meets certain prescribed conditions, if such an example is possible; and to be able to state that no such example will satisfy the conditions, if it is not possible. The prescribed conditions will be of a nature that can easily be understood by the student.

There are two sets of directions that apply to the items used for this level.

* (Stimulus attribute) "For items A through C, you are to produce an example of a sequence which meets certain prescribed specifications. (new line) Write the general expression for the nth term for some sequence: "will appear before one group of items.

  (Response attribute) A blank line for the answer will appear to the right of each item.

** (Stimulus attribute) "For items D through G, you are to state whether or not there exists a sequence which meets certain prescribed conditions." will appear before a second group of items. Directions to circle "yes" or "no" for each item will accompany this statement.

  (Response attribute) At the right of each item will appear the words "yes," "no."

Objective 1: The student can produce an example of a divergent sequence which meets certain prescribed conditions.

<u>Item specification</u>:

(General description)	The student can produce an example of a sequence which is not bounded.
(Stimulus attribute)	* applies.
(Response attribute)	* applies.
(Sample item--6A)	Write the general expression for the nth term for some sequence which does not have a finite limit because the terms of the sequence are not bounded.

<u>Item specifications</u>:

(General description)	The student can produce an example of a sequence which has two distinct cluster points.
(Stimulus attribute)	1. The term "cluster point" will not be used. 2. The example will be one for which infinitely many terms take on one value and infinitely many terms take on a different value. 3. * applies.
(Response attribute)	* applies.
(Sample item--6B)	Write the general expression for the nth term for some sequence which does not have a limit because there are two values taken on by infinitely many terms of the sequence.

Item specifications:

(General description)	The student can correctly state that it is possible to produce an example of a sequence for which infinitely many terms of the sequence take on a value that is not the limit of that sequence.
(Stimulus attribute)	** applies.
(Response attribute)	** applies.
(Sample item--6F)	Is there any sequence for which infinitely many terms are 7, but the limit of that sequence is <u>not</u> 7? yes no

Objective 2: The student can produce an example of a convergent sequence which meets certain prescribed conditions.

<u>Item specifications</u>:

(General description) The student can specify that the limit of a constant function is that constant.

(Stimulus attributes)

1. This item will appear with a group of items. This group will be preceded by the directions, "Write the limit of each sequence described below. If the <u>sequence has no finite limit, write</u> '<u>NONE</u>.'"

2. The constant will be a negative number..

3. The sequence will be described by presenting the first three terms of the sequence, separated by commas; followed by an ellipsis; followed by an explanation of how each term is determined.

(Response attribute) The student's response will be indicated below the description of the sequence in the space marked "Ans. __________."

Item specifications:

(General description)	The student can produce an example of a convergent sequence which has a finite number of terms which are negative (positive), but the preassigned limit is positive (negative).
(Stimulus attributes)	1. The prescribed conditions are that the initial terms are of one sign and the limit is a preassigned number of the opposite sign.
	2. * applies.
(Response attribute)	* applies.
(Sample item--6C)	Write the general expression for the nth term for some sequence for which the first five terms are negative, but the limit of that sequence is 7.

Item specifications:

(General description)	The student can correctly state that it is impossible to produce an example of a sequence which has infinitely many values of one sign, but whose limit is of the opposite sign.
(Stimulus attribute)	** applies.
(Response attribute)	** applies.
(Sample item--6D)	Is there any sequence whose limit is -3 and which has infinitely many positive terms? yes no

Item specifications:

(General description)	The student can correctly state that it is possible to produce an example of a sequence whose successive terms get closer to a given value, but the limit of that same sequence is a different value.
(Stimulus attribute)	** applies.
(Response attribute)	** applies.
(Sample item--6E)	Is there any sequence for which each successive term is closer to 2, but the limit of this sequence is <u>not</u> 2? yes no

Item specifications:

(General description)	The student can correctly state that it is possible to produce an example of a sequence for which there is a finite limit, but none of the terms of that sequence equal the limit.
(Stimulus attribute)	** applies.
(Response attribute)	** applies.
(Sample item--6G)	Is there any sequence whose limit is one but no term of the sequence is 1? yes no

COMPREHENSION LEVEL

This level of understanding requires that the student demonstrate knowledge of the general principles which characterize the convergence/divergence of a sequence.

Objective 1: The student can classify sequences in terms of their convergence or divergence when the rule of correspondence for the sequence is presented in general terms.

Item specifications:

(General description)	If a sequence which is formed by the quotient of two polynomials is presented to a student, along with a description of the degree of each polynomial (when compared with each other), then the student can classify the sequence in terms of its convergence or divergence.
(Stimulus attributes)	1. A formula for the quotient of two polynomials will be given.
	2. The degree of the numerator will be p and the denominator q. The relationship between p and q will be given.
	3. The coefficients of n^p and n^q will be natural numbers greater than 1 and will have no common factors.
	4. A phrase indicating that the *one correct response* relating to the limit of the

sequence is desired will be given.

(Response attributes) The responses will be phrases indicating that the limit

a) is the quotient of the coefficients of n^p and n^q.

b) is 0.

c) is not a real number because the values of the sequence are not bounded.

(Sample item--3E) For all real numbers p and q, if $p > q > 1$ then

$$\lim_{n \to \infty} \frac{3n^p - 5n^{p-3} - 7}{4n^q + 9n^{q-1} + 2}$$

a) is 3/4.

b) is 0.

c) is not a real number because the values of the sequence are not bounded.

<u>Item specifications</u>:

(General description) The student can classify sequences of the form r^n in terms of their convergence or divergence.

(Stimulus attributes)

1. Values for r will be given.
2. The stimulus will indicate that the one correct response concerning the $\underset{n\to\infty}{\text{limit}}\, r^n$ is desired.

(Response attributes) The responses will be:

a) is 1.
b) is 0.
c) is not a real number because the values of the sequence are not bounded.
d) may or may not be a real number, but more information is required for this to be determined.

(Sample item--3F) For each real number r that is between 0 and 2, $\underset{n\to\infty}{\text{limit}}\, r^n$

a) is 1.
b) is 0.

c) is not a real number because the values of the sequence are not bounded.

d) may or may not be a real number, but more information is required for this to be determined.

Objective 2: The student can recognize valid justification for the divergence of a sequence.

Item specifications:

(General description) Given an example of a sequence which is bounded and has two cluster points, the student can identify the reason that the sequence diverges.

(Stimulus attributes)

1. A sequence will be described which has two distinct positive cluster points, c and k.
2. The stimulus will be an incomplete sentence which will be made complete by adding the one correct response.

(Response attribute) The responses will be phrases that indicate:

a) the limit is $\frac{1}{2}$ (c + k).
b) the limits are both c and k.
c) there is no limit because there are infinitely many terms in the sequence.
d) there is no limit due to there being more than one cluster point. The term "cluster point" will not be used.

(Sample item--3D)

The sequence whose odd-numbered terms have the value 2.9999 and whose even-numbered terms have the value 3.0001

a) has limit 3.

b) has both limits 2.9999 and 3.0001.

c) has no limit because there are infinitely many terms in the sequence.

d) has no limit because infinitely many terms of the sequence have the value 2.9999 and infinitely many terms of the sequence have the value 3.0001.

Item specifications:

(General description) Given the graph of a sequence which does not have a finite limit and which has no cluster points, the student can identify that the sequence diverges because it is unbounded.

(Stimulus attributes)

1. This sequence will be presented using a two-dimensional graph.

2. At least seven points will appear on the graph, one of which will correspond to the nth term of the sequence.

3. This sequence will not be bounded above or below.

4. A phrase will refer to the graph and will be made complete by adding the one correct response.

(Response attributes) The responses will be phrases indicating that:

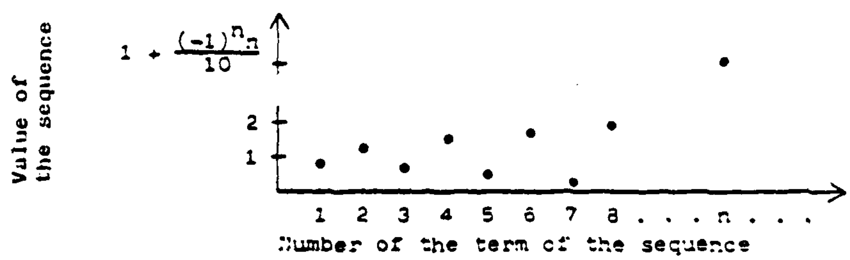
a) the limit is the value of the first term of the sequence rounded to its nearest integer.

b) the limit is the general expression for the nth term.

c) there is no limit because there are infinitely many positive values and infinitely many negative values.

d) there is no limit due to unboundedness.

(Sample item--3H)



The sequence whose graph is pictured above

a) has limit 1.

b) has limit $1 + \frac{(-1)^n n}{10}$.

c) has no finite limit because there are infinitely many positive terms and infinitely many negative terms.

d) has no finite limit because the values of the sequence are not bounded.

Objective 3: The student exhibits an understanding that the limit is a unique number and that infinity is not a number.

Item specifications:

(General description)	The student recognizes that the limit is unique.
(Stimulus attribute)	A statement will be made concerning the possibility of a sequence having more than one finite limit.
(Response attribute)	The student is to circle T for True or F for False.
(Sample item--5D)	T F A sequence can have more than one limit.

Item specifications:

(General description)	In the context of determining limits, the student can demonstrate the knowledge that infinity is not a very large number.
(Stimulus attributes)	1. A formula for a_n will be given in which the sequence values change after at least a million terms.
	2. The rule of correspondence will involve a sequence which is not bounded and a convergent sequence. However, one of these sequences will be terminated after a finite number of terms, while the other will be missing the corresponding number of initial terms.
	3. The stimulus will indicate that the one correct response concerning the limit of the sequence is desired.

(Response attribute) The responses will be phrases that indicate that the limit

a) is the number found by evaluating a_n at the value for which the rule of correspondence changes.

b) is the limit of the sequence, if it exists. If not, the response will be chosen to be the limit of the sequence from which the initial terms of the sequence in this item were chosen.

c) is not a real number because the values of the sequence are not bounded.

(Sample item--3I) Suppose that we have a sequence $\{a_n\}$ such that for each natural number n

$$a_n = \begin{cases} n^4 & \text{for } n \leq 1{,}000{,}000{,}000{,}000 \\ \frac{1}{n^4} & \text{for } n > 1{,}000{,}000{,}000{,}000 . \end{cases}$$

$\lim_{n \to \infty} a_n$

a) is $1{,}000{,}000{,}000{,}000^4$.

b) is 0.

c) is not a real number because the terms of the sequence are not bounded.

Objective 4: The student can classify sequences in terms of their convergence/divergence when the underlying principles involve basic limit theorems.

Item specifications:

(General description) The student can classify subsequences in terms of their convergence or divergence when relevant information about the sequence and subsequence is given.

(Stimulus attributes)

1. The stimulus will specify the convergence/divergence of the original sequence. In the case of convergence, the limit L will be given.

2. The method for obtaining the subsequence will be specified.

3. The stimulus will indicate that the one correct response concerning the limit of the subsequence is desired.

(Response attributes) The responses will be that the limit

a) is L.

b) is not a finite number.

c) may or may not be a finite number, but more information is required for this to be determined.

(Sample item--30)

For each sequence formed by choosing every fifth term of some sequence whose limit is $\sqrt{2}$, the limit of the newly formed sequence

a) is $\sqrt{2}$.

b) is not a finite number.

c) may or may not be a finite number, but more information is required for this to be determined.

Item specifications:

(General description)	The student can classify a sequence which is sandwiched between two sequences, both of which converge to the same limit, as one for which the limit exists.
(Stimulus attributes)	1. A description of three sequences will be given in which each term of the third sequence is between the corresponding terms of two convergent sequences.
	2. The terminology "corresponding terms" will be explained and illustrated with an example.
	3. The two convergent sequences will have the same limit and that limit will be given.
	4. A phrase indicating that the one correct response concerning the limit of the third sequence is desired.

(Sample item--30)

For each sequence f, for each sequence g, and for each natural number n, f(n) and g(n) are <u>corresponding terms</u> of f and g. For example, 1 and 5, 2 and 10, 3 and 15, ... are the corresponding terms of f = {(1,1), (2,2), (3,3),...} and g = {(1,5), (2,10), (3,15),... }.

For each sequence f, for each sequence g, and for each sequence h,

> if each term of h is between the corresponding terms of f and of g, and the limits of f and g are both π, then the limit of h
>
> a) is π.
>
> b) is not a finite number.
>
> c) may or may not be a finite number, but more information is required for this to be determined.

Objective 5: The student can make judgments concerning the behavior of the terms of a convergent sequence.

Item specifications:

(General description)	The student recognizes that the behavior of a finite number of terms of a sequence does not affect the limit.
(Stimulus attributes)	1. A situation describing two sequences will be given. One of the sequences will be formed by altering a finite number of terms of the other sequences. 2. A real number L will be given as the limit of the original sequence. 3. A statement regarding the limit of the newly formed sequence will be given.
(Response attribute)	The student is to circle T for True or F for False.
(Sample item--5A)	T F For each sequence S, if the limit of S is 17 and we alter S only by subtracting 5 from

each of its first
1000 terms, then the
limit of the newly
formed sequence is 12.

<u>Item specifications:</u>

(General description) The student recognizes that successive terms of a convergent sequence need not be closer to the limit than its immediate predecessor is.

(Stimulus attribute) A statement will be made indicating that a specific term of a convergent sequence is closer to its limit, than the preceding term is.

(Response attribute) The student is to circle T for True or F for False.

(Sample item--5C) T F For each sequence S, if the limit of S is 22.7, then the 875th term is closer to 22.7 than the 874th term is.

Item specifications:

(General description)	The student recognizes that the limit of a sequence need not be one of the values of the sequence.
(Stimulus attribute)	The statement will indicate that every sequence which has a limit L must have at least one term which is L.
(Response attribute)	The student is to circle T for True or F for False.
(Sample item--53)	T F Consider every sequence whose limit is 4. At least one term in each of these sequences is 4.

FORMAL LEVEL

This level of understanding requires that the student be able to communicate a precise definition of the limit of a sequence and demonstrate a knowledge of the relationships among the component parts of the definition.

AD-A139 399 DEVELOPING AND MEASURING AN UNDERSTANDING OF THE CONCEPT OF THE LIMIT OF A SEQUENCE(U) AIR FORCE INST OF TECH WRIGHT-PATTERSON AFB OH T A BRATINA 1983 3/5

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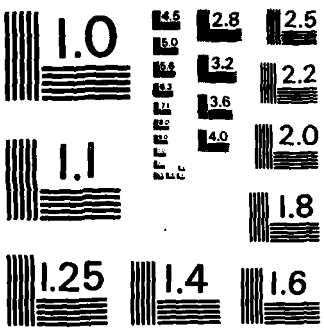


MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS–1963–A

Objective 1: The student can demonstrate a knowledge of the relationship among the component parts of the "ε/N definition."

Item specifications:

(General description) The student recognizes that N is a function of ϵ in the "ϵ/N definition."

(Stimulus attributes)

1. A statement will that each sequence which has a given limit is to be considered.
2. Another statement will be given that all terms of the sequence beyond a given term are within a given distance of that limit.
3. The question will address how many terms of each sequence are required before each term is within a given distance of that limit.

(Response attributes) The responses will be

a) Greater than or equal to the number of terms stated in stimulus attribute 2.

b) Less than the number of terms stated in stimulus attribute 2.

c) Cannot be determined without further information.

(Sample item--3C)

Consider each sequence whose limit is 8. Suppose every term after the 67th lies within 1/4 of 8. How many terms of each such sequence are required before each term is within 1/100 of 8?

a) Greater than or equal to 67 terms.

b) Less than 67 terms.

c) Cannot be determined without more information.

<u>Item specifications</u>:

(General description)	The student can demonstrate a knowledge that N in the "ε/N definition" depends on the particular sequence (as well as ε) under discussion.
(Stimulus attributes)	1. A statement will be given indicating that two different sequences have the same limit which will be given. 2. A second statement will be made that when the values of the first sequence all differ from that limit by a specified amount, then the corresponding values of the second sequence do the same. 3. "Corresponding values" will be explained to the student.
(Response attribute)	The student's response will be T for True or F for False.

(Sample item--5F)

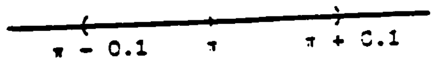
Suppose that two different sequences have the same limit, say 4. It must be the case that when the values of the first sequence all differ from 4 by less than 0.001, then the corresponding values of the second sequence also differ from 4 by less than 0.001.

(Recall that for each sequence f, for each sequence g, and for each natural number n, f(n) and g(n) are the *corresponding values* of f and g.)

Item specifications:

(General description)	The student can demonstrate a knowledge of the relationship between n and N in the "ϵ/N definition."
(Stimulus attributes)	1. A statement will be given in which each sequence whose limit is a given real number is to be considered.
	2. The student is to decide if it follows that "all but a finite number of terms" lie in a given interval containing this limit.
	3. A number line picturing this interval will accompany the statement.
(Response attribute)	The student's response will be T for True or F for False.
(Sample item--5E)	For each sequence S, if the limit of S is π, then all but a finite number of terms lie in the interval $(\pi - 0.1, \pi + 0.1)$

pictured below.



Item specifications:

(General description)	The student realizes that beyond a finite number of terms in a convergent sequence, the absolute value of the difference between the term and the limit will be less than any positive number.
(Stimulus attributes)	1. The stimulus will state what the limit of a particular sequence is. 2. A statement that the absolute value of the difference between the terms and the given limit is less than a specific positive number for all but a finite number of terms will be made.
(Response attribute)	The student's response will be T for True or F for False.

(Sample item--5I1)

The limit of a sequence is S 6.

True or False:
For each natural number n, there is a natural number m such that

if $n > m$ then $|S(n) - 6| < \sqrt{2}$.

(Circle here.) T F

Item specifications:

(General description) The student realizes that just because the absolute value of the difference of the terms of a convergent sequence and a given number is less than a preassigned small value, the given number is not necessarily the limit of that sequence.

(Stimulus attributes)

1. The student will be given the information that a particular sequence has a limit; and the absolute value of the difference of the terms of that sequence and a specific number will be given to be less than a preassigned positive number for all but a finite number of terms of that sequence.

2. A statement will be made that the limit of that sequence is equal to the given subtrahend.

(Response attribute) The student's response will be T for True or F for False.

(Sample item--5I2)

Now consider a sequen[illegible] such that T has a lim[illegible] [illegible]d for each natural number n, there exists a natural number m such that
$\lvert T(n) - 37 \rvert < 0.001$, whenever $n > m$.
True or False:
$\lim_{n \to \infty} T(n) = 37$.
(Circle here.) T F

Objective 2: The student can discriminate between right and wrong statements purporting to be definitions for the limit of a sequence.

Item specifications:

(General description) The student can determine if a statement relating to the limit of a sequence is equivalent to the statement that the limit of a particular sequence is a real number L.

(Stimulus attributes)

1. The stimulus will indicate that two statements are equivalent.
2. One statement will indicate that for each sequence S, the limit of S is a real number L. L will be given.
3. Another statement will be given.

(Response attribute) The student's response will be T for True or F for False.

(Sample item--50)

T F The following are equivalent statements:

a) For each sequence S, the limit of S is 9.3.

b) For each sequence S

and for each natural number n,
$|S(n) - 9.3| \geq |S(n+1) - 9.3|$.

<u>Item specifications</u>:

(General description)	The student can recognize the correct definition for the limit of a sequence.
(Stimulus attribute)	A correct definition of the limit of a sequence will be presented.
(Response attribute)	The student's response will be T for True or F for False.
(Sample item)	For each sequence f and for each real number ℓ, the limit of f is ℓ if and only if for each positive real number s, no matter how small s might be, there exists a natural number m such that for each natural number n whenever $n > m$, $\|\ell - f(n)\| < s$.

APPENDIX B

INITIAL OBJECTIVES AND CORRESPONDING ITEMS FOR UNDERSTANDING THE LIMIT OF A SEQUENCE

Please read the following pages carefully and respond according to the directions. For most of your responses, you will be using a five-point scale. The responses are intended to correspond to the following opinions:

1. Strongly disagree.
2. Disagree.
3. No opinion.
4. Agree.
5. Strongly agree.

For example, if you "agree with a statement" preceding the scale on which to mark your opinion, you would place a check mark above the 4 as indicated on the scale below:

Strongly Disagree				✓		Strongly Agree
	1	2	3	4	5	

If you find that you have specific comments (e.g., phraseology, symbolism, etc.) on the material and you do not believe that any statements address these concerns, please feel free to write them either at the place where they occur or on the back of the last page.

A few comments concerning the purpose of the instrument and introductory comments about the model I am using precede the sections on which I am asking you to comment. I hope these comments will serve to enlighten rather than confuse you.

Thank you for your generosity with your time in evaluating the material contained in this package. I will stop by in a week, but if you have any questions please call me (386-5637).

Sincerely,

INTRODUCTORY COMMENTS

The purpose of the instrument is to 1) operationally define what it means to understand the limit of a sequence, and 2) measure that understanding. (Particular instructional methodology is not of concern here.)

A model is being used for this study. It is simply a tool which aids in identifying the full range of behaviors required for understanding the limit concept. It is not intended to be unique. It was derived from reviewing the literature on models for understanding concepts and from experience regarding the topic of limits of sequences. The model should be easily understood by mathematicians and mathematics educators alike and should include the full range of behaviors required for understanding the limit concept.

The model consists of five levels of student behavior: intuition, identification, production, comprehension, and formal. The full description for each level is contained within. From each of these five levels evolved more descriptive objectives of desired student behaviors. Then from the objectives evolved item specifications. The levels and their corresponding objectives are all presented on the last page of this package. (The specifications are spelled out in great detail on the following pages.)

It should be noted that the instrument is not intended to measure an understanding of the application of the formal concept or of specialized techniques for complicated sequences, e.g., $(1 + 1/n)^n$. Such behavior might be considered as "extensions of the limit concept" and are not included here. Understanding of the limit concept is the key here--not understanding the behavior of complicated sequences.

INTUITIVE LEVEL

INTUITIVE LEVEL: This level of understanding requires the student to demonstrate the ability to comprehend situations which involve or imply the contexts from which the concept of limit of a sequence will develop. Vocabulary usage is nonrigorous, corresponding to pre-formal experiences with limits.

Objectives:

1. The student can demonstrate a knowledge of convergence in situations involving "a limit object" of an implied sequence.

2. The student can make appropriate judgments about situations involving divergence.

3. The student can demonstrate a knowledge of how successive approximations can be used in measurement.

The objectives are described in more detail (by the use of item specifications) on the following pages.

Objective: The student can demonstrate a knowledge of convergence in situations involving "a limit object" of an implied sequence.

Please react to the following statement by checking the response which most accurately reflects your opinion.

This objective is appropriate for understanding the limit of a sequence at the intuitive level.

Strongly Disagree	___	___	___	___	___	Strongly Agree
	1	2	3	4	5	

If you marked 1 or 2 (strongly disagree or disagree), please express your opinion why the objective is not appropriate for this level of understanding.

The following item specifications serve to elaborate on the objective presented on the previous page.

Item specifications:

(General description) The student can make judgments related to neighborhoods about the limit point of a "convergent" sequence of points.

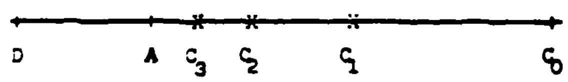
(Stimulus attributes)

1. The stimulus will describe a situation in which a sequence of points are identified on a line segment so that the points converge to a limit point. The limit point will not be a member of the sequence.
2. The stimulus will contain fewer than 70 words.
3. The item will be accompanied by a picture with appropriate points labelled.

(Response attributes)

1. The responses are statements that:
   a) After a large number of midpoints have been chosen, the midpoints are not distinct from the limit point.
   b) Involve placing a small circle around the limit point, and that there will be a point in the sequence inside of that circle regardless of how small it is.
   c) Suggesting that the sequence is not bounded by the limit point.

(Sample item)



Point C_1 is the midpoint of the line segment with endpoints A and C_0. Shown above are the first three midpoints C_1, C_2, and C_3 of a sequence of midpoints which are obtained by the following continuing process:

For each natural number n, point C_n is the midpoint of the segment with endpoints A and C_{n-1}.

Which statement is true regarding this sequence of midpoints?

a) Points C_n and A are the same for values of n greater than 100.

b) Regardless of how small a circle is placed around Point A, there will be a midpoint that will be inside of the circle.

c) The midpoints will eventually be between Points A and D.

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

<u>Item specifications</u>:

(General description) An irregularly shaped simple closed curve is given. A sequence of successively refined grids are overlaid one-by-one on the irregular figure. For each grid, the boundary of the figure formed by all squares entirely contained in the interior of the curve is shown. The student recognizes that the sequence of these boundaries approaches the simple closed curve.

(Stimulus attributes)

1. A picture of two transparencies of a grid of squares overlying an irregular figure will be presented with the item.
2. The development of a sequence of boundaries of the squares contained within the irregular figure will be described.
3. The irregular figure will completely contain at least one square of the grid. A second grid will be a refinement of the first and will appear on the right of the first.
4. The stimulus will ask which of several statements about the boundary of squares is consistently true.
5. The passage will contain less than 100 words.

(Response attributes)

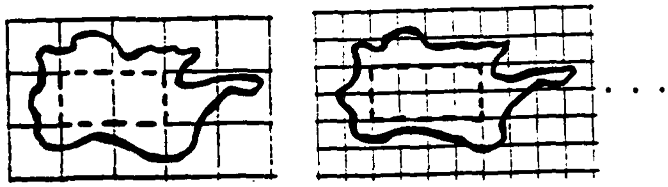
1. The responses will contain less than 20 words.
2. The three responses will be statements that

   a) indicate a lack of visualizing the continuing process.

   b) the boundary approaches the boundary of the irregular figure.

c) the boundary will eventually be the same as the boundary of the irregular figure.

(Sample item)



Grid of squares overlying boundary. Refinement of previous grid.

Pictured above are the first two of a series of transparencies of grids of squares placed on the boundary of an irregularly shaped figure (solid line). The length of each side of a grid square is 1/2 the length of each side of each square in the preceding grid. For each grid the boundary (dotted lines) of the figure formed by all squares entirely contained inside the irregular figure is established. Which of the following statements is consistently true? The boundary of all squares entirely contained inside the irregular figure

a) forms a rectangle.

b) shows a greater resemblance to the boundary of the irregular figure for each successive refinement of the grid.

c) will eventually be the same as the boundary of the irregular figure.

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree	1	2	3	4	5	Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree	1	2	3	4	5	Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree	1	2	3	4	5	Strongly Agree

Objective: The student can make appropriate judgments about situations involving divergence.

Please react to the following statement by checking the response which most accurately reflects your opinion.

This objective is appropriate for understanding the limit of a sequence at the intuitive level.

Strongly Disagree	___ 1	___ 2	___ 3	___ 4	___ 5	Strongly Agree

If you marked 1 or 2 (strongly disagree or disagree), please express your opinion why the objective is not appropriate for this level of understanding.

The following item specifications serve to elaborate on the objective presented on the previous page.

<u>Item specifications</u>:

(General description) Given a situation which involves a sequence whose terms alternate among more than one value, the student recognizes that such a sequence does not approach a "limit object."

(Stimulus attribute) The stimulus will ask for the sum of 1 + (-1) + 1 + (-1) + 1 + (-1) . . .

(Response attribute) The responses will indicate that the sum of 1 + (-1) + 1 + (-1) + . . .

a) is 0.

b) is 1.

c) cannot be determined because different regroupings of the addend result in different sums.

(Sample item) The sum of 1 + (-1) + 1 + (-1) + 1 + (-1) + . . .

a) is 0.

b) is 1.

c) cannot be determined because different regroupings of the addend result in different sums.

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The item specifications are appropriate for this objective.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The sample item is consistent with the item specifications.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

Item specifications:

(General description) Given a process which produces a sequence of triangles of unbounded area, the student recognizes that the area of the figures grows arbitrarily large.

(Stimulus attributes)

1. The stimulus will contain:

   a) first, a general description of the situation (sequential process).

   b) a drawing depicting the situation.

   c) a question asking which of several responses is true.

2. The stimulus will use 75 words or less to describe the construction of the sequence of triangles.

(Response attributes)

1. The responses will be no more than 25 words long.

2. There will be three responses as follows:

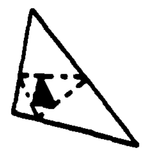
   a) An incorrect statement asserting that a preceding triangle in the sequence contains the vertices of a succeeding triangle.

   b) A statement that the area of any of the triangles in the sequence can (cannot) exceed the area of a circle with a specific large radius.

c) Based on the choice in b, a statement that the length of any one of the sides of the triangles in the sequence cannot (can) exceed a specific linear measure.

(Sample item)

A sequence of triangles are constructed so that the vertices of each triangle lie on the sides of the triangle which succeeds it. For example, in the picture at the right, the vertices of the first triangle (shaded) lie on the sides of the second triangle (dotted line). Which of the following statements about the sequence of triangles is true?



a) The sides of the eighth triangle contain the vertices of the ninth triangle.

b) The process suggested could produce a triangle which has greater area than a circle with radius 1,000,000 miles.

c) The process suggested cannot produce a triangle which has a side longer than 1,000,000 miles.

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

Objective: The student can demonstrate a knowledge of how successive approximations can be used in measurement.

Please react to the following statement by checking the response which most accurately reflects your opinion.

This objective is appropriate for understanding the limit of a sequence at the intuitive level.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

If you marked 1 or 2 (strongly disagree or disagree), please express your opinion why the objective is not appropriate for this level of understanding.

The following item specifications serve to elaborate on the objective presented on the previous page.

Item specifications:

(General description) Using sequences of regular polygons, the student can identify which of several methods yields the best approximation for the area of the circle.

(Stimulus attributes)

1. A drawing of three circles with equal radii will accompany the item. Three regular polygons (from left to right, a triangle, square and pentagon) will be circumscribed about the circle.

2. The stimulus will refer to the picture and describe the method of taking successively more circumscribed regular polygons and finding their areas.

3. The stimulus will indicate that the method for obtaining the best approximation for the area of the circle is desired.

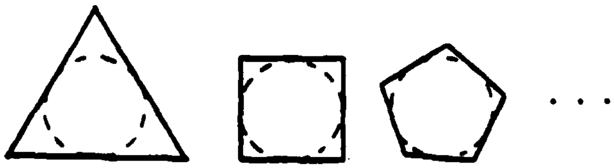
(Response attribute) The responses will indicate that:

a) the best approximation will result from averaging the areas of the polygons.

b) the best approximation will come from the area of a polygon with a specified number of sides (greater than ten).

c) the more sides, the better the approximation will be.

(Sample item)



A process of circumscribing regular polygons about a circle is shown in an attempt to approximate the area of a circle. The number of sides for each successive polygon increases by one. Which of the following statements concerning the area of the circle is true?

a) Averaging the areas of the polygons will provide the best approximation.

b) The area of the polygon with 15 sides will provide the best approximation.

c) The greater the number of sides, the better the approximation will be.

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

<u>Item specifications</u>:

(General description) If a successive approximation is presented among less effective methods for measuring a desired quantity, the student identifies the successive approximation as the preferred method.

(Stimulus attributes)

1. The stimulus will ask which of several responses provides the best approximation for the instantaneous velocity of a falling object at a specific time t.

2. A picture illustrating the setting will accompany this item.

3. Specific formulas will not be stated.

4. Thirty words or less will be used in the stimulus.

(Response attributes)

1. The responses will be statements that the method should be to:

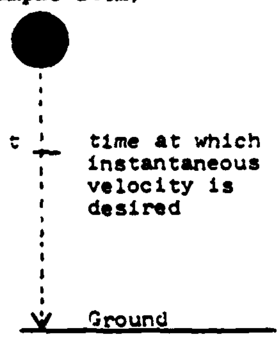
   a) divide the distance fallen by the time t.

   b) find the average velocity for successively shorter time intervals containing t and select the average velocity for the shortest of those intervals.

   c) take several average velocities for different times and average the average velocities.

2. Each choice will contain fewer than thirty words.

(Sample item)



Which of the following processes will consistently result in the best approximation for the instantaneous velocity of a falling object at a specific time t?

a) Divide the distance the object has fallen by the specific time t.

b) Find the average velocity of the falling object for successively shorter time intervals which contain t and select the average velocity corresponding to the shortest of those time intervals.

c) Take the average velocity of the falling object for different times preceding t. Repeat this process and take the average of the average velocities.

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The item specifications are appropriate for this objective.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The sample item is consistent with the item specifications.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

Recall that all of the objectives appear on the last page of this package. This page was intended for your quick use and to provide the "big picture" for understanding the concept of the limit of a sequence. You may want to glance at it before checking your response to the next statement.

The objectives are appropriate for describing the understanding of the limit of a sequence at the intuitive level as this level was described earlier.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

If you answered 1 or 2 (strongly disagree or disagree), please describe briefly your difficulty.

IDENTIFICATION LEVEL

IDENTIFICATION LEVEL: This level of understanding requires the student to be able to classify sequences in terms of their convergence or divergence, using only elementary methods (inspection or elementary arithmetic, algebraic, or trigonometric properties). In the case of convergence, the student can specify the limit.

Objectives:

1. The student can determine the limit of certain convergent sequences by using only inspection.

2. The student can classify examples of divergent sequence as sequences for which the limit does not exist.

3. The student can classify a sequence expressed as the ratio of two polynomials in terms of its convergence or divergence. In the case of convergence, the student can specify the limit.

4. The student can classify sequences involving the basic trigonometric functions, c^n and $c^{1/n}$, and $\sqrt{a_n}$ in terms of their convergence or divergence. In the case of convergence, the student can specify the limit.

For all items in this section:

(Stimulus attributes)*

1. The stimulus will be: "In the blank provided, write the limit(s) of each sequence that appears below. If the particular sequence does not have a limit, write DNE. (a_n represents the value of the nth term of a sequence, where n stands for a natural number)."

2. This statement will appear before a block of items (i.e., <u>not</u> repeated above each item).

(Response attribute)** A blank line will follow the example.

The objectives are described in more detail (by the use of item specifications) on the following pages.

Objective: The student can determine the limit of certain convergent sequences by using only inspection.

Please react to the following statement .y checking the response which most accurately reflects your opinion.

This objective is appropriate for understanding the limit of a sequence at the identification level.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

If you marked 1 or 2 (strongly disagree or disagree), please express your opinion why the objective is not appropriate for this level of understanding.

The following item specifications serve to elaborate on the objective presented on the previous page.

Item specifications:

(General description) The student can specify the limit of a convergent sequence whose limit is nonmonotonically approached by consecutive values of the terms of the sequence.

(Stimulus attributes) 1. * applies.

2. The general expression for the nth term of a sequence whose limit is nonmonotonically approached will be given.

(Response attribute) ** applies.

(Sample item) $a_n = \frac{2 + (-1)^n}{n}$-- ____________

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree __1__ __2__ __3__ __4__ __5__ Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree __1__ __2__ __3__ __4__ __5__ Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree __1__ __2__ __3__ __4__ __5__ Strongly Agree

Item specifications:

(General description) The student can specify that the limit of a constant function is that constant.

(Stimulus attributes) 1. * applies.

2. The general expression for the nth term of the constant sequence will be presented using the form $a_n =$.

3. The constant will be a negative number.

(Response attribute) ** applies.

(Sample item) $a_n = -1\frac{1}{2}$ ____________

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

<u>Item specifications</u>:

(General description) The student can specify the limit of a bounded monotonic sequence.

(Stimulus attributes) 1. * applies.

2. The general expression for the nth term of a sequence which converges to a non-zero number will be given.

3. The expression is written as
a_n = .

(Response attribute) ** applies.

(Sample item) $a_n = 98 + \frac{3}{n^2}$ ____________

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The item specifications are appropriate for this objective.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The sample item is consistent with the item specifications.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

Objective: The student can classify examples of divergent sequence as sequences for which the limit does not exist.

Please react to the following statement by checking the response which most accurately reflects your opinion.

This objective is appropriate for understanding the limit of a sequence at the identification level.

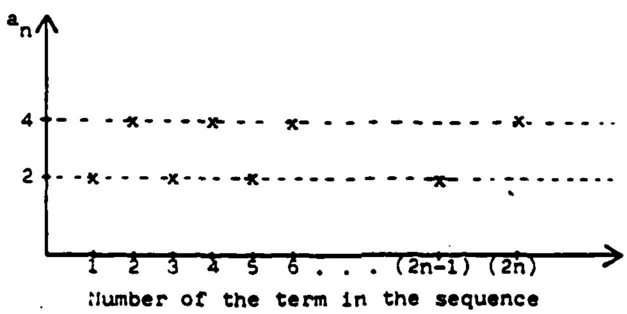
Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

If you marked 1 or 2 (strongly disagree or disagree), please express your opinion why the objective is not appropriate for this level of understanding.

The following item specifications serve to elaborate on the objective presented on the previous page.

<u>Item specifications</u>:

(General description)	The student can classify an example of a sequence which has more than one cluster point as one whose limit does not exist.
(Stimulus attributes)	1. * applies.
	2. A two-dimensional graph will be given on which labels will appear that depict a sequence whose odd-numbered terms take on one value and whose even-numbered terms take on a different value.
	3. The (2n-1)th and the (2n)th terms will appear on the graph.
(Response attribute)	** applies.
(Sample item)	



____

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item speci.'cations are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

Item specifications:

(General description) The student can classify an example of an unbounded sequence as one whose limit does not exist.

(Stimulus attributes) 1. * applies.

2. The first three terms of a sequence which diverges to $-\infty$, followed by an ellipsis, and a phrase explaining what the general expression for the nth term of the sequence is, will be provided.

(Response attribute) ** applies.

(Sample item) 8, 7, 6, . . . in which the general expression for the nth term of the sequence is $a_n = 9-n$ ______________

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The item specifications are appropriate for this objective.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The sample item is consistent with the item specifications.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

Objective: The student can classify a sequence expressed as the ratio of two polynomials in terms of its convergence or divergence. In the case of convergence, the student can specify the limit.

Please react to the following statement by checking the response which most accurately reflects your opinion.

This objective is appropriate for understanding the limit of a sequence at the identification level.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

If you marked 1 or 2 (strongly disagree of disagree), please express your opinion why the objective is not appropriate for this level of understanding.

The following item specifications serve to elaborate on the objective presented on the previous page.

<u>Item specifications</u>:

(General description) The student can classify a sequence expressed as the ratio of higher order polynomials, with the degree of the numerator exceeding that of the denominator, as one whose limit does not exist.

(Stimulus attributes)

1. * applies.
2. The example will be a sequence in which the degree of the polynomial in the numerator exceeds that of the denominator.
3. The coefficients of the terms whose power is greater than four will be different from one and not share a common factor.
4. The degree of each polynomial will be greater than four.

(Response attribute)

1. ** applies.

(Sample item) $a_n = \dfrac{4n^7 + 5}{3n^5 + 4n - 1}$ ____________

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

Item specifications:

(General description) The student can specify the limit of a sequence expressed as the ratio of two lower order polynomials with the degree of the denominator being greater than or equal to that of the numerator.

(Stimulus attributes)

1. * applies.

2. The example will be a sequence in which the degree of the polynominal in the denominator is greater than or equal to that of the numerator. (If a "greater than" case is represented in this section, then "equal to" will be used. Otherwise, "greater than" will apply to this item.)

3. Coefficients for the terms will be different from one and not share a common factor.

4. The degree of each polynomial will be less than or equal to two.

(Response attribute) ** applies..

(Sample item) $a_n = \frac{3n}{4n - 7}$

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

Item specifications:

(General description) The student can specify the limit of a sequence expressed as the ratio of higher order polynomials with the degree of the denominator being greater than or equal to that of the numerator.

(Stimulus attributes)

1. * applies.
2. The example will be a sequence in which the degree of the polynomial in the denominator is greater than or equal to that of the numerator.
3. The coefficients of the terms whose power is greater than four will be different from one and not share a common factor.
4. The degree of each polynomial will be greater than four.

(Response attribute) ** applies.

(Sample item)

$$a_n = \frac{7n^9 - 12n^8 + 15n - 1}{2n^{10} + n^3 + 2}$$

____

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

<u>Objective</u>: The student can classify sequences in terms of their convergence or divergence when the rule of correspondence is presented in general terms.

Please react to the following statement by checking the response which most accurately reflects your opinion.

This objective is appropriate for understanding the limit of a sequence at the comprehension level.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

If you marked 1 or 2 (strongly disagree or disagree), please express your opinion why the objective is not appropriate for this level of understanding.

The following item specifications serve to elaborate on the objective presented on the previous page.

Item specifications:

(General description) The student can classify an example of the form $\sqrt{a_n}$ (where a_n is positive) in terms of its convergence or divergence. In the case of convergence, the student can specify the limit.

(Stimulus attributes)

1. * applies.
2. The general expression for the nth term of a sequence will be given as $\sqrt{b_n}$ (where b_n is positive).

(Response attribute) ** applies.

(Sample item) $a_n = \sqrt{\frac{3n-1}{2}}$ __________

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

Item specifications:

(General description)	The student can classify an example of sequences of the form c^n (where c is a positive number) in terms of its convergence or divergence. In the case of convergence, the student can specify the limit.
(Stimulus attributes)	1. * applies.
	2. The general expression for the nth term of a sequence will be given as c^n for preassigned value of c (not equal to zero).
(Response attribute)	** applies.
(Sample item)	$a_n = (.7739)^n$ __________

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

Item specifications:

(General description)	The student can classify an example of sequences involving the basic trigonometric functions in terms of

its convergence or divergence. In the case of convergence, the student can specify the limit.

(Stimulus attributes) 1. * applies.

2. The general expression for the nth term of a sequence will be given.

3. Elementary properties of the sine, cosine, or tangent function will be used, but no more than one of these trigonometric functions will appear in the expression.

(Response attribute) ** applies.

(Sample item) $a_n = \cos(\frac{n\pi}{2})$ __________

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

<u>Item specifications</u>:

(General description) The student can classify an example of sequences of the form $\sqrt[n]{c}$ (where c is a positive number) in terms of its convergence or divergence. In the case of convergence, the student can specify the limit.

(Stimulus attributes)

1. * applies.

2. The general expression for the nth term of a sequence will be given as $\sqrt[n]{c}$ for a preassigned value of c (greater than zero).

3. Negative exponents will not be used.

(Response attribute) ** applies.

(Sample item) $a_n = \sqrt[n]{271.32}$ ____________

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

Recall that all of the objectives appear on the last page of this package. This page was intended for your quick use and to provide the "big picture" for understanding the concept of the limit of a sequence. You may want to glance at it before checking your response to the next statement.

The objectives are appropriate for describing the understanding of the limit of a sequence at the identification level as this level was described earlier.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

If you answered 1 or 2 (strongly disagree or disagree), please describe briefly your difficulty.

--

PRODUCTION LEVEL

PRODUCTION LEVEL: This level of understanding requires the student to be able to produce an example of a sequence which meets certain prescribed conditions, if such an example is possible; and to be able to state that no such example will satisfy the conditions, if it is not possible. (The prescribed conditions will be of a nature that can easily be understood by the student.)

Objectives:

1. The student can produce an example of a divergent sequence which meets certain prescribed conditions.

2. The student can produce an example of a convergent sequence which meets certain prescribed conditions.

The objectives are described in more detail (by the use of item specifications) on the following pages.

<u>Objective</u>: The student can produce an example of a divergent sequence which meets certain prescribed conditions.

Please react to the following statement by checking the response which most accurately reflects your opinion.

This objective is appropriate for understanding the limit of a sequence at the production level.

Strongly Diagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

If you marked 1 or 2 (strongly disagree or disagree), please express your opinion why the objective is not appropriate for this level of understanding.

The following item specifications serve to elaborate on the objective presented on the previous page.

Item specifications:

(General description)	The student can produce an example of sequence which is unbounded.
(Stimulus attributes)	1. The directions require the general expression for the nth term of a sequence to be written.
	2. The prescribed condition is that the limit does not exist because the sequence is unbounded.
	3. The directions specify to write "NONE" in the event of an impossible setup.
(Response attribute)	A blank line will appear below the stimulus. It will be indented.
(Sample item)	Write the general expression for the nth term of a sequence whose limit does not exist because the values of terms are unbounded. (If there is no sequence that will satisfy this condition, write NONE.)

____

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree __ 1 __ 2 __ 3 __ 4 __ 5 Strongly Agree

The item specifications are appropriate for this objective.

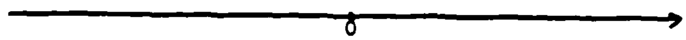
Strongly Disagree __ 1 __ 2 __ 3 __ 4 __ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

<u>Item specifications</u>:

(General description)	The student can draw the graph of a sequence which has two cluster points.
(Stimulus attributes)	1. The directions will be to provide a graph of a divergent sequence which has two cluster points. 2. The terms "divergent" and "cluster point" will not be used. 3. The stimulus will refer to the location of the graph. 4. The directions specify to write "NONE" in the event of an impossible setup.
(Response attributes)	A line, approximately five inches long, with an arrow facing to the right will appear below the stimulus. "0" will be labelled on this number line.
(Sample item)	On the graph below, plot the values of a sequence whose limit does not exist because there are two values taken on by infinitely many terms of the sequence. (If there is no sequence that will satisfy this condition, write NONE.)



Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The item specifications are appropriate for this objective.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The sample item is consistent with the item specifications.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

Objective: The student can produce an example of a convergent sequence which meets certain prescribed conditions.

Please react to the following statement by checking the response which most accurately relects your opinion.

This objective is appropriate for understanding the limit of a sequence at the production level.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

If you marked 1 or 2 (strongly disagree or disagree), please express your opinion why the objective is not appropriate for this level of understanding.

The following item specifications serve to elaborate on the objective presented on the previous page.

Item specifications:

(General description)	The student can produce an example of a convergent sequence which has a finite number of terms which have negative (positive) values, but the pre-assigned limit is positive (negative).
(Stimulus attributes)	1. The directions require the general expression for the nth term of a sequence to be written. 2. The prescribed conditions are that the initial terms are of one sign and the limit is a preassigned number of the opposite sign. 3. The directions specify to write "NONE" in the event of an impossible setup.
(Response attribute)	A blank line will appear below the stimulus. It will be indented.
(Sample item)	Give an example for a_n for which $a_n < 0$ when $n \leq 5$, but $\lim_{n\to\infty} a_n = 7$.

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

Item specifications:

(General description) The student can correctly state that it is impossible to produce an example of a sequence which has infinitely many values of one sign, but whose limit is of the opposite sign.

(Stimulus attributes)
1. The directions require the general expression for the nth term of a sequence to be written.
2. The prescribed conditions are that infinitely many terms are of one sign and the limit is a preassigned value of another sign.
3. The directions specify to write "NONE" in the event of an impossible setup.

(Response attribute) A blank line will appear below the stimulus. It will be indented.

(Sample item) Write the general expression for the nth term of a sequence with infinitely many negative values, but with limit 5. (If there is no sequence that will satisfy this condition, write NONE.)

____

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The item specifications are appropriate for this objective.

Strongly Disagree	1	2	3	4	5	Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree	1	2	3	4	5	Strongly Agree

Recall that all of the objectives appear on the last page of this package. This page was intended for your quick use and to provide the "big picture" for understanding the concept of the limit of a sequence. You may want to glance at it before checking your response to the next statement.

The objectives are appropriate for describing the understanding of the limit of a sequence at the production level as this level was described earlier.

Strongly Disagree ____ ____ ____ ____ ____ Strongly Agree

If you answered 1 or 2 (strongly disagree or disagree), please describe briefly your difficulty.

COMPREHENSION LEVEL

COMPREHENSION LEVEL: This level of understanding requires that the student demonstrate knowledge of the general principles which characterize the convergence/divergence of a sequence.

Objectives:

1. The student can classify sequences in terms of their convergence or divergence when the rule of correspondence is presented in general terms.

2. The student can recognize valid justification for the divergence of a sequence.

3. In the context of using limits, the student exhibits good "number sense."

4. The student can classify sequences in terms of their convergence/divergence when the underlying principles involve basic limit theorems.

5. The student can made judgments concerning the behavior of the terms of a convergent sequence.

The objectives are described in more detail (by the use of item specifications) on the following pages.

Objective: The student can classify sequences involving the basic trigonometric functions, c^n and $c^{1/n}$, and $\sqrt{a_n}$ in terms of their convergence or divergence. In the case of convergence, the student can specify the limit.

Please react to the following statement by checking the response which most accurately reflects your opinion.

This objective is appropriate for understanding the limit of a sequence at the identification level.

Strongly Disagree	___ 1	___ 2	___ 3	___ 4	___ 5	Strongly Agree

If you marked 1 or 2 (strongly disagree or disagree), please express your opinion why the objective is not appropriate for this level of understanding.

The following item specifications serve to elaborate on the objective presented on the previous page.

Item specifications:

(General description) If a sequence which is formed by the quotient of two polynomials is presented to a student, along with a description of the degree of each of the polynomials (when compared with each other), then the student can classify the sequence in terms of its convergence or divergence.

(Stimulus attributes)

1. A formula for the quotient of two polynomials will be given.
2. The degree of the numerator will be p and the denominator q. The relationship between p and q will be given.
3. The coefficients of n^p and n^q will be natural numbers greater than 1 and will have no common factors.
4. A phrase indicating that a response relating to the limit of the sequence is desired will be given.

(Response attribute) The responses will be phrases indicating that the limit

a) is the quotient of the coefficients of n^p and n^q.

b) is 0.

c) does not exist because the terms are not bounded.

d) cannot be determined from the information provided.

(Sample item) If $a_n = \dfrac{3n^p - 5n^{p-3} + \cdots + n - 7}{4n^q + n^{q-1} + \cdots - n + 2}$,

where $p > q$ (and the denominator is never 0), then $\lim\limits_{n \to \infty} a_n$

a) is 3/4.

b) is 0.

c) does not exist because the values of the terms of the sequence are not bounded.

d) cannot be determined from the information provided.

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The item specifications are appropriate for this objective.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The sample item is consistent with the item specifications.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

<u>Item Specifications</u>:

(General description) The student can classify sequences of the form r^n in terms of their convergence or divergence.

(Stimulus attributes)

1. $a_n = r^n$ will be given.
2. Values for r will be given.
3. The stimulus will indicate that a response concerning the limit of the sequence is desired.

(Response attribute) The responses will be :

a) is 1.

b) is 0.

c) does not exist because the values of the sequence are not bounded.

d) cannot be determined from the information provided above.

(Sample item) If $a_n = r^n$, where $0 < r < 2$, then $\lim_{n \to \infty} a_n$

a) is 1.

b) is 0.

c) does not exist because the values of the terms of the sequence are not bounded.

d) cannot be determined from the information provided above.

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

<u>Objective</u>: The student can recognize valid justification for the divergence of a sequence.

Please react to the following statement by checking the response which most accurately reflects your opinion.

This objective is appropriate for understanding the limit of a sequence at the comprehension level.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

If you marked 1 or 2 (strongly disagree or disagree), please express your opinion why the objective is not appropriate for this level of understanding.

The following item specifications serve to elaborate on the objective presented on the previous page.

Item specifications:

(General description)	Given an example of a sequence which is not bounded and has two cluster points, the student can identify the reason that the sequence diverges.
(Stimulus attributes)	1. The nth term for a sequence with infinitely many values equal to c and infinitely many values equal to k will be given (where c and k are constants). 2. The stimulus will be an incomplete sentence which will be made complete by adding the response.
(Response attribute)	The responses will be phrases that indicate that: a) the limit is c. b) the limit is k. c) the limits are c and k. d) there is no limit due to unboundedness. e) there is no limit due to there being more than one cluster point. (The term "cluster point" will not be used.)
(Sample item)	The sequence whose nth term is defined by cos $n\pi$ has a) a limit equal to 1. b) a limit equal to -1. c) limits of 1 and -1. d) has no limit because infinitely many terms of the sequence have the value 1 and infinitely many terms of the sequence have the value -1.

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The item specifications are appropriate for this objective.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The sample item is consistent with the item specifications.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

<u>Item specifications</u>:

(General description) Given the graph of a sequence whose limit does not exist and which has no cluster points, the student can identify that the sequence diverges because it is unbounded.

(Stimulus attributes)

1. A number line will be shown on which appears initial values of a sequence which is not bounded. The general expression for the nth term will appear.

2. The phrase will refer to the number line and will be made complete by adding the response.

(Response attribute) The responses will be phrases indicating that:

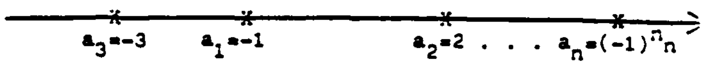
a) the limit is zero.

b) the limit is the general expression for the nth term.

c) there is no limit because there are infinitely many positive values and infinitely many negative values.

d) there is no limit due to unboundedness.

(Sample item)



The sequence whose graph is pictured above

a) has a limit of 0.

b) has a limit of $(-1)^n n$.

c) does not have a limit because there are infinitely many positive values and infinitely many negative values.

d) does not have a limit because the values of the terms are not bounded.

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The item specifications are appropriate for this objective.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The sample item is consistent with the item specifications.

Strongly Disagree	1	2	3	4	5	Strongly Agree

Objective: In the context of using limits, the student exhibits good "number sense."

Please react to the following statement by checking the response which most accurately reflects your opinion.

This objective is appropriate for understanding the limit of a sequence at the comprehension level.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

If you marked 1 or 2 (strongly disagree or disagree), please express your opinion why the objective is not appropriate for this level of understanding.

The following item specifications serve to elaborate on the objective presented on the previous page.

Item specifications:

(General description)	*The student recognizes that the limit is unique.*
(Stimulus attribute)	A statement that the limit of a sequence (if it exists) is or is not a unique number will be made.
(Response attribute)	The choices will be "True" or "False."
(Sample item)	True or False (Circle one). If $\lim_{n \to \infty} a_n = L_1$ and $\lim_{n \to \infty} a_n = L_2$, then $L_1 = L_2$.

Please react to the following statements by *checking* the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The item specifications are appropriate for this objective.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The sample item is consistent with the item specifications.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

Item specifications:

(General description)	In the context of determining limits, the student can demonstrate the

knowledge that infinity is not a very large number.

(Stimulus attributes)

1. A formula for a_n will be given in which the sequence values change after at least a million terms.

2. The rule of correspondence will involve a sequence which is not bounded and a convergent sequence. However, one of these sequences will be terminated after a finite number of terms, while the other will be missing the corresponding number of initial terms.

3. The stimulus will indicate that a response concerning the limit of the sequence is desired.

(Response attribute)

The responses will be phrases that indicate that the limit

a) is the number found by evaluating a_n at the value for which the rule of correspondence changes.

b) is the number found by incorrectly evaluating the formula at the value for which the rule of correspondence changes.

c) is the limit of the sequence, or in the case of the rule of correspondence for which the limit would have existed if the rule had not been changed, it is that limit.

d) does not exist because it is unbounded.

(Sample item)

If

$$a_n = \begin{cases} n^p & \text{for } n \leq 1{,}000{,}000{,}000{,}000 \\ \dfrac{1}{n^p} & \text{for } n > 1{,}000{,}000{,}000{,}000 \end{cases} \quad \text{(where } p > 0$$

then $\lim_{n \to \infty} a_n$

a) is $1{,}000{,}000{,}000{,}000^P$.

b) is $\frac{1}{1{,}000{,}000{,}000{,}000^P}$.

c) is 0.

d) does not exist because the sequence is unbounded.

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

<u>Objective</u>: The student can classify sequences in terms of their convergence/divergence when the underlying principles involve basic limit theorems.

Please react to the following statement by checking the response which most accurately reflects your opinion.

This objective is appropriate for understanding the limit of a sequence at the comprehension level.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

If you marked 1 or 2 (strongly disagree or disagree), please express your opinion why the objective is not appropriate for this level of understanding.

The following item specifications serve to elaborate on the objective presented in the previous page.

Item specifications:

(General description) The student can classify every bounded monotonic sequence in terms of convergence.

(Stimulus attributes)

1. The following conditions will be given: $a_n < a_{n+1}$ and $-M < a_n < M$. ("$\leq$" can be substituted for "<" in either or both cases.)

2. The stimulus will indicate that a response concerning the limit of the sequence is desired.

(Response attribute) The responses will indicate that the limit

a) is M.

b) is M and -M.

c) does not exist because there are infinitely many values near M and -M.

d) exists but cannot be determined from the information provided.

(Sample item) If $a_n \leq a_{n+1}$ and $-M \leq a_n < M$ for all values of n, then $\lim_{n\to\infty} a_n$

a) is M.

b) is M and -M.

c) does not exist because there are infinitely many terms of the sequence with values near M and -M.

d) exists but cannot be determined from the information provided.

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The item specifications are appropriate for this objective.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The sample item is consistent with the item specifications.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

<u>Item specifications</u>:

(General description) The student can classify sequences that are formed by adding/subtracting/multiplying/dividing the terms of two sequences when information concerning convergence/divergence of each of the two sequences is known.

(Stimulus attributes)

1. Information about the convergence/divergence of each of two sequences will be given.

2. The rule of correspondence for a third sequence will be given. It will involve only one operation which will be limited to addition, subtraction, multiplication, or division.

(Response attribute) The responses will indicate that

a) the limit of the third sequence exists.

b) the limit of the third sequence does not exist.

c) that the existence of the limit of the third sequence cannot be determined from the information provided.

(Sample item) c_n is the general expression for the nth term of a sequence. If $c_n = a_n + b_n$ and $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ do not exist, then

a) the $\operatorname{limit}_{n\to\infty} c_n$ exists.

b) the $\operatorname{limit}_{n\to\infty} c_n$ does not exist.

c) the existence of $\operatorname{limit}_{n\to\infty} c_n$ cannot be determined without further information.

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

Item specifications:

(General description) The student can classify subsequences in terms of their convergence or divergence when relevant information

	about the sequence and subsequence is given.
(Stimulus attributes)	1. The stimulus will specify the convergence/divergence of the original sequence. In the case of convergence, the limit L will be given.
	2. The method of obtaining the subsequence will be specified.
	3. The stimulus will indicate that a response concerning the limit of the subsequence is desired.
(Response attributes)	The responses will be that the limit
	a) is L.
	b) is a number different from L.
	c) does not exist.
	d) cannot be determined from the information provided.
(Sample item)	If $\lim_{n\to\infty} a_n = L$ and $b_n = a_{5n}$, then $\lim_{n\to\infty} b_n$
	a) is L.
	b) is a number different from L.
	c) does not exist.
	d) cannot be determined from the information provided above.

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree	1	2	3	4	5	Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree	1	2	3	4	5	Strongly Agree

Item specifications:

(general description) The student can classify a sequence which is sandwiched between two sequences, both of which converge to the same limit, as one for which the limit exists.

(Stimulus attributes)

1. The following information will be given:

   $a_n < c_n < b_n$ for all values of n and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$.

2. A phrase indicating that a response concerning these sequences is desired.

(Response attribute) The responses will be

a) $\lim_{n\to\infty} a_n < \lim_{n\to\infty} c_n$.

b) $\lim_{n\to\infty} c_n = L$.

c) $\lim_{n\to\infty} c_n$ does not exist because there are infinitely many values near a and b.

d) the existence of the $\lim_{n\to\infty} c_n$ cannot be determined from the information provided.

(Sample item) If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$ and if $a_n < c_n < b_n$ for all values of n, then

a) $\lim_{n\to\infty} a_n < \lim_{n\to\infty} c_n$.

b) $\lim_{n\to\infty} c_n = L$.

c) $\lim_{n\to\infty} c_n$ does not exist because there are infinitely many values near a and b.

d) the existence of $\lim_{n\to\infty} c_n$ cannot be determined from the information provided.

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

Objective: The student can make judgments concerning the behavior of the terms of a convergent sequence.

Please react to the following statement by checking the response which most accurately reflects your opinion.

This objective is appropriate for understanding the limit of a sequence at the comprehension level.

Strongly Disagree						Strongly Agree
	___ 1	___ 2	___ 3	___ 4	___ 5	

If you makred 1 or 2 (strongly disagree or disagree), please express your opinion why the objective is not appropriate for this level of understanding.

The following item specifications serve to elaborate on the objective presented on the previous page.

<u>Item specifications</u>:

(General description)	The student recognizes that the behavior of a finite number of terms of a sequence does not affect the limit.
(Stimulus attributes)	1. A situation describing two sequences will be given. One of the sequences will be formed by altering a finite number of terms of the other sequence. 2. The information that the original sequence is convergent will be given. 3. A statement that the limit of the newly formed sequence will be similarly altered will be given.
(Response attribute)	The choices will be "True" or "False."
(Sample item)	True or False (Circle one). Let $b_n = a_n - 5$ for $n < 100$ and $b_n = a_n$ for $n \geq 100$. If $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} b_n = L - 5$.

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The item specifications are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

Item specifications:

(General description) The student recognizes that the limit of a sequence need not be one of the values of the sequence.

(Stimulus attributes) 1. The statement will be that the existence of the limit of a sequence means that at least one of the values of the sequence must equal that limit.

2. Both symbolism and descriptive phrases will be used.

(Response attribute) The choices will be "True" or "False."

(Sample item) True or False (Circle one).

If $\lim_{n \to \infty} a_n = L$, then $L = a_i$ for some $i = 1, 2, 3, \ldots, n, \ldots$ (That is, if the limit exists, at least one term must have a value equal to the limit.)

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The sample item is consistent with the item specifications.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

Item specifications:

(General description)	The student recognizes that the a_{n+1} need not be closer to the limit of a convergent sequence than a_n.
(Stimulus attribute)	A statement will be made that it is (not) necessary for consecutive terms of a sequence to be closer and closer to the limit.
(Response attribute)	The choices will be "True" or "False."
(Sample item)	True or False (Circle one). For any sequence $\{a_n\}$ whose limit is L, a_{m+1} is closer to L than a_m.

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The item specifications are appropriate for this objective.

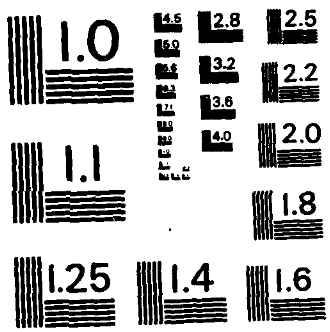
Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The sample item is consistent with the item specifications.

Strongly Disagree	1	2	3	4	5	Strongly Agree

AD-A139 399 DEVELOPING AND MEASURING AN UNDERSTANDING OF THE CONCEPT OF THE LIMIT OF A SEQUENCE(U) AIR FORCE INST OF TECH WRIGHT-PATTERSON AFB OH T A BRATINA 1983 4/5

UNCLASSIFIED AFIT/CI/NR-83-90D F/G 5/10 NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

Recall that all of the objectives appear on the last page of this package. This page was intended for your quick use and to provide the "big picture" for understanding the concept of the limit of a sequence. You may want to glance at it before checking your response to the next statement.

The objectives are appropriate for describing the understanding of the limit of a sequence at the comprehension level as this level was described earlier.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

If you answered 1 or 2 (strongly disagree or disagree), please describe briefly your difficulty.

FORMAL LEVEL

FORMAL LEVEL: This level of understanding requires that the student be able to communicate the precise definition of the limit of a sequence and demonstrate a knowledge of the relationships among the component parts of the definition.

Objectives:

1. The student can demonstrate a knowledge of the relationship among the component parts of the "ε/N definition."

2. The student can discriminate between right and wrong statements purporting to be definitions for $\lim_{n \to \infty} a_n = L$ and $\lim_{n \to \infty} a_n \neq L$.

3. The student can state the definitions of $\lim_{n \to \infty} a_n = L$ and $\lim_{n \to \infty} a_n \neq L$.

4. The student can prove or disprove that $\lim_{n \to \infty} a_n = L$, for cases in which the proofs involve only elementary arithmetic and/or algebraic techniques.

The objectives are described in more detail (by the use of item specifications) on the following pages.

<u>Objective</u>: The student can demonstrate a knowledge of the relationship among the component parts of the "ε/N definition."

Please react to the following statement by checking the response which most accurately reflects your opinion.

This objective is appropriate for understanding the limit of a sequence at the formal level.

Strongly Disagree						Strongly Agree
	___ 1	___ 2	___ 3	___ 4	___ 5	

If you marked 1 or 2 (strongly disagree or disagree), please express your opinion why the objective is not appropriate for this level of understanding.

The following item specifications serve to elaborate on the objective presented on the previous page.

Item specifications:

(General description)	The student can demonstrate a knowledge that N is a function of ϵ in the "ϵ/N definition."
(Stimulus attributes)	1. Statements will be made about N and two different values of ϵ. 2. A statement will be presented which is to be judged in terms of whether it can be validly derived from the given information. 3. Inequality statements will be used for this item.
(Response attribute)	The choices will be "True" or "False."
(Sample item)	True or False (Circle one). The following information is known about a particular sequence whose nth term has the value a_n: a) N, ϵ_1, and ϵ_2 are positive numbers with $\epsilon_1 > \epsilon_2$. b) If $n > N$, then $L - \epsilon_1 < a_n < L + \epsilon_1$. The following statement can be derived from this information: If $n > N$, then $L - \epsilon_2 < a_n < L + \epsilon_2$.

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

<u>Item specifications</u>:

(General description) The student can demonstrate a knowledge of the relationship between n and N in the "ε/N definition."

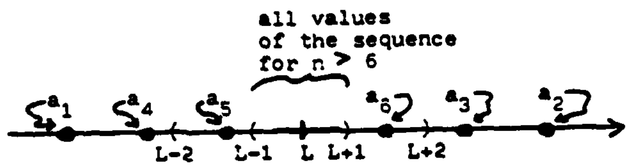
(Stimulus attributes)

1. A number line will be presented with the item showing two different intervals containing L.
2. A statement will be made concerning the number of terms required for a_n to be in the larger interval.
3. The statement will involve translating the graphical information into absolute value notation.

(Response attribute) The choices will be "True" or "False."

(Sample item) True or False (Circle one).

The limit of the sequence (pictured on the number line below) is given to be L. For this same sequence, whenever $n > 4$, then $|a_n - L| < 2$.



Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

<u>Item specifications</u>:

(General description) The student can demonstrate a knowledge that N in the "ε/N definition" depends on the particular sequence (as well as ε) under discussion.

(Stimulus attributes)

1. The statement will involve a situation in which two different sequences each converge to the same limit.

2. Statements will be made to the effect that the number of terms required for each sequence to be within a specified distance of the limit will be the same.

(Response attribute)

The choices will be "True" or "False."

(Sample item)

True or False (Circle one).

$\lim_{n\to\infty} a_n = L$ and $|a_n - L| < \varepsilon$ whenever $n > \frac{1}{\varepsilon}$. Therefore, if $\lim_{n\to\infty} b_n = L$ then $|b_n - L| < \varepsilon$ whenever $n > \frac{1}{\varepsilon}$.

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

Objective: The student can discriminate between right and wrong statements purporting to be definitions for $\lim_{n \to \infty} a_n = L$ and $\lim_{n \to \infty} a_n \neq L$.

Please react to the following statement by checking the response which most accurately reflects your opinion.

This objective is appropriate for understanding the limit of a sequence at the formal level.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

If you marked 1 or 2 (strongly disagree or disagree), please express your opinion why the objective is not appropriate for this level of understanding.

The following item specifications serve to elaborate on the objective presented on the previous page.

Item specifications:

(General description)	The student can determine if a statement relating to the limit of a sequence is equivalent to $\lim_{n \to \infty} a_n \neq L$.
(Stimulus attributes)	1. The stimulus will indicate that two statements are equivalent.
	2. One statement will be $\lim_{n \to \infty} a_n \neq L$.
	3. A second statement will be presented.
(Response attribute)	The choices will be "True" or "False."
(Sample item)	True or False (Circle one).
	The following are equivalent statements:
	a) $\lim_{n \to \infty} a_n \neq L$
	b) For some $\epsilon > 0$ and for all N, there exists an n such that $\lvert a_n - L\rvert \geq \epsilon$ whenever $n > N$.

Please react to the following statement by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this *objective*.

Strongly Disagree	1	2	3	4	5	Strongly Agree

The *sample* item is consistent with the item specifications.

Strongly Disagree	1	2	3	4	5	St[illegible]gly A[illegible]

<u>Item specifications</u>:

(General description) The student can dete[illegible] if a statement involving t[illegible]mit of a sequence is equivalent to $\text{limit}_{n \to \infty} a_n = L$.

(Stimulus attributes)

1. The stimulus will indicate that two statements are equivalent.
2. One statement will be $\lim_{n \to \infty} a_n = L$.
3. A second statement will be presented.

(Response attribute) The choices will be "True" or "False."

(Sample item) True or False (Circle one).

The following are equivalent statements:

a) $\text{limit}_{n \to \infty} a_n = L$

b) For each n, $|a_n - L| \geq |a_{n+1} - L|$ where L is the limit of the sequence whose nth term is a_n.

Please react to the following statement by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The item specifications are appropriate for this objective.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The sample item is consistent with the item specifications.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

Objective: The student can state the definitions of $\lim\limits_{n \to \infty} a_n = L$ and $\lim\limits_{n \to \infty} a_n \neq L$.

Please react to the following statement by checking the response which most accurately reflects your opinion.

This objective is appropriate for understanding the limit of a sequence at the formal level.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

If you marked 1 or 2 (strongly disagree or disagree), please express your opinion why the objective is not appropriate for this level of understanding.

The following item specifications serve to elaborate on the objective presented on the previous page.

Item specifications:

(General description)	The student can state the definition of $\lim_{n \to \infty} a_n \neq L$.
(Stimulus attributes)	1. The student will be directed to write the response in the "space provided below" the stimulus. 2. The stimulus will indicate that a precise definition for $\lim_{n \to \infty} a_n \neq L$ is desired.
(Response attribute)	A blank space (approximately two inches wide) will follow the stimulus.
(Sample item)	In the space provided below, give a precise definition for "$\lim_{n \to \infty} a_n \neq L$" where a_n is the general expression for the nth term of a sequence.

Please react to the following statement by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The item specifications are appropriate for this objective.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

The sample item is consistent with the item specifications.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

Item specifications:

(General description)	The student can state the definition of the limit of a sequence.
(Stimulus attributes)	1. The student will be directed to write the response in the "space provided below" the stimulus.
	2. The stimulus will indicate that a precise definition for the limit is desired.
(Response attribute)	A blank space (approximately two inches wide) will follow the stimulus.
(Sample item)	In the space provided below, give a precise definition for "$\lim_{n \to \infty} a_n = L$" where a_n is the general expression for the nth term of the sequence.

Please react to the following statement by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree	1	2	3	4	5	Strongly Agree

Objective: The student can prove or disprove that $\lim_{n \to \infty} a_n = L$, for cases in which the proofs involve only elementary arithmetic and/or algebraic techniques.

Please react to the following statement by checking the response which most accurately reflects your opinion.

This objective is appropriate for understanding the limit of a sequence at the formal level.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

If you marked 1 or 2 (strongly disagree or disagree), please express your opinion why the objective is not appropriate for this level of understanding.

The following item specifications serve to elaborate on the objective presented on the previous page.

Item specifications:

(General description) The student can formally prove $\lim_{n \to \infty} a_n \neq L$ for a particular sequence and a particular value of L.

(Stimulus attributes)

1. The directions will indicate that a "formal proof" is required.

2. The example that will be given will be of the form $\lim_{n \to \infty} a_n \neq L$, where $a_n = \frac{c}{n}$ for a natural number c and $0 < L < .1$.

(Response attribute) A blank space (approximately two inches wide) will be left below the stimulus.

(Sample item) Provide a formal proof that $\lim_{n \to \infty} \frac{5}{n} \neq \frac{1}{100}$.

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

Item specifications:

(General description) The student can prove $\lim_{n\to\infty} a_n = L$ for a particular sequence.

(Stimulus attributes) 1. The stimulus will be a statement saying to prove $\lim_{n\to\infty} a_n = L$.

2. a_n will be of the form $\frac{c}{n^2}$ or $\frac{c}{\sqrt{n}}$ where c is a positive number. The value for L will be given.

(Response attribute) A blank space (approximately two inches wide) will be left below the stimulus.

(Sample item) Prove: $\lim_{n\to\infty} \frac{3}{n^2} = 0$.

Please react to the following statements by checking the response which most accurately reflects your opinion. (When indicating your choice, realize that item specifications include the general description, stimulus attributes, response attributes, and sample item.)

The item specifications would be clearly understood by an item writer.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The item specifications are appropriate for this objective.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

The sample item is consistent with the item specifications.

Strongly Disagree ___ 1 ___ 2 ___ 3 ___ 4 ___ 5 Strongly Agree

Recall that all of the objectives appear on the last page of this package. This page was intended for your quick use and to provide the "big picture" for understanding the concept of the limit of a sequence. You may want to glance at it before checking your response to the next statement.

The objectives are appropriate for describing the understanding of the limit of a sequence at the formal level as this level was described earlier.

Strongly Disagree	___	___	___	___	___	Strongly Agree
	1	2	3	4	5	

If you answered 1 or 2 (strongly disagree or disagree), please describe briefly your difficulty.

The final statement to which you will respond regards the levels of understanding. Please express your opinion to the following:

The levels are appropriate for describing the full range of behaviors underlying the understanding of of the limit of a sequence.

Strongly Disagree ___ ___ ___ ___ ___ Strongly Agree
1 2 3 4 5

If you checked 1 or 2 (strongly disagree or disagree), please suggest any additional level(s) and corresponding objective(s) that should be included.

Thank you for your cooperation in completing this package:

INTUITIVE LEVEL: This level of understanding requires the student to demonstrate the ability to comprehend situations which involve or imply the contexts from which the concept of limit of a sequence will develop. Vocabulary usage is nonrigorous, corresponding to pre-formal experiences with limits.

Objectives:

1. The student can demonstrate a knowledge of convergence in situations involving "a limit object" of an implied sequence.
2. The student can make appropriate judgments about situations involving divergence.
3. The student can demonstrate a knowledge of how successive approximations can be used in measurement.

PRODUCTION LEVEL: This level of understanding requires the student to be able to produce an example of a sequence which meets certain prescribed conditions. If such an example is possible; and to be able to state that no such example will satisfy the conditions, if it is not possible. (The prescribed conditions will be of a nature that can easily be understood by the student.)

Objectives:

1. The student can produce an example of a divergent sequence which meets certain prescribed conditions.
2. The student can produce an example of a convergent sequence which meets certain prescribed conditions.

COMPREHENSION LEVEL: This level of understanding requires that the student demonstrate knowledge of the general principles which characterize the convergence/divergence of a sequence.

Objectives:

1. The student can classify sequences in terms of their convergence or divergence when the rule of correspondence is presented in general terms.
2. The student can recognize valid justification for the divergence of a sequence.
3. In the context of using limits, the student exhibits good "number sense."
4. The student can classify sequences in terms of their convergence/divergence when the underlying principles involve basic limit theorems.
5. The student can made judgments concerning the behavior of the terms of a convergent sequence.

IDENTIFICATION LEVEL: This level of understanding requires the student to be able to classify sequences in terms of their convergence or divergence, using only elementary methods (inspection or elementary arithmetic, algebraic, or trigonometric properties). In the case of convergence, the student can specify the limit.

Objectives:

1. The student can determine the limit of certain convergent sequences by using only inspection.
2. The student can classify examples of divergent sequence as sequences for which the limit does not exist.
3. The student can classify a sequence expressed as the ratio of two polynomials in terms of its convergence or divergence. In the case of convergence, the student can specify the limit.
4. The student can classify sequences involving the basic trigonometric functions, e^n and $e^{1/n}$, and $\sqrt{n}$ in terms of their convergence or divergence. In the case of convergence, the student can specify the limit.

FORMAL LEVEL: This level of understanding requires that the student be able to communicate the precise definition of the limit of a sequence and demonstrate a knowledge of the relationships among the component parts of the definition.

Objectives:

1. The student can demonstrate a knowledge of the relationship among the component parts of the "ϵ/N definition."
2. The student can discriminate between right and wrong statements purporting to be definitions for $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n \neq L$.
3. The student can state the definitions of $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n \neq L$.
4. The student can prove or disprove that $\lim_{n\to\infty} a_n = L$, for cases in which the proofs involve only elementary arithmetic and/or algebraic techniques.

APPENDIX C

RESPONSES FROM NATIONALLY RECOGNIZED EXPERTS REGARDING PREREQUISITE SKILLS FOR UNDERSTANDING LIMITS OF SEQUENCES

PREREQUISITES

for

Understanding Limits of Sequences

Please indicate your degree of agreement by circling one of the numbers at the right of each topic. The numbers correspond to:

1: I strongly agree that a student must understand this in order to understand limits of sequences.

2: I agree that a student must understand this in order to understand limits of sequences.

3: I have no opinion on whether this topic should be included as a prerequisite subskill for understanding limits of sequences.

4: I do not agree that a student must understand this in order to understand limits of sequences.

5: I strongly disagree with the statement that a student must understand this in order to understand limits of sequences.

Topic	Importance
Absolute Value related to:	
i) Distance	1 2 3 4 5
ii) Inequalities	1 2 3 4 5
iii) Segments or intervals	1 2 3 4 5
Sequences	
i) Terms	1 2 3 4 5
ii) Relation to functions	1 2 3 4 5

Algebraic Knowledge related to:

i)	Inequalities	1 2 3 4 5
ii)	Zero ?	1 2 3 4 5
iii)	Positive or negative numbers	1 2 3 4 5
iv)	Common manipulations	1 2 3 4 5
v)	Substituting for variables	1 2 3 4 5

Quantification

i)	Universalizations	1 2 3 4 5
ii)	Existential statements	1 2 3 4 5
iii)	Combinations of quantifiers, multiply quantified	1 2 3 4 5
iv)	Denials	1 2 3 4 5

Same for me. Not fancy stuff, though.

Conditional Sentences

i)	Denials	1 2 3 4 5
ii)	Judging truth values	1 2 3 4 5
iii)	Counterexamples	1 2 3 4 5

Nothing very fancy.

Are there any other topics that you consider prerequisite subskills for understanding the concept of the limit of a sequence? If so, please list them below.

Name R G Bartle

PREREQUISITES

for

Understanding Limits of Sequences

Please indicate your opinion concerning the relevance of each topic to the understanding of limits of sequences by circling one of the numbers at the right. The numbers correspond to:

1: The understanding of this topic is essential to understand limits of sequences.

2: The understanding of this topic is desirable to understand limits of sequences.

3: The understanding of this topic is irrelevant to the understanding of limits of sequences.

It would be especially helpful if you would comment about any topic which you reject as a prerequisite for understanding limits of sequences, i.e., for those topics for which you circled "3". There is a space provided at the end for such comments.

Topic	Relevance		
Absolute Value related to:			
i) Distance	1	2	3
ii) Inequalities	1	2	3
iii) Segments or intervals	1	2	3
Sequences			
i) Terms	1	2	3
ii) Relation to functions	1	2	3
Algebraic Knowledge related to:			
i) Inequalities	1	2	3
ii) Zero	1	2	3
iii) Positive or negative numbers	1	2	3
iv) Common manipulations	1	2	3
v) Substituting for variables	1	2	3

Quantification

i)	Universalizations	1	2	3
ii)	Existential statements	1	2	3
iii)	Combinations of quantifiers	1	2	3
iv)	Denials	1	2	3

Conditional Sentences

i)	Denials	1	2	3
ii)	Judging truth values	1	2	3

Counterexamples	1	2	3

Comments concerning those topics which are not relevant to understanding limits of sequences.

Are there any other topics that you consider prerequisite subskills for understanding the concept of the limit of a sequence? If so, please list them below.

I was frustrated in that I wanted to answer dependent on the level, but couldn't, so I changed the levels on [illegible] to speak as if all were desired. For example, absolute value would get a 3 on "Intuitive" but a 1 on "Formal Level"

Name

PREREQUISITES

for

Understanding Limits of Sequences

Please indicate your opinion concerning the relevance of each topic to the understanding of limits of sequences by circling one of the numbers at the right. The numbers correspond to:

1: The understanding of this topic is essential to understand limits of sequences.

2: The understanding of this topic is desirable to understand limits of sequences.

3: The understanding of this topic is irrelevant to the understanding of limits of sequences.

It would be especially helpful if you would comment about any topic which you reject as a prerequisite for understanding limits of sequences, i.e., for those topics for which you circled "3". There is a space provided at the end for such comments.

Topic	Relevance		
Absolute Value related to:			
i) Distance	1	2	3
ii) Inequalities	1	2	3
iii) Segments or intervals	1	2	3
Sequences			
i) Terms	1	2	3
ii) Relation to functions	1	2	3
Algebraic Knowledge related to:			
i) Inequalities	1	2	3
ii) Zero	1	2	3
iii) Positive or negative numbers	1	2	3
iv) Common manipulations	1	2	3
v) Substituting for variables	1	2	3

Quantification

i) Universalizations	1	2	3
ii) Existential statements	1	2	3
iii) Combinations of quantifiers	1	2	3
iv) Denials	1	2	3

Conditional Sentences

i) Denials	1	2	3
ii) Judging truth values	1	2	3

Counterexamples	1	2	3

Comments concerning those topics which are not relevant to understanding limits of sequences.

Are there any other topics that you consider prerequisite subskills for understanding the concept of the limit of a sequence? *If so, please list them below.*

Name Lexton Buchanan

PREREQUISITES

for

Understanding Limits of Sequences

Please indicate your degree of agreement by circling one of the numbers at the right of each topic. The numbers correspond to:

1: I strongly agree that a student must understand this in order to understand limits of sequences.

2: I agree that a student must understand this in order to understand limits of sequences.

3: I have no opinion on whether this topic should be included as a prerequisite subskill for understanding limits of sequences.

4: I do not agree that a student must understand this in order to understand limits of sequences.

5: I strongly disagree with the statement that a student must understand this in order to understand limits of sequences.

Topic	Importance
Absolute Value related to:	
i) Distance	1 2 3 4 5
ii) Inequalities	1 2 3 4 5
iii) Segments or intervals	1 2 3 4 5
Sequences	
i) Terms	1 2 3 4 5
ii) Relation to functions	1 2 3 4 5

Algebraic Knowledge related to:

i)	Inequalities	1 2 3 4 5
ii)	Zero	1 2 3 4 5
iii)	Positive or negative numbers	1 2 3 4 5
iv)	Common manipulations	1 2 3 4 5
v)	Substituting for variables	1 2 3 4 5

Quantification

i)	Universalizations	1 2 3 4 5
ii)	Existential statements	1 2 3 4 5
iii)	Combinations of quantifiers, multiply quantified	1 2 3 4 5
iv)	Denials	1 2 3 4 5

Conditional Sentences

i)	Denials	1 2 3 4 5
ii)	Judging truth values	1 2 3 4 5
iii)	Counterexamples	1 2 3 4 5

Are there any other topics that you consider prerequisite subskills for understanding the concept of the limit of a sequence? If so, please list them below.

Name Chiu Yeung Chan

PREREQUISITES

for

Understanding Limits of Sequences

Please indicate your degree of agreement by circling one of the numbers at the right of each topic. The numbers correspond to:

1: I strongly agree that a student must understand this in order to understand limits of sequences.

2: I agree that a student must understand this in order to understand limits of sequences.

3: I have no opinion on whether this topic should be included as a prerequisite subskill for understanding limits of sequences.

4: I do not agree that a student must understand this in order to understand limits of sequences.

5: I strongly disagree with the statement that a student must understand this in order to understand limits of sequences.

I don't feel absolute value is necessarily a prereq for limit. It can still be done in terms of intervals and neighborhoods.

Topic	Importance
Absolute Value related to:	
i) Distance	1 2 3 4 5
ii) Inequalities	1 2 3 4 5
iii) Segments or intervals	1 2 3 4 5
Sequences	
i) Terms	1 2 3 4 5
ii) Relation to functions	1 2 3 4 5

Algebraic Knowledge related to:

i)	Inequalities	1 2 3 4 5
ii)	Zero	1 2 3 4 5
iii)	Positive or negative numbers	1 2 3 4 5
iv)	Common manipulations	1 2 3 4 5
v)	Substituting for variables	1 2 3 4 5

Quantification

i)	Universalizations	1 2 3 4 5
ii)	Existential statements	1 2 3 4 5
iii)	Combinations of quantifiers, multiply quantified	1 2 3 4 5
iv)	Denials	1 2 3 4 5

Conditional Sentences

i)	Denials	1 2 3 4 5
ii)	Judging truth values	1 2 3 4 5
iii)	Counterexamples	1 2 3 4 5

Are there any other topics that you consider prerequisite subskills for understanding the concept of the limit of a sequence? If so, please list them below.

Name A. F Corford

PREREQUISITES

for

Understanding Limits of Sequences

Please indicate your opinion concerning the relevance of each topic to the understanding of limits of sequences by circling one of the numbers at the right. The numbers correspond to:

1: The understanding of this topic is essential to understand limits of sequences.

2: The understanding of this topic is desirable to understand limits of sequences.

3: The understanding of this topic is irrelevant to the understanding of limits of sequences.

It would be especially helpful if you would comment about any topic which you reject as a prerequisite for understanding limits of sequences, i.e., for those topics for which you circled "3". There is a space provided at the end for such comments.

Topic	Relevance
Absolute Value related to:	
i) Distance	1 2 3
ii) Inequalities	1 2 3
iii) Segments or intervals	1 2 3
Sequences	
i) Terms	1 2 3
ii) Relation to functions	1 2 3
Algebraic Knowledge related to:	
i) Inequalities	1 2 3
ii) Zero	1 2 3
iii) Positive or negative numbers	1 2 3
iv) Common manipulations	1 2 3
v) Substituting for variables	1 2 3

Quantification

i) Universalizations	1	2	3
ii) Existential statements	1	2	3
iii) Combinations of quantifiers	1	2	3
iv) Denials	1	2	3

Conditional Sentences

i) Denials	1	2	3
ii) Judging truth values	1	2	3

Counterexamples	1	2	3

Comments concerning those topics which are not relevant to understanding limits of sequences.

Are there any other topics that you consider prerequisite subskills for understanding the concept of the limit of a sequence? If so, please list them below.

Name Dwight B. Goodner

LEVELS OF UNDERSTANDING INTUITIVE IDENTIFICATION PRODUCTION COMPREHENSION FORMAL

Skills are classified for different levels.

PREREQUISITES

for

Understanding Limits of Sequences

Please indicate your opinion concerning the relevance of each topic to the understanding of limits of sequences by circling one of the numbers at the right. The numbers correspond to:

1: The understanding of this topic is <u>essential</u> to understand limits of sequences.

2: The understanding of this topic is <u>desirable</u> to understand limits of sequences.

3: The understanding of this topic is <u>irrelevant</u> to the understanding of limits of sequences.

It would be especially helpful if you would comment about any topic which you reject as a prerequisite for understanding limits of sequences, i.e., for those topics for which you circled "3". There is a space provided at the end for such comments.

Topic	Relevance
Absolute Value related to:	
i) Distance	1 2 3
ii) Inequalities	1 2 3
iii) Segments or intervals	1 2 3
Sequences	
i) Terms	1 2 3
ii) Relation to functions	1 2 3
Algebraic Knowledge related to:	
i) Inequalities	1 2 3
ii) Zero	1 2 3
iii) Positive or negative numbers	1 2 3
iv) Common manipulations	1 2 3
v) Substituting for variables	1 2 3

Essential for formal level

Irrelevant for all others

Essential for all levels

Essential for all levels

	Essential for formal		Irrelevant for all other levels
Quantification			
i) **Universalizations**	1	2	3
ii) **Existential statements**	1	2	3
iii) **Combinations of quantifiers**	1	2	3
iv) **Denials**	1	2	3
Conditional Sentences			
i) **Denials**	1	2	3
ii) **Judging truth values**	1	2	3
Counterexamples	1	2	3

Comments concerning those topics which are not relevant to understanding limits of sequences.

Quantification, absolute values and quantification are necessary only for the formal ϵ-δ level. Many bright students can find limits, produce examples and develop a good feel for limits without those skills

Are there any other topics that you consider prerequisite subskills for understanding the concept of the limit of a sequence? If so, please list them below.

1) GRAPHING SKILL (SOME PEOPLE LIKE TO "SEE" LIMITS).

2) ABILITY TO WORK HEURISTICALLY AND SENSE WHEN AN ESTIMATION APPROACH IS HELPFUL

Name MATT HASSETT

PREREQUISITES

for

Understanding Limits of Sequences

Please indicate your degree of agreement by circling one of the numbers at the right of each topic. The numbers correspond to:

1: I strongly agree that a student must understand this in order to understand limits of sequences.

2: I agree that a student must understand this in order to understand limits of sequences.

3: I have no opinion on whether this topic should be included as a prerequisite subskill for understanding limits of sequences.

4: I do not agree that a student must understand this in order to understand limits of sequences.

5: I strongly disagree with the statement that a student must understand this in order to understand limits of sequences.

Topic	Importance
Absolute Value related to:	
i) Distance	1 2 3 4 5
ii) Inequalities	1 2 3 4 5
iii) Segments or intervals	1 2 3 4 5
Sequences	
i) Terms	1 2 3 4 5
ii) Relation to functions	1 2 3 4 5

Algebraic Knowledge related to:

i)	*Inequalities*	1 2 3 4 5
ii)	Zero	1 2 3 4 5
iii)	Positive or negative numbers	1 2 3 4 5
iv)	Common manipulations	1 2 3 4 5
v)	Substituting for variables	1 2 3 4 5

Quantification

i)	Universalizations	1 2 3 4 5
ii)	Existential statements	1 2 3 4 5
iii)	Combinations of quantifiers, multiply quantified	1 2 3 4 5
iv)	Denials	1 2 3 4 5

Conditional Sentences

i)	Denials	1 2 3 4 5
ii)	Judging truth values	1 2 3 4 5
iii)	Counterexamples	1 2 3 4 5

Are there any other topics that you consider prerequisite subskills for understanding the concept of the limit of a sequence? If so, please list them below.

Name Wolfgang Heil

UNIVERSITY OF VIRGINIA
DEPARTMENT OF MATHEMATICS
[illegible]

[illegible]

Nov. 30, 1982

Mr. Tuiren A. Bratina
1827 Portland Ave.
Tallahassee, FL, 32303.

Dear Mr. Bratina:

I have given quite a bit of thought to your letter about limits of sequences, but what I have to say does not fit into the format of putting check-marks in boxes. So I'll write a fairly long letter, which I hope will be of some use to you.

If "understanding" limits of sequences means learning a definition that is logically adequate to define precisely what it means to say that "the limit of the sequence $x_1, x_2, x_3, \ldots$ is ℓ", practically no skills at all are necessary. If Y is a set, a sequence of members of Y is a function $n \mapsto x_n$ $(n \in \{1, 2, 3, \ldots\})$ such that for each natural number n, x_n is in Y. Let ℓ be a point in Y. To give precision to the vague idea of being "near to ℓ", we assume that $\mathcal{U}$ is a set of subsets of Y, each containing ℓ; these we call "neighborhoods" of ℓ. Then the statement $\lim_{n\to\infty} x_n = \ell$, or "$\ell$ is a limit

2

of x_n as n increases", is that to each neighborhood U of ℓ there corresponds a natural number n_U such that whenever n is a natural number and $n > n_U$, x_n belongs to U. This needs no skill for working with members of Y — in fact, we have not assumed anything about these members, so we can't do anything in this level of generality.

But I think that this is a degradation of the word "understand". For an understanding worthy of the name, one needs to have met an assortment of examples and reached an easy familiarity with them. In fact, it is fundamentally wrong to think of the "general case" as basic, with examples arising from it. Historically and functionally the examples came first. The service of the generalization is to let us comprehend a multitude of examples; it is the examples that have individuality and use. Moreover, there is little profit in just <u>having</u> a sequence. The only point in studying a sequence is to <u>do</u> something with it. To be able to <u>do</u> anything interesting with the x_n and ℓ, the set Y

3

has to have some useful properties; and, pedagogically speaking, it is wise to work with spaces Y whose properties are easily comprehended by ordinary students. The easiest – and, fortunately, the most useful – examples are: Y is the real number system. Y is the set of points in the plane. Y is the set of points in 3-space. Y is the complex number system. Y is the augmented real number system consisting of the real numbers and $+\infty$ and $-\infty$. For the reals, a set U is a neighborhood of l if for some positive k, U consists of all y such that $|y-l| < k$. Your first page of questions consists of quite sensible exercises in this example.

But I am opposed to the use of the symbol $\forall_x$. It cannot have a precise and fully general meaning without a deep study of the foundations of mathematics, and maybe not then! Take for example your 1a. Is "$\forall_x\ x^2 \geq 0$" true or false? Written out as $(\forall_x)(x \text{ real} \Rightarrow x^2 \geq 0)$ it is true

4

If it means "for all x, if x is complex no. then $x^2 \geq 0$" it is false. If it means "for all things x, $x^2 \geq 0$" it is false, but also confusing. Suppose x is the set {2, 5; 13} or is "Christmas" — what's x^2? This page reminds me painfully of the first attempt to eliminate the nonsense in the elementary mathematics of 40 or 50 years ago. There were several text-books (we used Cooley, Gans, Kline & Wahlert) that tried to clean the mess, & most of them devoted a chapter to abstract logic. It was anathema to the students, and the authors tacitly agreed, because nothing in that "important" chapter was ever mentioned again in the book. Your examples such as "For each real number x, there exists some real number y such that x + y = x" are by contrast easily intelligible. Avoid that "$\forall x$" and "$\exists x$".

5. Except for the symbolic quantifiers, the questions in your list should be answerable by any one studying sequences, and in fact by any one studying any part of mathematics or any precise science. But in my view, a student is better off if, when he sees a statment $\lim_{n\to\infty} x_n = 17$ he thinks "If I decide on any tolerable limit of error, I can replace all but finitely many x_n by 17 and not exceed allowed error", rather than if he can dress the statement in all the clothing of formal logic, with learned references to universal quantifiers and absolute values and neighborhoods but doesn't have any gut feeling about how the x_n are behaving.

Yours sincerely

E. J. McShane

I am glad to know that you found my comments useful, but I really did not expect you to bother to respond to me, but just to use what you found helpful and reject the rest. (after all, I am not directing your study!) I do find your responses helpful in understanding what you are doing.

Quarters 4310A
United States Air Force Academy
Colorado Springs, CO 80840

February 3, 1983

I wish you good progress and a successful outcome.

Please forgive me for replying by pencilled notes and comments. But right now I cannot take time to write an organized critique in letter form. Good luck AEM 2/15/83

Dr. Albert E. Meder
P.O. Box 417
Hampton, NJ 08827

Dear Dr. Meder:

I am a Florida State University student who wrote to you a few months ago concerning research I'm doing involving understanding limits of sequences. Your response was very helpful, and I'm trying to address some of the issues which concerned you. Let me apologize first for the long delay since receiving your correspondence. I moved from Florida back to Colorado (where I work), and am just now getting settled. I will be working on my dissertation here in Colorado.

I hope I can alleviate some of the problem areas you pointed out in your remarks to me. I erred in not describing my entire study to you, and I'd like to correct that deficiency in this letter.

There are three aspects to my research:

OK. This puts the matter in helpful perspective.

1. To determine what it means to understand limits of sequences.

By measuring understanding, I presume you hope to determine the level of understanding that the student possesses.

2. To develop an instrument which measures that understanding.

3. To investigate which subskills are prerequisite for understanding limits of sequences. Related to levels of understanding?

In order to accomplish the first "phase" of this research, a panel of experts critiqued objectives which were used to state what it means to understand limits of sequences. The following five levels were found to best categorize these objectives:

This is rather vague, but I think that it is getting at the right idea

INTUITIVE LEVEL—This level of understanding requires the student to demonstrate the ability to comprehend situations which involve or imply the contents from which the concept of limit of a sequence will develop. Vocabulary usage is nonrigorous, corresponding to pre-formal experiences with limits.

I think this means that he recognizes whether a sequence has a limit or not, but can't really communicate how he knows it.

IDENTIFICATION LEVEL—This level of understanding requires the student to be able to classify sequences in terms of their convergence or divergence, using only elementary methods (inspection or elementary arithmetic or algebraic properties). In the case of convergence, the student can specify the limit.

On this level, since he can specify the limit, he can presumably explain informally why it, and not something else, is the limit.

PRODUCTION LEVEL--This level of understanding requires the student to be able to produce an example of a sequence which meets certain prescribed conditions, if such an example is possible; and to be able to state that no such example will satisfy the conditions, if it is not possible. The prescribed conditions will be of a nature that can easily be understood by the student.

I question whether this level is really on a "ladder" with the other four

COMPREHENSION LEVEL--This level of understanding requires that the student demonstrate knowledge of the general principles which characterize the convergence/divergence of a sequence.

Should be something elementary: knowledge of the meaning of terms, without necessarily formalizing this in a definition

I had a very fine professor of mathematics in college, from whom I learned a great deal, who said he was always suspicious of "general principles" in mathematics! So am I

FORMAL LEVEL--This level of understanding requires, in addition to the above knowledge, that the student be able to communicate the precise definition of the limit of a sequence and demonstrate a knowledge of the relationships among the component parts of the definition. See below -

It might be well to mention here that although the importance of such areas as utilizing limit theorems cannot be denied, the definition of understanding limits of sequences was restricted to the five categories mentioned above. Also, it should be realized that students may understand the concept of limit, but not have the mathematical maturity required to deal with some complicated sequences. With that in mind, the types of sequences were those whose ranges were real-valued, and the functions were restricted to those which students of beginning algebra and geometry would be expected to encounter. (As a matter of fact, the panel encouraged me to eliminate some objectives I had which required the students to identify limits of sequences whose rule of correspondence utilized trigonometric functions, because deficiencies in their understanding of these functions would confound the issue of whether they understood limits.)

I concur completely with these decisions

I hope the comments I made concerning what is meant by "understanding limits of sequence", is clear. I'm sorry that I did not provide you with this information in my previous letter.

I believe I have benefitted by a second concern that you expressed in your letter. My checklist reflects changes that you suggested regarding what I should be asking the experts. I wholeheartedly agree that addressing whether the topic is essential, desirable, or irrelevant is the opinion I really went here.

You also wisely questioned the procedure of determining prerequisite skills by use of an opinion poll. Again, I omitted details which should have been provided to you. You suggested an experimental procedure. Indeed, this is what I have in mind. The judges who helped with the formulation of the definition of "understanding of limits of sequences" also helped with the instrument designed to measure this understanding. (Students of the topic also helped with the instrument by taking the test for time and expressed their opinions about directions, format, or anything that they believed to be a problem.) Eventually, the test measuring understanding of limits of sequences was completed, and data has already been collected which supports the validity and reliability of this instrument.

Re the Formal level: "able to formulate, communicate, understand and explain the precise definition --- Then I don't know what the last clause means.

Anybody can memorize an ϵ-δ definition, but understanding its significance and explaining it is something else.

This leads me to the third part of my study. I am seeking to find the correlations between "understanding of sequence limits" and a variety of topics that my major professor (Dr. Herbert Wills) and I and knowledgeable persons such as yourself have identified. To put this in a practical vein, it may turn out, for example, that students' "understanding of limits" scores do not have a high correlation with students' "counterexamples" scores. It would appear that students scoring high on "understanding limits" and low on a tested prerequisite skill would be evidence that that specific tested skill was, in fact, not a prerequisite. On the other hand, a high correlation between the "limits" scores and the scores for a particular suspected prerequisite skill would be ammunition for saying that the skill was indeed a prerequisite. Granted, high correlations don't always allow such conclusive results, but coupled with the opinions of persons with expertise in teaching limits of sequences, I believe a strong case can be made for declaring understanding of a topic as prerequisite for understanding limits of sequences.

At the risk of being overbearing I am sending you another checklist and the instruments that are being used to measure understanding of the prerequisite skills, with the hope that you will respond to the checklist. (You could certainly feel free to comment again on any test items, but I am really including the instruments in case you have any doubts about the meaning of some of the topics I have on the checklist--I found it so hard to describe some prerequisite skills by restricting myself to a single word or phrase.)

Hopefully, I have reacted to all of the concerns you expressed in your letter. I appreciate the generosity you have shown by offering such positive suggestions to me.

Sincerely,

Tuiren A. Bratina

Tuiren A. Bratina

I will add only one comment on your instruments. They seem to be good tests, I have not checked them, item by item, but "sampled" them. My only criticism is an overemphasis on notation and concepts of symbolic logic. (I say that as one who has taught courses in symbolic logic, and who is a long-time member of the Association for Symbolic Logic)

Enclosures:
- (1) Checklist
- (2) Instruments
- (3) Self-addressed, stamped envelope

* I agree that a high score on "understanding limits" and a low score on "a particular suspected prerequisite skill" is very good evidence that the skill is not a prerequisite. I disagree with the statement that a high score on "understanding limits" and a high score on the alleged prerequisite validates the prerequisite. The explanation might be that "understanding limits" enables the testee to score high on the other skill, which may not be a prerequisite at all. Or, indeed, the student may just be very bright, and neither skill is related to the other.

PREREQUISITES

for

Understanding Limits of Sequences

Please indicate your opinion concerning the relevance of each topic to the understanding of limits of sequences by circling one of the numbers at the right. The numbers correspond to:

1: The understanding of this topic is <u>essential</u> to understand limits of sequences.

2: The understanding of this topic is <u>desirable</u> to understand limits of sequences.

3: The understanding of this topic is <u>irrelevant</u> to the understanding of limits of sequences.

It would be especially helpful if you would comment about any topic which you reject as a prerequisite for understanding limits of sequences, i.e., for those topics for which you circled "3". There is a space provided at the end for such comments.

Topic	Relevance
Absolute Value related to:	
i) Distance	1 2 3
ii) Inequalities	1 2 3
iii) Segments or intervals	1 2 3
Sequences	
i) Terms	1 2 3
ii) Relation to functions	1 2 3
Algebraic Knowledge related to:	
i) Inequalities	1 2 3
ii) Zero	1 2 3
iii) Positive or negative numbers	1 2 3
iv) Common manipulations	1 2 3
v) Substituting for variables	1 2 3

Quantification			
i) Universalizations	1	2	3
ii) Existential statements	1	2	3
iii) Combinations of quantifiers	1	2	3
iv) Denials	1	2	3
Conditional Sentences			
i) Denials	1	2	3
ii) Judging truth values	1	2	3
Counterexamples	1	2	3

Comments concerning those topics which are not relevant to understanding limits of sequences.

One can understand limits of sequences on all of the five levels you set forth without having any formal knowledge whatsoever of formal logic.

Are there any other topics that you consider prerequisite subskills for understanding the concept of the limit of a sequence? If so, please list them below.

* The concept of absolute value is essential. The particular applications may or may not be essential. They surely will be helpful. I tried to indicate this by my 3, 1, 2 ratings.

Name [illegible]

I did this w.o. looking over the test!

PREREQUISITES

for

Understanding Limits of Sequences

Please indicate your degree of agreement by circling one of the numbers at the right of each topic. The numbers correspond to:

1: I strongly agree that a student must understand this in order to understand limits of sequences.

2: I agree that a student must understand this in order to understand limits of sequences.

3: I have no opinion on whether this topic should be included as a prerequisite subskill for understanding limits of sequences.

4: I do not agree that a student must understand this in order to understand limits of sequences.

5: I strongly disagree with the statement that a student must understand this in order to understand limits of sequences.

Topic	Importance
Absolute Value related to:	
i) Distance	1 2 3 4 5
ii) Inequalities	1 2 3 4 5
iii) Segments or intervals	1 2 3 4 5
Sequences	
i) Terms	1 2 3 4 5
ii) Relation to functions	1 2 3 4 5

"understand limits of sequences" can mean one or more of three things

Red : Conceptually

Black: Operationally — computing limits

Blue: Prove theorems about limits of sequences

Algebraic Knowledge related to:

i)	Inequalities	1 2 3 4 5
ii)	Zero	1 2 3 4 5
iii)	Positive or negative numbers	1 2 3 4 5
iv)	Common manipulations	1 2 3 4 5
v)	Substituting for variables	1 2 3 4 5

Quantification

i)	Universalizations	1 2 3 4 5
ii)	Existential statements	1 2 3 4 5
iii)	Combinations of quantifiers, multiply quantified	1 2 3 4 5
iv)	Denials	1 2 3 4 5

Conditional Sentences

i)	Denials	1 2 3 4 5
ii)	Judging truth values	1 2 3 4 5
iii)	Counterexamples	1 2 3 4 5

Are there any other topics that you consider prerequisite subskills for understanding the concept of the limit of a sequence? If so, please list them below.

I believe that a major prequisite in this, and many topics, is to not clutter it up with anything more than the minimum terminology, and otherwise to have had the prerequisite of frequently coming to grips with (and finding) the crux of a problem.

Name Dennis Sentilles

PREREQUISITES

for

Understanding Limits of Sequences

Please indicate your degree of agreement by circling one of the numbers at the right of each topic. The numbers correspond to:

1: I strongly agree that a student must understand this in order to understand limits of sequences.

2: I agree that a student must understand this in order to understand limits of sequences.

3: I have no opinion on whether this topic should be included as a prerequisite subskill for understanding limits of sequences.

4: I do not agree that a student must understand this in order to understand limits of sequences.

5: I strongly disagree with the statement that a student must understand this in order to understand limits of sequences.

Topic	Importance
Absolute Value related to:	
i) Distance	1 2 3 4 5
ii) Inequalities	1 2 3 4 5
iii) Segments or intervals	1 2 3 4 5
Sequences	
i) Terms	1 2 3 4 5
ii) Relation to functions	1 2 3 4 5

Algebraic Knowledge related to:

i)	Inequalities	1 2 3 4 5
ii)	Zero	1 2 3 4 5
iii)	Positive or negative numbers	1 2 3 4 5
iv)	Common manipulations	1 2 3 4 5
v)	Substituting for variables	1 2 3 4 5

Quantification

i)	Universalizations	1 2 3 4 5
ii)	Existential statements	1 2 3 4 5
iii)	Combinations of quantifiers, multiply quantified	1 2 3 4 5
iv)	Denials	1 2 3 4 5

Conditional Sentences

i)	Denials	1 2 3 4 5
ii)	Judging truth values	1 2 3 4 5
iii)	Counterexamples	1 2 3 4 5

Are there any other topics that you consider prerequisite subskills for understanding the concept of the limit of a sequence? If so, please list them below.

Name Carl Whitman

PREREQUISITES

for

Understanding Limits of Sequences

Please indicate your degree of agreement by circling one of the numbers at the right of each topic. The numbers correspond to:

1: I strongly agree that a student must understand this in order to understand limits of sequences.

2: I agree that a student must understand this in order to understand limits of sequences.

3: I have no opinion on whether this topic should be included as a prerequisite subskill for understanding limits of sequences.

4: I do not agree that a student must understand this in order to understand limits of sequences.

5: I strongly disagree with the statement that a student must understand this in order to understand limits of sequences.

Topic	Importance
Absolute Value related to:	
i) Distance	1 2 3 4 5
ii) Inequalities	1 2 3 4 5
iii) Segments or intervals	1 2 3 4 5
Sequences	
i) Terms	1 2 3 4 5
ii) Relation to functions	1 2 3 4 5

Algebraic Knowledge related to:

i)	Inequalities	1 2 3 4 5
ii)	Zero	1 2 3 4 5
iii)	Positive or negative numbers	1 2 3 4 5
iv)	Common manipulations	1 2 3 4 5
v)	~~Substitution~~ variables	1 2 3 4 5
vi)	Evaluation of expressions	1

Quantification

not necessarily ∀ + E

i)	Universalizations	1 2 3 4 5
ii)	Existential statements	1 2 3 4 5
iii)	Combinations of quantifiers, multiply quantified	1 2 3 4 5
iv)	Denials	1 2 3 4 5

Conditional Sentences 1 2 3 4 5

Only after conditionals are understood as part of the English (or other) language

i)	Denials	1 2 3 4 5
ii)	Judging truth values	1 2 3 4 5
iii)	Counterexamples	1 2 3 4 5

Are there any other topics that you consider prerequisite subskills for understanding the concept of the limit of a sequence? If so, please list them below.

I believe that graphing, particularly with parallel axes is very helpful and should be a prerequisite sub skill for understanding limit of sequences.

Ex

Name Alice G. Woolley

formerly - Alice G. Hart

(over)

I went through these very hurriedly since I am leaving for Thanksgiving. If you have questions concerning my reasons please feel free to write me.

I feel that you need to consider the way the language is used in the U.S. very carefully. Your meaning might not be clear to students just learning the subject.

I hope this is what you wanted.

Please give my regards to Herb Wills.

Alice G. Woodby

University of Illinois at Urbana-Champaign

DEPARTMENT OF MATHEMATICS · 273 ALTGELD HALL · 1409 WEST GREEN STREET · URBANA, ILLINOIS 61801 · (217) 333-3350

October 20, 1982

Tuiren A. Bratina
1827 Portland Ave.
Tallahassee, FL 32303

Dear Tuiren:

Please give my regards to Prof. Wills. It is nice to hear from a student of his. Since you are a student of Prof. Wills I suppose I am a kind of educational grandfather to you.

It has been a very long time since I have thought about the problems of teaching limits of sequences. But it is nice to be asked to rethink the matter. Instead of responding to your questionnaire let me simply record my thoughts in this letter.

First of all, I would say that before I can respond to your question on prerequisite knowledge I need to know how the subject of limits is to be approached. Different approaches presuppose different things. For example, here at Illinois we are offering two kinds of calculus courses. In addition to the traditional calculus we are also teaching a few sections that use nonstandard analysis. These sections will approach limits via the notion of infinitesimal. So the prerequisites are quite different from the traditional approach using standard analysis.

But, let us stay with standard analysis which is still the most common approach. Even with standard analysis one can do limits in different ways. One can use a straight ϵ/δ approach, or one can use a topological approach, and one can do a combination of the two. So the question then is do you mean by prerequisites the prerequisite information that is the intersection of all the conventional approaches or do you mean the union or something in between. For your purposes I would guess that you need to think about prerequisites that are sufficient for any classical approach. Given that, I can now tell you what I think a student should know in order to fully understand limits of sequences of real numbers.

The notion of limit derives from the completeness property of the real number system. Therefore a student should be very comfortable with the following ideas.

1) The algebraic properties of the real numbers; i.e. the properties of addition and multiplication.

2) The elementary order properties of the real numbers including the notion of absolute values.

3) The completeness property of the real numbers approached from the point of view of sup-infs, or Dedekind cuts, or nested intervals. Indeed I think a good student should be familiar with all three approaches.

4) The basic topology of the real line.

Which of these ideas is the most important? They are all important. Indeed I would avoid any effort to rank mathematical ideas according to importance. A particular notion is either prerequisite or it isn't. And whether it is prerequisite or not, cannot be settled with an opinion poll, but depends on whether or not the idea is needed.

I hope my thoughts have been helpful, and I wish you success with your project.

Sincerely yours,

Wilson M. Zaring

Wilson M. Zaring
Director of Graduate Studies

WMZ/lb

APPENDIX D

PREREQUISITE SKILLS

Each sentence has a "box" in it. Below the sentence appears a row of numerals. Your job is to find which of those numerals, when placed in each occurrence of the box, makes the resulting statement true.

You will need to address each numeral as a separate item. If it "works", circle it. If it doesn't work, do not mark it. It could be possible that all numerals, no numerals, or just some of the numerals should be circled.

For example:

$1 + \square \geq 6.$

3 6 9

1. Each sentence in this section is about the function f described below.

For each natural number n, $f(n) = 2^{(9-n)} + 3.$

A. If $\square > 7$ then $|f(\square) - 7| < 7.$

3 6 7 9

B. If $\square > 3$ then $|f(\square) - 13| < 8.$

1 5 6 7 11

C. For each natural number n,

if $n > \square$ then $|f(n) - 3| < 1.$

5 8 11

D. For each natural number n,
if $n > 8$ then $|f(n) - \square| < 2$.

1 2 3 5 6

E. For each natural number n,
if $n > 6$ then $|f(n) - 2| < \square$.

3 5 258

F. For each natural number n,
if $n > \square$ then $|f(n) - 5| < 1.6$.

5 7 9 11

2. A. If $\square > 6$ then $|\square^3 - 11| < 50$.

2 4 6 7

B. If $\square > 1$ then $|(\square - 8)^2 - 20| < 14$.

2 4 8 12 24

C. For each natural number n,
if $n > 6$ then $\left|\frac{32.1 - n}{|32.1 - n|} - \square\right| < 2$.

-1 0 1/2 1 2

D. For each natural number n,

if $n > 8$ then $|(-1)^n - 1| < \square$.

0 1 2 3

E. For each natural number n,

if $n > 2$ then $\left|\frac{5 + (-1)^n}{n} - 0\right| < \square$.

1 1.4 1.5 2

F. For each natural number n,

if $n > 5$ then $\left|\frac{(-1)^n}{4} - \square\right| < 3$.

-(1/4) 0 2.5 3 4

G. For each natural number n,

if $n > \square$ then $\left|\frac{36(-1)^n}{n} - 2\right| < 3$.

2 5 34 35

H. For each natural number n,

if $n > \square$ then $|\sqrt{3n} - 7| < 5$.

2 10 35

For the following items, you are to decide which statements are denials of the boxed statement. An example of a denial of the statement:

$$2 + 2 = 4$$

is:

It is not the case that $2 + 2 = 4$.

(1) Some natural numbers are odd.

(a) Some natural numbers are even.

(b) All natural numbers are odd.

(c) All natural numbers are even.

Circle T for TRUE or F for FALSE:

Statement (a) is a denial of statement (1).	T	F
Statement (b) is a denial of statement (1).	T	F
Statement (c) is a denial of statement (1).	T	F

(2) There is some real number x such that $x \cdot 0 = 0$.

(a) For each real number x, $x \cdot 0 \neq 0$.

(b) There is some real number x such that $x \cdot 0 \neq 0$.

(c) There is no real number x such that $x \cdot 0 = 0$.

Circle T for TRUE or F for FALSE:

Statement (a) is a denial of statement (2).	T	F
Statement (b) is a denial of statement (2).	T	F
Statement (c) is a denial of statement (2).	T	F

(3) $\boxed{\text{Some real number has a square which is negative.}}$

(a) Each real number has a square which is not negative.

(b) Some real number has a square which is not negative.

(c) No real number has a square which is negative.

Circle T for TRUE or F for FALSE:

Statement (a) is a denial of statement (3).	T	F
Statement (b) is a denial of statement (3).	T	F
Statement (c) is a denial of statement (3).	T	F

(4) $\boxed{\text{For each real number } x,\ x^2 + 9 = (x + 3)^2.}$

(a) For some real number x, $x^2 + 9 \neq (x + 3)^2$.

(b) For each real number x, $x^2 + 9 \neq (x + 3)^2$.

(c) For no real number x, $x^2 + 9 = (x + 3)^2$.

Circle T for TRUE or F for FALSE:

Statement (a) is a denial of statement (4).	T	F
Statement (b) is a denial of statement (4).	T	F
Statement (c) is a denial of statement (4).	T	F

(5) $\boxed{\text{For each real number } x,\ |x| = |-x|.}$

(a) There is no real number x such that $|x| \neq |-x|$.

(b) There is some real number x such that $|x| \neq |-x|$.

(c) For each real number x, $|x| \neq |-x|$.

Circle T for TRUE or F for FALSE:

Statement (a) is a denial of statement (5).	T	F
Statement (b) is a denial of statement (5).	T	F
Statement (c) is a denial of statement (5).	T	F

(6) $\boxed{\text{Some real number is one more than itself.}}$

(a) Some real number is not one more than itself.

(b) Each real number is not one more than itself.

(c) There does not exist a real number which is one more than itself.

Circle T for TRUE or F for False:

Statement (a) is a denial of statement (6).	T	F
Statement (b) is a denial of statement (6).	T	F
Statement (c) is a denial of statement (6).	T	F

(7) It is not the case that all rectangles are squares.

(a) All rectangles are squares.

(b) Some rectangles are squares.

(c) Some rectangles are not squares.

Circle T for TRUE or F for FALSE:

Statement (a) is a denial of statement (7).	T	F
Statement (b) is a denial of statement (7).	T	F
Statement (c) is a denial of statement (7).	T	F

(8) It is not the case that for each real number x, $(3^2)^x = 3^{2x}$.

(a) For each real number x, $(3^2)^x = 3^{2x}$.

(b) For some real number x, $(3^2)^x = 3^{2x}$.

(c) For some real number x, $(3^2)^x \neq 3^{2x}$.

Circle T for TRUE or F for FALSE:

Statement (a) is a denial of statement (8).	T	F
Statement (b) is a denial of statement (8).	T	F
Statement (c) is a denial of statement (8).	T	F

(9) $\boxed{\text{For all sequences, if a sequence has a finite limit then the sequence is bounded.}}$

(a) For all sequences, if the sequence is bounded then the sequence has a finite limit.

(b) For some sequences, the sequence has a finite limit but the sequence is not bounded.

(c) For some sequences, the sequence is not bounded and the sequence has a finite limit.

Circle T for TRUE *or* F *for* FALSE:

Statement (a) is a denial of statement (9).	T	F
Statement (b) is a denial of statement (9).	T	F
Statement (c) is a denial of statement (9).	T	F

(10) $\boxed{\text{For each real number } x, \text{ if } |x| < 1, \text{ then } x^2 > x.}$

(a) For each real number x, $x^2 < x$.

(b) For each real number x, if $|x| \geq 1$, then $x^2 \geq x$.

(c) For some real number x, $|x| < 1$ and $x^2 \leq x$.

Circle T for TRUE or F for FALSE:

Statement (a) is a denial of statement (10).	T	F
Statement (b) is a denial of statement (10).	T	F
Statement (c) is a denial of statement (10).	T	F

(11) $\boxed{\text{For each natural number } n, \text{ if } n > 5 \text{ then } \left|\frac{2}{n}\right| < 1.}$

(a) For each natural number n, n > 5 but $\left|\frac{2}{n}\right| \geq 1$.

(b) For some natural number n, n > 5 but $\left|\frac{2}{n}\right| \geq 1$.

(c) For some natural number n, $n \leq 5$ and $\left|\frac{2}{n}\right| \geq 1$.

Circle T for TRUE or F for FALSE:

Statement (a) is a denial of statement (11). T F

Statement (b) is a denial of statement (11). T F

Statement (c) is a denial of statement (11). T F

(12) $\boxed{\text{For each natural number } n, \text{ if } n = 5 \text{ then } n > 3.}$

(a) For some natural number n, if $n \leq 3$ then $n \neq 5$.

(b) For each natural number n, if $n \neq 5$ then $n \leq 3$.

(c) For some natural number n, n = 5 and $n \leq 3$.

Circle T for TRUE or F for FALSE:

Statement (a) is a denial of statement (12). T F

Statement (b) is a denial of statement (12). T F

Statement (c) is a denial of statement (12). T F

(13) There exists some real number x such that if $3x = 2$ then $2x = 3$.

(a) For each real number x, $3x = 2$ but $2x \neq 3$.

(b) There exists some real number x such that if $3x = 2$ then $2x \neq 3$.

(c) There exists some real number x such that $3x = 2$ and $2x = 3$.

Circle T for TRUE or F for FALSE:

Statement (a) is a denial of statement (13).	T	F
Statement (b) is a denial of statement (13).	T	F
Statement (c) is a denial of statement (13).	T	F

(14) For each real number x, if $\{\{x\}\} \neq 0$ then $[\![x]\!] < x$.

(a) For each real number x, if $\{\{x\}\} = 0$ then $[\![x]\!] \geq x$.

(b) There exists some real number x such that $\{\{x\}\} = 0$ and $[\![x]\!] < x$.

(c) There exists some real number x such that $\{\{x\}\} \neq 0$ and $[\![x]\!] \geq x$.

Circle T for TRUE or F for FALSE:

Statement (a) is a denial of statement (14).	T	F
Statement (b) is a denial of statement (14).	T	F
Statement (c) is a denial of statement (14).	T	F

Fill in each $\square$ and $\bigcirc$ below so that the resulting statement is true.

1. The set of all points which are a distance of not more than 1 away from 8 is the line segment whose endpoints are $\square$ and $\bigcirc$.

2. $\{x: |5 - x| \leq 2\} = \{x: \bigcirc \leq x \leq \square\}$.

3. Consider the set $\{x: |4 - x| < 1\}$. Each point in this set is at a distance of less than $\square$ away from $\bigcirc$.

4. $\{x: |x - 3| \leq 9\} = \{x: -\bigcirc \leq x - \square \leq \bigcirc\}$.

5. $\{x: |x - 5| \leq 2\}$ is the line segment whose endpoints are $\square$ and $\bigcirc$.

6. Consider a set of points. 5 is less than a distance of 3 away from each point in the set. The set is $\{x: |x - \square| < \bigcirc\}$.

7. $\{x: |x - 1| \leq \square\} = \{x: -1 \leq x \leq 3\}$.

8. The line segment whose endpoints are 4 and 10 is {x: -□ ≤ x - 7 ≤ □}.

9. {x: -2 ≤ x - 3 ≤ 2} is the line segment whose endpoints are □ and ○.

10. The line segment whose endpoints are 5 and 9 is the set of all points which are a distance of less than or equal to □ away from ○.

11. {x: -6 ≤ x - 4 ≤ 6} = {x: |x - □| ≤ ○}.

12. The set of all points which are a distance of less than 3 away from 4 is {x: -□ < ○ - x < □}.

13. The line segment whose endpoints are 2 and 4 is {x: |x - 3| ≤ □}.

14. {x: -3 < 5 - x < 3} is the set of all points which are a distance of less than □ away from ○.

Complete each of the exercises below.

1. For each natural number n, $f(n) = (n - 9)^2$.

   a) $f(10) = ?$ ____

   b) Circle <u>all</u> correct choices at the right:

      $f(?) = 25$ 2 4 8 14

   c) The 1st 3 terms of the sequence f are: ____, ____, ____

   d) The 89th term of the sequence f is ____.

2. For each natural number k, $h(k) = 7 - 3k$.

   a) $h(10) = ?$ ____

   b) Circle <u>all</u> correct choices at the right:

      $h(?) = -35$ 3 7 14 35

   c) The 1st 3 terms of the sequence h are: ____, ____, ____

   d) The 32nd term of the sequence h is ____.

3. For each natural number n, $g(n) = (-1)^n$.

a) g(10) = ? ____

b) Circle <u>all</u> correct choices at the right:

g(?) = -1 3 9 12 21

c) The 1st 3 terms of the sequence g are: ____, ____, ____

d) The 98th term of the sequence g is ____.

4. For each natural number j, f(j) = 1946.

a) f(10) = ? ____

b) Circle <u>all</u> correct choices at the right:

f(?) = 1946 6 13 18 22

c) The 1st 3 terms of the sequence f are: ____, ____, ____

d) The 36th term of the sequence f is ____.

Circle T for TRUE or F for FALSE.

1. a) For each real number x, there exists some real number y such that x + y = x. T F

   b) For each real number y, there exists some real number x such that x + y = x. T F

   c) There exists some real number x such that for each real number y, x + y = x. T F

   d) There exists some real number y such that for each real number x, x + y = x. T F

2. a) For each real number x, there exists some real number y such that x = y. T F

   b) For each real number y, there exists some real number x such that x = y. T F

   c) There exists some real number x such that for each real number y, x = y. T F

   d) There exists some real number y such that for each real number x, x = y. T F

3. a) For each real number x, there exists some real number y such that y - x = y. T F

   b) For each real number y, there exists some real number x such that y - x = y. T F

   c) There exists some real number x such that for each real number y, y - x = y. T F

   d) There exists some real number y such that for each real number x, y - x = y. T F

4. a) For each real number x, there exists some real number y such that 0 = x + y. T F

   b) For each real number y, there exists some real number x such that 0 = x + y. T F

   c) There exists some real number x such that for each real number y, 0 = x + y. T F

   d) There exists some real number y such that for each real number x, 0 = x + y. T F

5. a) For each real number x, there exists some real number y such that $y = -x$. T F

   b) For each real number y, there exists some real number x such that $y = -x$. T F

   c) There exists some real number x such that for each real number y, $y = -x$. T F

   d) There exists some real number y such that for each real number x, $y = -x$. T F

6. a) For each real number x, there exists some real number y such that $|x - 5| < y$. T F

   b) For each real number y, there exists some real number x such that $|x - 5| < y$. T F

   c) There exists some real number x such that for each real number y, $|x - 5| < y$. T F

   d) There exists some real number y such that for each real number x, $|x - 5| < y$. T F

7. a) For each real number x, there exists some real number y such that $|x - 5| > y$. T F

   b) For each real number y, there exists some real number x such that $|x - 5| > y$. T F

   c) There exists some real number x such that for each real number y, $|x - 5| > y$. T F

   d) There exists some real number y such that for each real number x, $|x - 5| > y$. T F

8. a) For each real number x, there exists some real number y such that $|x - y| < 2$ T F

   b) For each real number y, there exists some real number x such that $|x - y| < 2$ T F

   c) There exists some real number x such that for each real number y, $|x - y| < 2$. T F

   d) There exists some real number y such that for each real number x, $|x - y| < 2$. T F

Here are some true/false items concerning real numbers-
Circle T for TRUE or F for FALSE.

1\. a) $\forall_x\ x^2 \geq 0.$ T F   b) $\forall_x\ -x \leq 0.$ T F

c) $\forall_x\ |x| \geq 0.$ T F   d) $\forall_x\ |-x| = x.$ T F

e) $\forall_x\ \sqrt{x^2} = x.$ T F   f) $\forall_x\ x + 2 \geq x.$ T F

g) $\forall_x\ x/2 \leq x.$ T F   h) $\forall_x\ x - 2 \leq x.$ T F

i) $\forall_x\ 5x \geq -3x.$ T F   j) $\forall_x\ x^2 \geq x.$ T F

k) $\forall_x\ 2^{-x} < 0.$ T F   l) $\forall_x\ x \geq -x.$ T F

m) $\forall_x\ x^3 \geq x^2.$ T F   n) $\forall_x\ x^1 \cdot x^4 \neq x^4.$ T F

2\. a) $\forall_x \forall_y\ -(x - y) = y - x.$ T F

b) $\forall_{x>0} \forall_{y>0}\ \sqrt{x}\sqrt{y} = \sqrt{xy}$. T F

c) $\forall_x \forall_y\ (x + y)^2 = x^2 + 2xy + y^2$ T F

d) $\forall_x$ if x is a positive number then $x \geq 0$. T F

e) $\forall_x$ {x: x is a natural number} = {1, 2, 3,...} T F

f) 0 is a natural number. T F

g) 3/4 is a natural number. T F

h) $\forall_x \forall_y\ |x - y| \leq |x| - |y|.$ T F

i) $\forall_x \forall_y\ (x - y)^2 = (y - x)^2.$ T F

j) $\forall_x$ if x is a natural number then $(-1)^x = 1$. T F

k) $\forall_x$ if x is not positive then x is negative. T F

l) $\forall_x$ if x is positive then x is not negative. T F

For the following exercises you are to decide which, if any, of the numbers listed as choices makes the statement false. Such numbers are <u>counterexamples</u>.

Circle each numeral which names a counterexample of the statement. It is possible that all choices, no choices, or just some of the choices are counterexamples.

Here is a sample item:

For each real number x

$$x^2 = 9.$$

-3   2   3   4

Notice that 4 is a counterexample because the statement:

$$4^2 = 9$$

is false.

1. For each real number x

$9 < x \cdot 0.$

-11 1 9 83

2. For each real number w

$-w \le w.$

-6 $\frac{1}{4}$ 1 8

3. For each real number y

$y + (+6) \ge y.$

-4 1 5 12

4. For each real number z

$z + (+6) \ge +6.$

-14 -7 -4 32

5. For each real number x

$4x \ge 4 + x.$

-5 $\frac{1}{2}$ $1\frac{1}{3}$ 3

6. For each real number w

$w = 6$ and $w = 7.$

-1 6 7 13

7. For each real number y

$+3 = |y|.$

-3 2 3 72

8. For each real number z

if $z > 6$ then $z < 6.$

-86 4 6 9

9. For each real number x

$x \neq +4.$

-17 1 4 9

10. For each real number w

$w < w.$

-2 6 7 19

11. For each real number y

$y = +3$ or $y = -3.$

-3 0 +3 $+(\frac{1}{9})$

12. For each real number z

$(z + 5)^2 = z^2 + 5^2.$

-5 0 $\frac{1}{5}$ +5

13. For each real number x

$|x^2| = x^2.$

-21 1 2 3

14. For each real number w

$\frac{w + 3}{3} = w + 1.$

-1 0 3 8

15. For each real number y

if $y = 5$ then $3y = 6.$

-4 2 5 9

16. For each real number z

if $z < 9$ then $z > 7.$

4 6 8 10

APPENDIX E

FINAL VERSION OF THE INSTRUMENT

MEASURING UNDERSTANDING OF

LIMITS OF SEQUENCES

Directions: This is a closed book test. No calculators may be used either.

The phrase "limit of a sequence" (used extensively throughout this test) has the same meaning as "limit of a sequence as n increases without bound" and "limit of a sequence as n approaches (tends to) infinity."

This test is not necessarily arranged in terms of order of difficulty, so do not spend a large amount of time on any one problem. You have one class period to complete the test.

1. In the blank provided, write the limit of each sequence that appears below. If the particular sequence does not have a finite limit, write "NONE".

For each natural number n, each term of the sequence is:

A) $98 + \frac{3}{n^2}$ __________ B) $9 - n$ __________

C) $(-1)^n$ __________ D) $(0.7739)^n$ __________

E) $\frac{2 + (-1)^n}{n}$ __________ F) $\frac{7n^9 - 12n^8 + 15n - 1}{2n^{10} + n^3 + 2}$ __________

G) $\frac{3n}{4n - 7}$ __________ H) $\sqrt[n]{271.32}$ __________

I) $\sqrt{\frac{3n - 1}{2}}$ __________ J) 19 __________

2. Write the limit of each sequence described below. If the sequence has no finite limit, write "NONE".

A) $-1\frac{1}{4}, -1\frac{1}{4}, -1\frac{1}{4}, \ldots$ in which each term of the sequence is $-1\frac{1}{4}$.

Ans. __________

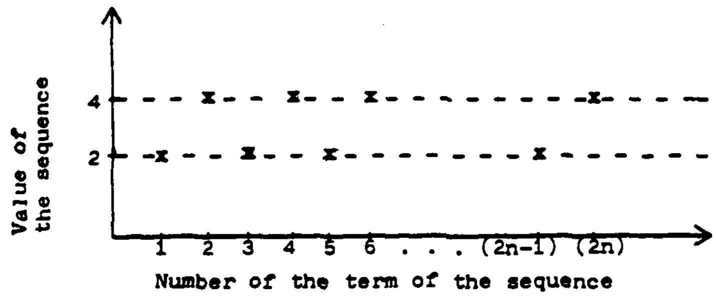
B) 1.9997, 1.9998, 1.9999, 2.0000, 2.0001, 2.0002, 2.0003, . . . , [1.9997 + (n/10000)], . . . for each natural number n.

Ans. __________

C) $\frac{9}{10}, \frac{99}{100}, \frac{999}{1000}, \ldots, \left[\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots + \frac{9}{10^n}\right], \ldots$ for each natural number n.

Ans. __________

D.



For each natural number n each term of the sequence appears on the graph above.

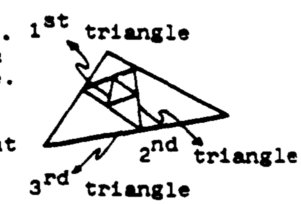
Ans. __________

E. $1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, 7, \frac{1}{8}, \ldots, \left\{\frac{1}{n}\right\}^{\overbrace{[(-1)^n]}^{\text{exponent}}}, \ldots$
for each natural number n.

Ans. __________

3. Circle the one correct answer for each of the following items:

A. A sequence of triangles is constructed. The sides of each new triangle contain the vertices of the preceding triangle. The first three triangles in this sequence are pictured at the right. Which of the following statements about this sequence is true?

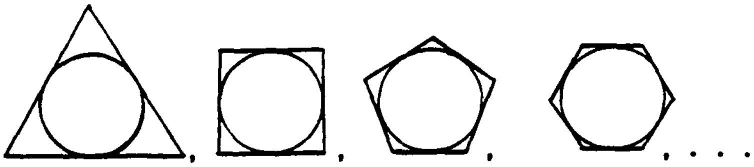


a) The perimeter of the fifth triangle is the average of the perimeters of the fourth and sixth triangles.

b) The process suggested could produce a triangle which has an area greater than that of a circle with a radius of 1,000,000 miles.

c) The process suggested cannot produce a triangle which has a side longer than 1,000,000 miles.

B.



A process of circumscribing regular polygons about a circle is shown in an attempt to approximate the area of that circle. The number of sides for each successive polygon increases by one. Which of the statements below is correct?

a) Averaging the areas of the polygons will provide the best approximation.

b) At some finite stage in the process, the area of the polygon will equal the area of the circle.

c) The greater the number of sides of the polygon, the better the approximation will be.

C. Consider each sequence whose limit is 8. Suppose every term after the 67th lies within ¼ of 8. How many terms of each such sequence are required before each term is within 1/100 of 8?

a) Greater than or equal to 67 terms.

b) Less than 67 terms.

c) Cannot be determined without more information.

D. The sequence whose odd-numbered terms have the value 2.9999 and whose even-numbered terms have the value 3.0001

a) has limit 3.

b) has both limits 2.9999 and 3.0001.

c) has no limit because there are infinitely many terms in the sequence.

d) has no limit because infinitely many terms of the sequence have the value 2.9999 and infinitely many terms of the sequence have the value 3.0001.

E. For all real numbers p and q,
if $p > q > 1$
then

$$\lim_{n \to \infty} \left\{\frac{3n^p - 5n^{p-3} - 7}{4n^q + 9n^{q-1} + 2}\right\}$$

a) is 3/4.

b) is 0.

c) is not a real number because the values of the sequence are not bounded.

F. For each real number r that is between 0 and 2,

$$\text{limit}_{n \to \infty} r^n$$

a) is 1.

b) is 0.

c) is not a real number because the values of the sequence are not bounded.

d) may or may not be a real number, but more information is required for this to be determined.

G. For each sequence f, for each sequence g, and for each natural number n, f(n) and g(n) are <u>corresponding terms</u> of f and g. For example, 1 and 5, 2 and 10, 3 and 15, ... are the corresponding terms of $f = \{(1,1), (2,2), (3,3), \ldots\}$ and $g = \{(1,5), (2,10), (3,15), \ldots\}$.

For each sequence f, for each sequence g, and for each sequence h,

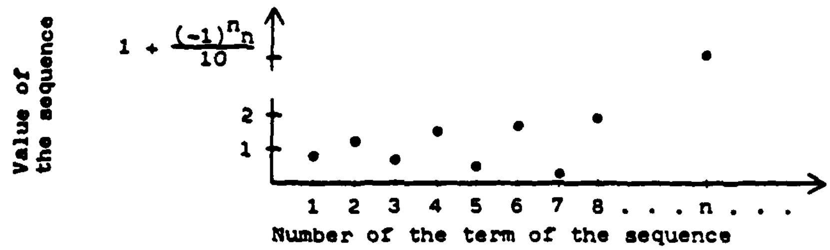
if each term of h is between the corresponding terms of f and of g, and the limits of f and g are both π, then the limit of h

a) is π.

b) is not a finite number.

c) may or may not be a finite number, but more information is required for this to be determined.

H.



The sequence whose graph is pictured above

a) has limit 1.

b) has limit $1 + \frac{(-1)^n n}{10}$.

c) has no finite limit because there are infinitely many positive terms and infinitely many negative terms.

d) has no finite limit because the values of the sequence are not bounded.

I. Suppose that we have a sequence $\{a_n\}$ such that for each natural number n

$$a_n = \begin{cases} n^4 & \text{for } n \leq 1{,}000{,}000{,}000{,}000 \\ \frac{1}{n^4} & \text{for } n > 1{,}000{,}000{,}000{,}000 . \end{cases}$$

$\underset{n \to \infty}{\text{Limit}}\ a_n$

a) is $1{,}000{,}000{,}000{,}000^4$.

b) is 0.

c) is not a real number because the terms of the sequence are not bounded.

J. For each sequence formed by choosing every fifth term of some sequence whose limit is $\sqrt{2}$, the limit of the newly formed sequence

a) is $\sqrt{2}$.

b) is not a finite number.

c) may or may not be a finite number, but more information is required for this to be determined.

4. Answer each "true/false" item that appears in A, B, and C below.

A. M. $1 + (-1) + 1 + (-1) + 1 + (-1) + 1 + \dots$

N. $1 + (-1) + 1 + (-1) + 1 + (-1) + \dots$

Circle "T" for true or "F" for false for each statement concerning the two expressions above.

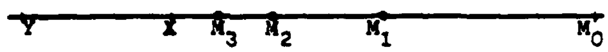
T F a) The expression in M has the value 1.

T F b) The expression in N has the value 0.

T F c) The expression in M has no real value.

T F d) The expression in M and N each have the values 0 and 1.

B.



Point M_1 is the midpoint of the line segment with endpoints X and M_0. Shown above are the first three midpoints M_1, M_2, and M_3 of a sequence of midpoints which are obtained by the following continuing process:

For each natural number n, Point M_n is the midpoint of the segment with endpoints X and M_{n-1}.

Are the following statements, concerning this process, true or false? (Circle "T" for true or "F" for false for each statement.)

T F a) For some natural number n, Point M_n and Point X are the same.

T F b) For each successive natural number n, the distance between Point M_n and Point X is decreasing.

T F c) For each successive natural number n, the distance between Point M_n and Point Y is decreasing.

T F d) Regardless of how small a circle is placed around Point X, after a certain midpoint in the sequence is reached, all of the following midpoints in the sequence will be inside of that circle.

T F e) Regardless of how small a circle is placed around Point Y, there will be some midpoint in the sequence which will be inside of that circle.

C. Circle "T" for true or "F" for false for each of the statements below.

T F 1) $0.999... = 0.9 + 0.09 + 0.009 + 0.0009 + \cdots$

T F 2) $0.999... = 1$

T F 3) $0.999... < 1$

T F 4) $0.999... > 1$

T F 5) $0.999 = 1$

T F 6) $0.999 = 1.000$

T F 7) $1/3 = 0.333...$

T F 8) $2/3 = 0.666...$

T F 9) $3/3 = 0.999...$

T F 10) $2(1/3) = 2(0.333...)$

T F 11) $2(0.333...) = 0.666...$

T F 12) $2(1/3) = 0.666...$

T F 13) $3(0.333...) = 0.999...$

T F 14) $3(1/3) = 0.999...$

T F 15) $3/3 = 1$

T F 16) $1 \div 9 = 0.111...$

T F 17) $4 \div 9 = 0.444...$

T F 18) $7 \div 9 = 0.777...$

T F 19) $2 \div 9 = 0.222...$

T F 20) $5 \div 9 = 0.555...$

T F 21) $8 \div 9 = 0.888...$

T F 22) $9 \div 9 = 0.999...$

T F 23) $6 \div 9 = 0.666...$

T F 24) $9 \div 9 = 1$

T F 25) 2/3 = 0.666

T F 26) 2/3 = 0.666...

T F 27) 2/3 = 0.667

T F 28) 0.666... = 0.667

T F 29) 0.666... = 0.6666...

5. Circle "T" for true or "F" for false for each of the following items.

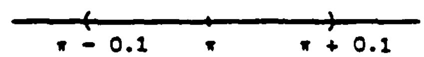
T F A. For each sequence S,
if the limit of S is 17 and we alter S only by subtracting 5 from each of its first 1000 terms, then the limit of the newly formed sequence is 12.

T F B. Consider every sequence whose limit is 4. At least one term in each of these sequences is 4.

T F C. For each sequence S,
if the limit of S is 22.7,
then the 875th term is closer to 22.7 than the 874th term is.

T F D. A sequence can have more than one finite limit.

T F E. For each sequence S,
if the limit of S is π,
then all but a finite number of terms lie in the interval $(\pi - 0.1, \pi + 0.1)$ pictured below.



T F F. Suppose that two different sequences have the same limit, say 4. It must be the case that when the values of the first sequence all differ from 4 by less than 0.001, then the corresponding values of the second sequence also differ from 4 by less than 0.001.
(Recall that for each sequence f, for each sequence g, and for each natural number n, f(n) and g(n) are the corresponding values of f and g.)

T F G. The following are equivalent statements:

a) For each sequence S, the limit of S is 9.3.

b) For each sequence S and for each natural number n,

$$|S(n) - 9.3| \geq |S(n+1) - 9.3| .$$

T F H. For each sequence f, for each real number ℓ, and for each natural number n,

the limit of f is ℓ

if and only if

for each positive real number s, no matter how small s might be, there exists a natural number m such that

whenever $n > m$, $|\ell - f(n)| < s$.

I. Circle "T" or "F" for each item below.

1.

The limit of a sequence is $\underset{S}{\wedge}$6.
True or False: For each natural number n, there is a natural number m such that if $n > m$ then $|S(n) - 6| < \sqrt{2}$.
(Circle here.) T F

2.

Now consider a sequence T such that T has a limit and for each natural number n, there exists a natural number m such that $|T(n) - 37| < 0.001$, whenever $n > m$.
True or False: $\lim_{n \to \infty} T(n) = 37$.
(Circle here.) T F

J. Consider the function f such that for each natural number n, $f(n) = n^4$.

It is the case that

$$\lim_{n \to \infty} f(n) = \infty.$$

True or False:

f does have a limit.

(Circle here.) T F

6. For items A through C, you are to produce an example of a sequence which meets certain prescribed specifications.

Write the general expression for the n^{th} term for some sequence:

A. Which does not have a finite limit because the terms of the sequence are not bounded. ____________

B. Which does not have a limit because there are two values taken on by infinitely many terms of the sequence. ____________

C. For which the first five terms are negative, but the limit of that sequence is 7. ____________

For items D through G, you are to state whether or not there exists a sequence which meets certain prescribed specifications. (Circle "yes" or "no" at the right of each item.)

D. Is there any sequence whose limit is -3 and which has infinitely many positive terms? yes no

E. Is there any sequence for which each successive term is closer to 2, but the limit of this sequence is <u>not</u> 2? yes no

F. Is there any sequence for which infinitely many terms are 7, but the limit of that sequence is <u>not</u> 7? yes no

G. Is there any sequence whose limit is one but no term of the sequence is 1? yes no

APPENDIX F

SUGGESTED LIMITS OF SEQUENCES INSTRUMENT

Directions: This is a closed book test. No calculators may be used either.

The phrase "limit of a sequence" (used extensively throughout this test) has the same meaning as "limit of a sequence as n increases without bound" and "limit of a sequence as n approaches (tends to) infinity."

This test is not necessarily arranged in terms of order of difficulty, so do not spend a large amount of time on any one problem. You have one class period to complete the test.

1. In the blank provided, write the limit of each sequence that appears below. If the particular sequence does not have a finite limit, write "NONE".

For each natural number n, each term of the sequence is:

A) $98 + \frac{3}{n^2}$ ________ B) $9 - n$ ________

C) $(-1)^n$ ________ D) $(0.7739)^n$ ________

E) $\frac{2 + (-1)^n}{n}$ ________ F) $\frac{7n^9 - 12n^8 + 15n - 1}{2n^{10} + n^3 + 2}$ ________

G) $\frac{3n}{4n - 7}$ ________ H) $\sqrt[n]{271.32}$ ________

I) $\sqrt{\frac{3n - 1}{2}}$ ________

2. Write the limit of each sequence described below. If the sequence has no finite limit, write "NONE".

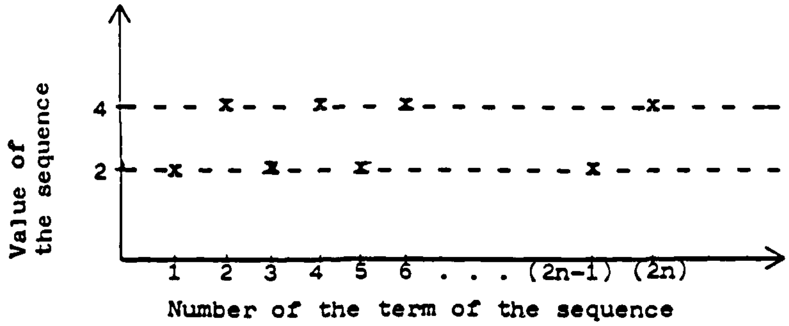
A) $-1\frac{1}{4}, -1\frac{1}{4}, -1\frac{1}{4}, \ldots$ in which each term of the sequence is $-1\frac{1}{4}$.

Ans. ________

B) 1.9997, 1.9998, 1.9999, 2.0000, 2.0001, 2.0002, 2.0003, . . . , [1.9997 + (n/10000)], . . . for each natural number n.

Ans. ________

C.



For each natural number n each term of the sequence appears on the graph above.

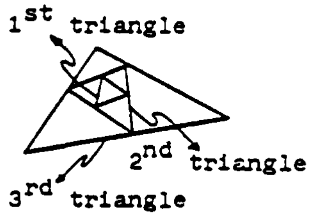
Ans. __________

D. $1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, 7, \frac{1}{8}, \ldots, \left\{\frac{1}{n}\right\}^{\overbrace{[(-1)^n]}^{\text{exponent}}}, \ldots$
for each natural number n.

Ans. __________

3. Circle the one correct answer for each of the following items:

A. A sequence of triangles is constructed. The sides of each new triangle contain the vertices of the preceding triangle. The first three triangles in this sequence are pictured at the right. Which of the following statements about this sequence is true?

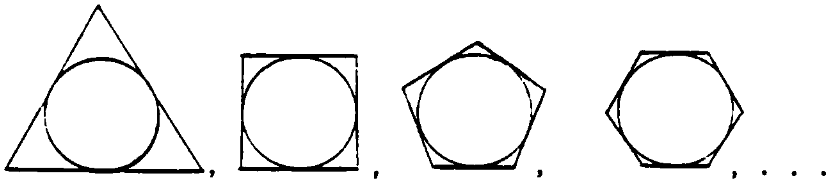


a) The perimeter of the fifth triangle is the average of the perimeters of the fourth and sixth triangles.

b) The process suggested could produce a triangle which has an area greater than that of a circle with a radius of 1,000,000 miles.

c) The process suggested cannot produce a triangle which has a side longer than 1,000,000 miles.

B.



A process of circumscribing regular polygons about a circle is shown in an attempt to approximate the area of that circle. The number of sides for each successive polygon increases by one. Which of the statements below is correct?

a) Averaging the areas of the polygons will provide the best approximation.

b) At some finite stage in the process, the area of the polygon will equal the area of the circle.

c) The greater the number of sides of the polygon, the better the approximation will be.

C. Consider each sequence whose limit is 8. It is not until the 67th term of each of these sequences that every term lies within 1/4 of 8. How many terms of each such sequence are required before each term is within 1/100 of 8?

a) Greater than or equal to 67 terms.

b) Less than 67 terms.

c) Cannot be determined without more information.

D. The sequence whose odd-numbered terms have the value 2.9999 and whose even-numbered terms have the value 3.0001

a) has limit 3.

b) has both limits 2.9999 and 3.0001.

c) has no limit because there are infinitely many terms in the sequence.

d) has no limit because infinitely many terms of the sequence have the value 2.9999 and infinitely many terms of the sequence have the value 3.0001.

AD-A139 399 DEVELOPING AND MEASURING AN UNDERSTANDING OF THE CONCEPT OF THE LIMIT OF A SEQUENCE(U) AIR FORCE INST OF TECH WRIGHT-PATTERSON AFB OH T A BRATINA 1983 5/5

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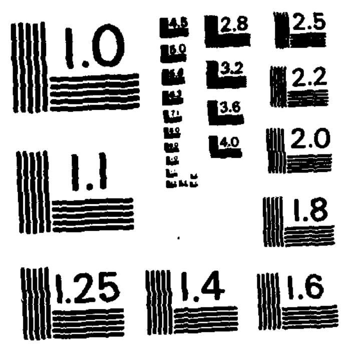


MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

E. For all real numbers p and q,
if $p > q > 1$
then

$$\lim_{n \to \infty} \left\{ \frac{3n^p - 5n^{p-3} - 7}{4n^q + 9n^{q-1} + 2} \right\}$$

a) is 3/4.

b) is 0.

c) is not a real number because the values of the sequence are not bounded.

F. For each real number r that is between 0 and 2,

$$\operatorname{limit}_{n \to \infty} r^n$$

a) is 1.

b) is 0.

c) is not a real number because the values of the sequence are not bounded.

d) may or may not be a real number, but more information is required for this to be determined.

G. For each sequence f, for each sequence g, and for each natural number n, f(n) and g(n) are <u>corresponding terms</u> of f and g. For example, 1 and 5, 2 and 10, 3 and 15, ... are the corresponding terms of $f = \{(1,1), (2,2), (3,3), \ldots\}$ and $g = \{(1,5), (2,10), (3,15), \ldots\}$.

For each sequence f, for each sequence g, and for each sequence h,

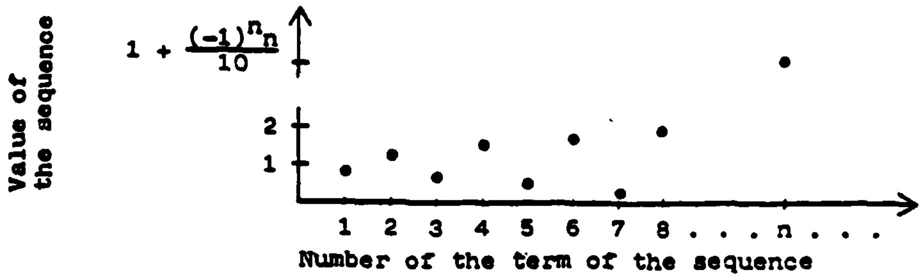
if each term of h is between the corresponding terms of f and of g, and the limits of f and g are both ∞, then the limit of h

a) is ∞.

b) is not a finite number.

c) may or may not be a finite number, but more information is required for this to be determined.

H.



The sequence whose graph is pictured above

a) has limit 1.

b) has limit $1 + \frac{(-1)^n n}{10}$.

c) has no finite limit because there are infinitely many positive terms and infinitely many negative terms.

d) has no finite limit because the values of the sequence are not bounded.

I. Suppose that we have a sequence $\{a_n\}$ such that for each natural number n

$$a_n = \begin{cases} n^4 & \text{for } n \le 1{,}000{,}000{,}000{,}000 \\ \frac{1}{n^4} & \text{for } n > 1{,}000{,}000{,}000{,}000 . \end{cases}$$

$\lim_{n \to \infty} a_n$

a) is $1{,}000{,}000{,}000{,}000^4$.

b) is 0.

c) is not a real number because the terms of the sequence are not bounded.

3. For each sequence formed by choosing every fifth term of some sequence whose limit is $\sqrt{2}$, the limit of the newly formed sequence

   a) is $\sqrt{2}$.

   b) is not a finite number.

   c) may or may not be a finite number, but more information is required for this to be determined.

4. Answer each "true/false" item that appears in A, B, and C below.

A. M. $1 + (-1) + 1 + (-1) + 1 + (-1) + 1 + \ldots$

   N. $1 + (-1) + 1 + (-1) + 1 + (-1) + \ldots$

Circle "T" for true or "F" for false for each statement concerning the two expressions above.

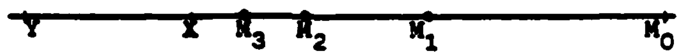
T F a) The expression in M has the value 1.

T F b) The expression in N has the value 0.

T F c) The expression in M has no real value.

T F d) The expression in M and N each have the values 0 and 1.

B.



Point M_1 is the midpoint of the line segment with endpoints X and M_0. Shown above are the first three midpoints M_1, M_2, and M_3 of a sequence of midpoints which are obtained by the following continuing process:

> For each natural number n, Point M_n is the midpoint of the segment with endpoints X and M_{n-1}.

Are the following statements, concerning this process, true or false? (Circle "T" for true or "F" for false for each statement.)

T F a) For some natural number n, Point M_n and Point X are the same.

T F b) For each successive natural number n, the distance between Point M_n and Point X is decreasing.

T F c) For each successive natural number n, the distance between Point M_n and Point Y is decreasing.

T F d) Regardless of how small a circle is placed around Point X, after a certain midpoint in the sequence is reached, all of the following midpoints in the sequence will be inside of that circle.

T F e) Regardless of how small a circle is placed around Point Y, there will be some midpoint in the sequence which will be inside of that circle.

C. Circle "T" for true or "F" for false for each of the statements below.

T F 1) $0.999... = 0.9 + 0.09 + 0.009 + 0.0009 + ...$

T F 2) $0.999... = 1$

T F 3) $0.999... > 1$

5. Circle "T" for true or "F" for false for each of the following items.

T F A. For each sequence S,
if the limit of S is 17 and we alter S only by subtracting 5 from each of its first 1000 terms, then the limit of the newly formed sequence is 12.

T F B. Consider every sequence whose limit is 4. At least one term in each of these sequences is 4.

T F C. For each sequence S,
if the limit of S is 22.7,
then the 875th term is closer to 22.7 than the 874th term is.

T F D. A sequence can have more than one finite limit.

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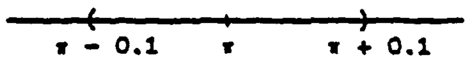

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a) For each sequence S, the limit of S is 9.3.

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T F H. For each sequence f and for each real number ℓ,

the limit of f is ℓ

if and only if

for each positive number s, no matter how small s might be, there exists a natural number m such that for each natural number n

whenever $n > m$, $|\ell - f(n)| < s$.

I. Circle "T" or "F" for each item below.

1.

> The limit of a sequence S is 6.
>
> True or False:
> For each natural number n, there is a natural number m such that
> if $n > m$ then $|S(n) - 6| < \sqrt{2}$.
>
> (Circle here.) T F

2.

> Now consider a sequence T such that T has a limit and for each natural number n, there exists a natural number m such that
> $|T(n) - 37| < 0.001$, whenever $n > m$.
>
> True or False:
>
> $\lim_{n \to \infty} T(n) = 37.$
>
> (Circle here.) T F

6. For items A through D, you are to produce an example of a sequence which meets certain prescribed specifications.

Write the general expression for the n^{th} term for some sequence:

A. Which does not have a finite limit because the terms of the sequence are not bounded. __________

B. Which does not have a limit because there are two values taken on by infinitely many terms of the sequence. __________

C. For which the first five terms are negative, but the limit of that sequence is 7. __________

D. For which each successive term is closer to 2, but the limit of this sequence is <u>not</u> 2. __________

For items E through G, you are to state whether or not there exists a sequence which meets certain prescribed specifications. (Circle "yes" or "no" at the right of each item.)

E. Is there any sequence whose limit is -3 and which has infinitely many positive terms? yes no

F. Is there any sequence for which infinitely many terms are 7, but the limit of that sequence is <u>not</u> 7? yes no

G. Is there any sequence whose limit is one but no term of the sequence is 1? yes no

VITA

Tuiren A. Bratina received all of her primary and secondary education from public schools in Angola, Indiana. Her bachelors and masters degrees were both from Ball State University in Muncie, Indiana.

She has taught mathematics in the civilian sector for more than five years. These teaching experiences range from junior high through college level courses. Indiana, New Mexico, and California were the locations of her employment.

In May, 1973, Tuiren A. Bratina entered the United States Air Force. She served as a communication-electronics maintenance officer in Mississippi, South Dakota, and California the first four years of her military career. In 1977 she was assigned to the Department of Mathematical Sciences at the United States Air Force Academy.

She began her doctoral program in mathematics education at The Florida State University, Tallahassee, Florida, in 1979. She returned to Colorado in 1983 where she currently is an assistant professor at the Air Force Academy.

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