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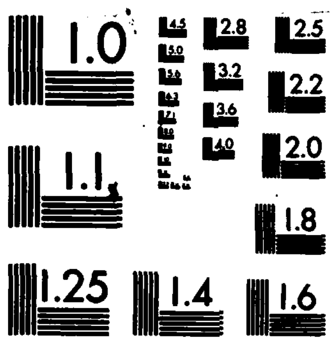
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THE AUTOCORRELATION FUNCTION OF SEASONAL ARMA MODELS

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Autoregressive Moving Average

ABSTRACT

This note obtains the theoretical autocorrelation function of an ARMA model with multiplicative seasonality. It is shown that this function can be interpretated as the result of the interaction between the seasonal and regular autocorrelation patterns of the ARMA model. The use of this result makes easier the identification of the structure of the model, is helpful in choosing between a multiplicative or additive seasonal component and leads to a better understanding of the properties of the estimated autocorrelation function of scalar ARMA processes.

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*Daniel Peña is Professor of Statistics at the Escuela Técnica Superior de Ingenieros Industriales, University of Madrid. The author is indebted to Arthur B. Treadway for many helpful comments on an earlier draft of this paper.

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SIGNIFICANCE AND EXPLANATION

In the process of building an ARIMA model for a time series, an initial model should be identified analyzing the patterns of the estimated autocorrelation function and partial autocorrelation function. Comparing these observed functions to the theoretical ones associated with different ARMA models an initial model can be entertained.

This identification stage was sometimes very difficult for seasonal models, because the theoretical structure of the autocorrelation function was not completely known. This function is obtained in this paper. The result is not only interesting from a theoretical point of view but has important practical implications as shown in an example.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

THE AUTOCORRELATION FUNCTION OF SEASONAL ARMA MODELS

Daniel Peña*

1. INTRODUCTION

It has often been stressed that the most difficult task in obtaining an ARIMA model for a given time series is the identification of the order of the process. Box and Jenkins (1970) developed broadly the theoretical properties of the autocorrelation function (acf) and partial autocorrelation function (paf) for processes without seasonal structure and outlined the pattern of the acf in some special seasonal processes when the regular part is moving average of order one or two. Cleveland (1972) proposed the inverse autocorrelation function as an alternative to the paf. Hamilton and Watts (1978) obtained the exact paf for the simplest ARIMA models and were able to show the general pattern of this function, as a simple composite of the autocorrelation and partial autocorrelation coefficients of the regular component. Finally, Cleveland and Tiao (1979) have proposed a broad class of seasonal models in which the dependence structure among the observation is not invariant to shifts in time, as assumed in the standard seasonal ARIMA representation.

One important difficulty in the identification of seasonal ARIMA processes is that the ignorance of the exact properties of the acf made this stage, according to Hamilton and Watts (1978), "a perplexing task because the correlation function is susceptible to confusing distortion in the case of seasonal time series". We will show in this note that the acf of a seasonal process can be interpreted readily as the result of the interaction between the seasonal and regular components of the model.

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2. THE SIMPLE AUTOCORRELATION FUNCTION FOR THE SEASONAL MODEL

Let us write a stationary seasonal ARMA model as:

$$W_t = \psi(B)\ \psi_s(B^s)\ a_t$$

where ψ and ψ_s are possible infinite operators. Let $\lambda(B)$ and $\Gamma_s(B^s)$ be the autocovariance generating functions for the regular and seasonal parts of W_t. Then, it can be shown (Box and Jenkins (1970)), that the autocovariance generating function of the process, $\gamma(B)$, is:

$$\gamma(B) = \sigma_a^2\ \lambda(B)\ \Gamma_s(B^s) = \sum_{-\infty}^{\infty} \gamma_i B^i \qquad (2.1)$$

where σ_a^2 is the variance of the white noise process, a_t, and

$$\lambda(B) = \psi(B)\ \psi(B^{-1}) = \sum_{-\infty}^{\infty} \lambda_i B^i$$

$$\Gamma_s(B^s) = \psi_s(B^s)\ \psi_s(B^{-s}) = \sum_{-\infty}^{\infty} \Gamma_{si} B^{si}$$

The autocorrelation generating function of W_t is:

$$a(B) = \gamma_0^{-1}\ \gamma(B) = \sigma_a^2\ \gamma_0^{-1}\lambda_0\Gamma_0\rho(B)\rho_s(B^s) \qquad (2.2)$$

where $\rho(B) = \lambda_o^{-1}\lambda(B)$ and $\rho_s(B^s) = \Gamma_0^{-1}\Gamma_s(B^s)$ are the autocorrelation generating functions of the regular and seasonal part.

Let us call r_i, R_i the theoretical autocorrelation coefficients of the regular and seasonal part of the model. Then, we can write

$$a(B) = K\rho(B)\rho_s(B^s) \qquad (2.3)$$

where

$$K = \gamma_0^{-1}\sigma_a^2\lambda_0\Gamma_0 = (1 + 2\sum_{i=1}^{\infty} r_{si}R_{si})^{-1}. \qquad (2.4)$$

Calling ρ_i the theoretical autocorrelation coefficient of order i for the overall process, we then have that

$$a(B) = \sum_{-\infty}^{\infty} \rho_h B^h = K(\sum_{-\infty}^{\infty} r_j B^j)(\sum_{-\infty}^{\infty} R_{si} B^{si}) = K \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} r_j R_{si} B^{si+j} \tag{2.5}$$

so that by using the value of K and equating powers of B,

$$\rho_j = \frac{r_j + \sum_{i=1}^{\infty} R_{si} (r_{si+j} + r_{si-j})}{1 + 2 \sum_{i=1}^{\infty} r_{si} R_{si}} . \tag{2.6}$$

Equation (2.6) displays in a readily understandable way the structure of the acf for multiplicative seasonal models. As far as the interpretation of (2.6) is concerned, there are two different cases. The first occurs when the regular structure has autocorrelation coefficients, r_i , that are nearly zero for $i > s/2$. In this case the denominator of (2.6) is approximately unity and:

$$\rho_j = r_j + \sum_{i=1}^{\infty} R_{si}(r_{si-j} + r_{si+j}) \tag{2.7}$$

so that we will observe: (a) in low order lags $(i < s/2)$, the exact regular pattern; (b) in seasonal lags, the exact seasonal structures; (c) in lags that are near multiples of the seasonal periods $(i = is \pm h;\ h<s/2)$ the reproduction of the regular structure, symetrically at both sides of the seasonal periods.

When the regular autocorrelation coefficients do not vanish even approximately for $i > s/2$, distortion can be expected in the above pattern. The problem will be especially acute if the seasonal period is low, for example for quarterly data, and when the autoregressive operator has one root near the unit circle.

3. AN EXAMPLE

In order to illustrate the kind of information that can be obtained by the use of the properties of the interactions between the regular and seasonal structure, we will consider the ozone data that has been widely modeled by different procedures. Box and Tiao (1975) fitted a $(0,0,1) \times (0,1,1)_{12}$ ARIMA model to this series. Abraham and Box

(1978) have shown how this model could be improved through a deterministic seasonal modelling. Cleveland and Tiao (1979) have presented another useful approach for these data.

Table 1 shows the estimated acf and paf for this series seasonally differenced for the period November, 1955, through November, 1969. The model fitted by Box and Tiao (1975) was:

$$(1-B^{12})Z_t = \underset{(.08)}{(1+.14B)}\underset{(.02)}{(1-.89B^{12})}a_t \qquad (3.1)$$

$\hat{\sigma}_a^2 = 0.974 \quad Q(37) = 36.9.$

Although the Ljung-Box statistic Q does not reject the model, the acf of the residuals shows significant values at lags 2, 22 and 24.

Using the theoretical properties of the acf, Table 1 strongly suggests that the regular part is autoregressive. This fact is clear from the specific structure of signs of both the acf and pacf at both sides of seasonal lags. As far as the seasonal structure is concerned, there is some evidence of AR structure because: (a) the values of the acf at lag 24 is not only significant but the pattern of signs around lag 24 suggests interactions and (b) the value of the acf at lag 36 is almost significant and at both sides of lag 36 we find the expected pattern of signs for a negative coefficient.

The simplest hypothesis for the seasonal part is then an ARMA (1,1) with negative autoregressive parameter. The two first coefficients of the paf suggest an AR(2) for the regular component. The estimated model is:

$$\underset{(.08)}{(1+.13B^{12})}(1-\underset{(.08)}{.15B}-\underset{(.08)}{.15B^2})\Delta_{12}Z_t = (1-.87B^{12})a_t \qquad (3.2)$$

$\sigma_a^2 = 0.907 \quad Q(34) = 23.8.$

The F statistic to test the reduction of variance in model (3.2) versus model (3.1) is 5.76 that is highly significant with $\alpha = 0.005$. Furthermore the acf and pacf of the residuals do not produce any doubts about the adequacy of model (3.2).

Table 1

lag	1	2	3	4	5	6	7	8	9	10	11	12	13	14
acf	0.27	0.22	0.15	-0.09	0.00	-0.04	-0.14	0.02	-0.21	-0.23	-0.19	-0.52	-0.16	-0.06
pacf	0.27	0.16	0.06	-0.18	0.03	0.00	-0.11	0.07	-0.20	-0.16	-0.09	-0.43	-0.06	0.11

lag	15	22	23	24	25	26	27	34	35	36	37	38	39
acf	-0.02	0.16	0.02	0.18	0.08	0.03	0.04	-0.11	-0.02	-0.14	-0.09	-0.07	-0.08
pacf	0.06	0.02	-0.11	-0.07	0.10	0.13	0.08	-0.04	-0.06	-0.07	0.04	-0.03	0.03

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This note obtains the theoretical autocorrelation function of an ARMA model with multiplicative seasonality. It is shown that this function can be interpretated as the result of the interaction between the seasonal and regular autocorrelation patterns of the ARMA model. The use of this result makes easier the identification of the structure of the model, is helpful in choosing between a multiplicative or additive seasonal component and leads to a better understanding of the properties of the estimated autocorrelation function of scalar ARMA processes.

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