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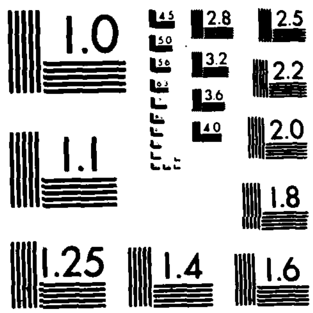

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THE LORENTZ TRANSFORMATION AND THE RADAR DOPPLER EQUATION

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ABSTRACT

The propagation of the electromagnetic waves in vacuum is discussed in terms of the Lorentz Transformation. A simple, but rigorous, derivation of the Lorentz Transformation is given. Then, the observed doppler shift, seen by a moving observer, of a moving target is derived in terms of inertial position and velocity. This radar doppler equation, given without approximation, is suitable for analysis of doppler data for satellite orbit determination.

TABLE OF CONTENTS

I. INTRODUCTION

Propagation of Electromagnetic Wave is rigorously described by Maxwell's equations. The description of Electromagnetic Wave propagation in one reference frame is related to the description of the same wave in a second reference frame by the General Lorentz Transformation (GLT), which is a direct consequence of the Special Theory of Relativity (TSR). The GLT encompasses both a change in the bases of the reference frames and a relative velocity of the two reference frames. The GLT is a unitary transformation of a four-vector, i.e., preserves the length of the vector. The GLT has all the usual properties, suitably generalized, of unitary transformations.

In the following development we will discuss the basic ideas that lead to the GLT, in a superficial but rigorous way. That is, the results are correct, but many of the fundamental consequences will not be addressed. These broader consequences of the GLT are addressed in many classical texts, and need not be repeated here. However, though the final results given here cannot be considered new, they are not explicitly given in any text. Therefore we hope to provide a complete and rigorous result, with sufficient detail that it can be applied directly in practical application and without ambiguity.

The radar Doppler equation involves four reference frames:

1. The frame of the transmitter at the time of transmission, in which the transmission frequency is observed.
2. The frame of the target (satellite) at the time of reflection or retransmission by a transponder, the same thing in the context of this discussion.
3. The frame of the receiver at the time of reception, in which the received frequency is observed.
4. The geocentric frame in which the target's trajectory (ephemeris) is described.

The Lorentz transformation is a mathematical description of the two basic tenents of the Theory of Special Relativity (TSR), as stated by Einstein in 1905:

1. All the laws of nature are identical in all inertial systems of reference. An inertial reference system is one in which a body, not acted upon by external forces, proceeds with constant velocity.
2. The velocity of propagation of interaction between particles, which is the same as the velocity of light in empty space, is the same in all inertial reference frames.

The TSR denies the existence of any fundamental reference frame. Therefore, none of these reference frames is any more fundamental than another. The principal objective of investigating the Radar Doppler Equation is for determination of the target's trajectory (orbit determination) based on the observed Doppler shift. Therefore, for the following discussion we will consider the geocentric frame as the basic frame, and seek to express the observed doppler shift as a function of the geocentric position and velocity.

II. THE LORENTZ TRANSFORMATION

We give here a simplified but complete derivation of the Lorentz transformation. There are a number of lucid developments of the Lorentz transformation [Refs. 1, 2, and 3] that reveal far more general properties. However for our purposes, precision metrology, what follows here allows us to be precisely clear about each quantity, when we come to practical calculation.

A reference frame is understood to consist of a system of meter sticks and clocks to define coordinates (x,y,z) and time (t). Now in a reference frame we can imagine relating the space and time coordinates by using the travel time $(t_2 - t_1)$ of electromagnetic propagation between two points (x_2, y_2, z_2) and (x_1, y_1, z_1) as:

$$c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 = 0 \quad (1)$$

A fundamental concept in the TSR is an event defined by the four quantities (t,x,y,z). Now we define the (proper) distance, i.e., the interval, between two events (s) as:

$$s^2 = c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 \quad (2)$$

or as differentials:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \ . \quad (3)$$

The electromagnetic propagation in a vacuum is said to have an interval s = 0 as given by (1). It follows from the second tenant of the TSR that if the interval between two events is zero in *one* inertial reference frame, then it is zero in all other inertial reference frames.

To make this more concrete, consider an inertial reference frame (K') moving with uniform velocity (V) with respect to an inertial reference frame (K) along the x axis as illustrated in Fig. 1. The differential interval between two events (ds) in the K frame is given by (3). The same two events viewed in the K' frame would have an interval (ds'):

$$(ds')^2 = c^2(dt')^2 - (dx')^2 - (dy')^2 - (dz')^2 \ . \quad (4)$$

We know that if ds = 0 in the K reference frame, then ds' = 0. Therefore, in general ds and ds' must be proportional, i.e.:

$$ds = a \ ds' \quad (5)$$

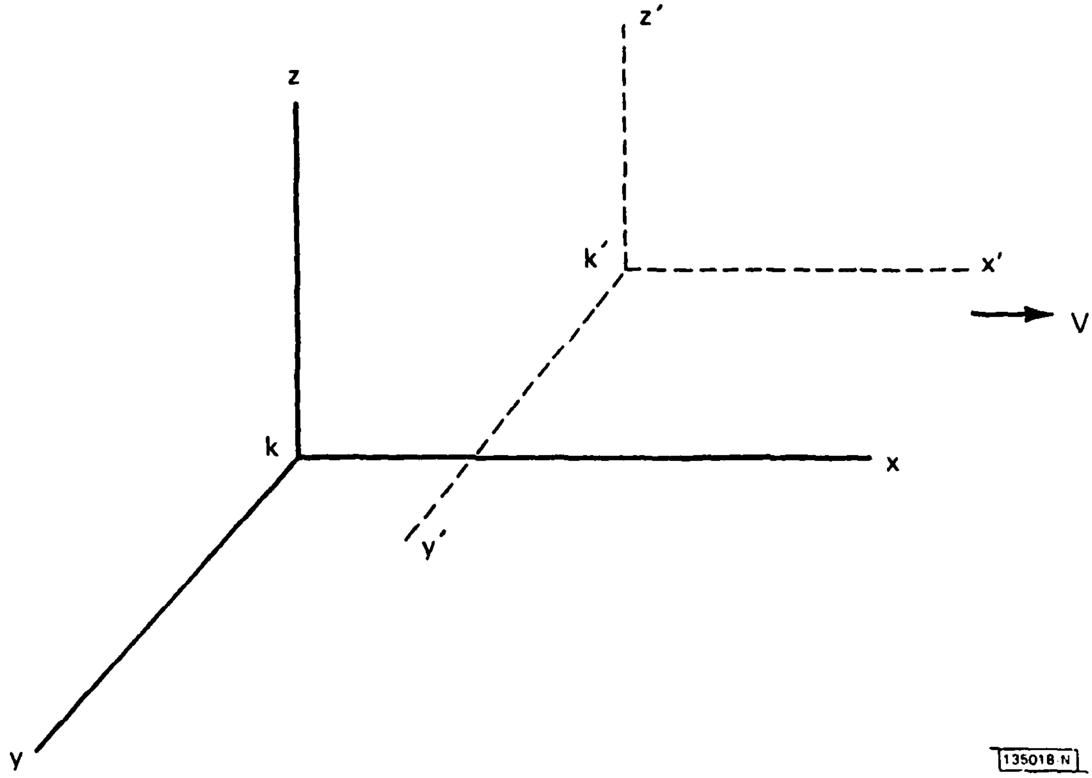

Fig. 1. K reference frame and K' reference frame.

where a can depend on V, i.e., a = a(V). But since the role of K and K' can be reversed, i.e., K is moving the K' is not, then

$$ds' = a\ ds \tag{6}$$

and therefore a = 1. Landau and Lifshitz [Ref. 2] give a more general discussion along these lines including among other things the possibility that a = -1. In any case we must have ds = ds' and therefore s = s'. Therefore, the interval between two ents (s or ds) is the same in all inertial reference frames, i. it is invariant under transformation from one inertial syster o another. In particular, this invariance is the mathematic expression of the constancy of the velocity of light. Finally we have:

$$\begin{aligned} ds^2 &= c^2dt^2 - dx^2 - dy^2 - dz^2 \\ &= c^2(dt')^2 - (dx')^2 - (dz')^2 \\ &= (ds')^2 \ . \end{aligned} \tag{7}$$

To obtain the mathematical form of the GLT we first consider a clock fixed in the K' system (dx' = dy' = dz' = 0) and therefore moving with respect to the K system along the x axis with velocity V. Now this clock, viewed in the K system would be at position dx, dy, dz, after time dt. The time dt would be measured with an identical clock. We have then:

$$dt' = dt \sqrt{1 - \frac{dx^2 + dy^2 t dz^2}{c^2 dt^2}} \; . \tag{8}$$

But

$$\frac{dx^2 + dy^2 + dz^2}{dt^2} = V^2 \tag{9}$$

so we have:

$$dt' = \sqrt{1 - (V/c)^2 \; dt} \tag{10}$$

This represents the "twin paradox," i.e., moving clocks "appear" to run slower than stationary clocks.

We now seek the transformation of an event recorded in the K system to the same event as it would be recorded in the K' system. We restrict the discussion to motion of the K' system along the x axis of the K system as one can always rotate the x, y, z coordinates to a system with the motion of the K' system along the x axis of the K system. In this case, the y and z coordinates are unaffected and we have y = y' and z = z'. Now we can seek a linear transformation:

$$\begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} a_{11}t' + a_{12}x' \\ a_{21}ct' + a_{22}x' \end{bmatrix} = [a_{ij}] \quad \begin{bmatrix} t' \\ x' \end{bmatrix} \tag{11}$$

Clearly, we must have:

$$[a_{ij}]^{-1} \; [a_{ij}] = I \tag{12}$$

where I is the unit matrix. From the moving clock experiment above we have immediately:

$$a_{11} = \frac{1}{\sqrt{1 - (V/c)^2}} = \frac{1}{\sqrt{1 - \beta^2}} = \gamma \ . \tag{13}$$

Now consider the motion of the K' origin as seen in the K system. We have $x' = y' = z' = 0.$ and

$$t = a_{11}t' = \gamma t'$$

$$x = a_{21}ct'$$

$$\therefore \quad a_{11} = \frac{\gamma}{c}\,\frac{x}{t} = \gamma\frac{V}{c} = \gamma\beta \ . \tag{14}$$

Since the roles of K a K' can be reversed, we must have:

$$[\alpha_{ij}(V)]^{-1} = [\alpha_{ij}(-V)] \ , \tag{15}$$

i.e.,

$$\begin{bmatrix} \gamma & a_{12}(-\beta) \\ -\beta\gamma c & a_{22}(-\beta) \end{bmatrix} \begin{bmatrix} \gamma & a_{12}(\beta) \\ \beta\gamma c & \alpha_{22}(\beta) \end{bmatrix} = I \tag{16}$$

which immediately gives:

$$\alpha_{12}(\beta) = \frac{\beta\gamma}{c}$$

$$a_{22} = \gamma \qquad (17)$$

Therefore the Lorentz transformation can be written:

$$\begin{bmatrix} t \\ x \end{bmatrix} = \begin{bmatrix} \gamma & \beta\gamma/c \\ c\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} t' \\ x' \end{bmatrix}$$

or

$$\begin{bmatrix} t' \\ x' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma/c \\ -\beta\gamma c & \gamma \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix} . \qquad (18)$$

We can write these expressions in a more symmetrical form by introducing a notation that considerably simplifies the writing of equations and makes quite clear the nature of the transformation. We deal with the three space variables as a (contravariant) vector:

$$\begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad (19)$$

(Since we are dealing with flat space-time, we need not be concerned with the distinction between covariant and contravariant vectors and tensors.) The time coordinate can be given in the same units as:

$$x^0 = ct \tag{20}$$

Therefore the Lorentz transformation to the K' system moving with velocity V along the x axis is:

$$\begin{bmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{21}$$

In the following we will label the spatial (three-vector) as $\bar{\bar{x}}$, and a Lorentz four-vector as $\ulcorner x \urcorner$. Therefore the Lorentz transormation along the x axis is:

$$\ulcorner x' \urcorner = L_1(\beta) \ulcorner x \urcorner \tag{22}$$

where

$$L_1(\beta) = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{23}$$

In general the velocity of K' will be in an arbitrary direction. Therefore, we must make a transformation by rotation of the three spatial axes such that the motion of K' is along the positive x axis. After the Lorentz transformation, we can then reverse the rotation to obtain the general transformation. Let $\hat{U}$ be the unit three-vector in the direction of motion seen in the K system. We can chose arbitrarily the other two unit three-vector bases for the transformed system. For our case with satellite trajectories we take the natural base vectors as the across track $(\hat{W})$ and approximate radial $(\hat{R})$ directions. If the satellite position and velocity three-vectors are $\bar{r}$ and $\dot{\bar{r}}$, then:

$$\hat{U} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \dot{\bar{r}} / \sqrt{\dot{\bar{r}} \cdot \dot{\bar{r}}}$$

$$\hat{W} = \begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} = \bar{r} \times \dot{\bar{r}} / \sqrt{(\bar{r} \times \dot{\bar{r}}) \cdot (\bar{r} \times \dot{\bar{r}})}$$

$$\hat{R} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = \hat{U} \times \hat{W} \quad . \qquad (24)$$

Then the desired transformation of a four-vector is:

$$\overline{x'} = A(\bar{r},\ \dot{\bar{r}})\ \overline{x} \tag{25}$$

where

$$A(\bar{r},\ \dot{\bar{r}}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & U_1 & U_2 & U_3 \\ 0 & W_1 & W_2 & W_3 \\ 0 & R_1 & R_2 & R_3 \end{bmatrix} \tag{26}$$

and the inverse is:

$$A^{-1}(\bar{r},\ \dot{\bar{r}}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & U_1 & W_1 & R_1 \\ 0 & U_2 & W_2 & R_2 \\ 0 & U_3 & W_3 & R_3 \end{bmatrix} \tag{27}$$

We are now in a position to write down the general Lorentz transformation to a moving frame, but a frame with its space basis three-vectors parallel to the original frame. This is called boost without rotation by Moller [Ref. 4], and is analagous to Fermi-Walker transport in the General Theory [Ref. 5]. This transformation is simply:

$$\overline{x'} = A^{-1}(\bar{r},\ \dot{\bar{r}}) L_1(\beta) A(\bar{r},\ \dot{\bar{r}})\ \overline{x}$$

$$= [S]\ \overline{x} \tag{28}$$

where $\beta^2 = \dot{\bar{r}} \cdot \dot{\bar{r}}/c^2$.

With a little algebra one can write:

$$S = S(\hat{U}, \beta) = \begin{bmatrix} \gamma & -\beta\gamma U_1 & -\beta\gamma U_2 & -\beta\gamma U_3 \\ -\beta\gamma U_1 & 1+(\gamma-1)U_1^2 & (\gamma-1)U_1U_2 & (\gamma-1)U_1U_3 \\ -\beta\gamma U_2 & (\gamma-1)U_1U_2 & 1+(\gamma-1)U_2^2 & (\gamma-1)U_2U_3 \\ -\beta\gamma U_3 & (\gamma-1)U_1U_3 & (\gamma-1)U_2U_3 & 1+(\gamma-1)U_3^2 \end{bmatrix} \tag{29}$$

Note that this transformation is in fact independent of the basis vectors $\hat{W}$ and $\hat{R}$, as it should be.

We now define the frequency four-vector that describes propagation of electromagnetic plane waves. A plane wave propagation in the direction of the unit three-vector $\hat{\ell}$ is a function of :

$$t - \frac{\bar{r} \cdot \hat{\ell}}{c} \tag{30}$$

The ℓ_i are the direction of cosines of the normal to the plane wave in the direction of propagation. The vector potential of such a wave can be written as the real part of a complex expression:

$$A = R_e\{A_0 \exp(-i\omega(t - \frac{\bar{r} \cdot \hat{\ell}}{c}))\}$$

$$= R_e\{A_0 \exp(-i\frac{\omega}{c}(x^0 - x^1\ell_1 - x^2\ell_2 - x^3\ell_3))\} \tag{31}$$

which can be written in terms of a frequency four-vector:

$$\overline{\nu} = \frac{\omega}{c} \begin{bmatrix} 1 \\ \ell_1 \\ \ell_2 \\ \ell_3 \end{bmatrix} = \begin{bmatrix} \nu^0 \\ \nu^1 \\ \nu^2 \\ \nu^3 \end{bmatrix} \tag{32}$$

as:

$$A = R_e \; A_0 \; \exp(i\phi) \tag{33}$$

where the phase is:

$$\phi = [\nu^0 \;\; \nu^1 \;\; \nu^2 \;\; \nu^3] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{34}$$

The matrix

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{35}$$

is in fact the flat space-time metric, and the occurrence of g_{ij} in (32) has a fundamental significance that is beyond the scope of this discussion. Now in a moving reference frame, the phase would be viewed as:

$$\phi = \phi' = \frac{\omega'}{c}[1 \;\; \ell^{1\prime} \;\; \ell^{2\prime} \;\; \ell^{3\prime}] \; [g_{ij}] \; \overrightarrow{x'}$$

$$= \frac{\omega}{c} \; [1 \;\; \ell^1 \;\; \ell^2 \;\; \ell^3] \; [g_{ij}] \; \overrightarrow{x} \tag{36}$$

which, using (28), can be written as:

$$\phi = \phi' = \frac{\omega}{c} \; [1 \;\; \ell^1 \;\; \ell^2 \;\; \ell^3] g_{ij} [S^{-1}] \; \overrightarrow{x'} \tag{37}$$

Now using the relation:

$$[g_{ij}][S]^{-1} = [S][g_{ij}] \tag{38}$$

we have

$$\phi = \phi' = \frac{\omega}{c} \; [1 \;\; \ell^1 \;\; \ell^2 \;\; \ell^3][S] g_{ij} \; \overrightarrow{x'}$$

$$= \frac{\omega'}{c}[1 \;\; \ell^{1\prime} \;\; \ell^{2\prime} \;\; \ell^{3\prime}][g_{ij}] \; \overrightarrow{x'}$$

$$= [\nu^{0\prime} \;\; \nu^{1\prime} \;\; \nu^{2\prime} \;\; \nu^{3\prime}][g_{ij}] \; \overrightarrow{x'} \tag{39}$$

This shows that the frequency four-vector (32) is indeed a four-vector and transforms like any four-vector.

The frequency four-vector contains all the information about the frequency and direction of propagation of a plane wave. The Lorentz transformation describes how this frequency four-vector is viewed in various reference frames. Before proceeding with the radar doppler we can now easily demonstrate some classical results, which can provide some insight into the deeper meaning of the Lorentz transformation.

If we consider the geometrically simple situation, where K' system is moving along the x axis of the K system. If the direction of transmission in the K system is θ, then the four-vector in the K system is:

$$\overline{\nu} = \nu \begin{bmatrix} 1 \\ \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \quad , \quad \nu = \frac{\omega}{c} \tag{40}$$

This four-vector transformed to the K' system is:

$$\overline{\nu'} = \gamma\nu \begin{bmatrix} 1 - \beta\cos\theta \\ -\beta + \cos\theta \\ \sin\theta/\gamma \\ 0 \end{bmatrix} = \nu' \begin{bmatrix} 1 \\ \cos\theta' \\ \sin\theta' \\ 0 \end{bmatrix} \tag{41}$$

The apparent frequency is:

$$\nu' = \gamma\nu(1 - \beta\cos\theta) \tag{42}$$

with the wave moving in the direction θ' given by:

$$\tan \theta' = \frac{\nu' \sin \theta'}{\nu' \cos \theta'} = \frac{\sin \theta}{\gamma(-\beta + \cos \theta)} \tag{43}$$

Of course the moving frame sees the wave coming from the direction $\theta_a' = \pi - \theta'$. The moving observer would deduce that the wave was actually coming from the direction $\theta_a = \pi - \theta$ using:

$$\tan \theta_a' = \frac{\sin \theta_a}{\gamma(\beta + \cos \theta_a)} \tag{44}$$

These are the classical expressions for the doppler shift and aberration as seen in a moving frame.

Now we sit in the moving (K') system, and reflect the incoming wave ($\overrightarrow{\nu'}$) back along the direction of propagation ($\overrightarrow{\nu_r}$). This could be done with a cube corner. Mathematically this is expressed as:

$$\overrightarrow{\nu'_r} = \nu' \begin{bmatrix} 1 \\ -\cos \theta' \\ -\sin \theta' \\ 0 \end{bmatrix} = \gamma\nu \begin{bmatrix} 1 - \beta \cos \theta \\ \beta - \cos \theta \\ -\sin \theta/\gamma \\ 0 \end{bmatrix} \tag{45}$$

If we then view this reflected wave in (i.e., transform to) the stationary (K) system we obtain:

$$\overline{\nu}_r = L_1(-\beta)\overline{\nu'}_r = \nu\gamma^2 \begin{bmatrix} 1 - 2\beta \cos\theta + \beta^2 \\ -\cos\theta(1 + \beta^2) + 2\beta \\ -\sin\theta/\gamma^2 \\ 0 \end{bmatrix} \tag{46}$$

Now the frequency is perceived to be shifted by:

$$\frac{\nu_r - \nu}{\nu} = \gamma^2(-2\beta \cos\theta + 2\beta^2) \tag{47}$$

and propagated in the direction θ_r:

$$\tan\theta_r = \frac{\sin\theta}{\gamma^2(\cos\theta(1 + \beta^2) - 2\beta)} \tag{48}$$

appearing to come from the direction:

$$\tan\theta_{ra} = \frac{\sin\theta_a}{\gamma^2(\cos\theta_a(1 + \beta^2) + 2\beta)} \tag{49}$$

Note that this is not the original direction of outward propagation (+180 deg) in the K system, though the wave was reflected along the transmitted direction in the K' system. This effect is known as velocity aberration. In this case the refected wave will not return to the transmitter. For a satellite at the height of Lageos (7503901, 8820) the return would be 120 m away from the transmitter. Therefore the cube corners mounted on Lageos and other satellites for laser ranging must be imperfect, spreading the return pulse such that some energy will return to the transmitter site.

From the point of view of the moving frame (K') one would have to reflect the plane wave along the direction:

$$\overline{\nu'_r} = \nu\gamma(1 - \beta \cos \theta) \begin{bmatrix} 1 \\ -(\cos \theta + \beta)/(1 + \beta \cos \theta) \\ -\sin \theta/(\gamma[1 + \beta \cos \theta]) \\ 0 \end{bmatrix} \qquad (50)$$

to return to the transmitter. In this case the apparent frequency would be:

$$\overline{\nu_r} = \nu\gamma^2(1 - 2\beta \cos \theta + 2\beta^2 \cos^2 \theta) \qquad (51)$$

One could also return the plane wave along the direction:

$$\overline{\nu'_r} = \gamma\nu(1 - \beta \cos \theta) \begin{bmatrix} 1 \\ -(\beta + \cos \theta(1 - 2\beta^2))/(1 - \beta \cos \theta) \\ -\sin \theta/\gamma \\ 0 \end{bmatrix} \qquad (52)$$

which would exactly cancel the effects of Special Relativity in observed frequency. One notes that all these effects are of second order in the small quantity β. This is why the effects of TSR are experimentally difficult to observe, and that some care must be used in the details of computation.

These examples assume the moving target controls the direction of reflection; more precisely the direction of retransmission. In the case of radar reflection, the target is expected to return some radiation to all directions in the half space normal to the incoming wave direction in the stationary system. One can express the returned four-vector in terms of the four-vector in the stationary system. That is how (50) was obtained.

Within this framework we can now formulate the radar Doppler equation. Recall that we ultimately need to write expressions for the transmitted and observed frequency in terms of the inertial geocentric position and velocity. Let us introduce the following notation. The time of transmission, the time of reflection at the target, and the time of reception are respectively: t1, t2, t3 as defined by a clock in our inertial geocentric system. From (10) we see that the (TSR) correction of the time measured at the transmitter and receiver to inertial geocentric time is negligible. The position and velocity of the transmitter at t1 are denoted are $\overline{R}_1$, $\dot{\overline{R}}_1$, the position and velocity of the target at t2 are $\overline{r}_2$, $\dot{\overline{r}}_2$, and the position and velocity of the receiver are $\overline{R}_3$, $\dot{\overline{R}}_3$; all expressed in the inertial geocentric reference frame. The unit three-vector of each velocity is denoted $\hat{U}$. the magnitude of each velocity vector is β in units of the speed of light (c). Therefore we have $\dot{\bar{R}}_1 = c\beta_1\hat{U}_1$, $\dot{\bar{r}}_2 = c\beta_2\hat{U}_2$ and $\dot{\bar{R}}_3 = c\beta_3\hat{U}_3$. We have the vector from transmitter to target $\bar{\rho}_{12} = \bar{r}_2 - \bar{R}_1$ and from target to receiver $\bar{\rho}_{23} = \bar{R}_3 - \bar{r}_2$. The ranges are $(\rho_{12})^2 = \bar{\rho}_{12} \cdot \bar{\rho}_{12}$ and $(\rho_{23})^2 = \bar{\rho}_{23} \cdot \bar{\rho}_{23}$. Now we can define the four-vector wave propagating between transmitter and target in the inertial reference frame as:

$$\overline{\nu_t} = \nu_t \begin{bmatrix} 1 \\ \hat{\ell}_{12} \end{bmatrix} \tag{53}$$

where

$$\hat{\ell}_{12} = \bar{\rho}_{12}/\rho_{12}$$

Now this wave as seen by the (moving) transmitter is:

$$\overline{\nu_t'} = S(\hat{U}_1,\ \beta_1)\overline{\nu_t} \tag{54}$$

and the target sees this wave as:

$$\overline{\nu_t''} = S(\hat{U}_2,\ \beta_2)\overline{\nu_t} \tag{55}$$

Using (54) we have:

$$\overline{\nu''_t} = S(\hat{U}_2,\ \beta_2)S(\hat{U}_1,\ -\beta_1)\overline{\nu'_t} \tag{56}$$

The same reasoning relates the received frequency four-vector as:

$$\overline{\nu''_r} = S(\hat{U}_2,\ \beta_2)S(\hat{U}_3,\ -\beta_3)\overline{\nu'_r} \tag{57}$$

Since the transmitted frequency seen by the target ($\nu_t{}^{0}{}''$) and the reflected frequency at the target ($\nu_r{}^{0}{}''$) are the same, we can equate the first element of these four-vectors and obtain the desired relation between the transmitted (ν'_t) and received (ν'_r) frequency in their respective frames.

For calculation we need not compute the whole Lorentz transformation matrix. We can write out the elements of

$$S(\hat{U},\ \beta)S(\hat{U}',\ \beta') = [S_{ij}] \tag{58}$$

as:

$$S_{oo} = \gamma\gamma' + \beta\beta'\gamma\gamma'(\hat{U} \cdot \hat{U}') \tag{59}$$

$$S_{oi} = -\gamma[\beta'\gamma'U'_i + \beta U_i + \alpha'\beta U_i(\hat{U} \cdot \hat{U}')] \qquad i = 1,2,3$$

$$S_{io} = -\gamma'[\beta\gamma U_i + \beta'U'_i + \alpha\beta'U_i(\hat{U} \cdot \hat{U}')] \qquad i = 1,2,3$$

$$S_{ii} = 1 + \alpha U_i^2 + \alpha'(U_i'^{\,2}) + U_iU'_i(\beta\gamma\beta'\gamma' + \alpha\alpha'(\hat{U} \cdot \hat{U}')] \qquad i = 1,2,3$$

$$S_{ij} = \beta\beta'\gamma\gamma'U_jU'_i + \alpha U_jU_i + \alpha'U'_jU'_i + \alpha\alpha'U'_jU_i(\hat{U} \cdot \hat{U}') \qquad i > j \geqslant 1$$

$$S_{ij} = \beta\beta'\gamma\gamma'U_iU'_j + \alpha U_iU_j + \alpha'U'_iU'_j + \alpha\alpha'U'_iU_j(\hat{U} \cdot \hat{U}') \qquad j > i \geqslant 1$$

where U_i are the *three components of* $\hat{U}$ and U'_i are the three components of $\hat{U}'$, and $\alpha = (\gamma - 1) \approx \beta^2/2$ is a small quantity of second order. Note that we have used positive β' in (58). Therefore in (56) $\beta' = -\beta_1$, and in (57) $\beta' = -\beta_3$.

Let us call $S^1{}_{ij} = S_{ij}(\hat{U}_2, \beta_2, \hat{U}_1, -\beta_1)$ and $S^3{}_{ij} = S_{ij}(\hat{U}_2, \beta_2, \hat{U}_3, -\beta_3)$. Then we have:

$$\nu'_r = [S^3_{oi}] \begin{bmatrix} \hat{\ell}^1_{23} \end{bmatrix} = \nu'_t \; [S^1_{oo} + \sum_{i=1}^{3} S^1_{oi} \; \ell^i_{12}] \tag{60}$$

or:

$$\nu'_r = \nu'_t \frac{S^1_{oo} + \sum_{i=1}^{3} S^1_{oi} \; \ell^i_{12}}{S^3_{oo} + \sum_{i=1}^{3} S^3_{oi} \; \ell^i_{23}} \tag{61}$$

The frequency shift seen is therefore:

$$\frac{\nu'_r - \nu'_t}{\nu'_t} = \frac{S^1_{oo} - S^3_{oo} + \sum_{i=1}^{3} S^1_{oi} \; \ell^i_{12} - \sum_{i=1}^{3} S^3_{oi} \; \ell^i_{23}}{S^3_{oo} + \sum_{i=1}^{3} S^3_{oi} \; \ell^i_{23}} \tag{62}$$

This expression is the desired result, without approximation, and no assumption that the transmitting and receiving site are the same. For purposes of computation (62) should be used. To see the meaning of the terms, we note the following. First, β' is the velocity of the transmitter (receiver or station) and is in general smaller than β for ground based observations. If we consider $\alpha' \approx \beta'^2/2$ negligible and if we therefore take $\alpha' = 1$, then:

$$S_{oi} = -\gamma\{\beta' U_i' + \beta U_i\} \qquad i = 1,2,3 \tag{63}$$

Recall that $\beta' = \beta_1$ or $-\beta_3$ and we see that the Lorentz transformation depends on the difference in velocity of the observer and target, which is consistent with the first tenent of TSR. Further, if we assume the station motion is rectilinear (i.e., unaccelerated) then $\beta_1 = \beta_3$, $\hat{U}_1 = \hat{U}_3$ and $S^1{}_{oo} = S^3{}_{oo}$ and $S'_{oi} = S^3{}_{oi}$. Therefore the observed frequency shift becomes:

$$\frac{\nu_r' - \nu_t'}{\nu_t'} = \frac{\sum_{i=1}^{3} S^1_{oi}(\ell^i_{12} - \ell^i_{23})}{S'_{oo} + \sum_{i=1}^{3} S'_{oi}\ \ell^i_{23}} \tag{64}$$

This approximation ignores the Lorentz boost due to the station velocity, but preserves the different pathes of transmission. If finally one takes the mean station-target vector

$$\hat{\ell}_{22} = (\bar{r}_2 - \bar{R}_2)/\rho_{22} \tag{65}$$

with

$$\hat{\ell}_{12} \approx \hat{\ell}_{22} \approx -\hat{\ell}_{23} \tag{66}$$

and

$$S_{oo} = \gamma, \; S_{oi} = -\gamma(\dot{\bar{r}} - \dot{\bar{R}})/c \tag{67}$$

we have:

$$\frac{\nu_r' - \nu_t'}{\nu_t'} = \frac{-2(\dot{\bar{r}} - \dot{\bar{R}}) \cdot \hat{\ell}_{22}}{c\left(1 + \frac{(\dot{\bar{r}} - \dot{\bar{R}}) \cdot \hat{\ell}_{22}}{\gamma c}\right)} \tag{68}$$

or

$$\frac{c}{2}\,\frac{\nu_r' - \nu_t'}{\nu_t'} = -(\dot{\bar{r}} - \dot{\bar{R}}) \cdot \ell_{22} + [(\dot{\bar{r}} - \dot{\bar{R}}) \cdot \hat{\ell}_{22}]^2/c \tag{69}$$

the classical approximation.

If one wants an approximation with the accuracy of (64), but in terms of the mean geometric station-target vector one has

$$\frac{c}{2}\,\frac{\nu_r' - \nu_t'}{\nu_t'} = -(\dot{\bar{r}} - \dot{\bar{R}}) \cdot \hat{\ell}_{22} + [(\dot{\bar{r}} - \dot{\bar{R}}) \cdot \hat{\ell}_{22}]^2/c$$
$$-(\dot{\bar{r}} - \dot{\bar{R}}) \cdot \overline{\Delta R}/\rho \tag{70}$$

where $\overline{\Delta R} = \bar{R}_2 - \bar{R}_1 = \bar{R}_3 - \bar{R}_2$.

Since the second and third terms of (70) are terms of second order they could be calculated with a predicted trajectory. We can then define the mean geometric range rate in terms of the observable as:

$$-(\dot{\bar{r}} - \dot{\bar{R}}) \cdot \hat{\ell}_{22} = \frac{c}{2} \frac{\nu_r' - \nu_t'}{\nu_t'} - [(\dot{\bar{r}} - \dot{\bar{R}}) \cdot \hat{\ell}_{22}]^2/c$$

$$+ (\dot{\bar{r}} - \dot{\bar{R}}) \cdot \overline{\Delta R}/\rho \qquad (71)$$

or

$$-(\dot{\bar{r}} - \dot{\bar{R}}) \cdot \hat{\ell}_{22} = \frac{c}{2} \frac{\nu_r' - \nu_t'}{\nu_t'} - \frac{c}{4} \left(\frac{\nu_r' - \nu_t'}{\nu_t'}\right)^2$$

$$+ (\dot{\bar{r}} - \dot{\bar{R}}) \cdot \overline{\Delta R}/\rho \qquad (72)$$

III. DISCUSSION

In all cases we have expressed the frequency shift as viewed in a moving frame (the observable) in terms of positions in our interial geocentric reference frame (the computable). Therefore, the vector quantities in (62), (64), and (69) are considered geometric quantities and are the three-space quantities used with Newtonian Mechanics. The correction terms for the Newtonian three-space are small. The elaboration of a trajectory is considerably simplified if done in three-space, allowances being made with small corrections. Therefore, the meaning of (62), (64), and (69) is as follows. A trajectory or orbit determination is made in convenient Newtonian three-space. Radar observations are used to determine the orbit state vector iteratively by, say, the method of Least Squares. In this calculation, the observed-computed difference is needed. In this case (62), (64) or (69) is used to obtain the computed frequency shift. Then approximating the difference as an error in the geometric range rate $(\dot{\bar{r}} - \dot{\bar{R}}) \cdot \hat{\imath}_{22}$ an observation equation is formed to improve the satellite state vector.

The discussion thus far has employed the TSR. We know that the space-time is in fact not flat but curved due to the presence of mass or energy. This is known as the General Theory of Relativity (GTR). For our purpose we can approximate the local space-time curvature with the Schwarzschild Metric [Ref. 5]. There will be effects due to the Sun and the Earth. We know that based on the GTR, there are a number of effects that must be considered. First, the dynamics of a particle must be modified. This is most easily accomplished with a small "correction" term to the Newtonian model. For close earth satellites, this is trivially accomplished [Ref. 6]. Second, the velocity of light is slower in the presence of a gravity field, than it is in flat space time. Using a light travel time of electromagnetic propagation to measure distance therefore requires a correction. In this case the effect of the earth's mass is very small, but the sun's mass is not. The correction can be estimated as follows. The space-time relation between the proper time of radial transit of a photon (dt) and the metric distance $(r - r_0)$ in the Schwarzschild metric can be shown to be

$$cdt = r - r_0 + 2\frac{GM}{c^2}\ln(r/r_0) \qquad (73)$$

where r_0 is the radius of the transmitter, r is the radius of the target, and GM is the gravitational constant multiplied by the mass. GM/c^2 has the units of length and is sometimes referred to as the Schwarzschild radius. The first two terms of the right hand side of (73) are the flat space-time (Newtonian) distance, and the last term can be viewed as a General Relativity term. Examples of the correction are given in Table I.

TABLE I

SPACE TIME CURVATURE EFFECT ON RANGE MEASUREMENT

Mass	Range Measurement To	Correction
Earth	Lageos	0.6 cm
Earth	Synchronous Satellite	1.6 cm
Earth	Moon	3.6 cm
Earth	Sun	8.9 cm
Sun	Lageos	12.5 cm
Sun	Synchronous Satellite	62.8 cm
Sun	Moon	741.0 cm
Sun	Sun ($1R_\odot$)	16.9 km

Now the conventional value of the speed of light in vacuum (c = 2.99792458 = 0.000000012 x 10^{10} cm/sec) is based on laboratory measurements at the earth's surface. Therefore measurements of distance near the earth implicitly use the appropriate value of c. in other words, a measurement of dt at the earth's surface correspond to $r - r_0$ if the absolute speed of light is reduced by

$$\frac{\delta c}{c} \approx 2\frac{GM}{c^2 r_0}$$

Therefore, the speed of light at infinity is greater by ths value. For $M_{\odot}$ and $a_{\odot}$, $\delta c/c = 2 \times 10^{-8}$. We can summarize the situation as follows. The choice of a particular value of c depends on the reference frame chosen. Discussion of data in the near earth environment, i.e., where one can ignore the change in c due to the change in distance from the sun, the constant obtained in terrestial laboratory measurements is correct. In this case, only the variation in the space time metric represented by the earth's mass should be used. Since ranges to the moon need be corrected by less than 5 cm, this can be ignored in most applications. However, when considering general solar system measurements, then a frame with c defined at infinity must be used. The metric distances in the earth's environment will of course then have to be expressed in this new system. Therefore. for earth orbiting bodies, within the moons orbit, the corrections given in Table I are not applicable. Finally, the frequency of our standard clocks will change in a potential field. This is potentially the most significant effect with Doppler measurements. The appropriate description includes a transformation from station to satellite. This transformation is the change in frequency along the geodesic path from station to satellite.

It is found that the frequency would change by a factor $1 + \Delta\phi$, where $\Delta\phi$ is the difference in potential between the transmitter and target. The reverse change takes place on the return path. If the transmitter and receiver are at the same potential (e.g., at the same geopotential height) then the change up and down would be exactly the same. However, if the measurement is of an oscillator onboard the target, then such a term should be included in (62) [Ref. 8]. The consequences of this effect for multistation observatories must be examined further.

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The propagation of the electromagnetic waves in vacuum is discussed in terms of the Lorentz Transformation A simple, but rigorous, derivation of the Lorentz Transformation is given. Then, the observed doppler shift, seen by a moving observer, of a moving target is derived in terms of inertial position and velocity. This radar doppler equation, given without approximation, is suitable for analysis of doppler data for satellite orbit determination.

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