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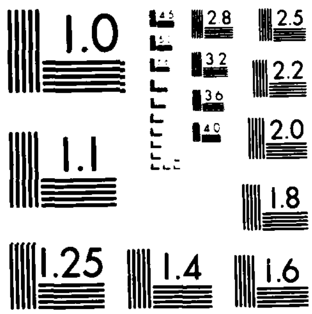
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THE LINEAR QUADRATIC OPTIMAL CONTROL PROBLEM FOR INFINITE DIMENSIONAL SYSTEMS WITH UNBOUNDED INPUT AND OUTPUT OPERATORS

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ABSTRACT

Part I of this paper deals with the problem of designing a feedback control for a linear infinite dimensional system in such a way that a given quadratic cost functional is minimized. The essential feature of this work is that:

a) it allows for unbounded control and observation, i.e. boundary control, point observation, input/output delays; and

b) the general theory is presented in such a way that it applies to both parabolic and hyperbolic ~~PDEs~~ as well as retarded and neutral ~~PDEs.~~

~~In~~ Part II the paper develops a state space approach for retarded systems with delays in both input and output. A particular emphasis is placed on the development of the duality theory by means of two different state concepts. The resulting evolution equations fit into the framework of Part I.

AMS(MOS) Subject Classification: 34K35, 93C20, 93C25, 93D15.

Key Words: Linear quadratic optimal control, infinite dimensional Riccati equation, unbounded control and observation, retarded systems.

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A. J. Pritchard* and D. Salamon**

1. INTRODUCTION

The control and observation processes for many dynamical systems are often severely limited. For example there may be delays in the control actuators and measurement devices. Also for systems described by partial differential equations it may not be possible to influence or sense the state of the system at each point of the spatial domain. Instead controls and sensors are restricted to a few points or parts of boundaries. Modelling such limitations results in unbounded input and output operators.

In the first part of this paper we develop a general theory for linear quadratic control which allows us to consider such operators. We then show how the theory applies to hyperbolic and parabolic partial differential equations and neutral systems with output delays.

In the second part of the paper we develop a state space theory for linear functional differential equations with general delays in the state, inputs and outputs. Then we show how the results of the general theory may be applied and hence solve the linear quadratic control problem for such systems.

PART I

2. FINITE TIME CONTROL

In a formal sense our basic model is

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t), \quad x(t_o) = x_o, \\ (2.1)\qquad &y(t) = Cx(t), \quad t_o \le t \le t_1, \end{aligned}$$

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where $u(\cdot) \in L^2(t_o,t_1;U)$, $y(\cdot) \in L^2(t_o,t_1;Y)$, U and Y are Hilbert spaces and A is the infinitesmal generator of a strongly continuous semigroup $S(t)$ on a Hilbert space H. In order to allow for possible unboundedness of the operators B and C, we assume that $B \in \mathcal{L}(U,V)$, $C \in \mathcal{L}(W,Y)$ where W,V are Hilbert spaces such that

$$(2.2) \quad W \subset H \subset V$$

with continuous dense injections. Of course, we interpret equation (2.1) in the mild form which means that its solution $x(t)$ is given by the variation-of-constants formula

$$(2.3) \quad x(t) = S(t-t_o)x_o + \int_{t_o}^{t} S(t-\sigma)Bu(\sigma)d\sigma, \quad t_o \le t \le t_1$$

In order to make this formula precise and to allow for trajectories in all three spaces W,H,V we have to assume that $S(t)$ is also a strongly continuous semigroup on W and V and that the following hypotheses are satisfied.

(H1) There exists some constant $b > o$ such that $\int_{t_o}^{t_1} S(t_1-\sigma)Bu(\sigma)d\sigma \in W$ and $\left\| \int_{t_o}^{t_1} S(t_1-\sigma)Bu(\sigma)d\sigma \right\|_W \le b \, \| u(\cdot) \|_{L^2(t_o,t_1;U)}$

for every $u(\cdot) \in L^2(t_o,t_1;U)$.

(H2) There exists some constant $c > o$ such that

$$\| C S(\cdot - t_o)\, x \|_{L^2(t_o,t_1;Y)} \le c \, \| x \|_V$$

for every $x \in W$.

Remarks 2.1

(i) For every $x_o \in W$ and every $u(\cdot) \in L^2(t_o,t;U)$ formula (2.3) defines a continuous function $x(\cdot)$ with values in W and the output function $y(\cdot)$ is continuous with values in Y. Alternatively if $x_o \in H$ (resp. V), then $x(\cdot)$ is only continuous with values in H (resp. V) and $y(\cdot) \in L^2(t_o,t_1;Y)$. Sometimes we will regard (2.3) as a dynamical system on the state space V.

(ii) In the following we identify the Hilbert spaces H,U and Y with their duals. Then it follows from (2.2) by duality that

$$V^* \subset H \subset W^*$$

with continuous dense embeddings. Moreover, $S^*(t)$ is a strongly continuous semigroup on all three spaces V^*,H,W^*.

(iii) The dual statements of (H1) and (H2) are the following

(H1*) For every $x \in V^*$ the following inequality holds

$$\| B^*S^*(t_1-\cdot)x\|_{L^2(t_o,t_1;U)} \le b\,\|x\|_{W^*}$$

(H2*) For every $y(\cdot) \in L^2[t_o,t_1;Y]$ we have

$$\Big\|\int_{t_o}^{t_1} S^*(\tau-t_o)C^*y(\tau)d\tau\Big\|_{V^*} \le c\,\|y(\cdot)\|_{L^2[t_o,t_1;Y]}$$

(see SALAMON [40]).

(iv) The expression $CS(t)x$ only makes sense when $x \in W$. However, if (H2) is satisfied, then for any $x \in V$ we will use the expression $CS(t)x$, $t_o \le t \le t_1$, to denote the function in $L^2(t_o,t_1;Y)$ which is obtained by continuous extension of the operator $W \ni x \to CS(\cdot)x \in L^2(t_o,t_1;Y)$ to $x \in V$. In the same manner we define $B^*S^*(t)x$, $t_o \le t \le t_1$, for $x \in W^*$ when (H1) is satisfied. In particular, the expressions $CS(t)Bu$ and $B^*S^*(t)C^*y$ have a well defined meaning as functions of t when (H1) and (H2) are satisfied.

Associated with the control system (2.3) is the performance index

$$J(u) = \langle x(t_1), Gx(t_1)\rangle_{V,V^*} \tag{2.4}$$
$$+ \int_{t_o}^{t_1} [\| Cx(t)\|_Y^2 + \langle u(t),Ru(t)\rangle_U]\, dt$$

where $G \in \mathcal{L}(V,V^*)$ is a positive semidefinite operator and $R \in \mathcal{L}(U)$ satisfies

$$\langle u,Ru\rangle_U \geq \varepsilon \| u \|_U^2$$

for some $\varepsilon > o$ and every $u \in U$.

Now let us consider system (2.3) with the feedback control

$$u_F(t) = F(t)x(t),\ t_o \leq t \leq t_1, \tag{2.5}$$

where $F(t) \in \mathcal{L}(V,U)$ is strongly continuous on the interval $[t_o,t_1]$. Then we may define a mild evolution operator $\Phi_F(t,s) \in \mathcal{L}(V)$, $t_o \leq s \leq t \leq t_1$, via

$$\Phi_F(t,s)x = S(t-s)x + \int_s^t S(t-\sigma)BF(\sigma)\Phi_F(\sigma,s)xd\sigma \tag{2.6}$$

(see CURTAIN-PRITCHARD [9]).

<u>Remarks 2.2</u>

(i) It follows from (2.6) that $\Phi_F(t,s)$ satisfies the equation

$$\Phi_F(t,s)x-x = \int_s^t \Phi_F(t,\sigma)[A+BF(\sigma)]xd\sigma,\quad t_o \leq s \leq t \leq t_1 \tag{2.7}$$

for every $x \in \mathcal{D}_V(A)$ (the domain of A regarded as an unbounded, closed operator on V), Equivalently the function $s \longrightarrow \Phi_F(t,s)x \in V$ is continuously differentiable on the interval $[t_o,t]$ for every $x \in \mathcal{D}_V(A)$ and satisfies

$$(2.8) \quad \frac{\partial \Phi_F(t,s)x}{\partial s} = - \Phi_F(t,s)[A + BF(s)]x, \quad t_o \leq s \leq t \leq t_1$$

(see CURTAIN-PRITCHARD [9]).

(ii) It is well known that the evolution operator satisfies the equation

$$(2.9) \quad \Phi_F(t,s)x = S(t-s)x + \int_s^t \Phi_F(t,\sigma)BF(\sigma)S(\sigma-s)x d\sigma$$

for $t_o \leq s \leq t \leq t_1$ and $x \in V$.

(see CURTAIN-PRITCHARD [9]).

(iii) Often we will consider the feedback system with an additional forcing input $v(\cdot)$ so that

$$(2.10) \quad u(t) = F(t)x(t) + v(t)$$

in (2.3). It follows easily from (2.9) that - for this control function - the corresponding solution of (2.3) is given by

$$(2.11) \quad x(t) = \Phi_F(t,t_o)x_o + \int_{t_o}^t \Phi_F(t,\sigma)Bv(\sigma)d\sigma, \; t_o \leq t \leq t_1.$$

(iv) Using (2.6) it is easy to see that $\Phi_F(t,s)$ is also a strongly continuous evolution operator on W and V and has the following properties.

(H1') Thereexists a constant $b' > o$ such that

$$\left\| \int_{t_o}^t \Phi_F(t,\sigma)Bu(\sigma)d\sigma \right\|_W \leq b' \; \| u(\cdot) \|_{L^2(t_o,t;U)}$$

for every $u(\cdot) \in L^2(t_o,t_1;U)$ and every $t \in [t_o,t_1]$.

(H2') There exists a constant $c' > o$ such that

$$\| C\Phi_F(\cdot,s)x \|_{L^2(s,t_1;Y)} \leq c' \; \| x \|_V$$

for every $x \in W$ and every $s \in [t_o,t_1]$.

The dual properties are the following

(H1'*) The inequality

$$\| B^*\phi_F^*(t,\cdot)x\|_{L^2(t_o,t;U)} \leq b' \| x \|_{W^*}$$

holds for every $x \in V^*$ and every $t \in [t_o,t_1]$.

(H2'*) The inequality

$$\| \int_s^{t_1} \phi_F^*(\tau,s)C^*y(\tau)d\tau \|_{V^*} \leq c' \| y(\cdot) \|_{L^2(s,t_1;Y)}$$

holds for every $y(\cdot) \in L^2(t_o,t_1;Y)$ and every $s \in [t_o,t_1]$.

Using the condition (H2') and its dual we can define a strongly continuous operator $P_F(t) \in \mathcal{L}(V,V^*)$, by

$$\begin{aligned} P_F(t)x &= \phi_F^*(t_1,t)G\phi_F(t_1,t)x \\ &+ \int_t^{t_1} \phi_F^*(\tau,t)[C^*C+F^*(\tau)RF(\tau)]\phi_F(\tau,t)xd\tau \end{aligned} \tag{2.12}$$

for $t_o \leq t \leq t_1$ and $x \in V$. Then the cost of the feedback control (2.5) corresponding to an initial state $x_o \in V$ is given by

$$J(u_F) = \langle x_o,P_F(t_o)x_o\rangle_{V,V^*} \, . \tag{2.13}$$

If the initial state is in H, then this expression can be interpreted via the inner product in H.

A formula comparing the cost of an arbitrary control $u(\cdot) \in L^2(t_o,t_1;U)$ with the cost of the feedback control (2.5) will play an important role in our analysis. In the proof of this result we will need to interchange some integrals. At some points this becomes a delicate problem since we will have to operate with terms like $C\Phi_F(t,s)B$. In order to make the results precise, we need a third hypothesis.

(H3) Suppose that

$$Z = \mathcal{D}_V(A) \subset W$$

with a continuous, dense embedding where the Hilbert space Z is endowed with the graph norm of A, regarded as an unbounded, closed operator on V.

This assumption is not very restrictive. It is satisfied by all known examples of systems which satisfy (H1) and (H2) if the spaces W and V are chosen appropriately. In the following we summarise some important consequences of (H3).

<u>Remarks 2.3</u>

(i) If (H3) is satisfied, then A can be regarded as a bounded operator from Z into V. Correspondingly A^* becomes a bounded operator from V^* into Z^*. On the other hand A can be restricted to a closed, densely defined operator on Z. It's adjoint in this sense coincides with the above operator $A^*: V^* \to Z^*$ (SALAMON [40, Lemma 1.3.2]) and moreover

$$\mathcal{D}_{W^*}(A^*) \subset \mathcal{D}_{Z^*}(A^*) = V^*.$$

(ii) It is a well known fact from semigroup theory that

$$T_t x = \int_0^t S(s)x\,ds \in \mathcal{D}_V(A) = Z$$

for every $x \in V$ and every $t \geq 0$. If (H3) is satisfied then T_t is a strongly continuous family of bounded, linear operators from V into W. It is easy to see that the adjoint operator $T_t^* \in \mathcal{L}(W^*,V^*)$ is given by

$$T_t^* x = \int_0^t S^*(s)x\,ds \in \mathcal{D}_{W^*}(A^*) \subset V^*$$

for $x \in W^*$ and $t \geq 0$.

(iii) if (H1), (H2) and (H3) are satisfied then the following equation holds for every $u \in U$ and every $t \geq 0$

$$C\int_0^t S(s)Bu\,ds = CT_t Bu = \int_0^t CS(s)Bu\,ds.$$

This seems like a trivial fact, however, we were not able to establish this identity without assuming (H3). Note that the LHS of the above equation has to be interpreted in terms of (H1) and the RHS in terms of (H2). For establishing the equation one must approximate $Bu \in V$ by a sequence of elements in W. Then the term on the LHS will not converge in general unless range $T_t \subset W$.

Lemma 2.4

Suppose that (H1), (H2), (H3) are satisfied, let $F(t) \in \mathcal{L}(V,U)$, $t_0 \leq t \leq t_1$, be strongly continuous and let $\Phi_F(t,s) \in \mathcal{L}(V) \cap \mathcal{L}(W)$ be defined by (2.6). Moreover, let $u(\cdot) \in L^2(t_0,t_1;U)$ and $y(\cdot) \in L^2(t_0,t_1;U)$ be given. Then

$$\text{(2.14)} \qquad \int_{t_0}^{t_0}\int_s^{t_1} \langle C\Phi_F(t,s)Bu(s),y(t)\rangle_Y dt\,ds = \int_{t_0}^{t_1} \langle C\int_{t_0}^{t} \Phi_F(t,s)Bu(s)ds,y(t)\rangle_Y dt$$

where the first expression must be interpreted in terms of (H2') and the second in terms of (H1').

Proof First note that, by (2.6) and (H1),

$$\Phi_F(t,s) - S(t-s) \in \mathcal{L}(V,W).$$

Hence it is enough to establish the desired equation with $\Phi_F(t,s)$ replaced by $S(t-s)$.

Secondly, let us assume that $u(\cdot) \in C^1(t_o,t_1;U)$ and $u(t_o) = 0$. Then

$$x(t) = \int_{t_o}^{t} S(t-s)Bu(s)ds$$

$$= \int_{t_o}^{t}\int_{t_o}^{s} S(t-s)B\dot{u}(\sigma)d\sigma ds$$

$$= \int_{t_o}^{t}\int_{\sigma}^{t} S(t-s)B\dot{u}(\sigma)dsd\sigma$$

$$= \int_{t_o}^{t} T_{t-\sigma}B\dot{u}(\sigma)d\sigma$$

Analogously, we get

$$z(s) = \int_{s}^{t_1} S^*(t-s)C^*y(t)dt = -\int_{s}^{t_1} T^*_{\tau-s}C^*\dot{y}(\tau)d\tau$$

for $y(\cdot) \in C^1(t_o,t_1;Y)$ with $y(t_1) = 0$. This implies

$$\int_{t_o}^{t_1} \langle Cx(t),y(t)\rangle_Y dt$$

$$= \int_{t_o}^{t_1}\int_{t_o}^{t} \langle CT_{t-s}B\dot{u}(s),y(t)\rangle_Y ds\, dt$$

$$= \int_{t_o}^{t_1}\int_{s}^{t_1} \langle \dot{u}(s),B^*T^*_{t-s}C^*y(t)\rangle_U dt\, ds$$

$$= \int_{t_o}^{t_1} \langle \dot{u}(s),B^* \int_{o}^{t_1-s} T^*_{\tau}\, C^*y(\tau+s)d\tau\rangle_U ds$$

$$= - \int_{t_o}^{t_1} \langle u(s),B^* \int_{o}^{t_1-s} T^*_{\tau}C^*\dot{y}(\tau+s)d\tau\rangle_U ds$$

$$= \int_{t_o}^{t_1} \langle u(s), B^* z(s)\rangle_U \, ds$$

Both sides of this equation depend continuously on $u(\cdot) \in L^2(t_o,t_1;U)$ and $y(\cdot) \in L^2(t_o,t_1;Y)$. Moreover

$$\int_{t_o}^{t_1} \langle u(s), B^* z(s)\rangle_U \, ds$$

$$= \int_{t_o}^{t_1} \langle Bu(s), \int_s^{t_1} S^*(t-s)C^* y(t)dt\rangle_{V,V^*} ds$$

$$= \int_{t_o}^{t_1} \int_s^{t_1} \langle CS(t-s)Bu(s), y(t)\rangle_Y \, dt\, ds.$$

This proves the statement of the lemma. $\boxtimes$

Now we are in the position to prove the desired comparison formula for the feedback control (2.5).

<u>Lemma 2.5</u>

Suppose that (H1), (H2), (H3) are satisfied, let $F(t) \in \mathcal{L}(V,U)$ be strongly continuous on the interval $[t_o,t_1]$ and let $P_F(t) \in \mathcal{L}(V,V^*)$ be defined by (2.12) and (2.6). Then the following equation holds for every $x_o \in V$ and every $u(\cdot) \in L^2(t_o,t_1;U)$

$$J(u) - \langle x_o, P_F(t_o)x_o\rangle_{V V^*}$$

$$(2.15) \quad = \int_{t_o}^{t_1} \langle R^{-1}B^*P_F(t)x(t)+u(t), R[R^{-1}B^*P_F(t)x(t)+u(t)]\rangle dt$$

$$- \int_{t_o}^{t_1} \langle R^{-1}B^*P_F(t)x(t)+F(t)x(t), R[R^{-1}B^*P_F(t)x(t)+F(t)x(t)]\rangle dt$$

where $x(t)$, $t_o \le t \le t_1$, is given by (2.3).

<u>Proof</u> We give a proof of (2.15) for the case $x_o \in W$ and $u(\cdot) \in C(t_o,t_1;U)$. Let $x(t)$ be the mild solution of (2.1) given by (2.3) and define $v(t) = u(t)-F(t)x(t)$ for $t \geq t_o$. Then, by Remark 2.2 (iii)

$$z(t) = \int_{t_o}^{t} \Phi_F(t-s)Bv(s)ds = x(t)-\Phi_F(t,t_o)x_o$$

for $t_o \leq t \leq t_1$. Let us first apply Lemma 2.4 to the functions.

$$v_s(t) = \begin{cases} v(t), & t_o \leq t \leq s, \\ 0, & s < t \leq t_1 \end{cases}$$

$$y_s(t) = \begin{cases} 0, & t_o \leq t < s, \\ C\Phi_F(t,s)Bv(s), & s \leq t \leq t_1 \end{cases}$$

and integrate over s. Then we get

$$\int_{t_o}^{t_1} [\int_{s}^{t_1} <C\Phi_F(t,s)z(s),C\Phi_F(t,s)Bv(s)>dt]ds$$

$$= \int_{t_o}^{t_1} [\int_{t_o}^{t_1} <C\int_{t_o}^{t} \Phi_F(t,\sigma)Bv_s(\sigma)d\sigma,y_s(t)>dt]\, ds$$

$$= \int_{t_o}^{t_1} [\int_{t_o}^{t_1} \int_{\sigma}^{t_1} <C\Phi_F(t,\sigma)Bv_s(\sigma),y_s(t)>dt\, d\sigma]\, ds \qquad \text{, by (2.14)}$$

$$= \int_{t_o}^{t_1} \int_{t_o}^{s} [\int_{s}^{t_1} <C\Phi_F(t,\sigma)Bv(\sigma),C\Phi_F(t,s)Bv(s)>dt]d\sigma\, ds$$

$$= \int_{t_o}^{t_1} [\int_{\sigma}^{t_1} \int_{s}^{t_1} <C\Phi_F(t,\sigma)Bv(\sigma),C\Phi_F(t,s)Bv(s)>dt\, ds]\, d\sigma$$

$$= \int_{t_o}^{t_1}\int_{\sigma}^{t_1} \langle C\Phi_F(t,\sigma)Bv(\sigma), C\int_{\sigma}^{t} \Phi_F(t,s)Bv(s)ds\rangle dt\, d\sigma \qquad \text{, by (2.14)}$$

$$= \int_{t_o}^{t_1}\int_{s}^{t_1} \langle C\Phi_F(t,s)Bv(s), C\int_{s}^{t} \Phi_F(t,\sigma)Bv(\sigma)ds\rangle dt\, ds$$

and hence, again using Lemma 2.4,

$$2\mathrm{Re}\int_{t_o}^{t_1}\int_{s}^{t_1} \langle C\Phi_F(t,s)x(s), C\Phi_F(t,s)Bv(s)\rangle dt\, ds$$

$$= 2\mathrm{Re}\int_{t_o}^{t_1}\int_{s}^{t_1} \langle C\Phi_F(t,t_o)x_o, C\Phi_F(t,s)Bv(s)\rangle dt\, ds$$

$$+ \mathrm{Re}\int_{t_o}^{t_1}\int_{s}^{t_1} \langle C\int_{t_o}^{s} \Phi_F(t,\sigma)Bv(\sigma)d\sigma, C\Phi_F(t,s)Bv(s)\rangle dt\, ds$$

$$+ \mathrm{Re}\int_{t_o}^{t}\int_{s}^{t_1} \langle C\int_{s}^{t} \Phi_F(t,\sigma)Bv(\sigma)d\sigma, C\Phi_F(t,s)Bv(s)\rangle dt\, ds$$

$$= 2\mathrm{Re}\int_{t_o}^{t_1} \langle C\Phi_F(t,t_o)x_o, C\int_{t_o}^{t} \Phi_F(t,s)Bv(s)ds\rangle dt \qquad \text{, by (2.14)}$$

$$+ \mathrm{Re}\int_{t_o}^{t_1}\int_{s}^{t_1} \langle Cz(t), C\Phi_F(t,s)Bv(s)\rangle dt\, ds$$

$$= 2\mathrm{Re}\int_{t_o}^{t_1} \langle C\Phi_F(t,t_o)x_o, Cz(t)\rangle dt$$

$$+ \int_{t_o}^{t_1} \langle Cz(t), Cz(t)\rangle dt \qquad \text{, by (2.14)}$$

$$= \int_{t_o}^{t_1} \| Cx(t)\|_Y^2\, dt - \int_{t_o}^{t_1} \| C\Phi(t,t_o)x_o\|_Y^2\, dt.$$

Analogous identities can be derived in a more straightforward way when $C^*C \in \mathcal{L}(W,W^*)$ is replaced by $G \in \mathcal{L}(V,V^*)$ or $F^*(t)RF(t) \in \mathcal{L}(V,V^*)$, respectively. This implies

$$2\,\mathrm{Re}\int_{t_o}^{t_1} \langle P_F(s)x(s),Bv(s)\rangle ds$$

$$= 2\mathrm{Re}\int_{t_o}^{t_1} \langle \Phi_F(t_1,s)x(s),G\Phi_F(t_1,s)Bv(s)\rangle ds$$

$$+ 2\,\mathrm{Re}\int_{t_o}^{t_1}\int_{s}^{t_1} \langle C\Phi_F(t,s)x(s),C\Phi_F(t,s)Bv(s)\rangle dt\, ds$$

$$+ 2\,\mathrm{Re}\int_{t_o}^{t_1}\int_{s}^{t_1} \langle F(t)\Phi_F(t,s)x(s),RF(t)\Phi_F(t,s)Bv(s)\rangle dt\, ds$$

$$= \langle x(t_1),Gx(t_1)\rangle - \langle \Phi_F(t_1,t_o)x_o,G\Phi_F(t_1,t_o)x_o\rangle$$

$$+ \int_{t_o}^{t_1} \langle Cx(t),Cx(t)\rangle dt - \int_{t_o}^{t_1} \langle C\Phi_F(t,t_o)x_o,C\Phi_F(t,t_o)x_o\rangle dt$$

$$+ \int_{t_o}^{t_1} \langle F(t)x(t),RF(t)x(t)\rangle dt - \int_{t_o}^{t_1} \langle F(t)\Phi_F(t,t_o)x_o,RF(t)\Phi_F(t,t_o)x_o\rangle dt.$$

Now the L H S of (2.15) equals

$$J(u) - \langle x_o, P_F(t_o)x_o\rangle$$

$$= \int_{t_o}^{t_1} \langle u(t), Ru(t)\rangle dt - \int_{t_o}^{t_1} \langle F(t)\Phi_F(t,t_o)x_o, RF(t)\Phi_F(t,t_o)x_o\rangle dt$$

$$+ \int_{t_o}^{t_1} \langle Cx(t), Cx(t)\rangle dt - \int_{t_o}^{t_1} \langle C\Phi_F(t,t_o)x_o, C\Phi_F(t,t_o)x_o\rangle dt$$

$$+ \langle x(t_1), Gx(t_1)\rangle - \langle \Phi_F(t_1,t_o)x_o, G\Phi_F(t_1,t_o)x_o\rangle$$

$$= \int_{t_o}^{t_1} \langle u(t), Ru(t)\rangle dt + 2\,\mathrm{Re} \int_{t_o}^{t_1} \langle P_F(t)x(t), Bv(t)\rangle dt$$

$$- \int_{t_o}^{t_1} \langle F(t)x(t), RF(t)x(t)\rangle dt .$$

It is easy to see that the final expression coincides with the R H S of (2.15). This proves Lemma 2.5. ▨

We are now able to prove the main result of this section

<u>Theorem 2.6</u> Let (H1), (H2) and (H3) be satisfied. Then there exists a unique strongly continuous self adjoint, non-negative operator $P(t) \in \mathcal{L}(V,V^*)$ $t_o \leq t \leq t_1$, solving the <u>integral Riccati equation</u>.

$$P(t)x = \Phi^*(t_1,t)G\Phi(t_1,t)x$$

$$+ \int_t^{t_1} \Phi^*(s,t)\,[C^*C+P(s)BR^{-1}B^*P(s)]\,\Phi(s,t)x\,ds \tag{2.16}$$

for $x \in W$ and $t_o \leq t \leq t_1$ where $\Phi(s,t) = \Phi_F(s,t)$ is the evolution operator defined by (2.6) with $F(t) = - R^{-1}B^*P(t) \in \mathcal{L}(V,U)$. Furthermore there is a unique optimal control which minimizes the performance index (2.4) subject to (2.3). This optimal control is given by the feedback control law

$$u_F(t) = - R^{-1}B^*P(t)x(t) \tag{2.17}$$

and the optimal cost is

$$J(u_F) = \langle x_o, P(t_o)x_o \rangle \ . \tag{2.18}$$

<u>*Proof.*</u> We regard equation (2.16) as a fixed point problem which is to be solved by iteration. Let us define the sequence $P_k(t) \in \mathcal{L}(V,V^*)$ recursively through

$$P_o(t) = 0$$

$$P_k(t) = P_F(t),\ F(t) = - R^{-1}B^*P_{k-1}(t)$$

for $k \in \mathbb{N}$ and $t_o \leq t \leq t_1$,

where $P_F(t)$ is given by (2.12). Let us also define

$$\Phi_k(s,t) = \Phi_F(s,t),\ F(t) = - R^{-1}B^*P_k(t),$$

so that

$$\begin{aligned} P_{k+1}(t)x &= \Phi_k^*(t_1,t)G\Phi_k(t_1,t)x \\ &+ \int_t^{t_1} \Phi_k^*(s,t)\ [C^*C+P_k(s)BR^{-1}B^*P_k(s)]\ \Phi_k(s,t)x\,ds \end{aligned} \tag{2.19}$$

holds for $t_o \leq t \leq t_1$ and $x \in W$. Applying Lemma 2.5 to $F(t) = - R^{-1}B^*P_{k-1}(t)$ and $u_k(t) = - R^{-1}B^*P_k(t)x(t)$, we obtain

$$\langle x_o, P_{k+1}(t_o)x_o\rangle$$

$$= J(u_k)$$

$$(2.20) \quad = \langle x_o, P_k(t_o)x_o\rangle - \int_{t_o}^{t_1} \langle [P_k(\tau)-P_{k-1}(\tau)]x(\tau), BR^{-1}B^*[P_k(\tau)-P_{k-1}(\tau)]x(\tau)\rangle d\tau$$

$$\leq \langle x_o, P_k(t_o)x_o\rangle$$

for $k \in \mathbb{N}$ and $x_o \in V$. Thus the sequence $\langle x_o, P_k(t_o)x_o\rangle_{V,V^*}$, $k \in \mathbb{N}$, is monotonically decreasing and positive. Hence $P_k(t_o)$ converges strongly to a non-negative, self adjoint operator $P(t_o) \in \mathcal{L}(V,V^*)$ (cf. KATO [25 , p.454, Theorem 3.3]). The same conclusion is valid for every $t \in [t_o,t_1]$ since $t_o \leq t_1$ can be chosen arbitrarily.

Moreover, (2.20) shows that the functions $P_k(t)x$, $t_o \leq t \leq t_1$, $k \in \mathbb{N}$, are uniformly bounded in V^*. Hence the limit function $P(t)x$, $t_o \leq t \leq t_1$, is strongly measurable in V^* and bounded. Therefore we can introduce a strongly continuous evolution operator $\Phi(s,t) = \Phi_F(s,t) \in \mathcal{L}(V)$ which is defined by (2.6) with $F(t) = -R^{-1}B^*P(t)$.

Our next step is to prove that $\Phi_k(s,t)x \in V$ converges for every $x \in V$ to $\Phi(s,t)x$ and that this convergence is uniform on the domain $t \leq s \leq t_1$ (t fixed). For this sake let us consider the identity

$$\Phi(s,t)x - \Phi_k(s,t)x$$

$$= \int_t^s S(s-\tau)BR^{-1}B^*[P_k(\tau)-P(\tau)]\,\Phi(\tau,t)x\,d\tau$$

$$- \int_t^s S(s-\tau)BR^{-1}B^*P_k(\tau)[\Phi(\tau,t)x-\Phi_k(\tau,t)x]\,d\tau$$

and apply Gronwall's lemma. Then the desired convergence of $\Phi_k(s,t)x$ follows from the pointwise strong convergence of $P_k(\tau)$ to $P(\tau)$ together with the dominated convergence theorem.

As a consequence of this convergence result we obtain a uniform bound on $\|\Phi_k(s,t)\|_{\mathcal{L}(V)}$ and thus on $\|\Phi_k(s,t)\|_{\mathcal{L}(W)}$ for $k \in \mathbb{N}$ and $t \le s \le t_1$. This allows us to apply the dominated convergance theorem to formula (2.19) and hence $P(t)$ satisfies the integral Riccati equation (2.16). Finally, it follows easily from (2.16) together with the strong continuity of $\Phi(s,t)$ in both variables and in both spaces V and W that the operator $P(t) \in \mathcal{L}(V,V^*)$ is strongly continuous on the interval $[t_o,t_1]$. Thus we have proved the existence of a solution to (2.16).

In order to prove the uniqueness for the solution of (2.16) together with the statements on the optimal control, let us assume that $P(t) \in \mathcal{L}(V,V^*)$ is any strongly continuous, non-negative solution of (2.16). Moreover, let $x_o \in V$, $u(\cdot) \in L^2(t_o,t_1;U)$ be given, let $x(t) \in V$ be the corresponding solution of (2.1) which is given by (2.3) and define $v(t) = u(t) + R^{-1}B^*P(t)x(t)$ for $t_o \le t \le t_1$. Then it follows from Lemma 2.5 that

$$(2.21) \qquad J(u) = \langle x_o,P(t_o)x_o\rangle + \int_{t_o}^{t_1} \langle v(t),Rv(t)\rangle dt.$$

Hence the optimal control is unique and given by the feedback law (2.17) and the optimal cost is given by (2.18). Moreover, we conclude from (2.21) that $\langle x_o,P(t_o)x_o\rangle = \langle x_o,\hat{P}(t_o)x_o\rangle$ for any two non-negative solutions $P(t),\hat{P}(t) \in \mathcal{L}(V,V^*)$ of (2.16) and any $x_o \in V$. Since $t_o \le t_1$ can be chosen arbitrarily, this proves the uniqueness of the solution to (2.16). ▨

The following result shows that the integral Riccati equation (2.16) can be converted into a differential Riccati equation.

Proposition 2.7

Suppose that (H1), (H2) and (H3) are satisfied and let $P(t) \in \mathcal{L}(V,V^*)$ be a non-negative, self adjoint, strongly continuous operator on the interval $[t_o,t_1]$. Moreover, let the evolution operator $\Phi(s,t) = \Phi_F(s,t) \in \mathcal{L}(V)$ be defined by (2.6) with $F(t) = -R^{-1}B^*P(t)$. Then the following statements are equivalent.

(i) Equation (2.16) holds for every $x \in W$ and every $t \in [t_o,t_1]$.

(ii) For every $x \in W$ and every $t \in [t_o,t_1]$ the following equation holds

$$(2.22) \quad P(t)x = \Phi^*(t_1,t)GS(t_1-t)x + \int_t^{t_1} \Phi^*(s,t)C^*CS(s-t)x\,ds$$

(iii) For every $x \in W$ and every $t \in [t_o,t_1]$ the following equation holds

$$P(t)x = S^*(t_1-t)GS(t_1-t)x$$
$$(2.23) \quad + \int_t^{t_1} S^*(s-t)[C^*C-P(s)BR^{-1}B^*P(s)]S(s-t)x\,ds$$

(iv) For every $x \in Z$ the function $P(t)x$, $t_o \le t \le t_1$ is continuously differentiable with values in Z^* and satisfies the differential Riccati equation

$$(2.24;1) \quad \frac{d}{dt}P(t)x + A^*P(t)x + P(t)Ax - P(t)BR^{-1}B^*P(t)x + C^*Cx = 0$$

$$(2.24;2) \quad P(t_1)x = Gx$$

Proof The equivalence of the statements (i), (ii) and (iii) can be established in a straightforward way by using the formulas (2.6) and (2.9). We will only show that (i) implies (ii). In fact, (i) implies that the following equation holds for every $x \in W$ and every $t \in [t_o,t_1]$

$$
\begin{aligned}
&P(t)x \\
&= \Phi^*(t_1,t)GS(t_1-t)x + \int_t^{t_1} \Phi^*(s,t)C^*CS(s-t)x\,ds \\
&+ \Phi^*(t_1,t)G\int_t^{t_1} \Phi(t_1,\tau)BF(\tau)S(\tau-t)x\,d\tau \\
&+ \int_t^{t_1} \Phi^*(s,t)C^*C\int_t^{s} \Phi(s,\tau)BF(\tau)S(\tau-t)x\,d\tau\,ds \\
&+ \int_t^{t_1} \Phi^*(s,t)F^*(s)RF(s)\Phi(s,t)x\,ds
\end{aligned}
$$

where $F(t) = -R^{-1}B^*P(t) \in \mathcal{L}(V,U)$ for $t_o \leq t \leq t_1$. The last three terms on the RHS of this equation cancel out since

$$
\begin{aligned}
&\int_t^{t_1} \Phi^*(s,t)F^*(s)RF(s)\Phi(s,t)x\,ds \\
&= \int_t^{t_1} \Phi^*(s,t)F^*(s)RF(s)S(s-t)x\,ds \\
&+ \int_t^{t_1}\int_t^{s} \Phi^*(s,t)F^*(s)RF(s)\Phi(s,\tau)BF(\tau)S(\tau-t)x\,d\tau\,ds \\
&= -\int_t^{t_1} \Phi^*(\tau,t)\{P(\tau)\,[BF(\tau)S(\tau-t)x] \\
&- \int_\tau^{t_1} \Phi^*(s,\tau)F^*(s)RF(s)\Phi(s,\tau)[BF(\tau)S(\tau-t)x]\,ds\}\,d\tau \\
&= -\int_t^{t_1} \Phi^*(\tau,t)\Phi^*(t_1,\tau)G\Phi(t_1,\tau)BF(\tau)S(\tau-t)x\,d\tau \\
&- \int_t^{t_1} \Phi^*(\tau,t)\int_\tau^{t_1} \Phi^*(s,\tau)C^*C\Phi(s,\tau)BF(\tau)S(\tau-t)x\,ds\,d\tau
\end{aligned}
$$

Applying Lemma 2.4 to the final term, we obtain the desired cancellation.

Now we prove the equivalence of the statements (iii) and (iv). Note that the equation

$$\begin{aligned} &\langle CS(t)z, CS(t)x\rangle - \langle Cz, Cx\rangle \\ (2.26) \qquad &= \int_0^t [\langle CS(s)Az, S(s)x\rangle + \langle CS(s)z, CS(s)Ax\rangle]ds \end{aligned}$$

holds for all $x, z \in \mathcal{D}_W(A)$ and every $t \geq 0$. It follows from (H3) and (H2) that both sides of this equation depend continuously on $x, z \in Z = \mathcal{D}_V(A) \subset W$ and that $\mathcal{D}_Z(A) \subset \mathcal{D}_W(A) \subset Z$. Consequently $\mathcal{D}_W(A)$ is dense in Z and hence (2.26) holds for all $x, z \in Z$.

From (2.26) we see that the function $\langle z, P(t)x\rangle$ - defined by (2.23) - is continuously differentiable on the interval $[t_o, t_1]$ for all $x, z \in Z$ and satisfies the equation

$$\begin{aligned} &\frac{d}{dt} \langle z, P(t)x\rangle \\ &= - \langle S(t_1-t)Az, GS(t_1-t)x\rangle - \langle S(t_1-t)z, GS(t_1-t)Ax\rangle \\ &\quad - \langle Cz, Cx\rangle + \langle z, P(t)BR^{-1}B^*P(t)x\rangle \\ &\quad - \int_t^{t_1} [\langle CS(s-t)Az, CS(s-t)x\rangle + \langle CS(s-t)z, CS(s-t)Ax\rangle]ds \\ &\quad + \int_t^{t_1} \langle S(s-t)Az, P(s)BR^{-1}B^*P(s)S(s-t)x\rangle ds \\ &\quad + \int_t^{t_1} \langle S(s-t)z, P(s)BR^{-1}B^*P(s)S(s-t)Ax\rangle ds \\ &= - \langle Az, P(t)x\rangle - \langle z, P(t)Ax\rangle \\ &\qquad - \langle Cz, Cx\rangle + \langle z, P(t)BR^{-1}B^*P(t)x\rangle \end{aligned}$$

This implies

$$\langle z, P(t)x\rangle_{Z,Z^*}$$

$$= \langle z, Gx + \int_t^{t_1} [A^*P(s)x+P(s)Ax-P(s)BR^{-1}B^*P(s)x+C^*Cx]ds\rangle_{Z,Z^*}$$

and hence (2.24;1). Thus we have proved that (iii) implies (iv).

Conversely, let us assume that P(t) satisfies (2.24). Then the following equation holds for every $x \in Z$ and every $t \in [t_o,t_1]$

$$S^*(t_1-t)GS(t_1-t)x - P(t)x$$

$$= \int_t^{t_1} \frac{d}{ds} S^*(s-t)P(s)S(s-t)xds$$

$$= \int_t^{t_1} S^*(s-t)[\dot{P}(s)+A^*P(s)+P(s)A]S(s-t)xds$$

$$= \int_t^{t_1} S^*(s-t)[C^*C-P(s)BR^{-1}B^*P(s)]S(s-t)xds$$

where the integral has to be understood in the Hilbert space Z^* and $\dot{P}(t)$ is the strong derivative of P(t), $t_o \leq t \leq t_1$ regarded as an operator in $\mathcal{L}(Z,Z^*)$. ▨

3. INFINITE TIME CONTROL

In this section we consider the control problem of minimizing the performance index

$$(3.1) \qquad J(u) = \int_o^\infty [\|y(t)\|_Y^2 + \langle u(t),Ru(t)\rangle_U]dt$$

where y(t) is again the output of (2.1) with $t_o = 0$, i.e.

(3.2) $$y(t) = CS(t)x_o + C\int_o^t S(t-s)Bu(s)ds, \quad t \geq 0.$$

For this infinite time problem it is not clear that the cost will be finite for any control input $u(\cdot) \in L^2(o,\infty;U)$. So we add this as another hypothesis.

(H4) For every $x_o \in V$ there exists a $u_{x_o}(\cdot) \in L^2[o,\infty;U]$ such that $J(u_{x_o})<\infty$.

We will derive the optimal control via the solution of an <u>algebraic Riccati equation</u> which is actually the stationary version of (2.24). For this sake we consider the finite time control problems of minimizing the cost functionals.

(3.3) $$J_T(u) = \int_0^T [\| y(t)\|_Y^2 + \langle u(t),Ru(t)\rangle_U]dt$$

subject to the constraint (3.2). The corresponding Riccati operator will be denoted by $P_T(t) \in \mathcal{L}(V,V^*)$ and satisfies the equation

(3.4) $$P_T(t)x = \int_t^T S^*(s-t)[C^*C-P_T(s)BR^{-1}B^*P_T(s)]S(s-t)xds$$

for every $x \in W$ and every $t \in [0,T]$.

<u>Lemma 3.1</u>

$$P_{T-\alpha}(t) = P_T(t+\alpha), \quad 0 \leq t \leq T-\alpha$$

<u>Proof</u>

The operator $P_T(t+\alpha)$ satisfies the equation

$$P_T(t+\alpha)x = \int_{t+\alpha}^T S^*(s-t-\alpha)[C^*C-P_T(s)BR^{-1}B^*P_T(s)]S(s-t-\alpha)xds$$

$$= \int_t^{T-\alpha} S^*(s-t)[C^*C-P_T(s+\alpha)BR^{-1}B^*P_T(s+\alpha)]S(s-t)xds$$

for $x \in W$ and $0 \leq t \leq T - \alpha$. Thus the statement of the Lemma follows from the equivalence of (2.16) and (2.23) (Proposition 2.7) together with the uniqueness result (Theorem 2.6). $\square$

We will derive the solution of the <u>algebraic Riccati equation</u> as the limit of the solutions to <u>integral Riccati equations</u> as T goes to infinity. For this we need the following preliminary result which is a special case of Proposition 2.7.

<u>Corollary 3.2</u>

Suppose that the hypotheses (H1), (H2) and (H3) are satisfied and let $P \in \mathcal{L}(V,V^*)$ be a non-negative, self adjoint operator. Moreover, let $S_p(t) \in \mathcal{L}(V) \cap \mathcal{L}(W)$ be the strongly continuous semigroup which is generated by $A-BR^{-1}B^*P: \mathcal{D}_V(A) \to V$, i.e. $S_p(t)$ satisfies the equation

$$S_p(t)x = S(t)x - \int_o^t S(t-s)BR^{-1}B^*PS_p(s)xds \tag{3.5}$$

for $x \in V$ and $t \geq o$. Then the following statements are equivalent.

(i) For every $x \in W$ and every $t \geq o$

$$\begin{aligned} Px = {} & S_p^*(t)PS_p(t)x \\ & + \int_o^t S_p^*(s)[C^*C+PBR^{-1}B^*P]S_p(s)xds \end{aligned} \tag{3.6}$$

(ii) For every $x \in W$ and every $t \geq o$

$$Px = S_p^*(t)PS(t)x + \int_o^t S_p^*(s)C^*CS(s)xds \tag{3.7}$$

(iii) For every $x \in W$ and every $t \geq o$

$$\begin{aligned} Px = {} & S^*(t)PS(t)x \\ & + \int_o^t S^*(s)[C^*C-PBR^{-1}B^*P]S(s)xds \end{aligned}$$

(iv) For every $x \in Z$ the following equation holds in Z^*

$$(3.9) \qquad A^*Px + PAx - PBR^{-1}B^*Px + C^*Cx = 0.$$

Now we are in the position to prove the main result of this section.

<u>Theorem 3.3</u>

Let (H1), (H2) and (H3) be satisfied. Then the following statements hold.

(i) The hypothesis (H4) is satisfied if and only if there exists a non-negative self adjoint solution $P \in \mathcal{L}(V,V^*)$ of (3.9).

(ii) If (H4) is satisfied, then there exists a unique optimal control $u_p(\cdot) \in L^2(o,\infty;U)$ which is given by the feedback law.

$$(3.10) \qquad u_p(t) = -R^{-1}B^*Px(t), \quad t \geq 0,$$

where $P \in \mathcal{L}(V,V^*)$ is the (unique) <u>minimal</u> solution of (3.9). Moreover, the optimal cost is given by

$$(3.11) \qquad J(u_p) = \langle x_o, Px_o \rangle.$$

(iii) If (H4) is satisfied, then the minimal solution $P \in \mathcal{L}(V,V^*)$ of (3.9) is strong limit of $P_T(o) \in \mathcal{L}(V,V^*)$ as T goes to infinity where $P_T(t)$ is defined by (3.4).

<u>Proof</u>

First recall that the optimal control of the finite time problem on the interval $[0,T]$ is given by $u_T(t) = R^{-1}B^*P_T(t)x(t)$, $0 \leq t \leq T$, and the optimal cost by $J_T(u_T) = \langle x_o, P_T(o)x_o \rangle$ (Theorem 2.4). So (H4) implies that

$$\langle x_o, P_T(o)x_o \rangle = J_T(u_T) \leq J_T(u_{x_o}) \leq J(u_{x_o}) < \infty$$

and thus there exists a limit of the increasing function $\langle x_o, P_T(o)x_o \rangle, T \geq 0$, for every $x_o \in V$. Hence there exists a non-negative, self adjoint operator $P \in \mathcal{L}(V,V^*)$ which is the strong limit of $P_T(o)$ (KATO [25, p.454, theorem 3.3]). By Lemma 3.1,

$$Px = \underset{T\to\infty}{\text{s-lim}}\ P_T(t)x \in V^* \tag{3.12}$$

exists uniformly in t on every compact time interval. Making use of formula (3.4), we obtain

$$\begin{aligned}
Px &= \lim_{T\to\infty} P_T(o)x \\
&= \lim_{T\to\infty} \int_o^T S^*(s)[C^*C-P_T(s)BR^{-1}B^*P_T(s)]S(s)x\,ds \\
&= \lim_{T\to\infty} \int_t^T S^*(t)S^*(s-t)[C^*C-P_T(s)BR^{-1}B^*P_T(s)]S(s-t)S(t)\,x\,ds \\
&+ \lim_{T\to\infty} \int_o^t S^*(s)[C^*C-P_T(s)BR^{-1}B^*P_T(s)]S(s)x\,ds \\
&= \lim_{T\to\infty} S^*(t)P_T(t)S(t)x \\
&+ \int_o^t S^*(s)[C^*C-PBR^{-1}B^*P]S(s)x\,ds \\
&= S^*(t)PS(t)x \\
&+ \int_o^t S^*(s)[C^*C-PBR^{-1}B^*P]S(s)x\,ds
\end{aligned}$$

and hence $P \in \mathcal{L}(V,V^*)$ is a solution of (3.6), (3.7), (3.8) and (3.9).

Conversely, let $Q \in \mathcal{L}(V,V^*)$ be any non-negative solution of (3.9) and let $u_Q(t) = -R^{-1}B^*Qx(t)$ be the corresponding feedback control law with the associated closed loop semigroup $S_Q(t) \in \mathcal{L}(V) \cap \mathcal{L}(W)$. Then the following inequality holds for every $x_o \in V$

$$\begin{aligned} &\langle x_o, Qx_o\rangle \\ &= \lim_{t\to\infty} \{ \langle S_Q(t)x_o, QS_Q(t)x_o\rangle \\ (3.13)\qquad &+ \int_o^t \langle S_Q(s)x_o, [C^*C+QBR^{-1}B^*Q]S_Q(s)x_o\rangle ds\} \\ &\geq \int_o^\infty \langle S_Q(t)x_o, [C^*C+QBR^{-1}B^*Q]\, S_Q(s)x_o\rangle\, ds \\ &= J(u_Q) \end{aligned}$$

and hence (H4) is satisfied. Moreover, the operator $P \in \mathcal{L}(V,V^*)$ defined by (3.12) satisfies the inequality

$$\begin{aligned} &\langle x_o, Px_o\rangle \\ &= \lim_{T\to\infty} \langle x_o, P_T(o)x_o\rangle \\ &\leq \lim_{T\to\infty} J_T(u) \\ &= J(u) \end{aligned}$$

for every admissible control $u(\cdot) \in L^2(o,\infty;U)$. This shows that P is the minimal positive semidefinite solution of (3.6). Finally, taking $Q = P$, we conclude that the unique optimal control is given by (3.10) with cost (3.11). ▨

Although the above theorem yields a solution to the infinite time problem, in a sense it is unsatisfactory. This is because we are not sure of a unique solution to the algebraic Riccati equation and also we cannot be sure that the semigroup $S_p(t)$ is exponentially stable. In order to resolve those difficulties, we need another hypothesis.

(H5) If $x_o \in V$ and $u(\cdot) \in L^2(o,\infty;U)$ are such that $J(u) < \infty$, then $x(\cdot) \in L^2(o,\infty;V)$ where $x(t)$, $t \geq o$, is given by (2.3) with $t_o = 0$.

__Theorem 3.4__

Let (H1), (H2), (H3) and (H5) be satisfied. Then the __algebraic Riccati equation__ (3.9) has at most one non-negative, self adjoint solution $P \in \mathcal{L}(V,V^*)$. Moreover, if P is such a solution, then the closed loop semigroup $S_p(t) \in \mathcal{L}(V)$ is exponentially stable.

__Proof__

If $P \in \mathcal{L}(V,V^*)$ is a positive semidefinite solution of (3.9), then the inequality (3.13) with Q=P shows that the closed loop control $u_p(t) = -R^{-1}B^*Px(t)$ has a finite cost for every initial state $x_o \in V$. By hypothesis (H5) this means that

$$\int_o^\infty \| S_p(t)x_o \|_V^2 \, dt < \infty$$

for every $x_o \in V$. Hence it follows from a result of DATKO [11] that the semigroup $S_p(t) \in \mathcal{L}(V)$ is exponentially stable (see also CURTAIN-PRITCHARD [9]). The stability of $S_p(t)$ shows that we have equality in (3.13) and hence

$$J(u_p) = \langle x_o, P\, x_o \rangle .$$

Now let $Q \in \mathcal{L}(V,V^*)$ be another non-negative solution of (3.9) and let us apply Lemma 2.5 to the performance index

$$J_{T,Q}(u) = \langle x(T), Qx(T) \rangle$$

$$+ \int_o^T [\| y(t) \|_Y^2 + \langle u(t), Ru(t) \rangle] \, dt$$

as well as the feedback $F(t) \equiv -R^{-1}B^*Q$ and the control input $u_p(t)$. Then $P_F(t) \equiv Q$ and hence the inequality

$$<x_o, Px_o>$$

$$= J(u_p)$$

$$= \lim_{T\to\infty} J_{T,Q}(u_p)$$

$$= \lim_{T\to\infty} [<x_o, Qx_o>$$

$$+ \int_o^T <R^{-1}B^*Qx(t)+u_p(t), R[R^{-1}B^*Qx(t)+u_p(t)]>dt$$

$$\geq <x_o, Qx_o>$$

holds for every $x_o \in V$. Interchanging the roles of P and Q, we conclude that $P = Q$. ▨

Finally, let us briefly discuss the hypotheses (H4) and (H5) which are chosen in a general sense but are difficult to check in concrete examples. In most cases it might be desirable to replace them by stronger assumptions which are easier to check.

<u>Remarks 3.5</u>

Let (H1) and (H2) be satisfied.

(i) Suppose that system (2.1) is <u>stabilizable</u> in the sense that there exists a feedback operator $F \in \mathcal{L}(V,U)$ such that the closed loop semigroup $S_F(t) \in \mathcal{L}(V)$ defined by

$$S_F(t)x = S(t)x + \int_o^t S(t-s)BFS_F(s)xds$$

for $t \geq o$ and $x \in V$ is exponentially stable. Then hypothesis (H4) is satisfied.

In fact, there is an instant $T > o$ and a constant $c_T > o$ such that the inequalities

$$\| S_F(t) \|_{\mathcal{L}(V)} < 1, \quad \| CS_F(\cdot)x \|_{L^2(0,T;Y)} \leq c_T \| x \|_V$$

hold for every $x \in W$. This implies that

$$\| CS_F(\cdot)x \|_{L^2(o,\infty;Y)} \leq c_T \sum_{k=o}^{\infty} \| S_F(T) \|^k_{\mathcal{L}(V)} \; \| x \|_V$$

for $x \in W$ and hence (H3) is satisfied.

(ii) Suppose that system (2.1) is <u>detectable</u> in the sense that there exists an operator $K \in \mathcal{L}(Y,V)$ such that the output injection semigroup $S_K(t) \in \mathcal{L}(V)$ defined by

$$S_K(t)x = S(t)x + \int_o^t S_K(t-s)KCS(s)xds$$

for $t \geq o$ and $x \in W$ (see SALAMON [40 , Theorem I.3.9]) is exponentially stable. Then hypothesis (H5) is satisfied.

In fact, if $x(t) \in V$ is defined by

$$x(t) = S(t)x_o + \int_o^t S(t-s)Bu(s)ds, \quad t \geq o,$$

and $y(t) = Cx(t)$ for $x_o \in V$ and $u(\cdot) \in L^2_{loc}(o,\infty;U)$, then it is easy to see that

$$x(t) = S_K(t)x_o + \int_o^t S_K(t-s)[Bu(s)-Ky(s)]ds, \quad t \geq o$$

Hence $J(u) < \infty$ implies that $x(\cdot) \in L^2(0,\infty;V)$.

(iii) If (H4) and (H5) are satisfied, then system (2.1) is stabilizable in the sense of (i). (Theorem 3.3 and Theorem 3.4).

(iv) For finite dimensional systems (H5) is equivalent to detectability in the sense of (ii). It seems to be an open problem whether this equivalence extends to the infinite dimensional situation.

4. EXAMPLES

4.1 Neutral systems with output delays

We consider the linear neutral functional differential equation (NFDE)

$$(4.1) \qquad \frac{d}{dt}(x(t)-Mx_t) = Lx_t+B_o u(t)$$

$$y(t) = Cx_t$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ and x_t is defined by $x_t(\tau) = x(t+\tau)$, $-h \le \tau \le o$, $h > o$. B_o is an $n \times m$ matrix and L, M, C are bounded linear functionals from $\mathcal{C} = C[-h,o;\mathbb{R}^n]$ into $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively. These can be represented by matrix-functions $\eta(\tau)$, $\mu(\tau)$, $\gamma(\tau)$ of bounded variation in the following way

$$L\phi = \int_{-h}^{o} d\eta(\tau)\phi(\tau), \qquad M\phi = \int_{-h}^{o} d\mu(\tau)\phi(\tau),$$

$$C\phi = \int_{-h}^{o} d\gamma(\tau)\phi(\tau), \qquad \phi \in \mathcal{C}.$$

In order to guarantee the existence and uniqueness of solutions of (4.1) we will always assume

$$(4.2) \qquad \mu(o) = \lim_{\tau \uparrow o} \mu(\tau)$$

Moreover, we will assume at some places that $M:\mathcal{C} \to \mathbb{R}^n$ is of the special form

$$(4.3) \qquad M\phi = \sum_{j=1}^{\infty} A_{-j}\phi(-h_j) + \int_{-h}^{o} A_{-\infty}(\tau)\phi(\tau)d\tau, \qquad \phi \in \mathcal{C},$$

where $o < h_j \le h$, $A_{-j} \in \mathbb{R}^{n \times n}$ for $j \in \mathbb{N}$, $A_{-\infty}(\cdot) \in L^1[-h,0;\mathbb{R}^{n \times n}]$ and

$$\sum_{j=1}^{\infty} \| A_{-j} \| < \infty .$$

A function $x(\cdot) \in L^2_{loc}(-h,\infty; \mathbb{R}^n)$ is said to be a solution of (4.1) if the function $w(t) = x(t) - Mx_t$ is absolutely continuous with an L^2 - derivative on every compact interval $[0,T]$, $T > 0$, and if $\dot{w}(t) = Lx_t + B_o u(t)$ for almost every $t \geq o$. It is well known (BURNS-HERDMAN-STECH [7], SALAMON [40]) that equation (4.1) admits a unique solution $x(t)$, $t \geq -h$, for every input $u(\cdot) \in L^2_{loc}(o,\infty; \mathbb{R}^m)$ and every initial condition

$$(4.4) \quad \lim_{t \downarrow o} x(t) - Mx_t = \phi^o, \quad x(\tau) = \phi^1(\tau), \quad -h \leq \tau < o,$$

where $\phi = (\phi^o, \phi^1) \in M^2 = \mathbb{R}^n \times L^2(-h,o; \mathbb{R}^n)$. Moreover it has been shown in [7], [40] that the evolution of the state

$$(4.5) \qquad \hat{x}(t) = (x(t) - Mx_t, x_t) \in M^2$$

of system (4.1), (4.4) can be described by the formula

$$(4.6) \qquad \hat{x}(t) = S(t)\phi + \int_o^t S(t-s)Bu(s)ds$$

where $B \in \mathcal{L}(\mathbb{R}^m, M^2)$ maps $u \in \mathbb{R}^m$ into the pair $Bu = (B_o u, o)$ and $S(t) \in \mathcal{L}(M^2)$ in the strongly continuous semigroup generated by A, where

$$D(A) = \{\phi \in M^2: \ \phi^1 \in W^{1,2}, \ \phi^o = \phi^1(0) - M\phi^1\}$$

$$A\phi = (L\phi^1, \ \dot{\phi}^1)$$

Here $W^{1,2}$ denotes the Sobolev space $W^{1,2}(-h,o; \mathbb{R}^n)$.

Obviously, the dense subspace

$$W = \{(\phi(o) - M\phi, \phi): \phi \in W^{1,2}\} = \mathcal{D}(A)$$

of M^2 - endowed with the $W^{1,2}$ norm- is invariant under $S(t)$ and $S(t)$ can be restricted to a strongly continuous semigroup on W.

The output of the system (4.1) may be described through the operator

$$C : W \to \mathbb{R}^p, \quad C\phi = \int_{-h}^{o} d\gamma(\tau)\phi^1(\tau), \quad \phi \in W.$$

Remarks 4.1

(i) The infinitesimal generator A of $S(t)$ can be interpreted as a bounded operator from W into M^2. By duality, M^2 can be regarded as a dense subspace of W^* and A^* extends to a bounded operator from M^2 into W^*.

(ii) It has been proved in BURNS-HERDMAN-STECH [7] and SALAMON [40] that system (4.1) satisfies the hypotheses (H1) and (H2) with $H = V = M^2$ and the subspace $W \subset M^2$ as defined above. Hypothesis (H1) says that the state $\hat{x}(T) \in M^2$ of (4.1) defined by (4.5) is in W for every input $u(\cdot) \in L^2(o,T; \mathbb{R}^m)$ and zero initial condition and that $\hat{x}(T) \in W$ depends continuously on $u(\cdot) \in L^2[o,T; \mathbb{R}^m]$. Hypothesis (H2) says that the output $y(\cdot)$ of the free system (4.1) (i.e. $u(t) \equiv 0$) is in $L^2(o,T; \mathbb{R}^p)$ and depends in this space continuously on the initial state $\phi \in M^2$.

(iii) If $M : \mathscr{C} \to \mathbb{R}^n$ is given by (4.3), then it is known that the semigroup $S(t) \in \mathscr{L}(M^2)$ is exponentially stable if and only if

$$\omega_o = \sup \{\mathrm{Re}\lambda : \det \Delta(\lambda) = 0\} < o$$

where $\Delta(\lambda) = \lambda[I - M(e^{\lambda\cdot})] - L(e^{\lambda\cdot})$, $\lambda \in \mathbb{C}$, is the characteristic matrix of the NFDE (4.1). A necessary condition for the exponential stability of $S(t)$ is the stability of the difference operator which means that

$$(4.7) \qquad \sup \{\mathrm{Re}\lambda: \det[I - \sum_{j=1}^{\infty} A_{-j} e^{-\lambda h_j}] = 0\} < 0.$$

These facts have been established by HENRY [21] for $S(t) \in \mathcal{L}(W)$. They extend to $S(t) \in \mathcal{L}(M^2)$ because of the similarity of these two semigroups through the transformation $\mu I - A: W \to M^2$ with $\mu \notin \sigma(A)$.

(iv) If $M: \mathcal{C} \to \mathbb{R}^n$ is given by (4.3) and if (4.7) holds, then system (4.1) is <u>stabilizable</u> in the sense that there exists a feedback operator $F \in \mathcal{L}(M^2, \mathbb{R}^m)$ such that the closed loop semigroup $S_F(t) \in \mathcal{L}(M^2)$ generated by $A + BF$ is exponentially stable if and only if

$$(4.8) \qquad \mathrm{rank}\left[\Delta(\lambda), B_o\right] = n \qquad \forall \lambda \in \mathbb{C}, \qquad \mathrm{Re}\lambda \geq 0.$$

(PANDOLFI [38], SALAMON [40]).

(v) If $M: \mathcal{C} \to \mathbb{R}^n$ is given by (4.3) and if (4.7) holds, then system (4.1) is detectable in the sense that there exists an output injection operator $K \in \mathcal{L}(\mathbb{R}^p, M^2)$ such that the closed loop semigroup $S_K(t) \in \mathcal{L}(M^2)$ generated by $A + KC$ is exponentially stable if and only if

$$(4.9) \qquad \mathrm{rank}\begin{bmatrix} \Delta(\lambda) \\ C(e^{\lambda \cdot}) \end{bmatrix} = n \qquad \forall \lambda \in \mathbb{C}, \qquad \mathrm{Re}\lambda \geq 0.$$

(SALAMON [40]).

Associated with the system (4.1) we consider the performance index

$$(4.10) \qquad J(u) = \int_0^{\infty} [\| y(t) \|^2_{\mathbb{R}^p} + \| u(t) \|^2_{\mathbb{R}^m}] dt$$

Then we have the following theorem.

Theorem 4.2

Assume $M: \mathcal{C} \to \mathbb{R}^n$ is given by (4.3) and (4.7) is satisfied, then the following statements hold.

(i) If (4.8) is satisfied, there exists, for every initial state $\phi \in M^2$, a unique optimal control which minimizes the cost functional (4.10). This optimal control is given by the feedback law

$$(4.11) \qquad u_\pi(t) = -B^*\pi\hat{x}(t)$$

where $\pi \in \mathcal{L}(M^2)$ is the minimal selfadjoint, non-negative operator

which satisfies the algebraic Riccati equation

$$(4.12) \qquad A^*\pi + \pi A + C^*C - \pi BB^*\pi = 0$$

(this equation must be understood in the space $\mathcal{L}(W,W^*)$). Moreover the optimal cost is given by

$$(4.13) \qquad J(u_\pi) = \langle \phi , \pi\phi \rangle_{M^2}$$

(ii) If (4.9) is satisfied, then there exists at most one non-negative self adjoint solution $\pi \in \mathcal{L}(M^2)$ of (4.12). Moreover if π is such a solution, the closed loop semigroup $S_\pi(t) \in \mathcal{L}(M^2)$ generated by $A-BB^*\pi$ is exponentially stable.

Apparently the paper of DATKO [10] and the thesis of ITO [25] have the only available results in the literature on the linear quadratic control problem for neutral systems. In DATKO [10] the optimal control is not shown to be of feedback form. ITO [25] considers neutral systems in the product space framework, however his detectability concept is very strong and unnatural. Moreover, the proofs appearing in both papers are quite complicated.

4.2 Parabolic systems

Consider the system

$$(4.14)\qquad \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

where A is a self adjoint operator on a real Hilbert space H. Let λ_n, $n \in \mathbb{N}$, be the (simple) eigenvalues of A with corresponding eigenvectors $\phi_n \in H$, $\|\phi_n\|_H = 1$, and assume $|\lambda_n| \to \infty$ as $n \to \infty$ and $\ldots \lambda_n < \ldots < \lambda_1 \leq$ constant. Then

$$(4.15)\qquad \begin{aligned} Ax &= \sum_{n=1}^{\infty} \lambda_n < x, \phi_n >_H \phi_n \\ D(A) &= \{x \in H : \sum_{n=1}^{\infty} \lambda_n^2 <x,\phi_n>_H^2 < \infty\} \end{aligned}$$

and A generates a strongly continuous semigroup S(t) on H, where

$$(4.16)\qquad S(t)x = \sum_{n=1}^{\infty} e^{\lambda_n t} <x,\phi_n>\phi_n .$$

A precise definition of the operators B and C will be given below. For now, we only assume that the expressions

$$(4.17)\qquad c_n = C\phi_n \in Y, \quad b_n = B^* \phi_n \in U,$$

are well defined for every $n \in \mathbb{N}$. The operators B and C are completely determined by these sequences. The remaining problem is to introduce - if possible - suitable spaces W and V such that $B \in \mathcal{L}(U,V)$, $C \in \mathcal{L}(W,Y)$ and that the hypotheses (H1) and (H2) are satisfied. It will turn out that there is in general some freedom in the choice of these spaces if they exist.

More precisely, for any positive sequence

$$\alpha = \{\alpha_n\}_{n \in \mathbb{N}}, \quad \alpha_n > 0 \quad \forall n \in \mathbb{N}$$

we define the Hilbert space

$$H_\alpha := \{x = \{x_n\}_{n \in \mathbb{N}} : \sum_{n=1}^{\infty} \alpha_n x_n^2 < \infty\}$$

with inner produce

$$\langle x, \hat{x} \rangle_\alpha = \sum_{n=1}^{\infty} \alpha_n x_n \hat{x}_n, \quad x, \hat{x} \in H_\alpha$$

Remarks 4.3

(i) We may identify H with $H_1 = \ell^2$ via the isometric isomorphism

$$i : H \to H_1$$
$$x \to \{\langle x, \phi_n \rangle_H\}_{n \in \mathbb{N}}$$

$$i^{-1} : H_1 \to H,$$
$$\{x_n\} \to \sum_{n=1}^{\infty} x_n \phi_n$$

(ii) $H_\beta \subset H_\alpha$ if and only if the sequence α_n / β_n is bounded.

(iii) $H_\alpha^* = H_{\alpha^{-1}} = \{ x : \sum_{n=1}^{\infty} \alpha_n^{-1} x_n^2 < \infty \}$

with respect to the duality pairing

$$\langle x^*, x \rangle = \sum_{n=1}^{\infty} x_n^* x_n, \quad x^* \in H_{\alpha^{-1}}, \; x \in H_\alpha .$$

(iv) $\phi_n \in H$ can be identified with the sequence $e_n = (0, \ldots, \overset{n}{1}, 0, \ldots)$ via the mapping i. This sequence is contained in any H_α.

Now we may associate with the operator A a whole family of operators A_α on H_α defined by

$$\mathcal{D}(A_\alpha) = \{x \in H_\alpha : \sum_{n=1}^{\infty} \alpha_n \lambda_n^2 x_n^2 < \infty\}$$

$$A_\alpha x = \{\lambda_n x_n\}_{n \in \mathbb{N}} .$$

Each of these operators generates a strongly continuous semigroup $S_\alpha(t) \in \mathcal{L}(H_\alpha)$ which is given by

$$S_\alpha(t)x = \{e^{\lambda_n t} x_n\}_{n \in \mathbb{N}} , \ t \geq o, \ x \in H_\alpha .$$

Associated with the sequences (4.17) we introduce the maps

$$\text{(4.18)} \qquad B_\beta : U \to H_\beta , \quad u \to \{\langle b_n, u\rangle_U\}_{n \in \mathbb{N}}$$

$$\text{(4.19)} \qquad C_\gamma : H_\gamma \to Y , \quad x \to \sum_{n=1}^{\infty} x_n c_n .$$

However, these operators are not well defined for every choice of β and γ respectively. Sufficient conditions of well posedness of (4.18) and (4.19) as well as for the Hypotheses (H1) and (H2) are given in the next Lemma.

<u>Lemma 4.4</u>

(i) If

$$\text{(4.20)} \qquad \sum_{n=1}^{\infty} \beta_n \|b_n\|_U^2 < \infty$$

then (4.18) defines a bounded operator $B_\beta \in \mathcal{L}(U,H_\beta)$.
In the case $U = \mathbb{R}$ condition (4.20) is also necessary.

(ii) If

$$(4.21) \qquad \sum_{n=1}^{\infty} \frac{1}{\gamma_n} \|c_n\|_Y^2 < \infty,$$

then (4.19) defines a bounded operator $C_\gamma \in \mathcal{L}(H_\gamma,Y)$. In the case $U = \mathbb{R}$ condition (4.21) is also necessary.

(iii) Let $n_o = \max \{n \in \mathbb{N} : \lambda_n \geq o\}$, let (4.20) be satisfied and suppose that

$$(4.22) \qquad \sum_{n=n_o+1}^{\infty} \frac{\gamma_n}{|\lambda_n|} \|b_n\|_U^2 < \infty.$$

Then, for every $T > o$, there exists a constant $b > o$ such that

$$\left\| \int_o^T S_\beta(T-s)B_\beta u(s)ds \right\|_Y \leq b \, \|u(\cdot)\|_{L^2[o,T;U]}$$

for every $u(\cdot) \in L^2[o,T;U]$.

(iv) If (4.21) holds and

$$(4.23) \qquad \sum_{n=n_o+1}^{\infty} \frac{1}{\beta_n|\lambda_n|} \|c_n\|_Y^2 < \infty,$$

then, for every $T > o$, there exists a constant $c > o$ such that

$$\|C_\gamma \; S_\gamma(\cdot)x\|_{L^2[o,T;Y]} \leq c \, \|x\|_\beta$$

for every $x \in H_\gamma$.

<u>Proof</u> First note that (ii) and (iv) are the dual statements of (i) and (iii) respectively. Statement (i) is trivial and (iii) follows from the inequality.

$$\left\| \int_o^T S_\beta(T-s)B_\beta u(s)ds \right\|_Y$$

$$= \sum_{n=1}^{\infty} \gamma_n \left(\int_o^T e^{\lambda_n(T-s)} b_n u(s)ds \right)^2$$

$$\leq \sum_{n=1}^{\infty} \gamma_n \int_o^T e^{2\lambda_n s} ds \cdot \|b_n\|^2 \cdot \|u(\cdot)\|^2_{L^2[o,T;\mathbb{R}^m]}$$

$$\leq \left[\sum_{n=1}^{n_o} \gamma_n \int_o^T e^{2\lambda_n s} ds \|b_n\|^2 + \sum_{n=n_o+1}^{\infty} \frac{\gamma_n}{2|\lambda_n|} \|b_n\|^2 \right] \|u(\cdot)\|^2_{L^2(o,T;\mathbb{R}^m)}$$

$\blacksquare$

So we have to consider the problem whether there exists sequences β_n, γ_n such that the inequalities (4.20-23) are satisfied. This problem has a simple solution.

<u>Lemma 4.5</u>

Given the sequences $b_n \in U$, $c_n \in Y$, $\lambda_n \in \mathbb{R}$ such that λ_n is strictly decreasing and tends to $-\infty$, there exist sequences $\beta_n > o$, $\gamma_n > o$ such that the inequalities (4.20-23) are satisfied if and only if

$$(4.24) \qquad \sum_{n=n_o+1}^{\infty} \frac{\|c_n\|_Y \cdot \|b_n\|_U}{|\lambda_n|^{\frac{1}{2}}} < \infty$$

Proof

Necessity It follows from (4.20) and (4.23) that

$$\sum_{n=n_o+1}^{\infty} \frac{\|c_n\|_Y \cdot \|b_n\|_U}{|\lambda_n|^{\frac{1}{2}}} \le \left[\sum_{n=n_o+1}^{\infty} \frac{\|c_n\|_Y^2}{\beta_n|\lambda_n|} \right]^{\frac{1}{2}} \left[\sum_{n=n_o+1}^{\infty} \beta_n \|b_n\|_U^2 \right]^{\frac{1}{2}} < \infty$$

<u>Sufficiency</u> Suppose that (4.24) is satisfied and define

$$(4.25) \qquad \beta_n = \begin{cases} \|c_n\|_Y / \|b_n\|_U |\lambda_n|^{\frac{1}{2}}, & b_n \neq o, \quad c_n \neq o, \quad \lambda_n \neq o, \\ n^2 \|c_n\|_Y^2 / |\lambda_n| , & b_n = o, \quad c_n \neq o, \quad \lambda_n \neq o, \\ 1/n^2 \|b_n\|_U^2 , & b_n \neq o, \quad c_n = o, \quad \lambda_n \neq o \\ 1 & \text{otherwise} \end{cases}$$

$$(4.26) \qquad \gamma_n = \begin{cases} |\lambda_n|^{\frac{1}{2}} \|c_n\|_Y / \|b_n\|_U , & b_n \neq o, \quad c_n \neq o, \quad \lambda_n \neq o, \\ n^2 \|c_n\|_Y^2 , & b_n = o, \quad c_n \neq o, \quad \lambda_n \neq o, \\ |\lambda_n| / n^2 \|b_n\|_U^2 , & b_n \neq o, \quad c_n = o, \quad \lambda_n \neq o, \\ \max \{1, |\lambda_n|\} , & \text{otherwise} \end{cases}$$

Then it is easy to see that (4.20-23) are satisfied. ▨

<u>Remarks 4.6</u>

(i) Let (4.24) be satisfied and let β_n and γ_n be given by (4.25) and (4.26) respectively. Then $\gamma_n = \beta_n|\lambda_n|$ for almost every $n \in \mathbb{N}$ (with at most a finite number of exceptions) and hence

$$\mathcal{D}(A_\beta) \subset \mathcal{D}(A_\beta^{\frac{1}{2}}) = \{x: \sum_{n=1}^{\infty} \beta_n|\lambda_n| \, x_n^2 < \infty \} = H_\gamma \subset H_\beta .$$

In particular, hypothesis (H3) is satisfied.

(ii) If the sequence β_n/γ_n is bounded, then we may assume without loss of generality that $H_\gamma \subset H_1 \subset H_\beta$ or equivalently that the sequences β_n and γ_n^{-1} are bounded. This can always be achieved by redefining b_n, c_n, β_n and γ_n.

(iii) The system (4.14) is <u>stabilizable</u> in the sense that there exists an $F_\beta \in \mathcal{L}(H_\beta,U)$ such that $A_\beta + B_\beta F_\beta$ generates an exponentially stable semigroup if and only if $b_n \neq o$ for $n = 1, \ldots, n_o$ where n_o is defined as in lemma 4.4(c) (see CURTAIN-PRITCHARD [9]).

(iv) Let (4.23) and (4.21) be satisfied. Then system (4.14) is <u>detectable</u> in the sense that there exists a $K_\beta \in \mathcal{L}(Y,H_\beta)$ such that the operator $A_\beta + K_\beta C_\gamma : \mathcal{D}(A_\beta) \to H_\beta$ generates an exponentially stable semigroup on H_β if and only if $c_n \neq o$ for $n = 1, \ldots, n_o$. This is not the dual problem of (iii) since $K_\beta C_\gamma$ is an unbounded perturbation of A_β. However, if $c_n \neq o$ for $n = 1, \ldots, n_o$, then we may choose $f_1, \ldots, f_{n_o} \in Y$ such that the matrix

$$\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_{n_o} \end{bmatrix} + \begin{bmatrix} \langle f_1,c_1\rangle & \cdots & \langle f_1,c_{n_o}\rangle \\ & & \\ \langle f_{n_o},c_1\rangle & \cdots & \langle f_{n_o},c_{n_o}\rangle \end{bmatrix}$$

is stable and define $K_\beta : Y \to H_\beta$

$$K_\beta y = \{\langle f_n,y\rangle\}_{n \in \mathbb{N}}$$

where $f_n = o$ for $n > n_o$. Then it follows from the finite-dimensionality of K_β that $A_\beta + K_\beta C_\gamma$ generates a semigroup (SALAMON [40]) and it is easy to see that this semigroup is exponentially stable.

Now we are in the position to apply the Theorems 3.3 and 3.4 to the Cauchy problem (4.14) with the performance index (4.10).

<u>Theorem 4.7</u>

Let the operators A, B, C be given as above. Suppose that (4.24) is satisfied, let β_n and γ_n be given by (4.25) and (4.26) respectively and assume that $H_\gamma \subset H_1 \subset H_\beta$. Finally, suppose that $b_n \neq o$ and $c_n \neq o$ for $n = 1, \ldots, n_o$.

Then the following statements hold.

(i) There exists a unique family $\{p_{n,m} \in \mathbb{R} : n,m \in \mathbb{N}\}$ defining a self-adjoint, non-negative operator $P \in \mathcal{L}(H_\beta, H_{\beta-1})$ via

$$Px = \{ \sum_{m=1}^{\infty} p_{n,m} x_m \}_{n \in \mathbb{N}} \quad , \quad x \in H_\beta,$$

and satisfying the following equation for all n,m

$$(4.27) \qquad \begin{aligned} &(\lambda_n + \lambda_m)p_{n,m} + \langle c_n, c_m \rangle_Y \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} p_{n,j} \langle b_j, b_k \rangle_U \; p_{k,m}. \end{aligned}$$

(ii) For every initial state $x_o \in H_\beta$ there exists a unique optimal control which minimizes the cost functional (4.10) subject to (4.14). This optimal control is given by the feedback law.

$$(4.28) \qquad u(t) = - B^* Px(t) = - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} b_n \, p_{n,m} \langle x(t), \phi_m \rangle_H$$

(note that $x(t) \in H$ for $t > o$) where $p_{n,m}$, $n,m \in \mathbb{N}$, are defined as above. The optimal cost is

$$(4.29) \qquad J(u) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} x_{on} \, p_{n,m} \, x_{om}.$$

Moreover, the closed loop semigroup on H_β, generated by $A_\beta - B_\beta B_\beta^* P$ is exponentially stable. ▨

As a specific example consider

$$(4.30;1) \qquad z_t = z_{\xi\xi}, \quad t > o, \quad o < \xi < 1,$$

(4.30;2) $\quad z_\xi(o,t) = u(t), \quad z_\xi(1,t) = o, \quad t > o,$

(4.30;3) $\quad z(\xi,o) = z_o(\xi), \qquad o < \xi < 1,$

(4.30;4) $\quad y(t) = \int_o^1 c(\xi)z(\xi,t)d\xi, \quad t > o.$

It can be shown (see CURTAIN-PRITCHARD [9]) that this system is equivalent to a Cauchy problem of the form (4.14) with $(Bu)(\xi) = -\delta(\xi)u$ (δ is the Dirac delta function), $\lambda_o = o$, $\phi_o(\xi) \equiv 1$, $\lambda_n = -n^2\pi^2$, $\phi_n(\xi) = \sqrt{2}\cos n\pi\xi$ for $n \in \mathbb{N}$ and $o \leq \xi \leq 1$. Hence

$$b_o = -1, \quad b_n = -\sqrt{2}, \qquad n \in \mathbb{N}$$

$$c_n = \int_o^1 c(\xi)\phi_n(\xi)d\xi, \qquad n = o, 1, 2, \ldots .$$

So condition (4.24) is satisfied if

$$\sum_{n=1}^{\infty} \frac{c_n}{n} < \infty$$

This allows for arbitrary bounded, linear output operators from $L^2[o,1]$ into $\mathbb{R}$ which means that $c(\cdot) \in L^2[o,1]$ or equivalently the sequence c_n is square summable. However, the output operator can also be unbounded. For example, if $c_n = n^{-2\varepsilon}$ then we may choose $\gamma_n = n^{1-2\varepsilon}$ and $\beta_n = n^{-1-2\varepsilon}$ so that W, V are the intermediate spaces

$$W = \left[H^1[o,1], L^2[o,1]\right]_{\frac{1}{2}+\varepsilon} = H^{\frac{1}{2}-\varepsilon}[0,1],$$

$$V = \left[H^1[o,1], L^2[o,1]\right]^*_{\frac{1}{2}-\varepsilon} = \left(H^{\frac{1}{2}+\varepsilon}[0,1]\right)^*.$$

In particular the solution operator P of the algebraic Riccati equation maps $\left(H^{\frac{1}{2}+\varepsilon}[o,1]\right)^*$ into $H^{\frac{1}{2}+\varepsilon}[o,1]$.

4.3 Hyperbolic systems

Consider the system

(4.31) $\ddot{z} = Az + Bu, \quad y = Cz,$

Where A is a selfadjoint operator on a Hilbert space H whose (simple) eigenvalues $\lambda_n = -\omega_n^2$ satisfy

(4.32) $\omega_1 \geq \delta, \quad \omega_{n+1} - \omega_n \geq \delta, \quad n \in \mathbb{N},$

for some $\delta > 0$. As before, let $\phi_n \in H$ be the corresponding eigenvectors of A with $||\phi_n||_H = 1$ so that A is given by (4.15). For the operators B and C we only assume that the expressions (4.17) are well defined.

Identifying H with its dual, we obtain

$$V \subset H \subset V^*,$$

$$V = \mathcal{D}((-A)^{\frac{1}{2}}) \quad \{x \in H : \sum_{n=1}^{\infty} |\lambda_n| < x, \phi_n >^2 < \infty\},$$

and A extends to a bounded operator from V into V*. In order to transform (4.31) into a first order system, we introduce the Hilbert space

$$\mathcal{H} = V \times H$$

$$<x,\hat{x}>_{\mathcal{H}} = -<x_0, A\hat{x}_0>_{V,V^*} + <x_1, \hat{x}_1>_H, \quad x, \hat{x} \in \mathcal{H}.$$

Then the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{H}$ defined by

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}, \quad \mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \times V,$$

is the infinitesimal generator of a strongly continuous group $\mathcal{S}(t) \in \mathcal{L}(\mathcal{H})$ which is given by

$$(4.33)\quad \mathcal{S}(t)\,x = \begin{bmatrix} \sum_{n=1}^{\infty} \left[(\cos\omega_n t)\langle x_o, \phi_n\rangle + \omega_n^{-1}(\sin\omega_n t)\langle x_1, \phi_n\rangle\right]\phi_n \\ \sum_{n=1}^{\infty} \left[-\omega_n(\sin\omega_n t)\langle x_o, \phi_n\rangle + (\cos\omega_n t)\langle x_1, \phi_n\rangle\right]\phi_n \end{bmatrix}$$

for $x = (x_o, x_1) \in \mathcal{H}$ and $t \geq 0$. Moreover, we introduce the operators

$$\mathcal{B} = \begin{bmatrix} 0 \\ B \end{bmatrix}, \qquad \mathcal{C} = \begin{bmatrix} C & 0 \end{bmatrix}$$

Then the second order system (4.31) can be formally associated with the first order Cauchy problem

$$(4.34)\qquad \begin{aligned} \dot{x} &= \mathcal{A}x + \mathcal{B}u \\ y &= \mathcal{C}x \end{aligned}$$

on the Hilbert space $\mathcal{H}$ by means of the identification $x = (z, \dot{z})$. In order to give a precise definition of the operators $\mathcal{B}$ and $\mathcal{C}$ we introduce, for any positive sequence $\alpha = \{\alpha_n\}, \alpha_n > 0$, the Hilbert space

$$\begin{aligned} \mathcal{H}_\alpha &= \{x = \{(x_{on}, x_{1n})\}_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} \alpha_n\left[|\lambda_n| x_{on}^2 + x_{1n}^2\right] < \infty\} \\ &= H_{\alpha|\lambda|} \times H_\alpha \end{aligned}$$

endowed with the inner product

$$\langle x, \hat{x}\rangle_\alpha = \sum_{n=1}^{\infty} \alpha_n\left[|\lambda_n|\, x_{on}\hat{x}_{on} + x_{1n}\hat{x}_{1n}\right], \quad x, \hat{x} \in \mathcal{H}_\alpha,$$

and we identify $\mathcal{H}$ with the Hilbert space $\mathcal{H}_1$ by means of the isometric isomorphism

$$\begin{aligned} i : \mathcal{H} &\longrightarrow \mathcal{H}_1, \\ x &\longrightarrow \{(\langle x_o, \phi_n\rangle, \langle x_1, \phi_n\rangle)\}_{n\in\mathbb{N}}. \end{aligned}$$

Then we may associate with the operator $\mathcal{A}$ the family of operators $\mathcal{A}_\alpha$ on $\mathcal{H}_\alpha$ defined by

$$\mathcal{A}_\alpha x = \{(x_{1n}, \lambda_n x_{on})\}_{n\in\mathbb{N}},$$

$$\mathcal{D}(\mathcal{A}_\alpha) = \{x \in \mathcal{H}_\alpha : \sum_{n=1}^{\infty} \alpha_n |\lambda_n| \left[|\lambda_n| x_{on}^2 + x_{1n}^2 \right] < \infty \}$$

Each of these operators generates a strongly continuous group $\mathcal{S}_\alpha(t) \in \mathcal{L}(\mathcal{H}_\alpha)$ which is given by

$$\mathcal{S}_\alpha(t)x = \begin{bmatrix} (\cos \omega_n t)\, x_{on} + \omega_n^{-1} (\sin \omega_n t)\, x_{1n} \\ - \omega_n (\sin \omega_n t)\, x_{on} + (\cos \omega_n t)\, x_{1n} \end{bmatrix}_{n \in \mathbb{N}}$$

for $x \in \mathcal{H}_\alpha$ and $t \geq 0$. Associated with the sequences (4.17) we introduce the maps

(4.35) $$\mathcal{B}_\beta : U \longrightarrow \mathcal{H}_\beta$$
$$u \longrightarrow \{(0, \langle b_n, u \rangle_U)\}_{n \in \mathbb{N}},$$

(4.36) $$\mathcal{C}_\gamma : \mathcal{H}_\gamma \longrightarrow Y,$$
$$x \longrightarrow \sum_{n=1}^{\infty} c_n x_{on} .$$

<u>Lemma 4.8</u>

(i) If

(4.37) $$\sum_{n=1}^{\infty} \beta_n ||b_n||_U^2 < \infty ,$$

then (4.35) defines a bounded operator $\mathcal{B}_\beta \in \mathcal{L}(U, \mathcal{H}_\beta)$. In the case $U = \mathbb{R}$ this condition is also sufficient.

(ii) If

(4.38) $$\sum_{n=1}^{\infty} \frac{||c_n||_Y^2}{\gamma_n |\lambda_n|} < \infty$$

then (4.36) defines a bounded operator $\mathcal{C}_\gamma \in \mathcal{L}(\mathcal{H}_\gamma, Y)$. In the case $Y = \mathbb{R}$ this condition is also sufficient.

(iii) If (4.37) holds and

(4.39) $$\sup_{n \in \mathbb{N}} \gamma_n ||b_n||_U^2 < \infty ,$$

then, for every $T > 0$, there exists a constant $b > 0$ such that

$$\left\| \int_0^T \mathcal{S}_\beta\,(T-s)\mathcal{B}_\beta\, u(s)ds \right\|_\gamma \leq b\, \|u(\cdot)\|_{L^2(0,T;U)}$$

for every $u(\cdot) \in L^2(0,T;U)$.

(iv) If (4.38) holds and

$$(4.40) \quad \sup_{n \in \mathbb{N}} \frac{\|c_n\|_Y^2}{\beta_n|\gamma_n|} < \infty ,$$

then, for every $T > 0$, there exists a constant $c > 0$ such that

$$\| \mathcal{C}_\gamma\, \mathcal{J}_\gamma(\cdot)x \|_{L^2(0,T;Y)} \leq c\, \|x\|_\beta$$

for every $x \in \mathcal{H}_\gamma$.

<u>Proof</u> The statements (i) and (ii) are trivial. In order to prove statement (iii) note that

$$\int_0^T \mathcal{S}_\beta(T-s)\mathcal{B}_\beta\, u(s)ds = \begin{bmatrix} \omega^{-1} \int_0^T (\sin\omega_n(T-s)) < b_n, u(s)>_U ds \\ \int_0^T (\cos\omega_n(T-s)) < b_n, u(s)>_U ds \end{bmatrix}_{n \in \mathbb{N}}$$

for every $u(\cdot) \in L^2(0,T;U)$ and hence

$$\left\| \int_0^T \mathcal{S}_\beta(T-s)\mathcal{B}_\beta\, u(s)\, ds \right\|_\gamma^2$$

$$= \sum_{n=1}^\infty \gamma_n \left\{ \left[\int_0^T (\sin\omega_n(T-s)) < b_n,\, u(s) >_U ds\right]^2 + \left[\int_0^T (\cos\omega_n(T-s)) < b_n, u(s) >_U ds\right]^2 \right\}$$

$$\leq (\sup \gamma_n \|b_n\|^2) \sum_{n=1}^\infty \left\{ \left\| \int_0^T (\sin\omega_n(T-s))u(s)\, ds \right\|_U^2 + \left\| \int_0^T (\cos\omega_n(T-s))u(s)\, ds \right\|_U^2 \right\}$$

$$\leq \text{const.} \ (\sup_{n\in\mathbb{N}} \gamma_n \|b_n\|^2) \|u(\cdot)\|^2_{L^2(0,T;U)}$$

For the final inequality we have to make use of condition (4.32) together with some properties of Fourier series (see INGHAM [24] and also RUSSELL [39, p.12-14]).

Finally, let us assume that (4.38) and (4.40) are satisfied. Then the following inequality holds for every $x \in \mathcal{H}_Y$ and every $y(\cdot) \in L^2(0,T;Y)$

$$\langle y(\cdot), \mathcal{C}_Y \mathcal{S}_Y(\cdot)x \rangle_{L^2(0,T;Y)}$$

$$= \sum_{n=1}^{\infty} \left\langle \int_0^T y(t)\cos\omega_n t\,dt\, x_{on} + \omega_n^{-1} \int_0^T y(t)\sin\omega_n t\,dt\, x_{1n}, c_n \right\rangle_Y$$

$$\leq \left[\sum_{n=1}^{\infty} \|c_n\|_Y^2 x_{on}^2 \right]^{\frac{1}{2}} \left[\sum_{n=1}^{\infty} \left\| \int_0^T y(t)\cos\omega_n t\,dt \right\|_Y^2 \right]^{\frac{1}{2}}$$

$$+ \left[\sum_{n=1}^{\infty} \omega_n^{-2} \|c_n\|_Y^2 x_{1n}^2 \right]^{\frac{1}{2}} \left[\sum_{n=1}^{\infty} \left\| \int_0^T y(t)\sin\omega_n t\,dt \right\|_Y^2 \right]^{\frac{1}{2}}$$

$$\leq \text{const.} \left[\sup_{n\in\mathbb{N}} \|c_n\|_Y^2 / \beta_n |\lambda_n| \right]^{\frac{1}{2}} \|y(\cdot)\|_{L^2(0,T;Y)}$$

$$\left\{ \left[\sum_{n=1}^{\infty} \beta_n |\lambda_n| x_{on}^2 \right]^{\frac{1}{2}} + \left[\sum_{n=1}^{\infty} \beta_n x_{1n}^2 \right]^{\frac{1}{2}} \right\}$$

$$\leq \text{const.} \|y(\cdot)\|_{L^2(0,T;Y)} \|x\|_\beta \quad .$$

This proves statement (iv). ▨

Lemma 4.9

Let the sequences $b_n \in U$, $c_n \in Y$, $\lambda_n = -\omega_n^2$ be given and suppose that (4.32) holds. Then there exist sequences $\beta_n > o$, $\gamma_n > o$ such that the inequalities (4.37-40) are satisfied and $\mathcal{D}(\mathcal{A}_\beta) \subset \mathcal{H}_\gamma \subset \mathcal{H}_\beta$ if and only if

$$(4.41) \quad \sum_{n=1}^{\infty} \frac{\|b_n\|_U^2 \, \|c_n\|_Y^2}{|\lambda_n|} < \infty$$

Proof The necessity of condition (4.41) is trivial. Conversely, if (4.41) is satisfied, then it is easy to see that the sequences

$$(4.42) \quad \beta_n = \begin{cases} \|c_n\|_Y^2/|\lambda_n|, & \|b_n\|_U \|c_n\|_Y \geq 1 \text{ or } b_n = o,\ c_n \neq o, \\ 1/\|b_n\|_U^2 |\lambda_n|, & \|b_n\|_U \|c_n\|_Y \leq 1 \text{ and } b_n \neq o, \\ 1/|\lambda_n|, & b_n = o,\ c_n = o, \end{cases}$$

$$(4.43) \quad \gamma_n = \begin{cases} 1/\|b_n\|_U^2, & b_n \neq o, \\ \|c_n\|_Y^2, & b_n = o,\ c_n \neq o, \\ 1, & b_n = o,\ c_n = o, \end{cases}$$

satisfy (4.37 - 4.40). Moreover, the sequence

$$\beta_n/\gamma_n = \begin{cases} \|b_n\|_U^2 \|c_n\|_Y^2/|\lambda_n|, & \|b_n\|_U \|c_n\|_Y \geq 1, \\ 1/|\lambda_n|, & \|b_n\|_U \|c_n\|_Y \leq 1 \end{cases}$$

is bounded and thus $\mathcal{H}_\gamma \subset \mathcal{H}_\beta$. Finally, $|\lambda_n|\beta_n \geq \gamma_n$ for every $n \in \mathbb{N}$ and hence $\mathcal{D}(\mathcal{A}_\beta) \subset \mathcal{H}_\gamma$. ▨

If (4.41) is satisfied and β_n and γ_n are given by (4.42) and (4.43), respectively, then it follows from Lemma 4.9 and Lemma 4.8 that system (4.34) satisfies the hypotheses (H1), (H2) and (H3) with

$$\mathcal{V} = \mathcal{H}_\beta, \qquad \mathcal{W} = \mathcal{H}_\gamma.$$

This time we will <u>not</u> assume that $\mathcal{H}_\gamma \subset \mathcal{H}_1 \subset \mathcal{H}_\beta$ so that we are not tied to an artificial identification of a certain Hilbert space with itself which sometimes leads to unnecessary complications. Nevertheless, we can apply Theorem 2.6 to our situation since the intermediate Hilbert space H does not play any role in the proof of that result.

By Theorem 2.6, there exists a unique positive semidefinite strongly continuous operator

$$\mathcal{P}(t) \in \mathcal{L}(\mathcal{H}_\beta, \mathcal{H}_{\beta-1}), \quad 0 \le t \le T,$$

which satisfies the differential Riccati equation associated with system (4.34) and the performance index

$$\mathcal{J}(u) = \int_0^T [\|y(t)\|_Y^2 + \|u(t)\|_U^2]\,dt. \tag{4.44}$$

This operator can be written in the form

$$\mathcal{P}(t)x = \begin{bmatrix} \sum_m (p_{nm}^{00}(t)x_{om} + p_{nm}^{01}(t)x_{1m}) \\ \sum_m (p_{nm}^{10}(t)x_{om} + p_{nm}^{11}(t)x_{1m}) \end{bmatrix}_{n \in \mathbb{N}} \tag{4.45}$$

for $x \in \mathcal{H}_\beta$ and $0 \le t \le T$. The fact that $\mathcal{P}(t)$ is self-adjoint results in the condition

$$\lambda_m p_{mm}^{00}(t) = \lambda_n p_{nm}^{00}(t), \quad p_{mn}^{10}(t) = -\lambda_n p_{nm}^{01}(t)$$

(4.46)

$$p_{mn}^{11}(t) = p_{nm}^{11}(t),$$

for $n, m \in \mathbb{N}$ and $0 \leq t \leq T$. In the next theorem we summarize our main conclusions for the Cauchy problem (4.34) with the cost functional (4.44).

Theorem 4.10

Let the operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be defined as above. Suppose that (4.41) is satisfied and let β_n and γ_n be defined by (4.42) and (4.43), respectively. Then the following statements hold.

(i) There exists a unique positive semidefinite operator $\mathcal{P}(t): \mathcal{H}_\beta \to \mathcal{H}_{\beta^{-1}}$ of the form (4.45), (4.46) whose coefficients $p_{nm}^{00}(t)$, $p_{nm}^{10}(t)$, $p_{nm}^{11}(t)$ are continuously differentiable on the interval $[0,T]$ and satisfy the differential Riccati equation

(4.47;1)

$$-\lambda_n \dot{p}_{nm}^{00}(t) + \lambda_n p_{nm}^{10}(t) + \lambda_m p_{mn}^{10}(t) + \langle c_n, c_m \rangle = \sum_j \sum_k \langle b_j, b_k \rangle_U \, p_{jm}^{10}(t) p_{kn}^{10}(t),$$

(4.47;2)

$$\dot{p}_{nm}^{10}(t) - \lambda_n p_{nm}^{00}(t) + \lambda_m p_{mm}^{11}(t) = \sum_j \sum_k \langle b_j, b_k \rangle_U \, p_{jm}^{10}(t) \, p_{kn}^{11}(t),$$

(4,47;3)

$$\dot{p}_{nm}^{11}(t) + p_{nm}^{10}(t) + p_{mn}^{10}(t) = \sum_j \sum_k \langle b_j, b_k \rangle_U \, p_{jm}^{11}(t) \, p_{kn}^{11}(t),$$

(4.47;4) $$p_{nm}^{00}(T) = p_{nm}^{10}(T) = p_{nm}^{11}(T) = 0,$$

for $n, m \in \mathbb{N}$ and $0 \leq t \leq T$.

(ii) For every initial state $(x_o, x_1) \in \mathcal{H}$ there exists a unique optimal control $u(t)$, $0 \leq t \leq T$, which minimizes the cost functional (4.44) subject to (4.34). This optimal control is given by the feedback law

$$
\begin{aligned}
u(t) &= -\mathcal{B}^* \mathcal{P}(t)x(t) \\
&= -\sum_n \sum_m b_n \, [p_{nm}^{10}(t)<x_o(t), \phi_m>_H + p_{nm}^{11}(t)<x_1(t), \phi_m>_H].
\end{aligned} \tag{4.48}
$$

The optimal cost is

$$
\begin{aligned}
J(u) &= <(x_o, x_1), \mathcal{P}(o)(x_o, x_1)> \\
&= \sum_n \sum_m \Big[-<x_o, \phi_n>\lambda_n \, p_{nm}^{00}(0)<x_o, \phi_m> \\
&\quad + 2 <x_1, \phi_n> p_{nm}^{10}(0)<x_o, \phi_m> + <x_1, \phi_n> p_{nm}^{11}(0)<x_1, \phi_m> \Big].
\end{aligned}
$$

As a specific example, consider the system

(4.50;1) $z_{tt} = z_{\xi\xi}$, $t > o$, $0 < \xi < 1$,

(4.50;2) $z(o,t) = u(t)$, $z(1,t) = 0$, $t > o$,

(4.50;3) $z(\xi,o) = x_o(\xi)$, $z_t(\xi,0) = x_1(\xi)$, $0 < \xi < 1$,

(4.50;4) $y(t) = \int_o^1 c(\xi)z(\xi,t)d\xi$, $t > o$,

in the Hilbert space

$$\mathcal{H} = H_0^1[0,1] \times L^2[0,1]$$

which we identify with its dual. Then the operator

$$A = \Delta, \quad \mathcal{D}(A) = H^2[0,1] \cap H_0^1[0,1],$$

has the eigenvalues $\lambda_n = -n^2\pi^2$ with corresponding eigenfunctions $\phi_n(\xi) = \sqrt{2}\sin n\pi\xi$, $0 \le \xi \le 1$, $n \in \mathbb{N}$, and $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{H}$ is given by

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix}, \quad \mathcal{D}(\mathcal{A}) = \mathcal{D}(A) \times H_0^1\,[0,1]$$

Moreover, the input operator for system (4.50) takes the form $\mathcal{B}u = (o, -\delta' u)$, $u \in \mathbb{R}$, where δ' is the distribitional derivative of the Dirac delta impulse at $\xi = 0$ (see CURTAIN-PRITCHARD [9]). Hence the following equations hold for $n \in \mathbb{N}$

$$b_n = \mathcal{B}^* \phi_n = \phi_n'(o) = \sqrt{2}\, n\pi,$$

$$c_n = \mathcal{C}\phi_n = \sqrt{2}\int_o^1 c(\xi)\sin n\pi\xi\, d\xi$$

So condition (4.41) is satisfied if and only if

$$(4.51) \qquad \sum_{n=1}^{\infty} |c_n|^2 < \infty$$

and we may choose

$$\beta_n = \max\left\{\frac{|c_n|^2}{|\lambda_n|}, \quad \frac{1}{|\lambda_n|^2}\right\}, \quad \gamma_n = \frac{1}{|\lambda_n|}, \quad n \in \mathbb{N},$$

(compare the formulae (4.42) and (4.43).

In particular, this means that the boundary control system (4.50) has continuous solutions in the space

$$\mathcal{W} = \mathcal{H}_\gamma = L^2\,[o,1] \times H^{-1}\,[o,1]$$

for every input $u(\cdot) \in L^2_{loc}\,(o,\infty)$. This result has also been established by

LASIECKA-TRIGGIANI [32]. Moreover, condition (4.51) shows that C can be chosen to be arbitrary bounded linear functional on $L^2[o,1]$. The space $\mathcal{V} = \mathcal{H}_\beta$ depends on this functional. For example if $c_n = n^{-1}$, then we can choose $\beta_n = n^{-4}$ and get

$$\mathcal{V} = \mathcal{H}_\beta = H^{-1}[o,1] \times \mathcal{D}(A)^*$$

In this situation it is usual to take $\mathcal{W}$ as the state space instead of $\mathcal{H}$ and identifying $\mathcal{W}$ with its dual we have

$$\mathcal{V}^* = H_o^1[o,1] \times L^2[o,1]$$

and (t) maps $\mathcal{V}$ into $\mathcal{V}^*$. Hence the Riccati operator has a smoothing effect <u>with respect to</u> $\mathcal{W} = L^2[o,1] \times H^{-1}[o,1]$.

<u>Remarks</u>

Boundary value control problems for parabolic and hyperbolic systems have been treated by a variety of authors, for example BALAKRISHNAN [2], WASHBURN [42], LIONS [34], CURTAIN-PRITCHARD [9], LASIECKA-TRIGGIANI [31], [32]. The weakest conditions imposed to generate a solution of the Riccati equation for hyperbolic systems are those of LASIECKA-TRIGGIANI [32]. They study precisely the boundary control problem (4.50) where the output operator is the identity on $L^2[o,1]$. This case cannot be treated within our framework, however, the Riccati operator in [32] is in $\mathcal{L}(\mathcal{W})$ and does not have a smoothing effect relative to $\mathcal{W} = L^2[o,1] \times H^{-1}[o,1]$.

<u>PART II</u>

In this part we develop a state space approach for linear retarded functional differential equations (RFDE) having general delays in the state - and input/output - variables. This will be done in the context of semigroup theory. In particular, we extend the concept of structural operators

(BERNIER-MANITIUS [4], MANITIUS [35], DELFOUR-MANITIUS [19], VINTER-KWONG [41], DELFOUR [16]) to RFDEs with delays in both input and output variables and we develop a duality theory for this class of systems (Section 5). In Section 6 we make use of these results in order to apply our general theory of the linear quadratic optimal control problem to RFDEs with input/output delays.

5. STATE SPACE THEORY FOR RETARDED SYSTEMS WITH DELAYS IN INPUT AND OUTPUT

5.1 Control systems with delays

We consider the linear RFDE

$$(5.1;1) \qquad \dot{x}(t) = Lx_t + Bu_t ,$$

$$(5.1;2) \qquad y(t) = Cx_t ,$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ and x_t, u_t are defined by $x_t(\tau) = x(t+\tau)$, $u_t(\tau) = u(t+\tau)$ for $-h \leq \tau \leq 0$ $(0 < h < \infty)$.

Correspondingly L, B, C are bounded linear functionals from $\mathcal{C} = C(-h,0; \mathbb{R}^n)$ respectively $C(-h,0; \mathbb{R}^m)$ into $\mathbb{R}^n$ respectively $\mathbb{R}^p$. These can be represented by matrix functions $\eta(\tau)$, $\beta(\tau)$, $\gamma(\tau)$ in the following way

$$L\phi = \int_0^h d\eta(\tau)\phi(-\tau), \qquad C\phi = \int_0^h d\gamma(\tau)\phi(-\tau), \; \phi \in C(-h,0; \mathbb{R}^n),$$

$$B\xi = \int_0^h d\beta(\tau)\xi(-\tau), \; \xi \in C(-h,0; \mathbb{R}^m).$$

Without loss of generality we assume that the matrix functions η, β and γ are normalised, i.e. vanish for $\tau \leq 0$, are constant for $\tau \geq h$ and left continuous for $0 < \tau < h$. A solution of (5.1;1) is a function $x \in L^2_{loc}(-h,\infty; \mathbb{R}^n)$ which is absolutely continuous with L^2- derivative on every compact interval $[0,T]$, $T > 0$, and satisfies (5.1;1) for almost every $t \geq 0$. It is well known that (5.1;1) admits a unique solution $x(t) = x(t;\phi,u)$, $t \geq -h$, for every input $u(\cdot) \in L^2_{loc}(o,\infty;\mathbb{R}^m)$ and every initial condition

$$(5.2;1) \quad x(o) = \phi^o, \qquad x(\tau) = \phi^1(\tau),$$

$$(5.2;2) \quad u(\tau) = \phi^2(\tau), \qquad -h \leq \tau < 0,$$

where $\phi = (\phi^o,\phi^1,\phi^2) \in \mathfrak{X} = \mathbb{R}^n \times L^2(-h,0;\mathbb{R}^n) \times L^2(-h,0;\mathbb{R}^m)$.
Moreover, $x(\cdot;\phi,u)$ depends continuously on ϕ and u on compact intervals, i.e. for any $T > 0$ there exists a $K > o$ such that

$$\|x(\cdot;\phi,u)\|_{W^{1,2}(0,T;\mathbb{R}^n)} \leq K\left[(\|\phi\| + \|u\|_{L^2(0,T;\mathbb{R}^m)}\right]$$

where $\|\phi\| = (\|\phi^o\|^2 + \|\phi^1\|^2_{L^2} + \|\phi^2\|^2_{L^2})^{1/2}$ for $\phi \in \mathfrak{X}$

(see e.g. BORISOVIC-TURBABIN [5], DELFOUR-MANITIUS [19], SALAMON [40]). The corresponding output $y(\cdot) = y(\cdot;\phi,u)$ is in $L^2_{loc}(o,\infty;\mathbb{R}^p)$ and depends - in this space - continuously on ϕ and u. The fundamental solution of (5.1;1) will be denoted by $X(t)$, $t \geq -h$, and is the $n \times n$ matrix valued solution of (5.1;1) which corresponds to $u \equiv o$ and satisfies $X(o) = I$, $X(\tau) = 0$ for $-h \leq \tau < 0$. Its Laplace transform is given by $\Delta^{-1}(\lambda)$, where

$$\Delta(\lambda) = \lambda I - L(e^{\lambda \cdot}) = \lambda I - \int_o^h d\eta(\tau) e^{-\lambda\tau} \quad , \quad \lambda \in \mathbb{C},$$

is the characteristic matrix of (5.1;1). It is well known that the forced motions of (5.1;1) can be written in the form

$$x(t;o,u) = \int_o^t X(t-s)Bu_s\,ds, \quad t \geq 0.$$

We also consider the transposed RFDE

$$(5.3;1) \quad \dot{z}(t) = L^T z_t + C^T v_t ,$$

$$(5.3;2) \quad w(t) = B^T z_t ,$$

with initial data

$$(5.4;1) \quad z(o) = \psi^o, \quad z(\tau) = \psi^1(\tau),$$

$$(5.4;2) \quad v(\tau) = \psi^2(\tau), \quad -h \leq \tau < 0,$$

where $\psi = (\psi^o,\psi^1,\psi^2) \in \mathcal{X}^T = \mathbb{R}^n \times L^2(-h,o;\ \mathbb{R}^n) \times L^2(-h,o;\ \mathbb{R}^p)$.
The unique solution of (5.3;1) and (5.4) will be denoted by $z(t) = z(t;\psi,v)$, $t \geq -h$, and the corresponding output by $w(t) = w(t;\psi,v)$, $t \geq o$.

5.2 State Concepts and Duality

The 'classical' way of introducing the state of a delay system is to specify an initial function of suitable length which describes the past history of the solution. This is due to the existence and uniqueness of the solution to the delay equation (in our case (5.1)) and its continuous dependence on the initial function (in our case (5.2)). Correspondingly, we may define the state of system (5.1) at time $t \geq o$ to be the triple.

$$\hat{x}(t) = (x(t), x_t, u_t) \in \mathcal{X}$$

and analogously, the state of the transposed system (5.3) at time $t \geq o$ will be given by

$$\tilde{\hat{z}}(t) = (z(t), z_t, v_t) \in \mathcal{X}^T.$$

The idea of including the input segment in the state of the system was first sugge ..d by ICHIKAWA [22], [23].

In order to describe the duality relation between the systems (5.1) and (5.3), we need an alternative state concept. For this we replace the initial functions ϕ^1 and ϕ^2 of the state- and input- variables by additional forcing terms of suitable length on the right-hand side of both equations in (5.1). These terms completely determine the future behaviour of the solution and the output. More precisely, we rewrite system (5.1), (5.2) as

$$(5.5;1) \quad \dot{x}(t) = \int_o^t d\eta(\tau)x(t-\tau) + \int_o^t d\beta(\tau)u(t-\tau) + f^1(t), x(o) = f^o,$$

$$(5.5;2) \quad y(t) = \int_o^t d\gamma(\tau)x(t-\tau) + f^2(t), \quad t \geq o$$

where the triple

$$f = (f^o, f^1, f^2) \in \mathcal{X}^{T^*} = \mathbb{R}^n \times L^2[0,h;\ \mathbb{R}^n] \times L^2[0,h;\ \mathbb{R}^p]$$

is given by

$$(5.6;1) \qquad f^o = \phi^o$$

$$(5.6;2) \quad f^1(t) = \int_t^h d\eta(\tau)\phi^1(t-\tau) + \int_t^h d\beta(\tau)\phi^2(t-\tau),$$

$$(5.6;3) \quad f^2(t) = \int_t^h d\gamma(\tau)\phi^1(t-\tau), \qquad 0 \leq t \leq h$$

Remarks 5.1

(i) The expressions on the right-hand side of (5.6;2) and (5.6;3) are well defined as square integrable functions on the interval [o,h] (see e.g. DELFOUR-MANITIUS [19] or SALAMON [40]). Each of them can be interpreted as the convolution of a Borel-measure on the interval [o,h] with an L^2-function on the interval [-h,o].

(ii) The product space $\mathcal{X}^{T^*} = \mathbb{R}^n \times L^2(o,h;\ \mathbb{R}^n) \times L^2(o,h;\ \mathbb{R}^p)$ can be identified with the dual space of $\mathcal{X}^T = \mathbb{R}^n \times L^2(-h,o;\ \mathbb{R}^n) \times L^2(-h,o;\ \mathbb{R}^p)$ via the duality pairing

$$\langle\psi,f\rangle_{\mathcal{X}^T,\mathcal{X}^{T^*}}$$

$$= \psi^{oT} f^o + \int_o^h \psi^{1^T}(-s)f^1(s)ds + \int_o^h \psi^{2^T}(-s)f^2(s)ds$$

for $\psi \in \mathcal{X}^T$ and $f \in \mathcal{X}^{T^*}$. In the same manner we can identify the product space $\mathcal{X}^* = \mathbb{R}^n \times L^2(o,h;\ \mathbb{R}^n) \times L^2(o,h;\ \mathbb{R}^m)$ with the dual space of $\mathcal{X} = \mathbb{R}^n \times L^2(-h,o;\ \mathbb{R}^n) \times L^2(-h,o;\ \mathbb{R}^m)$.

Now it is easy to see that the solution $x(t)$ and the output $y(t)$ of system (5.5) vanish for $t \geq o$ if and only if $f = o$. This fact motivates the definition of the initial state of system (5.5) to be the triple $f \in \mathcal{X}^{T^*}$. Correspondingly the state of (5.5) at time $t \geq o$ is the triple

$$\hat{x}(t) = (x(t),x^t,y^t) \in \mathcal{X}^{T^*}$$

where the function components $x^t \in L^2(o,h;\ \mathbb{R}^n)$ and $y^t \in L^2(o,h;\ \mathbb{R}^p)$ are the forcing terms of system (5.5) after a time shift. These are given by

$$(5.7;1) \qquad x^t(s) = \int_s^{t+s} d\eta(\tau)x(t+s-\tau) + \int_s^{t+s} d\beta(\tau)u(t+s-\tau) + f^1(t+s)\ ,$$

$$(5.7;2) \qquad y^t(s) = \int_s^{t+s} d\gamma(\tau)x(t+s-\tau) + f^2(t+s), \quad o \leq s \leq h,$$

where $f^1(t)$ and $f^2(t)$ are defined to be zero if $t \notin [o,h]$.

The idea of defining the state of a delay equation through the forcing term rather than the solution segment was first suggested by MILLER [37] for Volterra integro-differential equations. The corresponding duality relation has been discovered by BURNS and HERDMAN [6]. Further references in this direction can be found in SALAMON [40].

The same ideas as above can be applied to the transposed equation (5.3). For this we rewrite system (5.3), (5.4) in the following way.

$$(5.8;1)\quad \dot{z}(t) = \int_o^t d\eta^T(\tau)z(t-\tau) + \int_o^t d\gamma^T(\tau)v(t-\tau) + g^1(t), \; z(o) = g^o,$$

$$(5.8;2)\quad w(t) = \int_o^t d\beta^T(\tau)z(t-\tau) + g^2(t), \quad t \geq o,$$

where the triple

$$g = (g^o, g^1, g^2) \quad \mathcal{X}^* = \mathbb{R}^n \times L^2(o,h;\mathbb{R}^n) \times L^2(o,h;\mathbb{R}^m)$$

is given by

$$(5.9;1)\quad g^o = \psi^o,$$

$$(5.9;2)\quad g^1(t) = \int_t^h d\eta^T(\tau)\psi^1(t-\tau) + \int_t^h d\gamma^T(\tau)\psi^2(t-\tau),$$

$$(5.9;3)\quad g^2(t) = \int_t^h d\beta^T(\tau)\psi^1(t-\tau), \quad o \leq t \leq h.$$

The initial state of system (5.8) is the triple $g \in \mathcal{X}^*$ and the state at time $t \geq o$ is given by

$$\hat{z}(t) = z(t), z^t, w^t) \in \mathcal{X}^*$$

where the function components $z^t \in L^2(o,h;\mathbb{R}^n)$ and $w^t \in L^2(o,h;\mathbb{R}^m)$ are of the form

$$(5.10;1)\quad z^t(s) = \int_s^{t+s} d\eta^T(\tau)z(t+s-\tau) + \int_s^{t+s} d\gamma^T(\tau)v(t+s-\tau) + g^1(t+s),$$

$$(5.10;2)\quad w^t(s) = \int_s^{t+s} d\beta^T(\tau)z(t+s-\tau) + g^2(t+s), \quad o \leq s \leq h .$$

These expressions can be obtained from equation (5.8) through a time shift.

Summarizing out situation, we have introduced two different notions of the state both for the original RFDE (5.1) and for the transposed RFDE (5.3). A duality relation between these two equations involves both state concept.

The dual state concept (forcing terms) for the original system (5.1) is dual to the 'classical' state concept (solution segments) for the transposed system (5.3). More, precisely, we have the following result.

<u>Theorem 5.2</u> Let $u(\cdot) \in L^2_{loc}(o,\infty; \mathbb{R}^m)$ and $v(\cdot) \in L^2_{loc}(o,\infty; \mathbb{R}^p)$ be given.

(i) Let $f \in \mathcal{X}^{T*}$ and $\psi \in \mathcal{X}^T$. Moreover, suppose that $\hat{\hat{x}}(t) = (x(t),x^t,y^t) \in \mathcal{X}^{T*}$ is the corresponding state of (5.5) with output $y(t)$ and that $\hat{z}(t) = (z(t),z_t,v_t) \in \mathcal{X}^T$ is the state of (5.3), (5.4) at time $t \geq o$ with output $w(t)$. Then

$$<\psi,\hat{\hat{x}}(t)>_{\mathcal{X}^T,\mathcal{X}^{T*}} - <\hat{z}(t),f>_{\mathcal{X}^T,\mathcal{X}^{T*}}$$

$$= \int_o^t w^T(t-s)u(s)ds - \int_o^t v^T(t-s)y(s)ds, \quad t \geq o \ .$$

(ii) Let $\phi \in \mathcal{X}$ and $g \in \mathcal{X}^*$. Moreover, suppose that $\hat{x}(t) = (x(t),x_t,u_t)$ is the corresponding state of (5.1), (5.2) with output $y(t)$ and that $\hat{\hat{z}}(t) = (z(t),z^t,w^t) \in \mathcal{X}^*$ is the state of (5.8) at time $t \geq o$ with output $w(t)$. Then

$$<g,\hat{x}(t)>_{\mathcal{X}^*,\mathcal{X}} - <\hat{\hat{z}}(t),\phi>_{\mathcal{X}^*,\mathcal{X}}$$

$$= \int_o^t w^T(t-s)u(s)ds - \int_o^t v^T(t-s)y(s)ds, \quad t \geq o$$

<u>Proof</u> We will only give a proof of statement (i). For this let us assume that $z(t)$, $t \geq -h$, is the unique solution of (5.3), (5.4) with output $w(t)$, $t \geq o$, and that $x(t)$, $t \geq o$, is the unique solution of (5.5) with output $y(t)$, $t \geq o$. Moreover, let $x^t \in L^2[o,h; \mathbb{R}^n]$, $y^t \in L^2[o,h; \mathbb{R}^p]$ be given by (5.7) and define $x(t) = o$, $u(t) = o$ for $t < o$. Then it is easy to see that

$$\int_o^t (z^T(t-s)Lx_s - [L^T z_{t-s}]^T x(s))ds$$

$$= - \int_0^h \int_0^\tau \psi^{1T}(-s)d\eta(\tau)x(t+s-\tau)ds$$

and analogous equations hold for B and C. Moreover

$$\psi^{OT} x(t) - z^T(t)f^o = \int_0^t \frac{d}{ds} z^T(t-s)x(s)ds$$

$$= \int_0^t z^T(t-s)\dot{x}(s)ds - \int_0^t \dot{z}^T(t-s)x(s)ds$$

This implies

$$<\psi,\hat{x}(t)>_{\mathfrak{X}^T,\mathfrak{X}^{T*}} - <\hat{z}(t),f>_{\mathfrak{X}^T,\mathfrak{X}^{T*}}$$

$$= \psi^{OT}x(t) - z^T(t)f^o$$

$$+ \int_0^h \psi^{1T}(-s)x^t(s)ds + \int_0^h \psi^{2T}(-s)y^t(s)ds$$

$$- \int_0^h z^T(t-s)f^1(s)ds - \int_0^h v^T(t-s)f^2(s)ds$$

$$= \int_0^t z^T(t-s)\,[Lx_s+Bu_s+f^1(s)]\,ds$$

$$- \int_0^t [L^Tz_{t-s} + C^Tv_{t-s}]^T\, x(s)ds + \int_0^{h-t} \psi^{1T}(-s)f^1(t+s)ds$$

$$+ \int_0^h \int_0^\tau \psi^{1T}(-s)d\eta(\tau)x(t+s-\tau)ds + \int_0^h \int_0^\tau \psi^{1T}(-s)d\beta(\tau)u(t+s-\tau)ds$$

$$+ \int_0^h \int_0^\tau \psi^{2T}(-s)d\gamma(\tau)x(t+s-\tau)ds + \int_0^{h-t} \psi^{2T}(-s)f^2(t+s)ds$$

$$- \int_0^t z^T(t-s)f^1(s)ds - \int_t^h \psi^{1T}(t-s)f^1(s)ds$$

$$- \int_0^t v^T(t-s)f^2(s)ds - \int_0^h \psi^{2T}(t-s)f^2(s)ds$$

$$= \int_0^t [B^T z_{t-s}]^T u(s)ds - \int_0^t v^T(t-s)\left[Cx_s + f^2(s)\right]ds$$

$$= \int_0^t w^T(t-s)u(s)ds - \int_0^t v^T(t-s)y(s)ds. \qquad \blacksquare$$

5.3 Semigroups and structural operators

Throughout this section we restrict our discussion to the homogeneous systems (5.1) and (5.5) respectively (5.3) and (5.8) which means that $u(t) = 0$ respectively $v(t) = 0$ for $t \geq o$.

The evolution of the systems (5.1) and (5.3) in terms of the 'classical' state concept (solution segments) can be described by strongly continuous semigroups

$$\mathcal{S}(t) : \mathcal{X} \to \mathcal{X}, \qquad \mathcal{S}^T(t) : \mathcal{X}^T \to \mathcal{X}^T.$$

The semigroup $\mathcal{S}(t)$ on $\mathcal{X}$ associates with every $\phi \in \mathcal{X}$ the state

$$\mathcal{S}(t)\phi = \hat{x}(t) = (x(t), x_t, u_t) \in \mathcal{X}$$

of (5.1), (5.2) at time $t \geq o$ which corresponds to the input $u(s) = o$, $s \geq o$. Its infinitesimal generator is given by

$$\mathcal{D}(\mathcal{A}) = \{\phi \in \mathcal{X} \mid \phi^1 \in W^{1,2}(-h,0;\mathbb{R}^n), \phi^2 \in W^{1,2}(-h,0\ \mathbb{R}^m), \phi^o = \phi^1(o), \phi^2(o) = 0\}$$

$$\mathcal{A}\phi = (L\phi^1 + B\phi^2, \dot{\phi}^1, \dot{\phi}^2)$$

(SALAMON [40, Theorem I.2.6]). The semigroup $\mathcal{S}^T(t)$ is defined analogously and generated by the operator

$$\mathcal{D}(\mathcal{A}^T) = \{\psi \in \mathcal{X}^T \mid \psi^1 \in W^{1,2}(-h,o;\mathbb{R}^n), \psi^2 \in W^{1,2}(-h,o;\mathbb{R}^p);$$
$$\psi^o = \psi^1(o), \psi^2(o) = 0\}$$

$$\mathcal{A}^T\psi = (L^T\psi^1 + C^T\psi^2, \dot{\psi}^1, \dot{\psi}^2).$$

An interpretation of the adjoint semigroups $\mathcal{S}^{T*}(t)$, $\mathcal{X}^{T*} \to \mathcal{X}^{T*}$ and $\mathcal{S}^*(t)$: $\mathcal{X}^* \to \mathcal{X}^*$ can be given through the dual state concept (forcing terms) for the systems (5.1) and (5.3). More precisely, we have the following result which is a direct consequence to Theorem 5.2.

Corollary 5.3

(i) Let $f \in \mathcal{X}^{T*}$ be given and let $\hat{x}(t) = (x(t), x^t, y^t) \in \mathcal{X}^{T*}$ be the state of system (5.5) at time $t \geq o$ corresponding to the input $u(\cdot) \equiv 0$. Then $\hat{x}(t) = \mathcal{S}^{T*}(t)f$.

(ii) Let $g \in \mathcal{X}^*$ be given and let $z(t) = (z(t), z^t, v^t) \in \mathcal{X}^*$ be the state of system (5.8) at time $t \geq o$ corresponding to the input $v(\cdot) \equiv 0$. Then $\hat{z}(t) = \mathcal{S}^*(t)g$.

Our next result is an explicit characterization of the infinitesimal generators $\mathcal{A}^{T*}$ and $\mathcal{A}^*$ of the semigroups $\mathcal{S}^{T*}(t)$ and $\mathcal{S}^*(t)$.

Proposition 5.4

(i) Let $f, d \in \mathcal{X}^{T*}$ be given. Then $f \in \mathcal{D}(\mathcal{A}^{T*})$ and $\mathcal{A}^{T*}f = d$ if and only if the following equations hold

$$(5.11;1) \quad \eta(h)f^o = d^o + \int_o^h d^1(s)ds,$$

$$(5.11;2) \quad f^1(t) + [\eta(t)-\eta(h)]f^o = -\int_t^h d^1(s)ds, \quad o \leq t \leq h,$$

$$(5.11;3) \quad f^2(t) + [\gamma(t)-\gamma(h)]f^o = -\int_t^h d^2(s)ds, \quad o \leq t \leq h.$$

(ii) Let $g, k \in \mathcal{X}^*$ be given. Then $g \in \mathcal{D}(\mathcal{A}^*)$ and $\mathcal{A}^*g = k$ if and only if

$$(5.12;1) \quad \eta^T(h)g^o = k^o + \int_o^h k^1(s)ds$$

$$(5.12;2) \quad g^1(t) + [\eta^T(t)-\eta^T(h)]g^o = -\int_t^h k^1(s)ds, \quad o \leq t \leq h,$$

$$(5.12;3) \quad g^2(t) + [\beta^T(t)-\beta^T(h)]g^o = -\int_t^h k^2(s)ds, \quad o \leq t \leq h.$$

<u>Proof</u> Obviously it is enough to prove statement (i). First note that $f \in \mathcal{D}(\mathcal{A}^{T*})$ and $\mathcal{A}^{T*}f = d$ if and only if $\langle\psi,d\rangle = \langle\mathcal{A}^{T}\psi,f\rangle$ for every $\psi \in \mathcal{D}(\mathcal{A}^{T})$. Hence statement (i) is a consequence of

$$\langle\psi,d\rangle = \psi^{oT}d^{o} + \int_{o}^{h}\psi^{1T}(-s)d^{1}(s)ds + \int_{o}^{h}\psi^{2T}(-s)d^{2}(s)ds$$

$$= \psi^{1T}(o)[d^{o} + \int_{o}^{h}d^{1}(s)ds] - \int_{o}^{h}\dot{\psi}^{1T}(-t)\int_{t}^{h}d^{1}(s)dsdt$$

$$- \int_{t}^{h}\dot{\psi}^{2T}(-t)\int_{t}^{h}d^{2}(s)dsdt$$

and

$$\langle\mathcal{A}^{T}\psi,f\rangle$$

$$= \int_{o}^{h}\psi^{1T}(-\tau)d\eta(\tau)f^{o} + \int_{o}^{h}\psi^{2T}(-\tau)d\gamma(\tau)f^{o}$$

$$+ \int_{o}^{h}\dot{\psi}^{1T}(-s)f^{1}(s)ds + \int_{o}^{h}\dot{\psi}^{2T}(-s)f^{2}(s)ds$$

$$= \psi^{1T}(-h)\eta(h)f^{o} + \psi^{2T}(-h)\gamma(h)f^{o}$$

$$+ \int_{o}^{h}\dot{\psi}^{1T}(-s)\eta(s)f^{o}ds + \int_{o}^{h}\dot{\psi}^{2T}(-s)\gamma(s)f^{o}ds$$

$$+ \int_{o}^{h}\dot{\psi}^{1T}(-s)f^{1}(s)ds + \int_{o}^{h}\dot{\psi}^{2T}(-s)f^{2}(s)ds$$

$$= \psi^{1T}(o)\eta(h)f^{o} + \int_{o}^{h}\dot{\psi}^{1T}(-s)[f^{1}(s) + \eta(s)f^{o} - \eta(h)f^{o}]ds$$

$$+ \int_{o}^{h}\dot{\psi}^{2T}(-s)[f^{2}(s) + \gamma(s)f^{o} - \gamma(h)f^{o}]ds. \quad ▨$$

The duality relation between the systems (5.1) and (5.3) can now be described through the following four semigroups

$$\mathcal{S}(t) : \mathfrak{X} \to \mathfrak{X}, \qquad \mathcal{S}^T(t) : \mathfrak{X}^T \to \mathfrak{X}^T,$$

$$\mathcal{S}^{T*}(t) : \mathfrak{X}^{T*} \to \mathfrak{X}^{T*}, \quad \mathcal{S}^*(t) : \mathfrak{X}^* \to \mathfrak{X}^*$$

The semigroups on the left-hand side correspond to the RFDE (5.1) and those on the right-hand side to the transposed RFDE (5.3). On each side the upper semigroup describes the respective equation within the 'classical' state concept (solution segments) and the semigroup below within the dual state concept (forcing terms). A diagonal relation is actually given by functional analytic duality theory.

The relation between the two state concepts can be described by a so-called structural operator

$$\mathcal{F} : \mathfrak{X} \to \mathfrak{X}^{T*}$$

which associates with every $\phi \in \mathfrak{X}$ the corresponding triple

(5.13) $\quad \mathcal{F}\phi = f \in \mathfrak{X}^{T*} \quad$ (f given by (5.6)).

It is easy to see that this operator maps every state $\hat{x}(t) \in \mathfrak{X}$ of system (5.1) into the corresponding state $\hat{\hat{x}}(t) \in \mathfrak{X}^{T*}$ of system (5.5) which is given by (5.7) and (5.6). This fact together with corollary 5.3 shows that the following diagram commutes

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "\mathcal{S}(t)"] \arrow[d, "\mathcal{F}"'] & \mathfrak{X} \arrow[d, "\mathcal{F}"] \\
\mathfrak{X}^{T*} \arrow[r, "\mathcal{S}^{T*}(t)"] & \mathfrak{X}^{T*}.
\end{tikzcd}$$

Another important fact is that the adjoint operator $\mathfrak{F}^* : \mathfrak{X}^T \to \mathfrak{X}^*$ plays the same role for the transposed RFDE (5.3) as the structural operator $\mathfrak{F} : \mathfrak{X} \to \mathfrak{X}^{T*}$ does for the original RFDE (5.1). These properties are summarized in the theorem below.

Theorem 5.5

(i) $\mathfrak{F}\mathfrak{S}(t) = \mathfrak{S}^{T*}(t)\mathfrak{F}, \qquad \mathfrak{F}^* \mathfrak{S}^T(t) = \mathfrak{S}^*(t)\mathfrak{F}^*$

(ii) If $\phi \in \mathcal{D}(\mathcal{A})$, then $\mathfrak{F}\phi \in \mathcal{D}(\mathcal{A}^{T*})$ and

$$\mathcal{A}^{T*}\mathfrak{F}\phi = \mathfrak{F}\mathcal{A}\phi .$$

(iii) If $\psi \in \mathcal{D}(\mathcal{A}^T)$, then $\mathfrak{F}^*\psi \in \mathcal{D}(\mathcal{A}^*)$ and

$$\mathcal{A}^*\mathfrak{F}^*\psi = \mathfrak{F}^*\mathcal{A}^T\psi .$$

(iv) The adjoint operator $\mathfrak{F}^* : \mathfrak{X}^T \to \mathfrak{X}^*$ maps every $\psi \in \mathfrak{X}^T$ into the triple $\mathfrak{F}^*\psi = g \in \mathfrak{X}^*$ which is given by (5.9).

Proof Statement (i) follows from the above considerations, the statements (ii) and (iii) are immediate consequences of (i) and statement (iv) can be proved straight forwardly. ▨

A structural operator of the above type has first been introduced in BERNIER-MANITIUS [4], DELFOUR-MANITIUS [19] for retarded systems with state delays only and later on by VINTER-KWONG [41], DELFOUR [16] for RFDEs with delays in the state and control variables. An extension to neutral systems can be found in SALAMON [40].

5.4. Abstract Cauchy problems

In order to describe the action of the output operators for the RFDEs (5.1) and (5.3) - each within the two state concepts of section 5.2 - we introduce the following four subspaces

$$\mathcal{W} = \{\phi \in \mathcal{X} \mid \phi^1 \in W^{1,2}(-h,o;\ \mathbb{R}^n), \phi^o = \phi^1(o)\},$$

$$\mathcal{W}^T = \{\psi \in \mathcal{X}^T \mid \psi^1 \in W^{1,2}(-h,o;\ \mathbb{R}^n), \psi^o = \psi^1(o)\},$$

$$\mathcal{V}^{T*} = \{f \in \mathcal{X}^{T*} \mid f^2(\cdot)+\gamma(\cdot)f^o \in W^{1,2}(o,h;\mathbb{R}^p), f^2(h) = o\},$$

$$\mathcal{V}^* = \{g \in \mathcal{X}^* \mid g^2(\cdot)+\beta^T(\cdot)g^o \in W^{1,2}(o,h;\ \mathbb{R}^m), g^2(h) = o\}.$$

These have the following properties.

Remarks 5.6

(i) The subspaces $\mathcal{W}$, $\mathcal{W}^T$, $\mathcal{V}^{T*}$ and $\mathcal{V}^*$ are dense in $\mathcal{X}$, $\mathcal{X}^T$, $\mathcal{X}^{T*}$ and $\mathcal{X}^*$, respectively. Moreover, $\mathcal{W}$ and $\mathcal{V}^{T*}$ become Hilbert spaces if they are endowed with the norms

$$\|\phi\|^2_{\mathcal{W}} = \|\phi^o\|^2_{\mathbb{R}^n} + \int_{-h}^{o} \|\dot{\phi}^1(\tau)\|^2_{\mathbb{R}^n}\, d\tau + \int_{-h}^{o} \|\phi^2(\tau)\|^2_{\mathbb{R}^m}\, d\tau, \phi \in \mathcal{W},$$

$$\|f\|^2_{\mathcal{V}^{T*}} = \|f^o\|^2_{\mathbb{R}^n} + \int_o^h \|f^1(s)\|^2_{\mathbb{R}^n}\, ds$$

$$+ \int_o^h \left\| \frac{d}{ds}\,[f^2(s) + \gamma(s)f^o]\right\|^2_{\mathbb{R}^p} ds,\ f \in \mathcal{V}^{T*}.$$

Topologies on $\mathcal{W}^T$ and $\mathcal{V}^*$ can be defined analogously.

(ii) The dual spaces $\mathcal{V}$, $\mathcal{V}^T$, $\mathcal{W}^{T*}$ and $\mathcal{W}^*$ are extensions of $\mathcal{X}$, $\mathcal{X}^T$, $\mathcal{X}^{T*}$ and $\mathcal{X}^*$, respectively. Thus we obtain the inclusions

$$\mathcal{W} \subset \mathcal{X} \subset \mathcal{V}, \qquad \mathcal{W}^T \subset \mathcal{X}^T \subset \mathcal{V}^T,$$

$$\mathcal{V}^{T*} \subset \mathcal{X}^{T*} \subset \mathcal{W}^{T*}, \qquad \mathcal{V}^* \subset \mathcal{X}^* \subset \mathcal{W}^*.$$

with continuous, dense embeddings.

(iii) It is easy to see that $\mathcal{S}(h) \in \mathcal{L}(\mathcal{X}, \mathcal{W})$,

$$\mathcal{S}^T(h) \in \mathcal{L}(\mathcal{X}^T, \mathcal{W}^T), \quad \mathcal{S}^{T*}(h) \in \mathcal{L}(\mathcal{X}^{T*}, \mathcal{V}^{T*}) \text{ and}$$

$\mathcal{S}^*(h) \in \mathcal{L}(\mathcal{X}^*, \mathcal{V}^*)$. By duality, we obtain $\mathcal{S}(h) \in \mathcal{L}(\mathcal{V}, \mathcal{X})$
$\mathcal{S}^T(h) \in \mathcal{L}(\mathcal{V}^T, \mathcal{X}^T)$, $\mathcal{S}^{T*}(h) \in \mathcal{L}(\mathcal{W}^{T*}, \mathcal{X}^{T*})$ and $\mathcal{S}^*(h) \in \mathcal{L}(\mathcal{W}^*, \mathcal{X}^*)$

Before introducing the input - and output - operators, we prove that the spaces $\mathcal{W}$, $\mathcal{W}^T$, $\mathcal{V}^{T*}$ and $\mathcal{V}^*$ are invariant under the semigroups $\mathcal{S}(t)$, $\mathcal{S}^T(t)$, $\mathcal{S}^{T*}(t)$ and $\mathcal{S}^*(t)$, respectively. For this we need the following preliminary result.

<u>Lemma 5.7</u> Let $f \in \mathcal{V}^{T*}$ be given and let $d \in L^2(o,h;\mathbb{R}^p)$ satisfy

$$f^2(s) + [\gamma(s) - \gamma(h)]f^o = -\int_s^h d(\sigma)d\sigma, \quad o \leq s \leq h.$$

Moreover, let $x(\cdot) \in W^{1,2}(o,t;\mathbb{R}^n)$ be chosen such that $x(o) = f^o$ and let $y^t \in L^2(o,h;\mathbb{R}^p)$ be defined by (5.7;2). Then

$$y^t(s) + [\gamma(s) - \gamma(h)]\, x(t)$$

$$= -\int_s^h [\int_\sigma^{t+\sigma} d\gamma(\tau)\dot{x}(t+\sigma-\tau) + d(t+\sigma)]d\sigma, \quad o \leq s \leq h.$$

<u>Proof</u> Let us define $x(s) = f^o$ for $s \leq o$ and $d(\sigma) = 0$ for $\sigma \notin [o,h]$.

Then the equation

$$\int_{t+s}^h d\gamma(\tau)x(t+s-\tau) = [\gamma(h) - \gamma(t+s)]f^o$$

holds for all $t,s \geq o$. This implies

$$\int_s^h \Big[\int_\sigma^{t+\sigma} d\gamma(\tau)\dot{x}(t+\sigma-\tau) + d(t+\sigma)\Big] d\sigma$$

$$= \int_s^h \int_\sigma^h d\gamma(\tau)\dot{x}(t+\sigma-\tau)d\sigma + \int_s^{h-t} d(t+\sigma)d\sigma$$

$$= \int_s^h d\gamma(\tau) \int_s^\tau \dot{x}(t+\sigma-\tau)ds + \int_{t+s}^h d(\sigma)d\sigma$$

$$= \int_s^h d\gamma(\tau)[x(t)-x(t+s-\tau)] - f^2(t+s)$$

$$- [\gamma(t+s) - \gamma(h)]f^0$$

$$= [\gamma(h) - \gamma(s)]\ x(t) - \int_s^{t+s} d\gamma(\tau)x(t+s-\tau) + f^2(t+s)$$

$$= - y^t(s) - [\gamma(s) - \gamma(h)]x(t). \quad \blacksquare$$

Now we are in the position to prove the desired invariance properties of the subspaces $\mathcal{W}$, $\mathcal{W}^T$, $\mathcal{V}^{T*}$ and $\mathcal{V}^*$.

<u>Proposition 5.8</u>

(i) $\mathcal{S}(t)$ is a strongly continuous semigroup on $\mathcal{W}$ and $\mathcal{V}$.

(ii) $\mathcal{S}^T(t)$ is a strongly continuous semigroup on $\mathcal{W}^T$ and $\mathcal{V}^T$.

(iii) $\mathcal{S}^{T*}(t)$ is a strongly continuous semigroup on $\mathcal{V}^{T*}$ and $\mathcal{W}^{T*}$.

(iv) $\mathcal{S}^*(t)$ is a strongly continuous semigroup on $\mathcal{V}^*$ and $\mathcal{W}^*$.

(v) $\mathcal{F} \in \mathcal{L}(\mathcal{W}, \mathcal{V}^{T*})$ and $\mathcal{F} \in \mathcal{L}(\mathcal{V}, \mathcal{W}^{T*})$.

(vi) $\mathcal{J}^* \in \mathcal{L}(\mathcal{W}^T, \mathcal{V}^*)$ and $\mathcal{J}^* \in \mathcal{L}(\mathcal{V}^T, \mathcal{W}^*)$.

<u>Proof</u> First note that every solution $x(t)$ of (5.5) is absolutely continuous for $t \geq o$ and that its L^2-derivative depends continuously on $f \in \mathcal{X}^{T^*}$ (SALAMON [40, Theorem I.2.3 (i)]. This shows that $\mathcal{S}(t)$ is a strongly continuous semigroup on $\mathcal{W}$.

Now let $f \in \mathcal{V}^{T*}$ be given and let $x(\cdot) \in W^{1,2}_{loc}(o,\infty; \mathbb{R}^n)$ be the corresponding solution of (5.5) with $u(t) \equiv o$. Moreover let $y(t)$, $t \geq o$, be the output of (5.5) and let x^t and y^t be given by (5.7). Then $\mathcal{S}^{T*}(t)f = (x(t),x^t,y^t)$ (Corollary 5.3) and hence it follows from Lemma 5.7 that the function $t \rightarrow \mathcal{S}^{T*}(t)f$ is continuous with values in $\mathcal{V}^{T*}$ and depends in this space continuously on $f \in \mathcal{V}^{T*}$.

The same considerations - applied to the transposed system (5.3) - show that $\mathcal{S}^T(t)$ is a semigroup on $\mathcal{W}^T$ and that $\mathcal{S}^*(t)$ is s semigroup on $\mathcal{V}^*$. The remaining assertions in (i), (ii), (iii) and (iv) follow by duality.

In order to prove (v) and (vi), let $\phi \in \mathcal{W}$ be given. Then Lemma 5.7 - applied to $f = o$, $t = h$, and $x(s) = \phi^1(s-h)$ for $o \leq s \leq h$ - shows that $\mathcal{J}\phi \in \mathcal{X}^{T*}$ satisfies the equation

$$[\mathcal{J}\phi]^2(s) + [\gamma(s) - \gamma(h)][\mathcal{J}\phi]^0$$

$$= -\int_s^h \int_\sigma^h d\gamma(\tau)\dot{\phi}^1(\sigma-\tau)d\sigma, \qquad o \leq s \leq h .$$

Hence $\mathcal{J}\phi$ is in $\mathcal{V}^{T*}$ and depends in this space continuously on $\phi \in \mathcal{W}$. We conclude that $\mathcal{J} \in \mathcal{L}(\mathcal{W},\mathcal{V}^{T*})$. The remaining assertions of (v) and (vi) follow from this fact by analogy and duality. $\boxed{}$

Now let us introduce the output operators

$$\mathcal{C} : \mathcal{W} \rightarrow \mathbb{R}^p, \qquad \mathcal{B}^T : \mathcal{W}^T \rightarrow \mathbb{R}^m,$$
$$\mathcal{C}^{T*} : \mathcal{V}^{T*} \rightarrow \mathbb{R}^p, \qquad \mathcal{B}^* : \mathcal{V}^* \rightarrow \mathbb{R}^m.$$

by defining

$$\mathcal{C}\phi = \int_0^h d\gamma(\tau)\phi^1(-\tau), \qquad \phi \in \mathcal{W}$$

$$\mathcal{B}^T\psi = \int_0^h d\beta^T(\tau)\psi^1(-\tau), \qquad \psi \in \mathcal{W}^T$$

$$\mathcal{C}^{T*}f = f^2(o), \qquad f \in \mathcal{V}^{T*},$$

$$\mathcal{B}^*g = g^2(o), \qquad g \in \mathcal{V}^*.$$

Then the adjoint operators

$$\mathcal{B} : \mathbb{R}^m \to \mathcal{V}, \qquad \mathcal{C}^T : \mathbb{R}^p \to \mathcal{V}^T,$$

$$\mathcal{B}^{T*} : \mathbb{R}^m \to \mathcal{W}^{T*}, \qquad \mathcal{C}^* : \mathbb{R}^p \to \mathcal{W}^*$$

describe the input action for the systems (5.1) and (5.3). More precisely, we have the following result for the RFDE (5.1). The corresponding statements for the transposed RFDE (5.3) can be formulated analogously.

<u>Theorem 5.9</u> Let $u(\cdot) \in L^2_{loc}(o,\infty;\ \mathbb{R}^m)$ be given

(i) Let $\phi \in \mathcal{W}$ and let $\hat{x}(t) \in \mathcal{X}$ be the corresponding state of (5.1), (5.2) at time $t \geq o$. Then $\hat{x}(t)$, $t \geq o$, is a continuous function with values in $\mathcal{W}$ and depends in this space continuously on $\phi \in \mathcal{W}$ and $u(\cdot) \in L^2_{loc}(o,\infty;\ \mathbb{R}^m)$. Moreover

$$\text{(5.14;1)} \quad \hat{x}(t) = \mathcal{S}(t)\phi + \int_o^t \mathcal{L}(t-s)\,\mathcal{B}u(s)ds, \quad t \geq o,$$

where the integral is to be understood in the Hilbert space $\mathcal{V}$. The output $y(t)$ of (5.1) is given by

$$\text{(5.14;2)} \quad y(t) = \mathcal{C}\,\hat{x}(t), \qquad t \geq o.$$

(ii) Let $f \in \mathcal{V}^{T*}$ and let $\hat{\hat{x}}(t) \in \mathcal{X}^{T*}$ be the corresponding state of (5.5) at time $t \geq o$. Then $\hat{\hat{x}}(t)$, $t \geq o$, is a continuous function with values in $\mathcal{V}^{T*}$ and depends in this space continuously on $f \in \mathcal{V}^{T*}$ and $u(\cdot) \in L^2_{loc}(o,\infty;\ \mathbb{R}^m)$. Moreover

$$(5.15;1)\quad \hat{\hat{x}}(t) = \mathcal{S}^{T*}(t)f + \int_o^t \mathcal{S}^{T*}(t-s)\,\mathcal{B}^{T*}u(s)ds,\ t \geq o$$

where the integral is to be understood in the Hilbert space $\mathcal{W}^{T*}$. The output $y(t)$ of (5.5) is given by

$$(5.15;2)\quad y(t) = \mathcal{C}^{T*}\hat{\hat{x}}(t),\quad t \geq o.$$

Proof If $u(\cdot) \equiv o$, then the statements of the theorem follow immediately from Proposition 5.8 (i), (iii) together with the definition of the operators $\mathcal{C}$ and $\mathcal{C}^{T*}$. So we can restrict ourselves to the case $\phi = o$ and $f = o$.

First of all, the same arguments as in the beginning of the proof of Proposition 5.8 show that $\hat{x}(t)$ is continuous with values in $\mathcal{W}$ and depends in this space continuously on $u(\cdot) \in L^2_{loc}(o,\infty,\ \mathbb{R}^m)$. Secondly, we establish equation (5.14;1). For this let $g \in \mathcal{V}^*$ be given, let $\hat{\hat{z}}(t) \in \mathcal{X}^*$ be the corresponding state of (5.8) with $v(\cdot) \equiv o$ and let $w(t)$, $t \geq o$, be the output of (5.8). Then $\hat{\hat{z}}(t) = \mathcal{S}^*(t)g \in \mathcal{V}^*$ (Corollary 5.3 and Proposition 5.8 (iv)) and $w(t) = \mathcal{B}^*\hat{\hat{z}}(t)$, by definition of the operator $\mathcal{B}^*$. Hence it follows from Theorem 5.2 (ii) that the following equation holds for every $t \geq o$

$$\begin{aligned}
&\langle g,\hat{x}(t)\rangle_{\mathcal{V}^*,\mathcal{V}} \\
&= \langle g,\hat{x}(t)\rangle_{\mathcal{X}^*,\mathcal{X}} \\
&= \int_o^t w^T(t-s)u(s)ds \\
&= \int_o^t \langle \mathcal{B}^*\mathcal{S}^*(t-s)g,u(s)\rangle_{\mathbb{R}^m}\,ds
\end{aligned}$$

$$= \langle g, \int_0^t \mathcal{J}(t-s)\,\mathcal{B}u(s)ds\rangle_{\mathcal{V}^*,\mathcal{V}}.$$

This proves statement (i).

Now recall that $\hat{\hat{x}}(t) = \hat{\mathcal{J}}\hat{x}(t)$ as long as $f = o$ and $\phi = o$. Hence it follows from (i) and Proposition 5.8 (v) that $\hat{\hat{x}}(t)$, $t \geq o$, is continuous with values in $\mathcal{W}^{T*}$ and depends in this space continuously on $u(\cdot) \in L^2_{loc}(o,\infty; \mathbb{R}^m)$. Finally, equation (5.15;1) can be established in an analogous manner as (5.14;1). ▨

The previous theorem shows that the evolution of the state $\hat{x}(t)$ of the RFDE(5.1) in terms of the 'classical' state concept can be formally described through the abstract Cauchy problem

$$\Sigma \quad \begin{aligned} &\frac{d}{dt}\hat{x}(t) = \mathcal{A}\hat{x}(t) + \mathcal{B}u(t), \quad \hat{x}(o) = \phi, \\ &y(t) = \mathcal{C}\hat{x}(t), \end{aligned}$$

in the Hilbert space $\mathcal{W}$ respectively $\mathcal{V}$.

Analogously, the state $\hat{\hat{x}}(t) \in \mathcal{X}^{T*}$ of equation (5.5) in terms of the dual state concept defines a mild solution of the abstract Cauchy problem.

$$\Sigma^{T*} \quad \begin{aligned} &\frac{d}{dt}\hat{\hat{x}}(t) = \mathcal{A}^{T*}\hat{\hat{x}}(t) + \mathcal{B}^{T*}u(t), \ \hat{\hat{x}}(o) = f, \\ &y(t) = \mathcal{C}^{T*}\hat{\hat{x}}(t), \end{aligned}$$

in the Hilbert space $\mathcal{V}^{T*}$ respectively $\mathcal{W}^{T*}$.

If we consider the Cauchy problem Σ (respectively Σ^{T*}) in the smaller state space $\mathcal{W}$ (respectively $\mathcal{V}^{T*}$), then the output operator $\mathcal{C}$ (respectively $\mathcal{C}^{T*}$) will be bounded and the input operator $\mathcal{B}$ (respectively $\mathcal{B}^{T*}$) unbounded. Nevertheless, the solution of Σ (respectively Σ^{T*}) in the state space $\mathcal{W}$ (respectively $\mathcal{V}^{T*}$) is well defined, since the input operator satisfies the hypothesis (H1) of Section 2. More precisely, the operator $\mathcal{B}$ (respectively

$\mathcal{B}^{T*}$) has the following property which follows directly from Theorem 5.9.

Remark 5.10

For every $T > o$ there exists some constant $b_T > o$ such that the inequalities

$$\left\| \int_o^T \mathcal{S}(T-s)\, \mathcal{B}\, u(s)ds \right\|_{\mathcal{W}} \leq b_T \, \| u(\cdot) \|_{L^2(o,T;\, \mathbb{R}^m)}$$

$$\left\| \int_o^T \mathcal{S}^{T*}(T-s)\, \mathcal{B}^{T*} u(s)ds \right\|_{\mathcal{V}^{T*}} \leq b_T \| u(\cdot) \|_{L^2(o,T;\, \mathbb{R}^m)}$$

hold for every $u(\cdot) \in L^2(o,T;\, \mathbb{R}^m)$.

If we consider the Cauchy problem Σ (respectively Σ^{T*}) in the larger space $\mathcal{V}$ (respectively $\mathcal{W}^{T*}$), then the input operator will be bounded and the output operator unbounded. Nevertheless, the output of the system is well defined as a locally square integrable function since the output operator satisfies the hypothesis (H2) of Section 2. More precisely, the operator $\mathcal{C}$ (respectively $\mathcal{C}^{T*}$) has the following property.

Remark 5.11 For every $T > o$ there exists some constant $c_T > o$ such that the inequalities

$$\| \mathcal{C}\, \mathcal{S}(\cdot)\phi \|_{L^2(o,T;\, \mathbb{R}^p)} \leq c_T \, \| \phi \|_{\mathcal{V}},$$

$$\| \mathcal{C}^{T*} \mathcal{S}^{T*}(\cdot)f \|_{L^2(o,T;\, \mathbb{R}^p)} \leq c_T \, \| f \|_{\mathcal{W}^{T*}}$$

hold for every $\phi \in \mathcal{W}$ and every $f \in \mathcal{V}^{T*}$.

This follows by duality from the fact that the adjoint operators $\mathcal{C}^T$ and $\mathcal{C}^*$ are the input operators of the transposed equation (5.3) and hence satisfy analogous inequalities as those in Remark 5.10.

Now let us apply Theorem 5.9 to the transposed RFDE (5.3). Then we obtain that the state $\hat{z}(t) \in \mathcal{H}^T$ of (5.3) in terms of the 'classical' state concept defines a mild solution of the Cauchy problem

$$\Sigma^T \quad \begin{aligned} &\frac{d}{dt}\hat{z}(t) = \mathcal{A}^T\hat{z}(t) + \mathcal{C}^T v(t), \qquad \hat{z}(o) = \psi, \\ &W(t) = \mathcal{B}^T\hat{z}(t) \end{aligned}$$

(to be considered in the Hilbert spaces $\mathcal{W}^T$ and $\mathcal{V}^T$) whereas as the state $\hat{\hat{z}}(t) \in \mathcal{X}^*$ of (5.8) in terms of the dual state concept defines a mild solution of the Cauchy problem

$$\Sigma^* \quad \begin{aligned} &\frac{d}{dt}\hat{\hat{z}}(t) = \mathcal{A}^*\hat{\hat{z}}(t) + \mathcal{C}^* v(t), \qquad \hat{\hat{z}}(o) = g, \\ &v(t) = \mathcal{B}^*\hat{\hat{z}}(t) \end{aligned}$$

(to be considered in the Hilbert spaces $\mathcal{V}^*$ and $\mathcal{W}^*$).

Summarizing our situation we have to deal with the four Cauchy problems

$$\begin{array}{cc} \Sigma & \Sigma^T \\ \Sigma^{T*} & \Sigma^* \end{array}$$

These are related in the same manner as the semigroups $\mathcal{S}(t)$, $\mathcal{S}^T(t)$, $\mathcal{S}^{T*}(t)$ and $\mathcal{S}^*(t)$. More precisely, the Cauchy problems on the left-hand side corresponds to the RFDE (5.1) and those on the right-hand side to the transposed RFDE (5.3). On each side the upper Cauchy problem describes the respective equation with the 'classical' state concept (solution segments) and the Cauchy problem below within the dual state concept (forcing terms). A diagonal relation is actually given by functional analytic duality theory.

The vertical relations between the four Cauchy problems above may also be described through the structural operators $\mathfrak{F}$ and $\mathfrak{F}^*$. In particular, it follows from Theorem 5.9, that $\hat{\hat{x}}(t) = \mathfrak{F}\,\hat{x}(t)$, $t \geq o$, defines a mild solution of Σ^{T*} if $\hat{x}(t)$, $t \geq o$, is a mild solution of Σ. This fact is also a consequence of Theorem 5.5 together with the following relations between the various input/output operators by means of the structural operator $\mathfrak{F}$.

<u>Proposition 5.12</u>

$$\mathcal{B}^{T*} = \mathfrak{F}\mathcal{B}, \qquad \mathcal{C}^* = \mathfrak{F}^*\mathcal{C}^T$$

$$\mathcal{C} = \mathcal{C}^{T*}\mathfrak{F}, \qquad \mathcal{B}^T = \mathcal{B}^*\mathfrak{F}^* \quad .$$

<u>Proof</u> Let us first consider $\mathfrak{F}$ as an operator from $\mathcal{W}$ into $\mathcal{V}^{T*}$ (Proposition 5.8) and let $\phi \in \mathcal{W}$. Then $\mathfrak{F}\phi \in \mathcal{V}^{T*}$ and

$$\mathcal{C}^{T*}\mathfrak{F}\phi = [\mathfrak{F}\phi]^2(o) = \int_o^h d\gamma(\tau)\phi^1(-\tau) = C\phi^1 = \mathcal{C}\phi.$$

The equation $\mathcal{B}^T = \mathcal{B}^*\mathfrak{F}^*$ can be established analogously by the use of Theorem 5.5 (iv) and the remaining assertions of the proposition follow by duality. ▨

Finally, note that the Cauchy problems Σ, Σ^T, Σ^{T*} and Σ^* may also be understood in a strong sense. In particular, if $\phi \in \mathcal{W}$ and $u(\cdot) \in L^2_{loc}[o,\infty; \mathbb{R}^m]$, then it can be shown that the corresponding mild solution $\hat{x}(t)$ of Σ is in fact a strong solution. This means that $\hat{x}(t)$, $t \geq o$, is a continuous function with values in $\mathcal{W}$, that its derivative exists as a locally square (Bochner -) integrable function with values in the larger space $\mathcal{V}$, and that the first equation in Σ is satisfied in the Hilbert space $\mathcal{V}$ for almost every $t \geq o$ (SALAMON [40, Theorem I.3.4]). In order to make this rigorous, we need the fact that $\mathcal{A}$ can be interpreted as a bounded operator from $\mathcal{W}$ to $\mathcal{V}$. This means that $\mathcal{W}$ is the domain of $\mathcal{A}$ when $\mathcal{A}$ is regarded as an unbounded, closed operator on $\mathcal{V}$.

__Proposition 5.13__

$$\mathcal{W} = \mathcal{D}_{\mathcal{V}}(\mathcal{A}), \qquad \mathcal{W}^T = \mathcal{D}_{\mathcal{V}^T}(\mathcal{A}^T),$$

$$\mathcal{V}^{T*} = \mathcal{D}_{\mathcal{W}^{T*}}(\mathcal{A}^{T*}), \qquad \mathcal{V}^{*} = \mathcal{D}_{\mathcal{W}^{*}}(\mathcal{A}^{*}) \quad .$$

__Proof__ First note that $\mathcal{Z}^T := \mathcal{D}_{\mathcal{X}^T}(\mathcal{A}^T) \subset \mathcal{W}^T$ and hence $\mathcal{D}_{\mathcal{W}^{T*}}(\mathcal{A}^{T*}) \subset \mathcal{D}_{\mathcal{Z}^{T*}}(\mathcal{A}^{T*}) = \mathcal{X}^{T*}$ (see Remark 2.3)
Now let $f \in \mathcal{X}^{T*}$. Then $f \in \mathcal{D}_{\mathcal{W}^{T*}}(\mathcal{A}^{T*})$ if and only if the map

$$\psi \longrightarrow \langle \mathcal{A}^T\psi, f \rangle_{\mathcal{W}^T, \mathcal{W}^{T*}}, \quad \psi \in \mathcal{D}_{\mathcal{W}^T}(\mathcal{A}^T)$$

extends to a bounded linear functional on $\mathcal{W}^T$. But $\psi \in \mathcal{D}_{\mathcal{W}^T}(\mathcal{A}^T)$ if and only if $\psi^1 \in W^{2,2}(-h,o;\mathbb{R}^n)$, $\psi^2 \in W^{1,2}(-h,o;\mathbb{R}^p)$, $\psi^o = \psi^1(o)$, $\psi^2(o) = 0$, and $\dot{\psi}^1(o) = L^T\psi^1 + C^T\psi^2$; and the following equation holds for every $\psi \in \mathcal{D}_{\mathcal{W}^T}(\mathcal{A}^T)$

$$\begin{aligned}
&\langle \mathcal{A}^T\psi, f\rangle_{\mathcal{W}^T, \mathcal{W}^{T*}} \\
&= \langle \mathcal{A}^T\psi, f\rangle_{\mathcal{X}^T, \mathcal{X}^{T*}} \\
&= \int_o^h \psi^{1T}(-\tau)d\eta(\tau)f^o + \int_o^h \psi^{2T}(-\tau)d\gamma(\tau)f^o \\
&+ \int_o^h \dot{\psi}^{1T}(-s)f^1(s)ds + \int_o^h \dot{\psi}^{2T}(-s)f^2(s)ds \\
&= \int_o^h \psi^{1T}(-\tau)d\eta(\tau)f^o + \int_o^h \dot{\psi}^{1T}(-s)f^1(s)ds \\
&+ \int_o^h \dot{\psi}^{2T}(-s)[f^2(s) + \gamma(s)f^o - \gamma(h)f^o]ds
\end{aligned}$$

(compare the proof of Proposition 5.4). The latter expression defines a bounded linear functional on $\mathcal{W}^T = \{\psi \in \mathcal{X}^T \mid \psi^1 \in W^{1,2}(-h,o;\mathbb{R}^n),\ \psi^o = \psi^1(o)\}$ if and only if there exists a $d \in L^2(o,h;\mathbb{R}^p)$ such that the following equation holds for every $\psi^2 \in W^{1,2}(-h,o;\mathbb{R}^p)$ satisfying $\psi^2(o) = o$

$$\int_o^h \dot{\psi}^{2T}(-s)[f^2(s) + \gamma(s)f^o - \gamma(h)f^o]ds$$

$$= \int_o^h \psi^{2T}(-s)d(s)ds = -\int_o^h \dot{\psi}^{2T}(-s)\int_s^h d(\sigma)d\sigma ds.$$

This is equivalent to $f \in \mathcal{V}^{T*}$. We conclude that $\mathcal{V}^{T*} = \mathcal{D}_{\mathcal{W}^{T*}}(\mathcal{A}^{T*})$.

Analogous arguments show that $\mathcal{V}^* = \mathcal{D}_{\mathcal{W}^*}(\mathcal{A}^*)$. The remaining assertion of Proposition 5.13 follow by duality. ▨

6. THE LINEAR QUADRATIC OPTIMAL CONTROL PROBLEM

6.1 *The finite time case*

In the previous section we have developed two state space descriptions for the RFDE (5.1). Moreover, we have shown that the corresponding Cauchy problems Σ_1 and Σ_1^{T*} both satisfy the hypotheses (H1), (H2) and (H3) of Section 2 in suitably chosen Hilbert spaces (Remark 5.10 and Remark 5.11). This allows us to apply the results of Section 2 to the RFDE (5.1) within the state concepts of Sub-section 5.2 . For this sake we consider the cost functional

$$(6.1) \qquad \mathcal{J}_T(u) = \int_o^T [\|y(t)\|^2_{\mathbb{R}^p} + u^T(t)\, R\, u(t)]dt$$

associated with the systems (5.1) and (5.5) where $R \in \mathbb{R}^{m\times m}$ is a positive definite matrix.

Remarks 6.1 For simplicity, we assume there is no weight on the final state $\hat{x}(T)$ respectively $\hat{\hat{x}}(T)$ in the cost functional $\mathcal{J}_T(u)$. Such a weight could be introduced by means of a semi-definite operator $\mathcal{G}: \mathcal{W}^{T*} \to \mathcal{W}^{T}$ leading to the additional term $\langle \hat{\hat{x}}(t), \mathcal{G}\hat{\hat{x}}(t) \rangle_{\mathcal{W}^{T*}\mathcal{W}^{T}}$ in the performance index $\mathcal{J}_T(u)$. However, such a term could never be of the form $x^T(T)\,\mathcal{G}_o\,x(t)$ with some non-zero positive semi-definite matrix $\mathcal{G}_o \in \mathbb{R}^{n\times n}$ since the map $f \longrightarrow f^o$ from $\mathcal{X}^{T*}$ into $\mathbb{R}^n$ cannot be extended to a bounded linear functional on $\mathcal{W}^{T*}$.

The following result is now a direct consequence of Theorem 2.6 and Proposition 2.7.

<u>*Theorem 6.2*</u>

(i) There exists a unique *strongly continuous* operator family $\pi(t) \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$, $0 \leq t \leq T$, such that the function $\pi(t)\phi$ *is* continuously differentiable in $\mathcal{W}^*$ for every $\phi \in \mathcal{W}$ and satisfies the equation

$$\frac{d}{dt}\pi(t)\phi + \mathcal{A}^*\pi(t)\phi + \pi(t)\,\mathcal{A}\,\phi$$

$$(6.2) \qquad - \pi(t)\,\mathcal{B}R^{-1}\mathcal{B}^*\pi(t)\phi + \mathcal{C}^*\mathcal{C}\phi = 0,$$

$$\pi(T)\phi = 0.$$

(ii) There exists a unique strongly continuous operator family $\mathcal{P}(t) \in \mathcal{L}(\mathcal{W}^{T*}, \mathcal{W}^{T})$, $0 \leq t \leq T$, such that the function $\mathcal{P}(t)f$ is continuously differentiable in $\mathcal{V}^{T}$ for every $f \in \mathcal{V}^{T*}$ and satisfies the equation

$$\frac{d}{dt}\mathcal{P}(t)f + \mathcal{A}^T\mathcal{P}(t)f + \mathcal{P}(t)\,\mathcal{A}^{T*}f$$

$$(6.3) \qquad - \mathcal{P}(t)\,\mathcal{B}^{T*}R^{-1}\mathcal{B}^{T}\mathcal{P}(t)f + \mathcal{C}^{T}\mathcal{C}^{T*}f = 0,$$

$$\mathcal{P}(T)f = 0.$$

(iii) If $\pi(t) \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$ and $\wp(t) \in \mathcal{L}(\mathcal{W}^{T*}, \mathcal{W}^T)$, $0 \leq t \leq T$, are the solution operators of (6.2) and (6.3), then

$$\pi(t) = \mathcal{J}^* \wp(t) \mathcal{J}, \quad 0 \leq t \leq T. \tag{6.4}$$

(iv) There exists a unique optimal control which minimizes the performance index (6.1) subject to (5.1) and (5.2). This optimal control is given by the feedback control low

$$\begin{aligned} u(t) &= -R^{-1} \mathcal{B}^* \pi(t) \hat{x}(t) \\ &= -R^{-1} \beta^T \wp(t) \hat{\hat{x}}(t) \\ &= -R^{-1} \mathcal{B}^T \wp(t) \mathcal{J} \hat{x}(t) \end{aligned} \tag{6.5}$$

Where $\pi(t) \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$ and $\wp(t) \in \mathcal{L}(\mathcal{W}^{T*}, \mathcal{W}^T)$ are given by (6.2) and (6.3). The optimal cost corresponding to the initial state $\phi \in \mathcal{X}$ is

$$\begin{aligned} J_T(u) &= \langle \phi, \pi(0)\phi \rangle_{\mathcal{X}, \mathcal{X}^*} \\ &= \langle f, \wp(o) f \rangle_{\mathcal{X}^{T*}, \mathcal{X}^T} \end{aligned} \tag{6.6}$$

where $f = \mathcal{J}\phi \in \mathcal{X}^{T*}$ is the initial state of (5.5.1).

Proof

The statements (i), (ii) and (iv) follow immediately from Theorem 2.6 and Proposition 2.7. In order to prove (iii), let $\wp(t) \in \mathcal{L}(\mathcal{W}^{T*}, \mathcal{W}^T)$ be the unique solution of (6.3) and let $\pi(t) \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$ be defined by (6.4).

Moreover, let $\phi \in \mathcal{W}$ and $f := \mathcal{F}\phi \in \mathcal{V}^{T*}$. Then - since $\mathcal{F}^* \in \mathcal{L}(\mathcal{V}^T, \mathcal{W}^*)$ - the function $\pi(t)\phi = \mathcal{F}^* \mathcal{P}(t)f$, $0 \leq t \leq T$, is continuously differentiable with values in $\mathcal{W}^*$ and satisfies the following equation

$$
\begin{aligned}
&\frac{d}{dt}\pi(t)\phi + \mathcal{A}^*\pi(t)\phi + \pi(t)\mathcal{A}\phi \\
&- \pi(t)\mathcal{B}R^{-1}\mathcal{B}^*\pi(t)\phi + \mathcal{C}^*\mathcal{C}\phi \\
&= \frac{d}{dt}\mathcal{F}^*\mathcal{P}(t)f + \mathcal{A}^*\mathcal{F}^*\mathcal{P}(t)f + \mathcal{F}^*\mathcal{P}(t)\mathcal{F}\mathcal{A}\phi \\
&- \mathcal{F}^*\mathcal{P}(t)\mathcal{F}\mathcal{B}R^{-1}\mathcal{B}^*\mathcal{F}^*\mathcal{P}(t)\mathcal{F}\phi + \mathcal{F}^*\mathcal{C}^T\mathcal{C}^{T*}\mathcal{F}\phi \\
&= \mathcal{F}^*\Big[\frac{d}{dt}\mathcal{P}(t)f + \mathcal{A}^T\mathcal{P}(t)f + \mathcal{P}(t)\mathcal{A}^{T*}f \\
&- \mathcal{P}(t)\mathcal{B}^{T*}R^{-1}\mathcal{B}^T\mathcal{P}(t)f + \mathcal{C}^T\mathcal{C}^{T*}f\Big] \\
&= 0
\end{aligned}
$$

(See Theorem 5.5 and Proposition 5.12). Now statement (iii) follows from the uniqueness of the solution of (6.2). $\boxed{}$

Note that an analogous relation as (6.4) has been shown in DELFOUR-LEE-MANITIUS [14] and VINTER-KWONG [41] for RFDES with undelayed input/output - variables.

6.2 Stabilizability and detectability

In this section we investigate the sufficient conditions (H4), (H5) for the unique solvability of the algebraic Riccati equation (Chapter 3) in the case of the systems Σ and Σ^{T*}. We will not consider these hypotheses in their weakest form but have a look at the slightly stronger properties of stabilizability and detectability.

Definition 6.3

(i) System Σ is said to be stabilizable if there exists a feedback operator $K \in \mathcal{L}(\mathcal{V}, \mathbb{R}^m)$ such that the closed loop semigroup $\mathcal{S}_K(t) \in \mathcal{L}(\mathcal{V})$ defined by

$$(6.7) \qquad \mathcal{S}_K(t)\phi = \mathcal{S}(t)\phi + \int_0^t \mathcal{S}(t-s)\,\mathcal{B}\,\mathcal{S}_K(s)\phi\,ds$$

for $t \geq 0$ and $\phi \in \mathcal{V}$ is exponentially stable.

(ii) System Σ^{T*} is said to be stabilizable if there exists a feedback operator $\mathcal{K}^{T*} \in \mathcal{L}(\mathcal{W}^{T*}; \mathbb{R}^m)$ such that the closed loop semigroup $\mathcal{S}_K^{T*}(t) \in \mathcal{L}(\mathcal{W}^{T*})$ defined by

$$(6.8) \qquad \mathcal{S}_K^{T*}(t)f = \mathcal{S}^{T*}(t)f + \int_0^t \mathcal{S}^{T*}(t-s)\,\mathcal{B}^{T*}\mathcal{K}^{T*}\mathcal{S}_K^{T*}(s)f\,ds$$

for $t \geq 0$ and $f \in \mathcal{W}^{T*}$ is exponentially stable.

Remarks 6.4

(i) Note that the integral term in (6.7) is a bounded linear operator from $\mathcal{V}$ to $\mathcal{W}$ (Remark 5.10) and hence $\mathcal{S}_K(t)$ is also a strongly continuous semigroup on $\mathcal{X}$ and $\mathcal{W}$.

(ii) It follows from Remark (5.6) (iii) that for every $t \geq h$.

$$\mathcal{S}_K(t) \in \mathcal{L}(\mathcal{V},\mathcal{X}) \cap \mathcal{L}(\mathcal{X},\mathcal{W}).$$

(iii) The stability of the semigroup $\mathcal{S}_K(t)$ is independent of the choice of the state space $\mathcal{V}$, $\mathcal{X}$ or $\mathcal{W}$. In order to see this, note that the operator $\mu I - \mathcal{A} - \mathcal{B}\mathcal{K} : \mathcal{W} \to \mathcal{V}$ provides a similarity action between $\mathcal{S}_K(t) \in \mathcal{L}(\mathcal{W})$ and $\mathcal{S}_K(t) \in \mathcal{L}(\mathcal{V})$ if $\mu > 0$ is sufficiently large. Moreover, it follows from (ii) that the stability of $\mathcal{S}_K(t)$ on the Hilbert space $\mathcal{W}$ implies the stability on $\mathcal{X}$ and the stability on $\mathcal{X}$ implies the stability on $\mathcal{V}$.

(iv) The same arguments as above show that the closed loop semigroup $\mathcal{S}_K^{T*}(t) \in \mathcal{L}(\mathcal{W}^{T*})$ can be restricted to a semigroup on $\mathcal{X}^{T*}$ or $\mathcal{V}^{T*}$ and that its stability is independent of the choice of the state space

$\mathcal{W}^{T*}$, $\mathcal{X}^{T*}$ or $\mathcal{V}^{T*}$.

(v) Let $\mathcal{K}^{T*} \in \mathcal{L}(\mathcal{W}^{T*}, \mathbb{R}^m)$ be given and define

$$\mathcal{K} = \mathcal{K}^{T*}\mathcal{F} \in \mathcal{L}(\mathcal{V}, \mathbb{R}^m). \tag{6.9}$$

Then the following equation holds for every $t \geq 0$

$$(6.10) \qquad \mathcal{F}\, \mathcal{S}_K(t) = \mathcal{S}_K^{T*}(t)\, \mathcal{F}\,.$$

In fact, it follows from Theorem 5.5 and Proposition 5.12 that for every $\phi \in \mathcal{V}$ the function $\hat{\hat{x}}(t) = \mathcal{F}\, \mathcal{S}_K(t)\phi \in \mathcal{W}^{T*}$, $t \geq o$, defines a solution of (6.8) with $f = \mathcal{F}\,\phi$.

(vi) Every $\mathcal{K}^{T*} \in \mathcal{L}(\mathcal{W}^{T*}, \mathbb{R}^m)$ can be represented as

$$(6.11) \qquad \mathcal{K}^{T*}f = K_o f^o + \int_o^h K_1(-s)f(s)ds + \int_o^h K_2(-s)f^2(s)ds$$

where $K_2(\cdot) \in L^2(-h,o;\, \mathbb{R}^{m\times p})$, $\quad K_1(\cdot) \in W^{1,2}(-h,o;\, \mathbb{R}^{m\times n})$ and $K_o = K_1(o)$. Moreover, let us again suppose that $\mathcal{K} = \mathcal{K}^{T*}\mathcal{F}$ and consider the control law

$$(6.12) \qquad \begin{aligned} u(t) &= \mathcal{K}^{T*}\mathcal{F}\, \hat{x}(t) \\ &= K_o x(t) + \int_o^h\int_o^\tau K_1(s-\tau)d\eta(\tau)x(t-s)ds \\ &+ \int_o^h\int_o^\tau K_2(s-\tau)d\gamma(\tau)x(t-s)ds \\ &+ \int_o^h\int_o^\tau K_1(s-\tau)d\beta(\tau)u(t-s)ds \end{aligned}$$

for system (5.1). Then it follows from Equation (6.7) and Theorem 5.9 that for every solution pair $x(\cdot) \in L^2_{loc}(-h,\infty;\, \mathbb{R}^n) \cap W^{1,2}_{loc}(o,\infty;\, \mathbb{R}^n)$, $u(\cdot) \in L^2_{loc}(o,\infty;\, \mathbb{R}^m)$ of (5.1), (5.2), (6.12) the corresponding state

$\hat{x}(t) = (x(t), x_t, u_t) \in \mathcal{X}$ at time $t \geq o$ is given by

(6.13) $\hat{x}(t) = \mathcal{S}_K(t)\phi$

By (6.10), this implies that $\hat{\hat{x}}(t) = \mathcal{F}\,\hat{x}(t) \in \mathcal{X}^{T*}$ is given by $\hat{\hat{x}}(t) = \mathcal{S}_K^{T*}(t)\mathcal{F}\phi$.

(vii) If $\mathcal{K} \in \mathcal{L}(\mathcal{V}, \mathbb{R}^m)$ is given by (6.9) then the exponential stability of $\mathcal{S}_K(t)$ on $\mathcal{X}$ is equivalent to that of $\mathcal{S}_K^{T*}(t)$ on $\mathcal{X}^{T*}$. In fact , it follows from Equation (6.7) and Theorem 5.9 that

(6.14) range $\mathcal{S}_K^{T*}(h) \subset$ range $\mathcal{F}$

and hence Equation (6.10) shows that the stability of $\mathcal{S}_K(t)$ implies that of $\mathcal{S}_K^{T*}(t)$. The converse implication is a consequence of the fact that $\mathcal{S}_K(t)\phi = (x(t), x_t, u_t)$ and

$$x(t) = [\,\mathcal{S}_K^{T*}(t)\mathcal{F}\phi]^0, \quad u(t) = \mathcal{K}^{T*}\mathcal{S}_K^{T*}(t)\phi, \quad t \geq 0,$$

for every solution pair $x(t), u(t)$, $t \geq -h$, of the closed loop system (5.1), (5.2), (6.12) with $\phi \in \mathcal{X}$.

Having collected the basic properties of the feedback semigroups $\mathcal{S}_K(t)$ and $\mathcal{S}_K^{T*}(t)$, we are now in the position to prove the following stabilizability criterion.

<u>Theorem 6.5</u>

The following statements are equivalent.

(i) System Σ is stabilizable.

(ii) There exists a feedback operator $\mathcal{K} \in \mathcal{L}(\mathcal{W}, \mathbb{R}^m)$ such that the closed loop semigroup $\mathcal{S}_K(t) \in \mathcal{L}(\mathcal{W})$ defined by (6.7) for $t \geq 0$ and $\phi \in \mathcal{W}$ is exponentially stable.

(iii) System Σ^{T*} is stabilizable.

(iv) There exists a feedback operator $\mathcal{K}^{T*} \in \mathcal{L}(\mathcal{V}^{T*}, \mathbb{R}^m)$ such that the closed loop semigroup $\mathcal{S}_K^{T*}(t) \in \mathcal{L}(\mathcal{V}^{T*})$, defined by (6.8) for $t \geq o$ and $f \in \mathcal{V}^{T*}$ is exponentially stable.

(v) For every $\lambda \in \mathbb{C}$, $\mathrm{Re}\,\lambda \geq o$,

$$\mathrm{rank}\,[\Delta(\lambda),\ \mathcal{B}(e^{\lambda\cdot})] = n.$$

<u>Proof</u>

The implications "(iii) $\Rightarrow$ (i) $\Rightarrow$ (ii)" and "(iii) $\Rightarrow$ (iv)" follow from Remark 6.4.

Now we will prove that (v) implies (iii). Note that it has been shown in SALAMON [40 , Theorem 5.2.11 and Corollary 5.3.3] that (v) implies the existence of a stabilizing control law of the form (6.12) for the system (5.1) where $K_1(\cdot) \in W^{1,2}[-h,o;\mathbb{R}^{mxn}]$, $K_o = K_1(o)$ and $K_2(\tau) \equiv 0$. This means that every solution pair $x(t)$, $u(t)$, $t \geq -h$, of (5.1), (5.2), (6.12) with $\phi \in \mathcal{X}$ tends to zero with an exponential decay rate which is independent of ϕ. This shows that the semigroup $\mathcal{S}_K^{T*}(t)$ is stable on $\mathcal{X}^{T*}$ (Remark 6.4 (vi)) and hence on $\mathcal{W}^{T*}$ (Remark 6.4 (iv)).

It remains to show that (ii) and (iv) imply (v). For this sake assume that there exists a $\lambda \in \mathbb{C}$, $\mathrm{Re}\,\lambda \geq o$, and a non-zero vector $x_o \in \mathbb{C}^n$ such that $x_o^T\Delta(\lambda) = o$ and $x_o^T B(e^{\lambda\cdot}) = o$ and define $\psi := (x_o, e^{\lambda\cdot}x_o, 0) \in \mathcal{W}^T$. Then it is easy to see that $\mathcal{A}^T\psi = \lambda\psi$, $\mathcal{B}^T\psi = 0$ and hence $\mathcal{A}^*\mathcal{Y}^*\psi = \lambda\,\mathcal{Y}^*\psi$, $\mathcal{B}^*\mathcal{Y}^*\psi = 0$ (Theorem 5.5 and Proposition 5.12). Now equation (6.8) and (6.7) show that $\mathcal{S}_K^T(t)\psi = \mathcal{S}^T(t)\psi = e^{\lambda t}\psi$ and $\mathcal{S}_K^*(t)\mathcal{Y}^*\psi = \mathcal{S}^*(t)\mathcal{Y}^*\psi = e^{\lambda t}\mathcal{Y}^*\psi$ for every $\mathcal{K}^{T*} \in \mathcal{L}(\mathcal{V}^{T*}, \mathbb{R}^m)$ and every $\mathcal{K} \in \mathcal{L}(\mathcal{W}, \mathbb{R}^m)$. Since $\psi \neq o$ and $\mathcal{Y}^*\psi \neq o$, this shows that (ii) and (iv) are not satisfied. $\boxed{/\!/\!/}$

The next result is obtained by dualising Theorem 6.5.

Corollary 6.6

The following statements are equivalent

(i) System Σ is detectable in the sense that there exists an output injection operator $\mathcal{H} \in \mathcal{L}(\mathbb{R}^p, \mathcal{V})$ such that the closed loop semigroup $\mathcal{S}_{\mathcal{H}}(t) \in \mathcal{L}(\mathcal{V})$ generated by $\mathcal{A} + \mathcal{H}\mathcal{C}: \mathcal{W} \to \mathcal{V}$ is exponentially stable.

(ii) System Σ^{T*} is detectable in the sense that there exists an output injection operator $\mathcal{H}^{T*} \in \mathcal{L}(\mathbb{R}^p, \mathcal{W}^{T*})$ such that the closed loop semigroup $\mathcal{S}_{\mathcal{H}}^{T*}(t) \in \mathcal{L}(\mathcal{W}^{T*})$ generated by $\mathcal{A}^{T*} + \mathcal{H}^{T*}\mathcal{C}^{T*}: \mathcal{V}^{T*} \to \mathcal{W}^{T*}$ is exponentially stable.

(iii) For every $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda \geq o$,

$$\operatorname{rank}\begin{bmatrix} \Delta(\lambda) \\ C(e^{\lambda\cdot}) \end{bmatrix} = n$$

Note that $\mathcal{S}_{\mathcal{H}}(t) \in \mathcal{L}(\mathcal{V})$ satisfies the integral equation.

$$\mathcal{S}_{\mathcal{H}}(t)\phi = \mathcal{S}(t)\phi + \int_o^t \mathcal{S}_{\mathcal{H}}(t-s)\,\mathcal{H}\,\mathcal{C}\,\mathcal{S}(s)\phi ds \tag{6.15}$$

for every $t \geq o$ and every $\phi \in \mathcal{W}$ (see SALAMON [40 , Theorem I.3.9]) and hence can be restricted to a semigroup on $\mathcal{X}$ if $\mathcal{H} \in \mathcal{L}(\mathbb{R}^p, \mathcal{X})$. At the end of this section we give a concrete representation of the output injection semigroup $\mathcal{S}_{\mathcal{H}}(t) \in \mathcal{L}(\mathcal{X})$ by means of a closed loop functional differential equation. For this sake note that every $\mathcal{H} \in \mathcal{L}(\mathbb{R}^p, \mathcal{X})$ can be represented as

$$\mathcal{H}y = (H_o y, H_1(\cdot)y,\ H_2(\cdot)y) \in \mathcal{X}, \qquad y \in \mathbb{R}^p, \tag{6.16}$$

where $H_o \in \mathbb{R}^{nxp}$, $H_1(\cdot) \in L^2(-h,o;\ \mathbb{R}^{nxp})$, $H_2(\cdot) \in L^2(-h,o;\ \mathbb{R}^{mxp})$. Moreover, we introduce the abbreviating notation

$$H_i * \phi^2(\tau) = \int_\tau^o H_i(\tau-\sigma)\phi^2(\sigma)ds, \quad -h \leq \tau \leq o,$$

for $i = 1,2$ and $\phi^2 \in L^2(-h,o;\ \mathbb{R}^p)$.

Theorem 6.7

(i) Let $x(\cdot) \in L^2_{loc}(-h,\infty; \mathbb{R}^n) \cap W^{1,2}_{loc}(o,\infty; \mathbb{R}^n)$ satisfy the RFDE

$$(6.17) \qquad \dot{x}(t) = L(x_t + H_1 * y_t) + B(u_t + H_2 * y_t) + H_o y(t)$$

where $u(\cdot) \in L^2_{loc}(-h,\infty; \mathbb{R}^m)$ and $y(\cdot) \in L^2_{loc}(-h,\infty; \mathbb{R}^p)$. Then

$$(6.18) \quad \hat{x}(t) = (x(t), x_t + H_1 * y_t, u_t + H_2 * y_t) \in \mathcal{X}, \quad t \geq o,$$

is given by the variation-of-constants formula

$$(6.19) \qquad \hat{x}(t) = \mathcal{S}(t)\hat{x}(o) + \int_o^t \mathcal{S}(t-s)\mathcal{B}u(s)ds + \int_o^t \mathcal{S}(t-s)\mathcal{H}y(s)ds.$$

(ii) Let $x(\cdot) \in L^2_{loc}(-h,\infty; \mathbb{R}^n) \cap W^{1,2}_{loc}(o,\infty; \mathbb{R}^n)$ and $y(\cdot) \in L^2_{loc}(-h,\infty; \mathbb{R}^p)$ satisfy the equations

$$(6.20;1) \quad \dot{x}(t) = L(x_t + H_1 * y_t) + B(H_2 * y_t) + H_o y(t),$$

$$(6.20;2) \quad y(t) = C(x_t + H_1 * y_t), \quad t \geq o,$$

and let $\hat{x}(t) \in \mathcal{X}$, $t \geq o$ be defined by (6.18) with $u(t) \equiv o$. Then

$$(6.21) \qquad \hat{x}(t) = \mathcal{S}_{\mathcal{H}}(t)\hat{x}(o)$$

Proof In order to prove statement (i), let us first assume that $y(t) = o$ for $t \geq o$ and define $z(t) \in \mathbb{R}^n$, $v(t) \in \mathbb{R}^m$ for $t \geq -h$ by $z(t) = x(t)$, $v(t) = u(t)$ for $t \geq o$ and

$$z(\tau) = x(\tau) + \int_\tau^o H_1(\tau-\sigma)y(\sigma)d\sigma, \quad v(\tau) = u(\tau) + \int_\tau^o H_2(\tau-\sigma)y(\sigma)d\sigma,$$

for $-h \leq \tau \leq o$. Then it is easy to see that $\hat{x}(t) = (z(t), z_t, v_t)$ for all $t \geq o$ and hence the following equation holds

$$\begin{aligned}\dot{z}(t) &= \dot{x}(t)\\ &= L(x_t+H_1*y_t) + B(u_t+H_2*y_t) + H_o y(t)\\ &= L\, z_t + Bv_t\,, \quad t \geq o.\end{aligned}$$

This implies

$$\hat{x}(t) = (z(t), z_t, v_t) = \mathcal{J}(t)\hat{x}(o) + \int_o^t \mathcal{J}(t-s)\,\mathcal{B}\,u(s)ds.$$

Secondly, let $u(t) \equiv o$ and $x(\tau) = o$, $y(\tau) = o$ for $-h \leq \tau \leq o$. Moreover, let $Z(t) \in \mathbb{R}^{nxp}$, $V(t) \in \mathbb{R}^{mxp}$, $t \leq -h$, be the unique solution of $\dot{Z}(t) = LZ_t + BV_t$ corresponding to the input $V(t) = 0$, $t \geq o$, and the initial condition $Z(o) = H_o$, $Z(\tau) = H_1(\tau), V(\tau) = H_2(\tau)$, $-h \leq \tau < o$. Then

$$(6.22) \qquad (Z(t), Z_t, V_t) = \mathcal{J}(t)\,\mathcal{H} \in \mathcal{L}(\mathbb{R}^p, \mathcal{Y}), \quad t \geq o.$$

Now let us define

$$\begin{aligned} z(t) &= \int_o^t Z(t-s)y(s)ds\,, \qquad z(\tau) = o\,,\\ z(t,\tau) &= \int_o^t Z(t-s+\tau)y(s)ds,\\ v(t,\tau) &= \int_o^t V(t-s+\tau)y(s)ds,\end{aligned}$$

for $t \geq o$ and $-h \leq \tau \leq o$. Then we obtain

$$(6.23) \qquad \begin{aligned} z(t,\cdot) &= z_t + H_1*y_t \in \mathcal{C}(-h,o;\mathbb{R}^n)\\ V(t,\cdot) &= H_2*y_t \in L^2(-h,o;\mathbb{R}^m)\end{aligned}$$

and hence

$$\dot{z}(t) = \int_o^t \dot{Z}(t-s)y(s)ds + Z(o)y(t)$$

$$= \int_o^h d\eta(\tau) \int_o^t Z(t-s-\tau)y(s)ds$$

$$+ \int_o^h d\beta(\tau) \int_o^t V(t-s-\tau)y(s)ds + Z(o)y(t)$$

$$= L(z_t + H_1 * y_t) + B(H_2 * y_t) + H_o y(t)$$

for $t \geq o$. This implies that $x(t) = z(t)$ for $t \geq -h$. Thus it follows from (6.22) and (6.23) that

$$\hat{x}(t) = (x(t), x_t + H_1 * y_t, H_2 * y_t)$$

$$= (z(t), z(t,\cdot), v(t,\cdot))$$

$$= \int_o^t (Z(t-s), Z_{t-s}, V_{t-s})y(s)ds$$

$$= \int_o^t \mathcal{S}(t-s)\, \mathcal{H}\, y(s)ds.$$

This proves statement (i)

In order to prove statement (ii), let us assume that $x(\cdot) \in L^2_{loc}(-h,\infty; \mathbb{R}^n) \cap W^{1,2}_{loc}(o,\infty\ \mathbb{R}^n)$ and $y(\cdot) \cdot L^2_{loc}(-h,\infty; \mathbb{R}^p)$ satisfy (6.20) and that $\hat{x}(t) \in \mathcal{X}$ is defined by (6.18) with $u(\cdot) \equiv 0$. Moreover suppose that $\phi = \hat{x}(o) \in \mathcal{V}$. Then $y(t)$, $t \geq o$, satisfies the Volterra integral equation

$$
\begin{aligned}
y(t) &= C(x_t + H_1 * y_t) \\
&= \int_t^h d\gamma(\tau)\, \phi^1\,(t-\tau) \int_0^t d\gamma(\tau)\, x\,(t-\tau) \\
&+ \int_0^t \Big[\int_s^t d\gamma(\tau)\, H_1\,(s-\tau)\Big]\, y(t-s)ds, \quad t \geq o,
\end{aligned}
$$

with forcing term in $W^{1,2}_{loc}\,(0, \infty; \mathbb{R}^p)$. This implies that $y(\cdot) \in W^{1,2}_{loc}\,(0, \infty; \mathbb{R}^p)$ and hence, by (i),

$$
\hat{x}(t) = \mathcal{J}(t)\,\hat{x}(o) + \int_0^t \mathcal{J}(t-s)\mathcal{H}\,y(s)ds \in \mathcal{W} = \mathcal{D}_{\mathcal{V}}(\mathcal{A}),
$$

for every $t \geq o$. Moreover it follows from a general semigroup theoretic result that $\hat{x}(t)$ is continuously differentiable in $\mathcal{V}$ and satisfies

$$
\frac{d}{dt}\,\hat{x}(t) = \mathcal{A}\,\hat{x}(t) + \mathcal{H}\,y(t) = (\,\mathcal{A} + \mathcal{H}\,\mathcal{C}\,)\hat{x}(t), \quad t \geq o
$$

This proves (6.21) for the case $\hat{x}(o) \in \mathcal{W}$. In general (6.21) follows from the fact that both sides of this equation depend continuously on the initial functions $(x(o), x_o, y_o) \in \mathcal{X}^T$ of (6.20). (for existence and uniqueness results for this type of equations see SALAMON [40, section 1.2]). $\boxed{//}$

Finally, note that the transposed equation of (6.20) takes the form

$$\dot{z}(t) = L^T z_t + \mathcal{C}^T v_t, \tag{6.24;1}$$

$$v(t) = H_o^T z(t) + \int_0^h \int_0^\tau H_1^T(-s)\, d\eta^T(\tau) z(t+s-\tau) ds$$

$$+ \int_0^h \int_0^\tau H_2^T(-s)\, d\beta^T(\tau) z(t+s-\tau) ds \tag{6.24;2}$$

$$+ \int_0^h \int_0^\tau H_1^T(-s)\, d\gamma^T(\tau)\, v(t+s-\tau) ds\ .$$

This is nothing more than the transposed RFDE (5.3) with a control law which is analogous to (6.12).

6.3. The infinite time case

In this subsection we consider the performance index

$$\mathcal{J}(u) = \int_0^\infty [\,||\, y(t)\,||^2 + u^T(t)\, Ru(t)\,] dt \tag{6.25}$$

associated with the Cauchy problems Σ and Σ^{T*} where $R \in \mathbb{R}^{m \times m}$ is a positive definite matrix.

Combining the results of section 3 (Theorem 3.3 and Theorem 3.4) and of the previous subsection (Theorem 6.5 and Corollary 6.6) we obtain the facts which are summarised in the theorem below.

Theorem 6.8

(i) If

$$\operatorname{rank}\,[\Delta(\lambda),\, B(e^{\lambda \cdot})] = n \quad \forall \lambda \in \mathbb{C},\ \mathrm{Re}\lambda \geq 0, \tag{6.26}$$

then there exist positive semi-definite operators $\pi \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$ and $\mathcal{P} \in \mathcal{L}(\mathcal{W}^{T*}, \mathcal{W}^T)$ satisfying the algebraic Riccati equations.

$$\text{(6.27)} \qquad \pi\phi + \mathcal{A}^*\pi\phi + \pi\mathcal{A}\phi - \pi\mathcal{B}R^{-1}\mathcal{B}^*\pi\phi + \mathcal{C}^*\mathcal{C}\phi = 0$$

$(\phi \in \mathcal{W})$, respectively

$$\text{(6.28)} \qquad \mathcal{P} f + \mathcal{A}^T\mathcal{P} f + \mathcal{P}\mathcal{A}^{T*} f - \mathcal{P}\mathcal{B}^{T*}R^{-1}\mathcal{B}^T\mathcal{P} f + \mathcal{C}^T\mathcal{C}^{T*} f = 0$$

$(f \in \mathcal{W}^{T*})$. The minimal solutions π of (6.27) and $\mathcal{P}$ of (6.28) satisfy the relation

$$\text{(6.29)} \qquad \pi = \mathcal{F}^*\mathcal{P}\mathcal{F}$$

(ii) If (6.26) is satisfied, then there exists a unique optimal control $u(\cdot) \in L^2_{loc}(o, \infty; \mathbb{R}^m)$ which minimizes the performance index (6.25) subject to (5.1), (5.2). This optimal control is given by the feedback law

$$\begin{aligned} u(t) &= -R^{-1}\mathcal{B}^*\pi\hat{x}(t) \\ &= -R^{-1}\mathcal{B}^T\mathcal{P}\mathcal{F}\hat{x}(t) \qquad \text{(6.30)} \\ &= -R^{-1}\mathcal{B}^T\mathcal{P}\hat{\hat{x}}(t) \end{aligned}$$

where π respectively $\mathcal{P}$ is the minimal solution of (6.27) respectively (6.28). The optimal cost corresponding to the initial state $\phi \in \mathcal{X}$ is given by

$$\text{(6.31)} \qquad \mathcal{J}(u) = \langle\phi, \pi\phi\rangle = \langle f, \mathcal{P} f\rangle$$

where $f = \mathcal{F}\phi \in \mathcal{X}^{T*}$ is the initial state of (5.5).

AD-A139 333 THE LINEAR QUADRATIC OPTIMAL CONTROL PROBLEM FOR INFINITE DIMENSIONAL SYS..(U) WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER A J PRITCHARD ET AL. JAN 84 MRC-TSR-2624 DAAG29-80-C-0041 F/G 12/1

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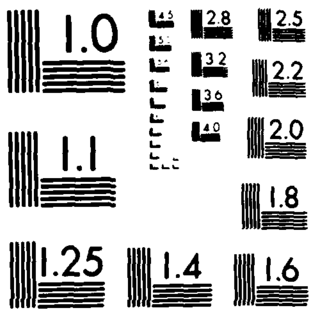

MICROCOPY RESOLUTION TEST CHART
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(iii) If

(6.32) $\quad \operatorname{rank} \begin{bmatrix} \Delta(\lambda) \\ C(e^{\lambda \cdot}) \end{bmatrix} = n \quad \forall \lambda \in \mathbb{C},\ \mathrm{Re}\lambda \geq 0$, then the algebraic

Riccati equation (6.27) respectively (6.28) has at most one self adjoint, nonnegative solution $\pi \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$ respectively $\mathcal{P} \in \mathcal{L}(\mathcal{W}^{T*}, \mathcal{W}^T)$. Moreover, if π respectively $\mathcal{P}$ is such a solution, then the closed loop semigroup $\mathcal{S}_\pi(t) \in \mathcal{L}(\mathcal{V})$, generated by $\mathcal{Q} - \beta R^{-1}\beta^*\pi$ respectively $\mathcal{S}_{\mathcal{P}}^{T*}(t) \in \mathcal{L}(\mathcal{W}^{T*})$ generated by $\mathcal{Q}^{T*} - \beta^{T*}R^{-1}\beta^T\mathcal{P}$ is exponentially stable.

REMARKS ON THE LITERATURE

The linear quadratic control problem for RFDEs with undelayed i/o-variables has been extensively studied by many authors, see e.g. KUSHNER-BARNEA [30], ALEKAL-BRUNOVSKY-CHYUNG-LEE [1], CURTAIN [8], DELFOUR-MITTER [11], MANITIUS [36], DELFOUR-McCALLA-MITTER [13], DELFOUR-LEE-MANITIUS [14], DELFOUR [15], BANKS-BURNS [3]. First results on systems with a single point delay in the state and control variables can be found in KOIVO-LEE [27], KWONG [28] and some further ideas in this direction in ICHIKAWA [22], [23]. However, an evolution equation approach to this problem has only recently been developed by VINTER and KWONG [41] for RFDEs with distributed delays in the control variable. This approach has been generalized to RFDEs with general delays in the state-and-control variables by DELFOUR [16], [17], [18]. Some results on the finite time linear quadratic control problem for RFDEs with a single point delay in state, control and observation can be found in the recent paper of LEE [33] and FERNANDEZ-BERDAGUER-LEE [20]. However, they use different methods and have much more restrictive assumptions.

We have derived the solution to our infinite time optimal control problem via the positive semi-definite solution $\mathcal{P} \in \mathcal{L}(\mathcal{W}^{T*}, \mathcal{W}^{T})$ respectively $\pi \in \mathcal{L}(\mathcal{V}, \mathcal{V}^{*})$ of the algebraic Riccati equation (6.28) respectively (6.27). Therefore it would be extremely interesting to have a detailed characterization of the structure of the operators $\mathcal{P}$ and π which arises from the product space structure of the state space. In the case of RFDEs with state delays only such a characterization has been given in KWONG [29]; VINTER-KWONG [41] for the operator $\mathcal{P}$ and in DELFOUR McCALLA-MITTER [13] for the operator π (note that in this special case the operators $\mathcal{P}$ and π may be defined on the state space $M^2 = \mathbb{R}^n \times L^2[-h,o; \mathbb{R}^n]$). An analogous result for general systems of the type (5.1) seems to be unknown since the Riccati equations (6.27) and (6.28) are apparently new.

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Linear quadratic optimal control, infinite dimensional Riccati equation, unbounded control and observation, retarded systems.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Part I of this paper deals with the problem of designing a feedback control for a linear infinite dimensional system in such a way that a given quadratic cost functional is minimized. The essential feature of this work is that:

a) it allows for unbounded control and observation, i.e. boundary control, point observation, input/output delays, and

20. ABSTRACT (cont.)

b) the general theory is presented in such a way that it applies to both parabolic and hyperbolic PDEs as well as retarded and neutral FDEs.

In Part II the paper develops a state space approach for retarded systems with delays in both input and output. A particular emphasis is placed on the development of the duality theory by means of two different state concepts. The resulting evolution equations fit into the framework of Part I.

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