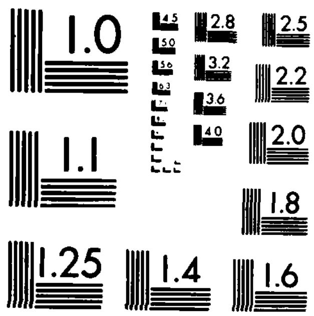
AD-A139 299 ESTIMATION OF LATENT GROUP EFFECTS(U) NATIONAL OPINION RESEARCH CENTER CHICAGO IL R J MISLEVY AUG 83 TR-2 N00014-83-C-0283 1/1

UNCLASSIFIED F/G 12/1 NL

END
DATE
FILMED
4 84
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

NORC

UNIVERSITY OF CHICA

Estimation of Latent Group Effects

Robert J. Mislevy

Psychometric Technical
Report No. 2

August 1983

This research was sponsored by the Personnel and Training Research Program, Psychological Sciences Division, Office of Naval Research under Contract No. N00014-83-C-0283, Contract Authority Identification No. NR 475-018,

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Psychometric Technical Report No.2	2. GOVT ACCESSION NO. [illegible]	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Estimation of Latent Group Effects		5. TYPE OF REPORT & PERIOD COVERED 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Robert J. Mislevy		8. CONTRACT OR GRANT NUMBER(s) N00014-83-C-0283
9. PERFORMING ORGANIZATION NAME AND ADDRESS NORC 6030 S. Ellis Avenue Chicago, IL 60637		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 62763N RF63521803 RF63521 NR 475-018
11. CONTROLLING OFFICE NAME AND ADDRESS Personnel and Training Research Programs Office of Naval Research (Code 442PT) Arlington, VA 22217		12. REPORT DATE August 1983 13. NUMBER OF PAGES 20
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Conventional methods of multivariate normal analysis do not apply when the variables of interest are not observed directly, but must be inferred from fallible or incomplete data. For example, responses to mental test items may depend upon latent aptitude variables, which modeled in turn as functions of demographic effects in the population. A method of estimating such effects by means of marginal maximum likelihood, implemented by means of an EM algorithm, is proposed. Asymptotic standard errors, likelihood ratio tests of alternative

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-LF-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) models, and computing approximations are provided. The procedures are illustrated with data for tests from the Armed Services Vocational Aptitude Battery administered to a national probability sample of American youth.

ESTIMATION OF LATENT GROUP EFFECTS

Robert J. Mislevy

National Opinion Research Center
Chicago, Illinois

AUGUST 1983

ESTIMATION OF LATENT GROUP EFFECTS

Abstract

Conventional methods of multivariate normal analysis do not apply when the variables of interest are not observed directly, but must be inferred from fallible or incomplete data. For example, responses to mental test items may depend upon latent aptitude variables, which modeled in turn as functions of demographic effects in the population. A method of estimating such effects by means of marginal maximum likelihood, implemented by means of an EM algorithm, is proposed. Asymptotic standard errors, likelihood ratio tests of alternative models, and computing approximations are provided. The procedures are illustrated with data for tests from the Armed Services Vocational Aptitude Battery administered to a national probability sample of American youth.

ESTIMATION OF LATENT GROUP EFFECTS

1. INTRODUCTION

Consider a number of multivariate normal populations in the random variable $\underset{\sim}{\theta}$, with a common dispersion matrix and with means given by linear functions of the fixed group-effect parameters $\underset{\sim}{\Gamma}$. Consumer attitudes in the cells of a multi-way demographic design, for example, might be modeled in terms of only main effects and selected interactions. Maximum likelihood (ML) estimation of $\underset{\sim}{\Gamma}$ from samples of $\underset{\sim}{\theta}$ from each population is well known, if it can be assumed that $\underset{\sim}{\theta}$ values are measured either without error or with iid normal and unbiased error components (Anderson, 1958).

Less familiar, however, are procedures to be followed when these assumptions are not tenable. If observations are of counts of favorable responses on an opinion survey, for example, the conditional distribution of observed score cannot be independent of expected score under any model with unbiased measurement errors (Lord and Novick, 1968:509). Or, as a second example, observed data may consist of subjects' responses to test items which depend stochastically on latent aptitude parameters through a quantal response model. More generally, we wish to consider situations in which it is not values of $\underset{\sim}{\theta}$ that are observed, but values of a secondary random variable $\underset{\sim}{x}$ whose distributions depend on $\underset{\sim}{\theta}$ through known density functions $p(\underset{\sim}{x}|\underset{\sim}{\theta})$.

This paper, then, presents a marginal maximum likelihood (MML) solution for $\underset{\sim}{\Gamma}$ from $\underset{\sim}{x}$, along the lines employed by Bock and Aitkin (1981) to estimate parameters in item response models. The results extend those of Andersen and Madsen (1977) and Sanathanan and Blumenthal (1978), who estimate the mean and variance of a univariate normal latent distribution when $p(\underset{\sim}{x}|\underset{\sim}{\theta})$ is the one-parameter logistic (Rasch) item response model, and of Andersen (1980), who tests the equality of latent means and variances in the same context.

We begin in Section 2 with a brief review of ML estimation of $\underset{\sim}{\Gamma}$ and $\underset{\sim}{\Sigma}$, the common dispersion matrix, when values of $\underset{\sim}{\theta}$ are observed, or, in the terminology of Dempster, Laird, and Rubin (1977), the "complete data" problem. Section 3 considers the case in which values of $\underset{\sim}{x}$ are observed instead, or the "incomplete data" problem. The resulting likelihood equations can be solved by means of cycles of an EM algorithm, which, since the unknown population density belongs to the exponential family, is guaranteed to converge to a local maximum. Computing approximations are presented in Section 4, asymptotic standard errors in Section 5, and likelihood ratio tests of fit in Section 6. Section 7 illustrates the procedures with data from the Profile of American Youth survey (U.S. Department of Defense, 1982).

2. THE "COMPLETE DATA" SOLUTION

We assume K homoscedastic p-variate normal distributions in the random variable $\underset{\sim}{\theta}$, with common dispersion matrix $\underset{\sim}{\Sigma}$ and means $\underset{\sim}{\mu}_k$ given as linear functions of M fixed group-effect parameters $\underset{\sim}{\gamma}_m$; that is,

$$\begin{bmatrix} \underset{\sim}{\mu}_1' \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{\mu}_M' \end{bmatrix} = \begin{bmatrix} \underset{\sim}{\xi}_1' \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{\xi}_K' \end{bmatrix} \begin{bmatrix} \underset{\sim}{\gamma}_1' \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{\gamma}_M' \end{bmatrix}$$

or, more compactly,

$$\underset{K\times p}{\underset{\sim}{M}} = \underset{K\times M}{\underset{\sim}{T}} \; \underset{M\times p}{\underset{\sim}{\Gamma}} ,$$

where $\underset{\sim}{T}$ is a known basis matrix, the k'th row of which specifies the dependence of $\underset{\sim}{\mu}_k$ on the parameters $\underset{\sim}{\Gamma}$.

Suppose that samples of $\underset{\sim}{\theta}$ of size N_k have been obtained from the K populations. Let $N = \sum_k N_k$ and let I_{ik} be indicators that take the value 1 when observation i is associated with population k and 0 when it is not. The likelihood of the sample is then given as

$$L = \prod_i \prod_k \left[g_k(\underset{\sim}{\theta}_i)\right]^{I_{ik}} ,$$

where

$$g_k(\underset{\sim}{\theta}_i) = \frac{|\underset{\sim}{\Sigma}|^{-1/2}}{(2\pi)^{p/2}} \exp\left[-1/2\,(\underset{\sim}{\theta}_i - \underset{\sim}{\Gamma}'\underset{\sim}{t}_k)'\ \underset{\sim}{\Sigma}^{-1}(\underset{\sim}{\theta}_i - \underset{\sim}{\Gamma}'\underset{\sim}{t}_k)\right] .$$

For reference in a following section, we digress briefly to demonstrate that with population membership known, this density belongs to the exponential family. Considering its parameters to be $\underset{\sim}{\Gamma}$ and $\underset{\sim}{\Sigma}^{-1}$ for convenience, we must show that it can be written in the form

$$f(\underset{\sim}{\theta}) = \exp\left\{\sum_w A_w(\underset{\sim}{\Gamma}, \underset{\sim}{\Sigma}^{-1})B_w(\underset{\sim}{\theta}) + C(\underset{\sim}{\theta}) + D(\underset{\sim}{\Gamma}, \underset{\sim}{\Sigma}^{-1})\right\} ,$$

where the summation runs over the unique elements of $\underset{\sim}{\Gamma}$ and $\underset{\sim}{\Sigma}^{-1}$. Letting (σ^{uv}) represent $\underset{\sim}{\Sigma}^{-1}$, this can be done by taking

$$A_{\sigma^{uv}}(\underset{\sim}{\Gamma}, \Sigma^{-1}) = \begin{cases} \sigma^{uv} & \text{if } u = v \\ 2\sigma^{uv} & \text{if } u \neq v \end{cases}$$

and

$$B_{\sigma^{uv}}(\underset{\sim}{\theta}) = \theta_u\theta_v ,$$

and for each element γ_{su} of Γ, taking

$$A_{\gamma_{su}}(\underset{\sim}{\Gamma}, \underset{\sim}{\Sigma}^{-1}) = \sum_v \sum_m \sigma^{uv}\gamma_{mv}\gamma_{su}$$

and

$$B_{\gamma_{su}}(\theta) = \sum_k \sum_m I_k t_{ks} t_{km} \quad .$$

Finally,

$$C(\underset{\sim}{\theta}) = 0$$

and

$$D(\underset{\sim}{\Gamma}, \underset{\sim}{\Sigma}^{-1}) = \log^{-1}\left[|\underset{\sim}{\Sigma}|^{1/2}(2\pi)^{p/2}\right] \quad .$$

Continuing to the main argument, we obtain the log likelihood as

$$\begin{aligned}\log L &= \sum_i \sum_k I_{ik} \log g_k(\underset{\sim}{\theta}_i) \\ &= C - N/2 \log|\underset{\sim}{\Sigma}| - 1/2 \sum_i \sum_k I_{ik}(\underset{\sim}{\theta}_i - \underset{\sim}{\Gamma}'\underset{\sim}{t}_k)' \underset{\sim}{\Sigma}^{-1}(\underset{\sim}{\theta}_i - \underset{\sim}{\Gamma}'\underset{\sim}{t}_k)\end{aligned} \tag{2.1}$$

where C does not depend on $\underset{\sim}{\Sigma}$ or $\underset{\sim}{\Gamma}$.

ML estimation proceeds by differentiating (2.1) with respect to $\underset{\sim}{\Gamma}$ and $\underset{\sim}{\Sigma}$, then equating the results to zero to obtain the likelihood equations:

$$\frac{\partial \log L}{\partial \underset{\sim}{\Gamma}} = \sum_i \sum_k I_{ik} \underset{\sim}{\Sigma}^{-1}(\underset{\sim}{\theta}_i - \underset{\sim}{\Gamma}'\underset{\sim}{t}_k)\underset{\sim}{t}_k' = 0$$

or

$$\sum_i \sum_k I_{ik}\underset{\sim}{\theta}_i \underset{\sim}{t}_k' = \sum_i \sum_k I_{ik}\underset{\sim}{\Gamma}'\underset{\sim}{t}_k\underset{\sim}{t}_k' \quad ,$$

then

$$\sum_k N_k \hat{\underset{\sim}{\mu}}_k \underset{\sim}{t}_k' = \sum N_k \underset{\sim}{\Gamma}'\underset{\sim}{t}_k\underset{\sim}{t}_k' \tag{2.2}$$

where

$$\hat{\underset{\sim}{\mu}}_k = N_k^{-1} \sum_i I_{ik}\underset{\sim}{\theta}_i \quad . \tag{2.3}$$

Rewriting (2.2) more compactly, we obtain the likelihood equation $\underset{\sim}{\Gamma}$ as

$$\underset{\sim}{T}'\underset{\sim}{D}\hat{\underset{\sim}{M}} = \underset{\sim}{T}'\underset{\sim}{D}\underset{\sim}{T}\underset{\sim}{\Gamma} \quad ,$$

where

$$\underset{\sim}{D} = \text{diag}(N_1, \ldots, N_K) \quad .$$

Assuming $\underset{\sim}{T}$ to be of full column rank M (a condition which if not satisfied initially can always be met by reparameterizing in terms of contrast among the original $\underset{\sim}{\gamma}$'s), we obtain

$$\hat{\underset{\sim}{\Gamma}} = (\underset{\sim}{T}'\underset{\sim}{D}\underset{\sim}{T})^{-1}\underset{\sim}{T}'\underset{\sim}{D}\hat{\underset{\sim}{M}} \quad . \tag{2.4}$$

Likelihood equations for $\underset{\sim}{\Sigma}$ are similarly obtained:

$$\frac{\partial \log L}{\partial \underset{\sim}{\Sigma}} = N/2(2\, \underset{\sim}{\Sigma}^{-1} - \text{diag}\, \underset{\sim}{\Sigma}^{-1})$$

$$+ \tfrac{1}{2} \sum_i \sum_k I_{ik} \{2\, \underset{\sim}{\Sigma}^{-1} (\underset{\sim}{\theta}_i - \underset{\sim}{\Gamma}'\underset{\sim}{t}_k)(\underset{\sim}{\theta}_i - \underset{\sim}{\Gamma}'\underset{\sim}{t}_k)'\, \underset{\sim}{\Sigma}^{-1}$$

$$- \text{diag}[\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{\theta}_i - \underset{\sim}{\Gamma}'\underset{\sim}{t}_k)(\underset{\sim}{\theta}_i - \underset{\sim}{\Gamma}'\underset{\sim}{t}_k)'\, \underset{\sim}{\Sigma}^{-1}]\} \quad ;$$

equating to zero and simplifying yields

$$\text{diag}\, \underset{\sim}{\Sigma} - 2\, \underset{\sim}{\Sigma} = \text{diag}\, \underset{\sim}{S} - 2\underset{\sim}{S} \tag{2.5}$$

where

$$S = N^{-1} \sum_i \sum_k I_{ik}(\underset{\sim}{\theta}_i - \underset{\sim}{\Gamma}'\underset{\sim}{t}_k)(\underset{\sim}{\theta}_i - \underset{\sim}{\Gamma}'\underset{\sim}{t}_k)' \quad . \tag{2.6}$$

After replacing $\underset{\sim}{\Gamma}$ by $\hat{\underset{\sim}{\Gamma}}$, we see from the form of (2.5) that

$$\hat{\underset{\sim}{\Sigma}} = \underset{\sim}{S} \quad . \tag{2.7}$$

It is well known that $\hat{\underset{\sim}{\Gamma}}$ and $\hat{\underset{\sim}{\Sigma}}$ are the unique zeros of the likelihood equations, and that they maximize the log likelihood function (2.1).

Note that (2.4) and (2.7) imply that $\underset{\sim}{M}$ and $\underset{\sim}{S}$ are jointly sufficient statistics for $\underset{\sim}{\Gamma}$ and $\underset{\sim}{\Sigma}$. In anticipation of the incomplete data problem, it is instructive to recognize their computation in (2.3) and (2.6) as standard formulas for means and dispersions, Stieltjes integrals over not the unknown true density but over an approximation of it, namely, the discrete distribution given by a finite sample of points from the distribution of interest.

3. THE "INCOMPLETE DATA" SOLUTION

Suppose that rather than values of $\underset{\sim}{\theta}$, we observe values of $\underset{\sim}{x}$ which depend on $\underset{\sim}{\theta}$ through $p(\underset{\sim}{x}|\underset{\sim}{\theta})$, densities of known form which may vary from one observation to the next. For example, $\underset{\sim}{x}$ you may be a vector of discrete values depending on the continuous latent variable $\underset{\sim}{\theta}$ through a quantal response model; or, as a second example, $\underset{\sim}{x}$ may be equal to $\underset{\sim}{\theta}$ plus a random error component, the distributions of which are known but need not be either iid nor normally distributed. Under these assumptions, the marginal likelihood of response $\underset{\sim}{x}_i$ obtained from population k is given as

$$h(\underset{\sim}{x}_i|\underset{\sim}{\Gamma}, \underset{\sim}{\Sigma}) = \prod_k \left[\int_{\underset{\sim}{\theta}} p(\underset{\sim}{x}_i|\underset{\sim}{\theta}) g_k(\underset{\sim}{\theta}|\underset{\sim}{\Gamma}, \underset{\sim}{\Sigma}) d\underset{\sim}{\theta}\right]^{I_{ik}} . \tag{3.1}$$

For notational convenience, we write simply $h(\underset{\sim}{x}_i)$ and $g_k(\underset{\sim}{\theta})$ hereafter, the dependence on $\underset{\sim}{\Gamma}$ and $\underset{\sim}{\Sigma}$ implicit.

From (3.1), the log marginal likelihood of samples of $\underset{\sim}{x}$ of size N_k is given by

$$\log L^* = \sum_i \log h(\underset{\sim}{x}_i)$$

$$= \sum_i \sum_k I_{ik} \log \int_{\underset{\sim}{\theta}} p(\underset{\sim}{x}_i|\underset{\sim}{\theta}) g_k(\underset{\sim}{\theta}) d\underset{\sim}{\theta} . \tag{3.2}$$

The derivative of log L* with respect to $\underset{\sim}{\Gamma}$ is then obtained as

$$\frac{\partial \log L^*}{\partial \underset{\sim}{\Gamma}} = \sum_i \frac{\partial \log h(\underset{\sim}{x}_i)}{\partial \underset{\sim}{\Gamma}}$$

$$= \sum_i \sum_k I_{ik} h^{-1}(\underset{\sim}{x}_i) \int_{\underset{\sim}{\theta}} p(\underset{\sim}{x}_i|\underset{\sim}{\theta}) \frac{\partial g_k(\underset{\sim}{\theta})}{\partial \underset{\sim}{\Gamma}} d\underset{\sim}{\theta}$$

$$= \sum_i \sum_k I_{ik} h^{-1}(\underset{\sim}{x}_i) \int_{\underset{\sim}{\theta}} p(\underset{\sim}{x}_i|\underset{\sim}{\theta}) g_k(\underset{\sim}{\theta}) \underset{\sim}{\Sigma}^{-1}(\underset{\sim}{\theta} - \underset{\sim}{\Gamma}'\underset{\sim}{t}_k)\underset{\sim}{t}_k' d\theta \quad . \tag{3.3}$$

Equating to zero yields

$$\sum_i \sum_k I_{ik} h^{-1}(\underset{\sim}{x}_i) \int_{\underset{\sim}{\theta}} p(\underset{\sim}{x}_i|\underset{\sim}{\theta}) g_k(\underset{\sim}{\theta}) \underset{\sim}{\theta} d\underset{\sim}{\theta} \underset{\sim}{t}_k'$$

$$= \sum_i \sum_k I_{ik} h^{-1}(\underset{\sim}{x}_i) \int_{\underset{\sim}{\theta}} p(\underset{\sim}{x}_i|\underset{\sim}{\theta}) g_k(\underset{\sim}{\theta}) d\underset{\sim}{\theta} \underset{\sim}{\Gamma}' \underset{\sim}{t}_k \underset{\sim}{t}_k' \quad ,$$

then

$$\sum_k N_k \hat{\underset{\sim}{\mu}}{}_k^* \underset{\sim}{t}_k' = \sum_k N_k \underset{\sim}{\Gamma}' \underset{\sim}{t}_k \underset{\sim}{t}_k' \quad , \tag{3.4}$$

where

$$\hat{\underset{\sim}{\mu}}{}_k^* = \int_{\underset{\sim}{\theta}} \underset{\sim}{\theta}\, p_k(\underset{\sim}{\theta}|(\underset{\sim}{X}))d\underset{\sim}{\theta} \tag{3.5}$$

with

$$p_k(\underset{\sim}{\theta}|\underset{\sim}{X})) = N_k^{-1} \sum_i I_{ik} p(\underset{\sim}{\theta}|\underset{\sim}{x}_i)$$

$$= N_k^{-1} \sum_i I_{ik} h^{-1}(\underset{\sim}{x}_i) p(\underset{\sim}{x}_i|\underset{\sim}{\theta}) g_k(\underset{\sim}{\theta}) \tag{3.6}$$

being the posterior density of $\underset{\sim}{\theta}$ in population k given $\underset{\sim}{\Gamma}$, $\underset{\sim}{\Sigma}$, and the observed data ($\underset{\sim}{X}$) via Bayes theorem. Rewriting (3.4) more compactly,

$$\underset{\sim}{T}'\underset{\sim}{D}\hat{\underset{\sim}{M}}{}^* = \underset{\sim}{T}'\underset{\sim}{D}\underset{\sim}{T}\underset{\sim}{\Gamma} \quad ,$$

from which

$$\hat{\Gamma} = (T'DT)^{-1}T'D\hat{M}^* \quad . \tag{3.7}$$

Similarly, differentiating (3.2) with respect to Σ yields

$$\frac{\partial \log L^*}{\partial \Sigma} = \sum_i \frac{\partial \log h(x_i)}{\partial \Sigma}$$

$$= \sum_i \sum_k I_{ik}\{-\tfrac{1}{2}(2\,\Sigma^{-1} - \operatorname{diag}\Sigma^{-1}) + \tfrac{1}{2}h^{-1}(x_i)\int_\theta p(x_i|\theta)g_k(\theta)$$

$$\times \{2\,\Sigma^{-1}(\theta - \Gamma' t_k)(\theta - \Gamma' t_k)'\,\Sigma^{-1} \tag{3.8}$$

$$- \operatorname{diag}[\Sigma^{-1}(\theta - \Gamma' t_k)(\theta - \Gamma' t_k)'\,\Sigma^{-1}\}]d\theta\} \quad .$$

Equating to zero then simplifying leads to

$$\operatorname{diag}\Sigma - 2\,\Sigma = \operatorname{diag}S^* - 2S^* \quad ,$$

where

$$S^* = \sum_k \int_\theta (\theta - \Gamma' t_k)(\theta - \Gamma' t_k)' p_k(\theta|x_i))d\theta \quad . \tag{3.9}$$

Again it is clear that

$$\hat{\Sigma} = S^* \quad .$$

Like (2.3) and (2.6), (3.5) and (3.9) are standard formulas for computing means and dispersions from an approximation of an unknown density. Now the approximation is not based on a discrete set of sample points from the distribution but on an average over observations of the posterior density of θ given each observation. These posterior densities, however, are computed via Bayes theorem in (3.6) with the true densities assumed known. Thus, the likelihood equations (3.7) and (3.10) constitute a system of implicit equations in Γ and Σ, since they are defined in terms of $\hat{M}^*$ and S^* which depend in turn on Γ and Σ through h and g_k.

One approach to solving (3.7) and (3.10), thereby obtaining zeros of the log likelihood, is the so-called method of successive approximations. That is, $\hat{\underset{\sim}{M}}^*$ and $\underset{\sim}{S}^*$ are computed through (3.5) and (3.9) with provisional estimates $\hat{\underset{\sim}{\Gamma}}^t$ and $\hat{\underset{\sim}{\Sigma}}^t$; improved estimates $\hat{\underset{\sim}{\Gamma}}^{t+1}$ and $\hat{\underset{\sim}{\Sigma}}^{t+1}$ are then obtained by evaluating (3.7) and (3.10) with respect to these new values. This procedure will be recognized as an application of the EM algorithm, as described by Dempster, Laird, and Rubin (1977), who demonstrated convergence to a maximum of the likelihood function when the complete data density is a member of the exponential family, as it is in the problem at hand. The notoriously slow convergence of the EM algorithm, which worsens as the densities $p(\underset{\sim}{x}|\underset{\sim}{\theta})$ become more diffuse, can be largely ameliorated by the use of acceleration techniques such as those described by Ramsey (1975).

4. COMPUTING APPROXIMATIONS

Because closed-form expressions for the integrals in (3.1), (3.5), and (3.9) are not generally available, numerical approximations are required in applying the foregoing solution. Three approaches are outlined in this section: Gauss-Hermite quadrature, quadrature over fixed points, and Monte Carlo integration.

For accuracy and stability, Gauss-Hermite quadrature is the preferred method of numerical integration over the normal distribution when p is small. Stroud and Sechrest (1966) provide tables of optimal points and weights for the univariate standard normal density, which will be denoted (Z_q) and $(W(Z_q))$, for $q = 1, \ldots, Q$. A grid of points for the p-variate standard normal is obtained as the Cartesian product of p univariate sets of points, with weights equal to the products of weights associated with each element in the vector defining a grid point. That is, a typical point in the grid will take the form

$$\underset{\sim}{Z}_q' = (Z_{q1}, \ldots, Z_{qp})$$

and have an associated weight of

$$W(\underset{\sim}{Z}_q) = \prod_{t=1}^{p} W(Z_{qt}) \quad .$$

The integrations in (3.1), (3.5), and (3.9) take place over a general p-variate normal distribution, necessitating a change of variables of integration in order for Gauss-Hermite quadrature to be employed. We illustrate with (3.1). Let $\underset{\sim}{z}_k = (\underset{\sim}{\theta} - \underset{\sim}{\Gamma}'\underset{\sim}{t}_k)\underset{\sim}{V}$, where $\underset{\sim}{V}\underset{\sim}{V}' = \underset{\sim}{\Sigma}$ is the Cholesky factorization of $\underset{\sim}{\Sigma}$ (implying that $|\underset{\sim}{V}| = |\underset{\sim}{\Sigma}|^{1/2}$). Then

$$h(\underset{\sim}{x}_i) = \prod_k \left\{\int_{\underset{\sim}{\theta}} p(\underset{\sim}{x}_i|\underset{\sim}{\theta})\frac{|\underset{\sim}{\Sigma}|^{-1/2}}{(2\pi)^{p/2}} \exp\left[-1/2(\underset{\sim}{\theta} - \underset{\sim}{\Gamma}'\underset{\sim}{t}_k)' \underset{\sim}{\Sigma}^{-1}(\underset{\sim}{\theta} - \underset{\sim}{\Gamma}'\underset{\sim}{t}_k)\right]d\underset{\sim}{\theta}\right\}^{I_{ik}}$$

$$= \prod_k \left\{\int_{\underset{\sim}{z}_k} p(\underset{\sim}{x}_i|\underset{\sim}{\theta}_k(\underset{\sim}{z}_k))\frac{|\underset{\sim}{\Sigma}|^{-1/2}}{(2\pi)^{p/2}} \exp(-\underset{\sim}{z}_k'\underset{\sim}{z}_k/2)|\underset{\sim}{V}|d\underset{\sim}{z}_k\right\}^{I_{ik}}$$

$$= \prod_k \left\{\int_{\underset{\sim}{z}_k} p(\underset{\sim}{x}_i|\underset{\sim}{\theta}_k(\underset{\sim}{z}_k))(2\pi)^{-p/2} \exp(-\underset{\sim}{z}_k'\underset{\sim}{z}_k/2)d\underset{\sim}{z}_k\right\}^{I_{ik}}$$

where

$$\underset{\sim}{\theta}_k(\underset{\sim}{z}) = \underset{\sim}{V}\underset{\sim}{z} + \underset{\sim}{\Gamma}'\underset{\sim}{t}_k \quad .$$

Then

$$h(\underset{\sim}{x}_i) \approx \prod_k \left\{\sum_q p(\underset{\sim}{x}_i|\underset{\sim}{X}_{qk})W(\underset{\sim}{X}_{qk})\right\}^{I_{ik}}$$

where

$$\underset{\sim}{X}_{qk} = \underset{\sim}{\theta}_k(\underset{\sim}{Z}_q) \quad \text{and} \quad W(\underset{\sim}{X}_{qk}) = W(\underset{\sim}{Z}_q) \quad .$$

Computing approximations of $\hat{\underset{\sim}{\mu}}_k^*$ and $\underset{\sim}{S}^*$ are obtained similarly as

$$\hat{\underset{\sim}{\mu}}_k^* \approx N_k^{-1} \sum_i I_{ik}h^{-1}(\underset{\sim}{x}_i) \sum_q \underset{\sim}{X}_{qk}p(\underset{\sim}{x}_i|\underset{\sim}{X}_{qk})W(\underset{\sim}{X}_{qk}) = \sum_q \underset{\sim}{X}_{qk}P_{qk}^* \qquad (4.1)$$

where

$$P^*_{qk} = N_K^{-1} \sum_i I_{ik} h^{-1}(\underset{\sim}{x}_i) p(\underset{\sim}{x}_i | \underset{\sim}{X}_{qk}) W(\underset{\sim}{X}_{qk})$$

and

$$\underset{\sim}{S}^* = \sum_k \sum_q (\underset{\sim}{X}_{qk} - \underset{\sim}{\Gamma}' \underset{\sim}{t}_k)(\underset{\sim}{X}_{qk} - \underset{\sim}{\Gamma}' \underset{\sim}{t}_k)' P^*_{qk} \qquad (4.2)$$

The similarity of the computing approximations (4.1) and (4.2) to the complete data solution given as (2.3) and (2.6) are immediately apparent; means and dispersions are again computed with respect to a discrete approximation of the distribution of interest. This time, however, the discrete approximation is based not on sample points from that distribution but on posterior estimates of its density at selected quadrature points, given the observed data ($\underset{\sim}{x}$). In contrast to (2.3) and (2.6), (4.1) and (4.2) constitute a system of implicit equations because of the dependence of the weights p^*_{qk} on the unknown parameters of the distribution.

An alternative approximation that can offer considerable computational advantage is quadrature over a fixed grid of points. Whereas Gauss-Hermite quadrature computes points anew each cycle in accordance with provisional estimates of $\underset{\sim}{\Gamma}$ and $\underset{\sim}{\Sigma}$, it is possible to retain the same grid of points for all cycles and thereby avoid computing $p(\underset{\sim}{x}|\underset{\sim}{X}_{qk})$ every cycle. A grid of points $\underset{\sim}{X}_q$ is selected a priori to span a region where the preponderance of the population distributions is believed to lie. New weights are computed in each cycle from provisional estimates of $\underset{\sim}{\Gamma}$ and $\underset{\sim}{\Sigma}$ as follows:

$$W_k(\underset{\sim}{X}_q) = \exp[-\tfrac{1}{2}(\underset{\sim}{X}_q - \underset{\sim}{\Gamma}' \underset{\sim}{t}_k)' \underset{\sim}{\Sigma}^{-1}(\underset{\sim}{X}_q - \underset{\sim}{\Gamma}' \underset{\sim}{t}_k)] \quad .$$

The computing aproximations (4.1) and (4.2) remain unchanged except for the substutitions of $\underset{\sim}{X}_q$ for $\underset{\sim}{X}_{qk}$ and $W_k(\underset{\sim}{X}_q)$ for $W(\underset{\sim}{X}_{qk})$. When the grid is well chosen, estimates of Σ and Γ will agree well with those computed via

Gauss-Hermite quadrature. When the points are poorly chosen, however, loss of accuracy and/or stability can result.

A second alternative that can prove useful when large p renders quadrature over a grid cumbersome is Monte Carlo integration. In each cycle, Q random points $\underset{\sim}{X}_{qk}$ are generated for each population k in accordance with provisional estimates of $\underset{\sim}{\Gamma}$ and $\underset{\sim}{\Sigma}$. The computing formulas (4.1) and (4.2) remain unchanged except that

$$W(\underset{\sim}{X}_{qk}) = 1/Q \qquad k = 1, \ldots, K \ .$$

5. ASYMPTOTIC STANDARD ERRORS

Following Bock and Lieberman (1970), we may approximate the inverse of the asymptotic covariance matrix of the estimators $\hat{\underset{\sim}{\Gamma}}$ and $\hat{\underset{\sim}{\Sigma}}$ by

$$\underset{\sim}{H} = \sum_i \left[\frac{\partial \log h(\underset{\sim}{x}_i)}{\partial \underset{\sim}{\xi}}\bigg|_{\hat{\underset{\sim}{\xi}}}\right]\left[\frac{\partial \log h(\underset{\sim}{x}_i)}{\partial \underset{\sim}{\xi}'}\bigg|_{\hat{\underset{\sim}{\xi}}}\right] ,$$

where $\underset{\sim}{\xi}$ represents the Mp elements of $\underset{\sim}{\Gamma}$ and the $p(p+1)/2$ nonredundant elements of $\underset{\sim}{\Sigma}$ written as a single vector. Large sample standard errors are obtained as the square roots of the diagonal elements of $\underset{\sim}{H}^{-1}$. Expressions necessary for the evaluation of $\underset{\sim}{H}$ are found in (3.3) and (3.8). Using the univariate case as an illustration, the required gradient vectors, gramian products of which are summed over observations to produce $\underset{\sim}{H}$, are shown below:

$$\frac{\partial \log h(\underset{\sim}{x}_i)}{\partial \gamma_m} = \sum_k I_{ik} h^{-1}(\underset{\sim}{x}_i) \int_\theta p(\underset{\sim}{x}_i|\theta)\sigma^{-2}(\theta - \underset{\sim}{\Gamma}'\underset{\sim}{t}_k)t_{km}d\theta$$

$$\approx \sigma^{-2} \sum_k I_{ik} h^{-1}(\underset{\sim}{x}_i) \sum_k p(\underset{\sim}{x}_i|X_{qk})W(X_{qk})(X_{qk} - \underset{\sim}{\Gamma}'\underset{\sim}{t}_k)t_{km}$$

$$= \sigma^{-2} \sum_k I_{ik} \sum_q p(X_{qk}|\underset{\sim}{x}_i)(X_{qk} - \underset{\sim}{\Gamma}'\underset{\sim}{t}_k)t_{km}$$

and

$$\frac{\partial \log h(\underset{\sim}{x}_i)}{\partial \sigma^2} = -\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_k I_{ik} h^{-1}(\underset{\sim}{x}_i) \int_\theta p(\underset{\sim}{x}_i|\theta) g_k(\theta)(\theta - \underset{\sim}{\Gamma}'\underset{\sim}{t}_k)(\theta - \underset{\sim}{\Gamma}'\underset{\sim}{t}_k)' d\theta$$

$$\approx -\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_k I_{ik} \sum_q p(X_{qk}|\underset{\sim}{x}_i)(X_{qk} - \underset{\sim}{\Gamma}'\underset{\sim}{t}_k)(X_{qk} - \underset{\sim}{\Gamma}'\underset{\sim}{t}_k)' \quad .$$

6. TESTS OF FIT

Consider two competing models for a given data set, with Model 1 nested within Model 2. In large samples, the fit of the two models can be compared by means of the statistic

$$\chi^2 = -2\log(L_1^*/L_2^*) \tag{6.1}$$

which, when Model 1 is correct, follows a chi-square distribution with degrees of freedom equal to the number of additional parameters in Model 2.

When the number of potential responses $\underset{\sim}{x}$ is small compared to the sample size, it is possible to compare the fit of a given model to a general multinominal alternative. First the universe of potential responses $\underset{\sim}{x}_\ell$ is partitioned into mutually exclusive and exhaustive classes such that the potential responses of a given observation constitute exactly one class. If a test with several parallel forms is administered, for example, each class of responses will consist of all possible response vectors to the items in a given test form. Let $r(\underset{\sim}{x}_{\ell k})$ be the count of *response* $\underset{\sim}{x}_\ell$ *observed in* population k, and let $N(\underset{\sim}{x}_{\ell k})$ be the total number of responses from the same class as $\underset{\sim}{x}_\ell$ that are observed in population k. Then the statistic

$$\chi^2 = -2\sum\sum r(\underset{\sim}{x}_{\ell k})\log[N(\underset{\sim}{x}_{\ell k})h(\underset{\sim}{x}_{\ell k})/r(\underset{\sim}{x}_{\ell k})] \tag{6.2}$$

[with terms for which $r(\underset{\sim}{x}_{\ell k}) = 0$ set to zero] will follow a chi-square distribution in large samples when the model is correct, with degrees of freedom equal to the number of non-zero $r(\underset{\sim}{x}_{\ell k})$ terms minus the number of parameters estimated in the model minus K. It will be noted that the difference between the values of (6.2) for two nested models takes the same value and has the same degrees of freedom as a direct comparison via (6.1).

Following Andersen (1980), we may test the equality of dispersion matrices across groups in a two-step procedure. First, means and dispersion matrices are estimated in all groups separately. The product of the likelihoods resulting from these separate analyses is accumulated. Second, separate means and a common dispersion matrix are estimated by employing an identity matrix as the basis matrix $\underset{\sim}{T}$ and proceding as described in Section 3. The resulting likelihood may be compared with the first via (6.2) to obtain a large-sample chi-square test of the equality of dispersion matrices over groups, with the number of degrees of freedom equal to $(K - 1)p(p + 1)/2$.

7. A NUMERICAL EXAMPLE

Item response models in psychometrics express the probability of a given response to a test item as a function of a subject's latent ability parameter $\underset{\sim}{\theta}$ and one or more parameters that characterize the regression of the item response on ability. The three-parameter logistic item response model (Birnbaum, 1968) for dichotomous items, for example, gives the probability of a correct response to item j from subject i as

$$\begin{aligned} P(x_{ij} &= 1 | \theta_i, a_j, b_j, c_j) \\ &= P_{ij} \\ &= c_j + (1 - c_j)\frac{\exp[1.7a_j(\theta_i - b_j)]}{1 - \exp[1.7a_j(\theta_i - b_j)]} , \qquad (7.1a) \end{aligned}$$

and the probability of an incorrect response as

$$P(x_{ij} = 0 | \theta_i, a_j, b_j, c_j) = 1 - P_{ij} \quad , \qquad (7.1b)$$

where x_{ij} denotes the response, 1 if correct and 0 if not, and where θ_i is the ability of subject i and a_j, b_j, and c_j are parameters that characterize item j: a_j, the slope parameter, reflects the reliability of the item; b_j, the threshold, reflects its difficulty; and c_j, the lower asymptote, reflects the minimal probability of a correct response from even subjects with extremely low abilities. Under the usual assumption of local or conditional independence, the probability of a pattern of responses from subject i to a number of items is given by the product over items of expression like (7.1):

$$P(\underset{\sim}{x}_i | \theta) = \prod_j P(x_{ij} | \theta_i, a_j, b_j, c_j) \quad . \qquad (7.2)$$

In most applications, item response models are used to estimate the latent abilities of individuals. With values of item parameters assumed known (generally estimated from a large sample of subjects), one may obtain a maximum likelihood estimate of θ with respect to a given response vector by maximizing (7.2) as a function of θ, and a large-sample standard error by taking the negative reciprocal of the second derivative of the natural logarithm of (7.2) evaluated at the mle $\hat{\theta}$.

There are several reasons _not_ to approximate the distribution of θ in a population from the distribution of $\hat{\theta}$, or to carry out ANOVA procedures on values of $\hat{\theta}$ to estimate group effects on means of θ in various subpopulations. First, values of θ are estimated with varying precision, thereby violating the assumptions upon which standard ANOVA procedures are based. Second, estimation of θ from certain response patterns is problematic; patterns with all correct or all incorrect responses, along with most patterns with total scores below chance level (the sum of the c_j's over the items a subject has been presented) yield infinite mle's. Deleting the data of subjects with such patterns biases estimates of the population means and variances, while assigning them finite values either arbitrarily or by incorporating prior information introduces biases into the estimation of the θ's themselves. Third, stable estimation of individuals' θ's requires at least 15 or 20 responses per subject, thereby proscribing the use of more efficient sampling designs that would be preferred when only population-level parameters are of interest. The methods introduced in the preceding sections suffer none of these deficiencies.

As an example, we consider data from the Profile of American Youth, a survey of the aptitudes of a sample of the population of Americans aged 16 through 23 in July 1980 (U.S. Department of Defense, 1982). Table 1 presents counts of the sixteen possible response patterns observed to four items from the Arithmetic Reasoning test of the Armed Services Vocational Aptitude Battery (ASVAB), Form 8A, as observed in samples of white males and females and black males and females. The parameters of these items under the three-parameter logistic item response model, shown in Table 2, were estimated from a sample of 1,178 cases from the 11,787 available using the BILOG computer program (Mislevy and Bock, 1982).

Tables 3 and 4 presents the results of fitting a series of nested models to the data of Table 1. Examination of the differences between likelihood ratio chi-squares against the general multinominal alternative suggests, to begin with, that within-group variation may not be homogeneous. Continuing the example for purposes of illustration, we find strong evidence for a race effect and, to a lesser extent, for sex and interaction effects.

The males' mean exceeds that of the females for whites, but the females' mean appears to equal or exceed that of the males for blacks.

INSERT TABLES 1-4

REFERENCES

Andersen, E. B. (1980), "Comparing Latent Populations," Psychometrika, 45, 121-134.

Andersen, E. B. and Madsen, M. (1977), "Estimating the Parameters of a Latent Population," Psychometrika, 42, 357-374.

Andersen, T. W. (1958), An Introduction to Multivariate Statistical Analysis, New York: John Wiley.

Birnbaum, A. (1968), "Some Latent Trait Models and Their Use in Inferring an Examinee's Ability." In F. M. and M. R. Novick, Statistical Theories of Mental Test Scores, Reading, Mass.: Addison-Wesley.

Bock, R. D. and Aitkin, M. (1981), "Marginal Maximum Likelihood Estimation of Item Parameters: Application of an EM Algorithm," Psychometrika, 46, 443-459.

Bock, R. D. and Lieberman, M. (1970), "Fitting a Model for n Dichotomously Scores Items," Psychometrika, 3, 179-197.

Dempster, A. P. Laird, N. M., and Rubin D. B. (1977), "Maximum Likelihood from Incomplete Data via the EM Algorithm (with Discussion)," Journal of Royal Statistical Society, Series B, 39, 1-38.

Lord, F. M. and Novick, M. R. (1968), Statistical Theories of Mental Test Scores, Reading, Mass.: Addison-Wesley.

Mislevy, R. J. and Bock, R. D. (1982), BILOG: Item Analysis and Test Scoring with Binary Logistic Models. Chicago: National Education Resources.

Ramsey, J. O. (1975), "Solving Implicit Equations in Psychometric Data Analysis," Psychometrika, 40, 361-372.

Sanathanan, L. and Blumenthal, N. (1978), "The Logistic Model and Estimation of Latent Structure," Journal of the American Statistical Association, 73, 794-798.

Stroud, A. H. and Sechrest, D. (1966), Gaussian Quadrature Formulas, Englewood Cliffs, New Jersey: Prentice-Hall.

U.S. Department of Defense, Office of the Assistant Secretary of Defense (Manpower, Reserve Affairs, and Logistics), (1982) Profile of American Youth, Washington, D.C.

TABLE 1

COUNTS OF OBSERVED RESPONSE PATTERNS

ITEM RESPONSE 1	2	3	4	WHITE MALES	WHITE FEMALES	BLACK MALES	BLACK FEMALES
0	0	0	0	23	20	27	29
0	0	0	1	5	8	5	8
0	0	1	0	12	14	15	7
0	0	1	1	2	2	3	3
0	1	0	0	16	20	16	14
0	1	0	1	3	5	5	5
0	1	1	0	6	11	4	6
0	1	1	1	1	7	3	0
1	0	0	0	22	23	15	14
1	0	0	1	6	7	10	10
1	0	1	1	19	6	1	2
1	1	0	0	21	18	7	19
1	1	0	1	11	15	9	5
1	1	1	0	23	20	10	8
1	1	1	1	86	42	2	4
TOTAL				264	227	141	147

TABLE 2

ITEM PARAMETERS

ITEM	a	b	c
1	1.27	-.13	.22
2	1.45	.42	.34
3	2.49	.71	.31
4	2.27	.62	.20

TABLE 3

PARAMETER ESTIMATES AND FIT STATISTICS

EFFECTS IN MODEL	GRAND MEAN	RACE	SEX	INTERACTION	VARIANCE	CHI-SQUARE	DF
GRAND MEAN	.02 (.05)	--	--	--	.85 (.12)	223.77	57
GRAND MEAN, SEX	.02 (.05)	--	.29 (.09)	--	.83 (.12)	213.31	56
GRAND MEAN, RACE	-.11 (.06)	.92 (.11)	--	--	.66 (.11)	124.14	56
GRAND MEAN, RACE, SEX	-.11 (.06)	.91 (.11)	.24 (.09)	--	.65 (.10)	115.92	55
GRAND MEAN, RACE, SEX, INTERACTION	-.11 (.06)	.90 (.11)	.13 (.11)	.42 (.21)	.65 (.10)	111.12	54
UNCONSTRAINED MEANS, UNCONSTRAINED VARIANCES			(VARIANCES = 1.06,	.63, .39,	.27)	100.57	51

TABLE 4

FITTED MEANS

EFFECTS IN MODEL	WHITE MALES	WHITE FEMALES	BLACK MALES	BLACK FEMALES
GRAND MEAN	.02	.02	.02	.02
GRAND MEAN, SEX	.16	-.13	.16	-.13
GRAND MEAN, RACE	.35	.35	-.57	-.57
GRAND MEAN, RACE, SEX	.47	.22	-.44	-.69
GRAND MEAN, RACE, SEX, INTERACTION	.51	.16	-.60	-.52
UNCONSTRAINED MEANS, UNCONSTRAINED VARIANCES	.49	.17	-.46	-.37

DISTRIBUTION LIST

MR. PAUL FOLEY
NAVY PERSONNEL R&D CENTER
SAN DIEGO, CA 92152

DR. JIM HOLLAN
CODE 14
NAVY PERSONNEL R&D CENTER
SAN DIEGO, CA 92152

DR. ED HUTCHINS
NAVY PERSONNEL R&D CENTER
SAN DIEGO, CA 92152

DR. NORMAN J. KERR
CHIEF OF NAVAL TECH. TRAINING
NAVAL AIR STATION MEMPHIS (75)
MILLINGTON, TN 38054

DR. LEONARD KROEKER
NAVY PERSONNEL R&D CENTER
SAN DIEGO, CA 92152

DR. WILLIAM L. MALOY (20)
CHIEF OF NAVAL ED. & TRAINING
NAVAL AIR STATION
PENSACOLA, FL 32508

DR. JAMES McBRIDE
NAVY PERSONNEL R&D CENTER
SAN DIEGO, CA 92152

DR. WILLIAM MONTAGUE
NPRDC CODE 13
SAN DIEGO, CA 92152

DR. WILLIAM E. NORDBROCK
FMC-ADCO BOX 25
APO, NY 09710

Library, Code P201L
Navy Personnel R&D Center
San Diego, CA 92152

TECHNICAL DIRECTOR
NAVY PERSONNEL R&D CENTER
SAN DIEGO, CA 92152

H. WILLIAM GREENUP
EDUCATION ADVISOR (E031)
EDUCATION CENTER, MCDEC
QUANTICO, VA 22134

DIR., OFFICE OF MANPOWER UTILIZATION
HQ. MARINE CORPS (MPU)
BCB, BLDG. 2009
QUANTICO, VA 22134

Headquarters, US Marine Corps
Code MPI-20
Washington, DC 20380

SPEC. ASS. FOR MARINE MATTERS
CODE 100M
OFFICE OF NAVAL RESEARCH
800 N. QUINCY STREET
ARLINGTON, VA 22217

DR. A.L. SLAFKOSKY
SCIENTIFIC ADVISOR (CODE RD-1)
HQ. U.S. MARINE CORPS
WASHINGTON, DC 20380

MAJOR FRANK YOHANNAN, USMC
HEADQUARTERS, MARINE CORPS
(CODE MPI-20)
WASHINGTON, DC 20380

TECHNICAL DIRECTOR
U.S. ARMY RESEARCH INST.
BEHAVORIAL & SOCIAL SCIENCES
5001 EISENHOWER AVENUE
ALEXANDRIA, VA 22333

MR. JAMES BAKER
ARMY RESEARCH INSTITUTE
5001 EISENHOWER AVENUE
ALEXANDRIA, VA 22333

DR. KENT EATON
ARMY RESEARCH INSTITUTE
5001 EISENHOWER BLVD.
ALEXANDRIA, VA 22333

DR. BEATRICE J. FARR
U.S. ARMY RESEARCH INST.
5001 EISENHOWER AVENUE
ALEXANDRIA, VA 22333

DR. MYRON FISCHL
U.S. ARMY RESEARCH INST.
SOCIAL & BEHAVIORAL SCIENCES
5001 EISENHOWER AVENUE
ALEXANDRIA, VA 22333

DR. MILTON S. KATZ
TRAINING TECHNICAL AREA
U.S. ARMY RESEARCH INST.
5001 EISENHOWER AVENUE
ALEXANDRIA, VA 22333

DR. HAROLD F. O'NEIL, JR.
DIRECTOR, TRAINING RES. LAB
ARMY RESEARCH INSTITUTE
5001 EISENHOWER AVENUE
ALEXANDRIA, VA 22333

COMMANDER, US ARMY RES. INST. FOR BEHAVIORAL & SOCIAL SCI.
ATTN: PERI-BR(DR. J. ORASANU)
5001 EISENHOWER AVENUE
ALEXANDRIA, VA 22333

JOSEPH PSOTKA, PH.D
ATTN: PERI-IC
ARMY RESEARCH INSTITUTE
5001 EISENHOWER AVENUE
ALEXANDRIA, VA 22333

MR. ROBERT ROSS
US ARMY RESEARCH INST. FOR THE SOCIAL & BEHAVIORAL SCIENCES
5001 EISENHOWER AVENUE
ALEXANDRIA, VA 22333

DR. ROBERT SASMOR
US ARMY RES. INST. FOR THE BEHAVIORAL & SOCIAL SCIENCES
5001 EISENHOWER AVENUE
ALEXANDRIA, VA 22333

DR. JOYCE SHIELDS
ARMY RESEARCH INST. FOR THE BEHAVIORAL & SOCIAL SCIENCES
5001 EISENHOWER AVENUE
ALEXANDRIA, VA 22333

DR. ROBERT J. SEIDEL
US ARMY RESEARCH INST.
5001 EISENHOWER AVINUE
300 N. WASHINGTON ST.
ALEXANDRIA, VA 22333

DR. HILDA WING
ARMY RESEARCH INST.
5001 EISENHOWER AVENUE
ALEXANDRIA, VA 22333

DR. ROBERT WISHER
ARMY RESEARCH INST.
5001 EISENHOWER AVE.
ALEXANDRIA, VA 22333

Lowell Schoer
Psycho. & Quanti. Foundations
College of Education
Univesity of Iowa
Iowa City, IA 52242

Dr. Edwin Shirkey
Department of Psychology
Univer. of Central Florida
Orlando, FL 32816

Dr. William Sims
Center for Naval Analysis
200 North Beauregard Street
Alexandria, VA 22311

Dr. H. Wallace Sinaiko
Smithsonian Institution
801 North Pitt Street
Alexandria, VA 22314

Dr. Kathryn T. Spoehr
Psychology Department
Brown University
Providence, RI 02912

Dr. Robert Sternberg
Department of Psychology
Yale University
Box 11A, Yale Station
New Haven, CT 06520

Dr. Peter Stoloff
Center for Naval Analysis
200 North Beauregard Street
Alexandria, VA 22311

Dr. William Stout
University of Illinois
Department of Mathematics
Urbana, IL 61801

Dr. Patrick Suppes
Insti. for Mathematical Studies
in the Social Sciences
Stanford University
Stanford, CA 94305

Dr. Hariharan Swaminathan
Lab. of Psy. & Eval. Research
School of Education
University of Massachusetts
Amherst, MA 01003

Dr. Kikumi Tatsuoka
Comp. Based Educ. Research
252 Eng. Research Laboratories
Urbana, IL 61820

Dr. Maurice Tatsuoka
220 Education Building
1310 South Sixth Street
Champaign, IL 61820

Dr. David Thissen
Department of Psychology
University of Kansas
Lawrence, KS 66044

Dr. Douglas Towne
Univer. of So. California
Behavioral Technology Labs
1845 S. Elena Avenue
Redondo Beach, CA 90277

Dr. Robert Tsutakawa
Department of Statistics
University of Missouri
Columbia, MO 65201

Dr. J. Uhlaner
Uhlaner Consultants
4258 Bonavita Drive
Encino, CA 91436

Dr. V. R. R. Uppuluri
Union Carbide Corporation
Nuclear Division
P.O. Box Y
Oak Ridge, TN 37830

Dr. David Vale
Assessment Systems Corporation
2233 University Avenue
Suite 310
St. Paul, MN 55114

Dr. Kurt Van Lehn
Xerox PARC
3333 Coyote Hill Road
Palo Alto, CA 94304

Dr. Howard Wainer
Division of Psy. Studies
Educational Testing Service
Princeton, NJ 08540

Dr. Michael T. WAller
Department of Educ. Psychology
Univeristy of Wis.-Milwaukee
Milwaukee, WI 53201

Dr. Brian Waters
HumRRO
300 North Washington
Alexandria, VA 22314

Dr. Phyllis Weaver
2979 Alexis Drive
Palo Alto, CA 94304

Dr. David J. Weiss
N660 Elliott Hall
University of Minnesota
75 E. River Road
Minneapolis, MN 55455

Dr. Donald O. Weitzman
Mitre Corporation
1820 Dolley Madison Blvd
McLean, VA 22102

Dr. Rand R. Wilcox
University of Southern Califor.
Department of Psychology
Los Angeles, CA 90007

Wolfgang Wildgrube
Streitkraefteamt
Box 20 50 03
D-5300 Bonn 2
WEST GERMANY

Dr. Bruce Williams
Depart. of Educ. Psychology
University of Illinois
Urbana, IL 61801

Dr. Wendy Yen
CTB/McGraw Hill
Del Monte Research Park
Monterey, CA 93940

H. William Greenup
Education Advisor (E031
Education Center MCDEC
Quantico, VA 22134

Dr. Douglas H. Jones
Advanced Statistical Technologies Corporation
10 Trafalgar Court
Lawrenceville, NJ 08148

Professor John A. Keats
Department of Psychology
The University of Newcastle
N.S.W. 2308
AUSTRALIA

Dr. Marcy Lansman
University of North Carolina
Davie Hall 013A
Chapel Hill, NC 27514

Dr. Alan Lesgold
Learning R&D Center
University of Pittsburgh
3939 O'Hara Street
Pittsburgh, PA 15260

Dr. Michael Levine
Depart. of Educa. Psychology
210 Education Bldg.
University of Illinois
Champaign, IL 61801

Dr. Charles Lewis
Fac. Soc. Wetenschappe
Rijksuniversiteit Groningen
Oude Boteringestraat 23
9712GC Groningen, NETHERLANDS

Dr. Robert Linn
College of Education
University of Illinois
Urbana, IL 61801

Mr. Phillip Livingston
Systems & Applied Sci. Corp.
6811 Kenilworth Avenue
Riverdale, MD 20840

Dr. Robert Lockman
Center for Naval Analysis
200 North Beauregard Street
Alaexandria, VA 22311

Dr. Frederic M. Lord
Educational Testing Service
Princeton, NJ 08541

Dr. James Lumsden
Department of Psychology
University of Western Australia
Nedlands W.A. 6009
AUSTRALIA

Dr. Gary Marco
Stop 31-E
Educational Testing Service
Princeton, NJ 08451

Dr. Scott Maxwell
Department of Psychology
University of Notre Dame
Notre Dame, IN 46556

Dr. Samuel T. Mayo
Loyola University of Chicago
820 North Michigan Avenue
Chicago, IL 60611

Mr. Robert McKinley
American College Testing Pro.
P.O. Box 168
Iowa City, IA 52243

Dr. Barbara Means
Human Resources Research Organ.
300 North Washington
Alexandria, VA 22314

Dr. Allen Munro
Behavioral Tech. Laboratories
1845 Elena Ave., Fourth Floor
Redondo Beach, CA 90277

Dr. W. Alan Nicewander
University of Oklahoma
Department of Psychology
Oklahoma City, OK 73069

Dr. Donald A. Norman
Cognitive Science, C-015
Univ. of California, San Diego
La Jolla, CA 92093

Dr. Melvin R. Novick
356 Lindquist Cnt. for Measure.
University of Iowa
Iowa City, IA 52242

Dr. James Olson
WICAT, Inc.
1875 Wouth State Street
Orem, UT 84057

Wayne M. Patience
American Council on Education
GED Testing Service, Suite 20
One Dupon Cirle, NW
Washington, DC 20036

Dr. James A. Paulson
Portland State University
P.O. Box 751
Portland, OR 97207

Dr. James W. Pellegrino
University of California, Santa Barbara
Department of Psychology
Santa Barabara, CA 93106

Dr. Mark D. Reckase
ACT
P.O. Box 168
Iowa City, IA 52243

Dr. Lauren Resnick
LRDC
University of Pittsburgh
3939 O'Hara Street
Pittsburg, PA 15260

Dr. Thomas Reynolds
University of Texas-Dallas
Marketing Department
P.O. Box 688
Richardson, TX 75080

Dr. Andrew M. Rose
Amer. Insti. for Research
1055 Thomas Jefferson St. NW
Washington, DC 20007

Dr. Lawrence Rudner
403 Elm Avenue
Takoma Park, MD 20012

Dr. J. Ryan
Department of Education
Univeristy of South Carolina
Columbia, SC 29208

Prof. Fumiko Samejima
Department of Psychology
Univeristy of Tennessee
Knoxville, TN 37916

Dr. Walter Schneider
Psychology Department
603 E. Daniel
Champaign, IL 61820

Dr. Kazuo Shigemasu
7-9-24 Kugenuma-Kaigan
Fujusawa 251
JAPAN

Dr. Erling B. Andersen
Department of Statistics
Studiestraede 6
1455 Copenhagen
DENMARK

Dr. Isaac Bejar
Educational Testing Service
Princeton, NJ 08450

Dr. Menucha Birenbaum
School of Education
Tell Aviv University
Tel Aviv, Ramt Aviv 60078
ISRAEL

Dr. Werner Birke
Personalstammamt der Bundeswehr
D-5000 Hoeln 90
WEST GERMANY

Dr. Robert Brennan
American College Testing Programs
P.O. Box 168
Iowa City, IA 52243

Dr. Glenn Bryan
6208 Poe Road
Bethesda, MD 20817

Dr. Ernest R. Cadotte
307 Stokely
University of Tennnessee
Knoxville, TN 37916

Dr. John B. Carroll
409 Elliott Road
Chapel Hill, NC 27514

Dr. Norman Cliff
Department of Psychology
Univ. of So. California
Univesity Park
Los Angeles, CA 90007

Dr. Lloyd Humphreys
Department of Psychology
University of Illinois
603 East Daniel Street
Champaign, IL 61820

Dr. William Koch
University of Texas-Austin
Measurement and Evaluation Center
Austin, TX 78703

Dr. Allan M. Collins
Bolt Beranek & Newman, Inc.
50 Moulton Street
Cambridge, MA 02138

Dr. Hans Crombag
Education Research Center
University of Leyden
Boerhaavelaan 2
2334 EN Leyden, NETHERLANDS

CTB/McGraw-Hill Library
2500 Garden Road
Monterey, CA 93940

Dr. Dattpradad Divgi
Syracuse University
Department of Psychology
Syracuse, NE 33210

Dr. Fritz Drasgow
Department of Psychology
Univeristy of Illinois
603 E. Daniel St.
Champaign, IL 61820

Dr. Susan Embertson
Psychology Department
University of Kansas
Lawrence, KS 66045

ERIC Facility-Acquisitions
4833 Rugby Avenue
Bethesda, MD 20014

Dr. Benjamin A. Fairbank, Jr.
McFann-Gray & Associates, Inc.
5825 Callaghan
Suite 225
San Antonio, TX 78228

Dr. Leonard Feldt
Lindquist Center for Measurement
Univesity of Iowa
Iowa City, IA 52242

Dr. Earl Hunt
Department of Psychology
University of Washington
Seattle, WA 98105

Dr. Jack Hunter
2122 Coolidge Street
Lansing, MI 48906

Dr. Richard L. Ferguson
The American College Testing Program
P.O. Box 168
Iowa City, IA 52240

Univ. Prof. Dr. Gerhard Fischer
Liebiggasse 5/3
A 1010 Vienna
AUSTRIA

Professor Donald Fitzgerald
University of New England
Armidale, New South Wales 2351
AUSTRALIA

Dr. Dexter Fletcher
University of Oregon
Department of Computer Science
Eugene, OR 97403

Dr. John R. Frederiksen
Bolt Beranek & Newman
50 Moulton Street
Cambridge, MA 02138

Dr. Janice Gifford
University of Massachusetts
School of Education
Amherst, MA 01002

Dr. Robert Glaser
Learning Research & Dev. Center
University of Pittsburgh
3939 O'Hara Street
Pittsburgh, PA 15260

Dr. Bert Green
Johns Hopkins University
Department of Psychology
Charles & 34th Street
Baltimore, MD 21218

Dr. Ron Hambleton
School of Education
University of Massachusetts
Amherst, MA 01002

Dr. Paul Horst
677 G Street, #184
Chula Vista, CA 90010

Dr. Huynh Huynh
College o´ Education
University of South Carolina
Columbia, SC 29208

AIR FORCE HUMAN RESOURCES LAB
AFHRL/MPD
BROOKS AFB, TX 78235

Technical Documents Center
AF Human Resources Lab
WPAFB, OH 45433

US AF OFFICE OF SCI. RES.
LIFE SCI. DIRECTORATE, NL
BOLLING AIR FORCE BASE
WASHINGTON, DC 20332

AIR UNIVERSITY LIBRARY
AUL/LSE 76/443
MAXWELL AFB, AL 36112

DR. EARL A. ALLUISI
HQ, AFHRL (AFSC)
BROOKS AFB, TX 78235

MR. RAYMOND E. CHRISTAL
AFHRL/MOE
BROOKS AFB, TX 78235

DR. ALFRED R. FREGLY
AFOSR/NL
BOLLING AFB, DC 20332

DR. GENEVIEVE HADDAD
PROGRAM MANAGER
LIFE SCIENCES DIRECTORATE
AFOSR
BOLLING AFB, DC 20332

DR. T. M. LONGRIDGE
AFHRL/OTE
WILLIAMS AFB, AZ 85224

MR. RANDOLPH PARK
AFHRL/MOAN
BROOKS AFB, TX 78235

DR. ROGER PENNELL
AF HUMAN RESOURCES LAB
LOWRY AFB, CO 80230

DR. MALCOM REE
AFHRL/MP
BROOKS AFB, TX 78235

3700 TCHTW/TTGHR
2LT TALLARIGO
SHEPPARD AFB, TX 76311

LT. COL JAMES E. WATSON
HQ USAF/MPXOA
THE PENTAGON
WASHINGTON, DC 20330

MAJOR JOHN WELSH
AFHRL/MOAN
BROOKS AFB, TX

DR. JOSEPH YASATUKE
AFHRL/LRT
LOWRY AFB, CO 80230

DEFENSE TECHNICAL INFO CEN.
CAMERON STATION, BLDG 5
ALEXANDRIA, VA 22314
ATTN: TC

DR. CRAIG I. FIELDS
ADVANCED RES. PROJECTS AGENCY
1400 WILSON BLVD.
ARLINGTON, VA 22209

DR. WILLIAM GRAHAM
TESTING DIRECTORATE
MEPCOM/MEPCT-P
FT. SHERIDAN, IL 60037

Jerry Lehnus
HQ MEPCOM
Attn: MEPCT-P
Fort Sheridan, Il 60037

MILITARY ASS. FOR TRAINING &
PERSONNEL TECH.
OFF. OF THE UNDER SEC. FOR RES
ROOM 3D129, the PENTAGON
WASHINGTON, DC 20301

DR. WAYNE SELLMAN
OFF. OF THE ASS. SEC. OF
DEFENSE (MRA & L)
2B269 the Pentagon
WASHINGTON, DC 20301

MAJOR JACK THORPE
DARPA
1400 WILSON BLVD.
ARLINGTON, VA 22209

DR. SUSAN CHIPMAN
LEARNING & DEVELOPMENT
NAT'L INST. OF EDUCATION
1200 19th STREET NW
WASHINGTON, DC 20208

DR. ARTHUR MELMED
724 BROWN
US DEPARTMENT OF ED.
WASHINGTON, DC 20208

DR. ANDREW R. MOLNAR
OFF OF SCI & ENG PER & ED
NAT'L SCIENCE FOUNDATION
WASHINGTON, DC 20415

DR. VERN W. URRY
PERSONNEL R&D CENTER
OFF OF PERSONNEL MANAGEMENT
1900 E STREET NW
WASHINGTON, DC 20415

MR. THOMAS A. WARM
US COAST GUARD INST.
P.O. SUBSTATION 18
OKLAHOMA CITY, OK 73169

DR. JOSEPH L. YOUNG, DIR.
MEMORY & COGNITIVE PROCESSES
NAT'L SCIENCE FOUNDATION
WASHINGTON, DC 20550

DR. HAMES ALGINA
UNIVERSITY OF FLORIDA
GAINESVILLE, FL 326

MR. BRAD SYMPSON
NAVY PERSONNEL R&D CENTER
SAN DIEGO, CA 92152

DR. MARTIN A. TOLCOTT
ASS. DIR. FOR LIFE SCIENCE
800 NORTH QUINCY STREET
ARLINGTON, VA 22217

DR. JAMES TWEEDDALE
TECHNICAL DIRECTOR
NAVY PERSONNEL R&D CENTER
SAN DIEGO, CA 92152

PERSONNEL & TRAINING
RESEARCH GROUP
CODE 442PT
OFFICE OF NAVAL RESEARCH
ARLINGTON, VA 22217

SPECIAL ASST. FOR EDUCATION
AND TRAINING (OP-01E)
RM. 2705 ARLINGTON ANNEX
WASHINGTON, DC 20370

LT FRANK C. PETHO, MSC,
USN (PH.D)
CNET (N-432)
NAS
PENSACOLA, FL 32508

OFFICE OF THE CHIEF OF NAVAL
OPERATIONS
RES. DEV. & STUDIES BRANCH
OP 115
WASHINGTON, DC 20350

DR. BERNARD RIMLAND (01C)
NAVY PERSONNEL R&D CENTER
SAN DIEGO, CA 95152

DR. CARL ROSS
CNET-PDCD
BUILDING 90
GREAT LAKES NTC, IL 60088

DR. ROBERT G. SMITH
OFFICE OF CHIEF OF NAVAL OP.
OP-987H
WASHINGTON, DC 20350

DR. ALFRED F. SMODE, DIR.
TRAINING ANALYSIS &
EVALUATION GROUP
DEPT. OF THE NAVY
ORLANDO, FL 32813

DR. R. SNOW, LIAISON SCI.
OFFICE OF NAVAL RESEARCH
BRANCH OFFICE, LONDON
BOX 39
FPO NEW YORK, NY 09510

DR. RICHARD SORENSEN
NAVY PERSONNEL R&D CENTER
SAN DIEGO, CA 92152

DR. FREDERICK STEINHEISER
CNO - OP115
NAVY ANNEX
ARLINGTON, VA 20370

DR. FRANK VICINO
NAVY PERSONNEL R&D CENTER
SAN DIEGO, CA 92152

DR. EDWARD WEGMAN
OFFICE OF NAVAL RESEARCH
800 NORTH QUINCY STREET
ARLINGTON, VA 22217

DR. RONALD WEITZMAN
NAVAL POSTGRADUATE SCHOOL
DEPT. OF ADM. SCIENCES
MONTEREY, CA 93940

DR. DOUGLAS WETZEL
CODE 12
NAVY PERSONNEL R&D CENTER
SAN DIEGO, CA 92152

DR. MARTIN F. WISKOFF
NAVY PERSONNEL R&D CENTER
SAN DIEGO, CA 92152

MR. JOHN H. WOLFE
NAVY PERSONNEL R&D CENTER
SAN DIEGO, CA 92152

DR. WALLACE WULFECK, III
NAVY PERSONNEL R&D CENTER
SAN DIEGO, CA 92152

DR. NICK BOND
OFFICE OF NAVAL RESEARCH
LIAISON OFFICE, FAR EAST
APO SAN FRANCISCO, CA 96503

LT. ALEXANDER BORY
APPLIED PSYCHOLOGY
MEASUREMENT DIVISION
NAMRL
NAS PENSACOLA, FL 32508

DR. ROBERT CARROLL
NAVOP 115
WASHINGTON, DC 20370

DR. ROBERT BREAUX
NAVTRAEQUIPCEN
CODE N-095R
ORLANDO, FL 32813

Chief of Naval Ed. & Training
Liason Office
Air Force Human Resource Lab
Operations Training Division
WILLIAMS AFB, AZ 85224

DR. STANLEY COLLYER
OFFICE OF NAVAL TECHNOLOGY
800 N. QUINCY STREET
ARLINGTON, VA 22217

CDR MIKE CURRAN
OFFICE OF NAVAL RESEARCH
800 N. QUINCY ST.
CODE 270
ARLINGTON, VA 22217

MIKE DURMEYER, INSTRUCTIONAL
PROGRAM DEVELOPMENT
BUILDING 90
NET-PDCD
GREAT LAKES NTC, IL 60088

DR. RICHARD ELSTER
DEPT. OF ADM. SCIENCES
NAVAL POSTGRADUATE SCHOOL
MONTEREY, CA 93940

DR. PAT FEDERICO
CODE P13
NPRDC
SAN DIEGO, CA 92152

DR. CATHY FERNANDES
NAVY PERSONNEL R&D CENTER
SAN DIEGO, CA 92152

LMED

-8