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UNCLASSIFIED MRC-TSR-2632 DAAG29-80-C-0041 F/G 12/1 NL

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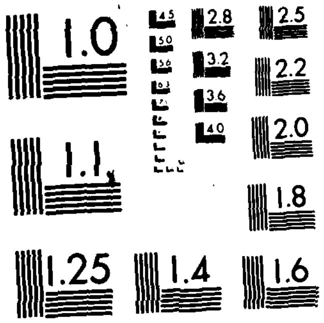


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MRC Technical Summary Report # 2632

# ON THE EQUIDECOMPOSABILITY OF A REGULAR TRIANGLE AND A SQUARE OF EQUAL AREAS

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January 1984

(Received December 16, 1983)





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## ABSTRACT

This paper deals with the subject of the first chapter of the second author's book "Mathematical Time Exposures": The equidecomposability of a regular triangle and a square of equal areas. A new solution of the problem is given, which also shows that the solution of the problem as given in the third edition of Hogo Steinhaus' "Mathematical Snapshots", is not correct.

AMS (MOS) Subject Classifications: 51-01, 51N20

Key Words: Equidecomposability of plane polygons

Work Unit Number 6 (Miscellaneous Topics)

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Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

## SIGNIFICANCE AND EXPLANATION

It is shown that the solution of the problem of the title, as given in the first snapshot of H. Steinhaus' "Mathematical Snapshots" is not correct. This is derived from a new solution of the problem.



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ON THE EQUIDECOMPOSABILITY OF A REGULAR TRIANGLE AND
A SQUARE OF EQUAL AREAS

D. W. Crowe and I. J. Schoenberg

Two plane polygons are said to be equidecomposable if it is possible to decompose one of them into a finite number of parts which can be rearranged to form the second polygon (see [1] and [2, Chapter 1]). The proposition of our title is the subject of [2, Chapter 1, pp. 1-6], where Chapter 1 is entitled "On Steinhaus's first mathematical snapshot of 1939". The reason why we return to it now is to point out that the solution as sketched in Figure 2 on page 4 of [3] is not correct. This is shown by the following different solution of the problem.

Let $T = ABC$ be a regular triangle having sided $= 2$ (Figure 1), where D, E, O are the midpoints of its sides. We choose the lines OB and OA as coordinate axes. Let $0 < \xi < 1$ and $F = (\xi, 0)$; on CB we take the segment GF of unit length, hence $G = (\xi - 1, 0)$. Since $DE = GF = 1$, it follows that DEFG is a parallelogram. We join

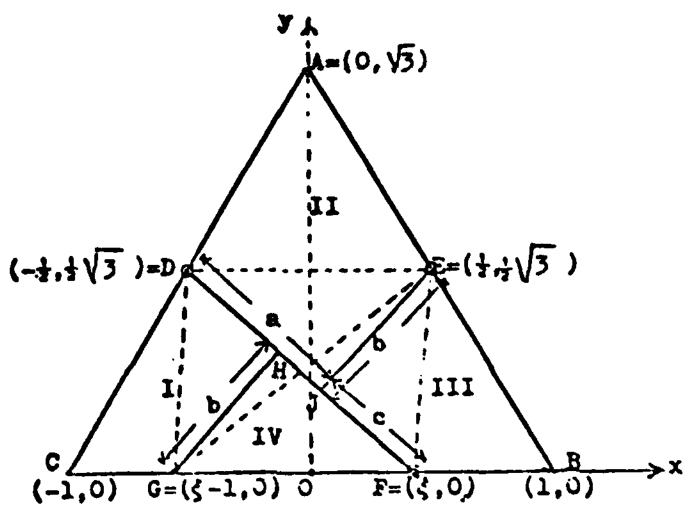


Fig.1

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D to F, and draw GH and EJ perpendicular to DF. Finally, let T be dissected into three quadrilaterals and one triangle as follows:

(1) I:= CGHD, II:= ADJE, III:= BFJE, IV:= HGF.

Because DEFG is a parallelogram, we have the segments a, c, b, defined by

(2) a = DJ = HF, c= DH = JF, b = GH = JE,

as indicated in Figure 1.

We claim: The triangle ABC = I ∪ II ∪ III ∪ IV is equidecomposable with a

(3) rectangle R of dimensions a + c and 2b.

To prove this we assume T = ABC to be made of stiff paper, and we dissect it into the four polygons (1). These we assume to be hinged at the three points F, E, and D.

We now perform three turning operations as follows:

1° We turn IV about the hinge F by +180°, obtaining IV' of Figure 2.

2° We turn the union III ∪ IV' about the hinge E by +180°, obtaining III' IV" of Figure 3.

3° We turn the union II ∪ III' ∪ IV" about the hinge D by +180°, obtaining the final union I ∪ II' ∪ III" ∪ IV''' of Figure 4.

If we observe the various segments of length

$1, \xi, 1 - \xi, a, b,$ and $c,$

as indicated in Figure 1, and also in Figures 2, 3, and 4, we find the final figure of Figure 4 to be a rectangle R of dimensions as described by (3). This proves our claim.

So far the position of the point $F = (\xi, 0)$, of Figure 1, has remained arbitrary. Now we wish to determine $\xi$ so that the rectangle R should become a square. By (3) this will be the case, provided that

(4) $a + c = 2b.$

We evidently need the length a, b, and c, as functions of $\xi$. From Figure 1 we find that

$$(a + c)^2 = (DF)^2 = (\xi + 1/2)^2 + 1/4\, 3 = \xi^2 + \xi + 1,$$

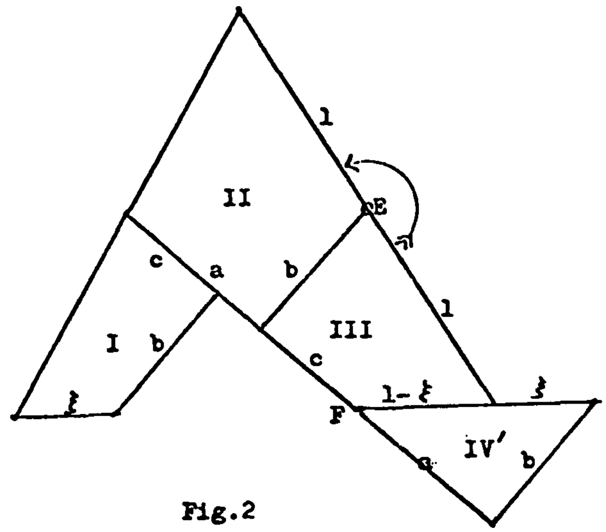
1
II
E
c
a
b
1
I
b
III
c
ξ
1- ξ
ξ
F
a
IV′
b

Fig.2

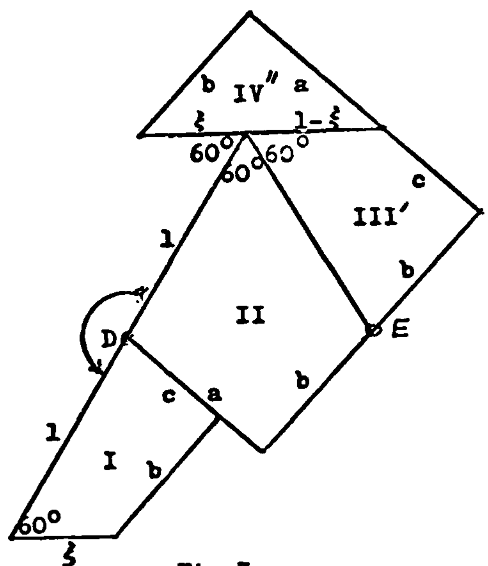
b
IV″
a
ξ
1- ξ
60°
60°
60°
c
III′
1
b
D
II
E
c
a
b
1
I
b
60°
ξ

Fig.3

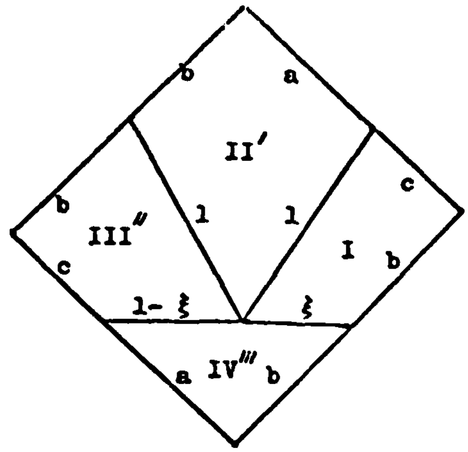


**Fig.4**

whence

(5) $a + c = \sqrt{\xi^2 + \xi + 1}.$

Observe next that the normal form of the equation of the line DF is found to be

$$\text{DF:} \quad \frac{\sqrt{3}\,x + (1+2\xi)y - \sqrt{3}\,\xi}{2\sqrt{\xi^2 + \xi + 1}}$$

This gives the distance b from E to the line DF to be

$$b = \tfrac{1}{2}\sqrt{3/(\xi^2 + \xi + 1)},$$

whence

(6) $2b = \sqrt{3/(\xi^2 + \xi + 1)}.$

The desired equation (4) becomes

(7) $\xi^2 + \xi + 1 - \sqrt{3} = 0,$

its solution being

(8) $= \dfrac{-1 + \sqrt{1 + 4(\sqrt{3} - 1)}}{2} = .490985 .$

This accuracy should be sufficient for the realization of the ingenious linkage of four polygons (1) as described by Steinhaus (See [2, Figure 1.3 on page 2]).

Returning to the incorrect solution as given in [3, Figure 2 on p. 4], we observe that it assumes that the parallelogram DEFG of our Figure 1 is a rectangle. This gives the value $\xi = .5$, which represents a relative error of 1.8%.

That the "solution" $\xi = 1/2$ can not possibly be correct is immediately apparent from our Figure 1: Indeed, it would imply that DEFG is a non-square rectangle and therefore

$$a + c = DF = EG > GH + JE = b + b = 2b,$$

and the equation (4) does not hold.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

| REPORT DOCUMENTATION PAGE | | READ INSTRUCTIONS BEFORE COMPLETING FORM |
|---|---|---|
| 1. REPORT NUMBER 2632 | 2. GOVT ACCESSION NO. AD-A139270 | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle) ON THE EQUIDECOMPOSABILITY OF A REGULAR TRIANGLE AND A SQUARE OF EQUAL AREAS | | 5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period |
| | | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s) D. W. Crowe and I. J. Schoenberg | | 8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706 | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 6 - Miscellaneous Topics |
| 11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office, P.O. Box 12211, Research Triangle Park, North Carolina 27709 | | 12. REPORT DATE January 1984 |
| | | 13. NUMBER OF PAGES 5 |
| 14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) | | 15. SECURITY CLASS. (of this report) UNCLASSIFIED |
| | | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Equidecomposability of plane polygons

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