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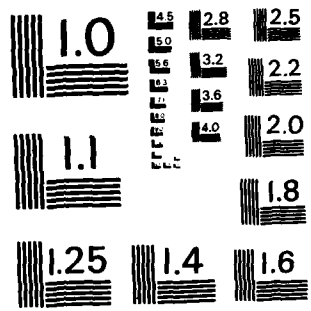
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SYSTEMS NOTE 83

A SOLUTION TECHNIQUE FOR COMPLEX
MATRIX GAMES

APPROVED
FOR PUBLIC RELEASE

by
G. J. MURRAY

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**A SOLUTION TECHNIQUE FOR COMPLEX
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SUMMARY

Recently Mond and Murray introduced the concept of a zero sum two person game in complex space and proved two minimax theorems. In this note a method is provided for the solution of such games. A numerical example is given.



POSTAL ADDRESS: Director, Aeronautical Research Laboratories,
Box 4331, Melbourne, Victoria, 3001, Australia

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NOTATION

N is the set $\{1, 2, \dots, n\}$, the first n positive integers. $x \geq y$ means that $x_j \geq y_j$ for all $j \in N$ (where x and y are n -dimensional column vectors). $x > y$ means that $x_j > y_j$ for all $j \in N$. $R^{m \times n}$ ($C^{m \times n}$) is the set of real (complex) $m \times n$ matrices. For any $A \in C^{m \times n}$, A^t is the transpose and A^H the conjugate transpose. $\operatorname{Re} w$ means the real part of w , e.g., for any $w \in C^n$, $\operatorname{Re} w$ denotes the vector whose j th component is $\operatorname{Re}(w_j)$. e is a column vector in which every element is 1, the dimension of the vector to be clear from the context. For any nonempty set $S \subset C^m$, $S^* = \{z \in C^m: w \in S \text{ implies that } \operatorname{Re}(z^H w) \geq 0\}$ is the *polar* of S , and $\operatorname{int} S^* = \{z \in C^m: 0 \neq w \in S \text{ implies that } \operatorname{Re}(z^H w) > 0\}$ is the *interior* of S^* .

The vectors common to S and S^* are denoted by $S = S \cap S^*$. A nonempty set $S \subset C^m$ is

- (i) *convex* if $0 \leq \lambda \leq 1$ implies that $\lambda S + (1-\lambda)S \subset S$
- (ii) a *cone* if $0 \leq \lambda$ implies that $\lambda S \subset S$
- (iii) a *convex cone* if (i) and (ii) are true
- (iv) a *pointed convex cone* if (iii) is true and $S \cap (-S) = \{0\}$
- (v) a *polyhedral cone* if $S = BR_+^k$ for some $B \in C^{m \times k}$.

1. INTRODUCTION

The concept of a zero sum two person game in complex space was introduced in [1]. Complex analogues of payoffs, probability vectors and strategy sets were defined and a minimax theorem established. A simpler proof of a more general minimax theorem was subsequently given in [2]. Both of the above papers provided numerical examples of 2×2 matrix games in complex space and their solutions. Solution of matrix games in complex space has so far been quite difficult, even for 2×2 matrices. In this note a method is derived for transforming a matrix game in complex space into a matrix game in real space having the same value. This latter game can then be solved by the standard techniques involving linear programming. Optimal strategies for the original game in complex space are easily obtained from the optimal strategies computed for the game in real space.

The technique outlined in this note can be used to solve any feasible matrix game in complex space. The only constraint on the size of the game matrix which can be accommodated is that which is imposed by the computer or linear programming package to be used.

2. DEFINITIONS AND PRELIMINARIES

The development of the theory of complex matrix games requires some generalization of real space game concepts.

An n -dimensional complex probability vector is a column vector $z \in C^n$ satisfying

$$(i) \operatorname{Re}(z_j) \geq 0 \quad (j = 1, \dots, n)$$

$$(ii) \sum_{j=1}^n z_j = 1.$$

Note that the sum of the imaginary parts of the components of a complex probability vector is zero.

A strategic cone $S \subset C^n$ is a pointed polyhedral (closed) convex cone such that $S \subset (R^n)^*$ and $S \cap (R^n)^* = \{0\}$. The requirement that $S \subset (R^n)^*$ assures that the real parts of all components of any vector $s \in S$ are non-negative while the requirement that $S \cap (R^n)^* = \{0\}$ means that, except for the origin, the complex axes are not in S . The significance of a strategic cone is that any complex probability vector $w \in S$, where S is a strategic cone, will have bounded imaginary components.

Let $S \subset C^n$ be a strategic cone. The set of complex probability vectors $w \in S$, if such vectors exist, will be contained within a strategic cone whose generators $p^j = c^j + id^j$ ($j = 1, \dots, n$) satisfy $\sum_{k=1}^m c_k^j = 1$ and $\sum_{k=1}^m d_k^j = 0$, and are thus complex probability vectors themselves.

The strategic cone generated by p^j ($j = 1, \dots, n$) will be denoted by P_S . P_S is the convex conical hull of all complex probability vectors $w \in S$. Since all of these vectors p^j ($j = 1, \dots, n$) belong to S , $P_S \subset S$. If S contains R^n then so does P_S . P_S is nonempty if and only if there exists at least one complex probability vector $w \in S$.

Note that $P_S \neq P_S^*$. By definition P_S does not contain any vectors whose imaginary parts do not sum to zero. P_S^* does contain such vectors.

For a complex two person game a strategy is defined to be a complex probability vector. Each player selects a strategy from his own strategy set. Generally each player makes his choice without any prior knowledge of the other player's choice.

A complex matrix game is completely specified by a payoff matrix A and strategic cones S and T . Let $A \in C^{m \times n}$, $S \subset C^m$, $T \subset C^n$. Player 1 chooses a strategy (complex probability vector) $w \in S$ and player 2 chooses a strategy $z \in T$. Player 1 receives a payoff $\text{Re}(w^H A z)$. Correspondingly player 2 receives a payoff $-\text{Re}(w^H A z)$. The game is *feasible* if there exists at least one strategy $w \in S$ and at least one strategy $z \in T$, i.e. P_S and P_T are nonempty.

A *solution* to complex matrix game $A \in C^{m \times n}$ with strategic cones $S \subset C^m$, $T \subset C^n$ is a pair of complex probability vectors $w_0 \in S$ and $z_0 \in T$ and a real number v such that

$$\text{Re } w_0^H A z \geq v \text{ for all probability vectors } z \in T, \text{ and}$$

$$\text{Re } w^H A z_0 \leq v \text{ for all probability vectors } w \in S.$$

w_0 and z_0 are *optimal strategies* for players 1 and 2 respectively and v is the *value* of the game.

Note that the existence of a solution (w_0, z_0, v) implies

$$\text{Re}(w^H A z_0) \leq \text{Re}(w_0^H A z_0) \leq \text{Re}(w_0^H A z)$$

and

$$\max_w \min_z \text{Re}(w^H A z) = \min_z \max_w \text{Re}(w^H A z) = \text{Re}(w_0^H A z_0) = v.$$

3. THE SOLUTION TECHNIQUE

Consider a matrix game specified by the payoff matrix $A \in C^{m \times n}$ and strategic cones $S \subset C^m$ and $T \subset C^n$. Player 1 chooses a strategy (complex probability vector) $w \in S$ and player 2 chooses a strategy $z \in T$. Player 1 receives a payoff of $\text{Re}(w^H A z)$ from player 2.

Let the complex probability vectors $p^j (j = 1, \dots, n_1)$ be the generators of the convex conical hull of all complex probability vectors $w \in S$. Similarly, let the complex probability vectors $q^k (k = 1, \dots, n_2)$ be the generators of the convex conical hull of all complex probability vectors $z \in T$. Let a and b be real n_1 and n_2 -dimensional probability vectors respectively, and let $P = [p^1 p^2 \dots p^{n_1}]$ and $Q = [q^1 q^2 \dots q^{n_2}]$.

Any strategies $w \in S$, $z \in T$ may be written in the form $w = Pa$, $z = Qb$. Note that $w^H = a^H P^H$. Let $M = \text{Re}(P^H A Q)$. Then, since a and b are real,

$$\text{Re}(w^H A z) = \text{Re}(a^H P^H A Q b) = a^H M b.$$

Now, $a^H M b$ is the payoff obtained for a matrix game in real space with player 1 choosing probability vector a while player 2 chooses probability vector b . This game can be solved using linear programming. Suppose that it has value v and optimal strategies a_0, b_0 . Then the original matrix game in complex space has value v and optimal strategies $w_0 = Pa_0$ and $z_0 = Qb_0$. Also $v = a_0^H M b_0 = \text{Re}(w_0^H A z_0)$. Note that the p^j 's and q^k 's represent complex analogues of pure strategies.

4. A NUMERICAL EXAMPLE

Consider a matrix game in complex space with the following payoff matrix:

		Player 2	
		1	2
Player 1	1	2	$1+i$
	2	$1-i$	$3i$
	3	i	$1-2i$
		2	

Let player 1 choose his strategy from strategic cone S and player 2 choose his strategy from strategic cone T . Let S be generated by the set of complex probability vectors

$$\begin{bmatrix} 1+\frac{1}{2}i \\ 0 \\ -\frac{1}{2}i \end{bmatrix}, \begin{bmatrix} 1-\frac{1}{2}i \\ \frac{1}{2}i \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2}i \\ 1+\frac{1}{2}i \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1-\frac{1}{2}i \\ \frac{1}{2}i \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{2}i \\ 1+\frac{1}{2}i \end{bmatrix}, \begin{bmatrix} \frac{1}{2}i \\ 0 \\ 1-\frac{1}{2}i \end{bmatrix}.$$

Let T be generated by the set of complex probability vectors

$$\begin{bmatrix} 1+\frac{1}{2}i \\ -\frac{1}{2}i \end{bmatrix}, \begin{bmatrix} \frac{1}{2}i \\ 1-\frac{1}{2}i \end{bmatrix}, \begin{bmatrix} 1-\frac{1}{2}i \\ \frac{1}{2}i \end{bmatrix}, \begin{bmatrix} -\frac{1}{2}i \\ 1+\frac{1}{2}i \end{bmatrix}.$$

For this game P_S (the convex conical hull of S) = S and $P_T = T$.

$$\text{Hence } P = \begin{bmatrix} 1+\frac{1}{2}i & 1-\frac{1}{2}i & -\frac{1}{2}i & 0 & 0 & \frac{1}{2}i \\ 0 & \frac{1}{2}i & 1+\frac{1}{2}i & 1-\frac{1}{2}i & -\frac{1}{2}i & 0 \\ -\frac{1}{2}i & 0 & 0 & \frac{1}{2}i & 1+\frac{1}{2}i & 1-\frac{1}{2}i \end{bmatrix}$$

$$\text{and } Q = \begin{bmatrix} 1+\frac{1}{2}i & \frac{1}{2}i & 1-\frac{1}{2}i & -\frac{1}{2}i \\ -\frac{1}{2}i & 1-\frac{1}{2}i & \frac{1}{2}i & 1+\frac{1}{2}i \end{bmatrix}.$$

$$M = \text{Re}(P^H A Q) = \frac{1}{10} \begin{bmatrix} 19 & 29 & 11 & 21 \\ 17 & 22 & 13 & 18 \\ 13 & 18 & -3 & 2 \\ 26 & -19 & 14 & -31 \\ 2 & -23 & 18 & -7 \\ -9 & 21 & -1 & 29 \end{bmatrix}$$

After adding 31 to each element in the matrix, the usual linear programming technique yields optimal strategies for the real matrix game M . These are

$$a_0 = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} \text{ for player 1 and } b_0 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \text{ for player 2.}$$

Optimal strategies for the original complex matrix game are thus

$$w_0 = \begin{bmatrix} \frac{1}{2} - \frac{1}{10}i \\ \frac{1}{2}i \\ \frac{1}{2} + \frac{1}{10}i \end{bmatrix} \text{ for player 1 and } z_0 = \begin{bmatrix} \frac{1}{2} - \frac{1}{10}i \\ \frac{1}{2} + \frac{1}{10}i \end{bmatrix} \text{ for player 2.}$$

The common value of the two games is $\frac{1}{10}$.

5. REMARKS

The complex space conversion of a game to a linear program has been demonstrated in [3]. There is in fact an equivalence between matrix games and linear programs in complex space [4]. Hence the above solution technique for matrix games in complex space suggests that it may be possible to convert a linear programming problem in complex space into a real linear program. The conversion is readily accomplished [5], thus providing a method for solving linear programs in complex space.

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