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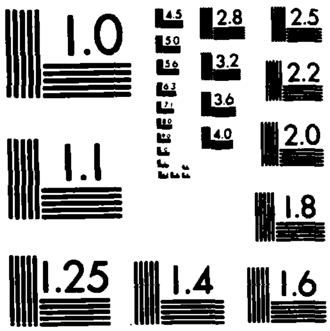

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The use of cross-correlation for detecting and tracking the source of signals received at spatially separated receiving sites has received considerable attention in the research community during recent years. The limited understanding of the statistical nature of ambiguity surfaces is based on the statistics of a single cell in the absence of signal and noise power fluctuations and for equal signal and processing bandwidths (herein called matched containment). The effects of power level fluctuations and signal overcontainment, where the processing bandwidth is larger than the signal bandwidth, must be quantitied (CONT.)

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20. Abstract (cont.)

in order to fully understand the statistical nature of ambiguity surfaces under realistic operational conditions. The effects of signal overcontainment have been quantified in the absence of fluctuations. Signal and noise power level fluctuations and the rate at which power levels fluctuate can adversely affect the signal-to-noise ratios required to attain a desired performance. The study results presented in this report address the effects of signal and noise power fluctuations on detection performance.

An ambiguity surface is a two-dimensional function, $\gamma^2(\tau, f_D)$, which is the sample magnitude-squared of the normalized cross-correlation between the observations received at two spatially separated sites as a function of the relative time delay (τ) and relative Doppler shift (f_D) between the observations. The surface is generated for a specific integration time (T) and processing bandwidth ($2W_p$) as shown in Figure 1-1. In actual practice, the processing bandwidth is always larger than or equal to the signal bandwidth.

It is concluded that fluctuations require a 3-4 dB increase in SNR for rapid fluctuations and a 4-6 dB increase in SNR for slow fluctuations over the SNR required to achieve comparable performance in the absence of fluctuations. In all fluctuation cases, the required SNR decreases as the fluctuation becomes "less" random (i.e., the variance decreases). However, the effects of slow fluctuation are basically independent of the signal time-bandwidth product (N_T), while the effects of rapid fluctuation can be reduced by increasing N_T.

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I. INTRODUCTION

The use of cross-correlation for detecting and tracking the source of signals received at spatially separated receiving sites has received considerable attention in the research community during recent years. It is necessary to understand the statistical nature of ambiguity surfaces and the interdependence of the cells in the surface under realistic operational conditions in order to accurately evaluate the tracking accuracies and the detection performance achievable with cross-correlation. The limited understanding of the statistical nature of ambiguity surfaces is based on the statistics of a single cell in the absence of signal and noise power fluctuations and for equal signal and processing bandwidths (herein called matched containment, refs. 1-3). The effects of power level fluctuations and signal overcontainment, where the processing bandwidth is larger than the signal bandwidth, must be quantified in order to fully understand the statistical nature of ambiguity surfaces under realistic operational conditions. The effects of signal overcontainment have been quantified in the absence of fluctuations (refs. 4-5).

It is well known that signal noise power levels do fluctuate during reasonable observation intervals (refs. 6-11). Signal and noise power level fluctuations and the rate at which power levels fluctuate can adversely affect the signal-to-noise ratios required to attain a desired performance (ref. 6). The effects of fluctuations on ambiguity surface statistics are unknown. The study results presented in this report address the effects of signal and noise power fluctuations on detection performance.

An ambiguity surface is a two-dimensional function, $\gamma^2(\tau, f_D)$, which is the sample magnitude-squared of the normalized cross-correlation between the observations received at two spatially separated sites as a function of the relative time delay (τ) and relative Doppler shift (f_D) between the observations. The surface is generated for a specific integration time (T) and processing bandwidth ($2W_P$) as shown in Figure 1-1. In actual practice, the processing bandwidth is always larger than or equal to the signal bandwidth.

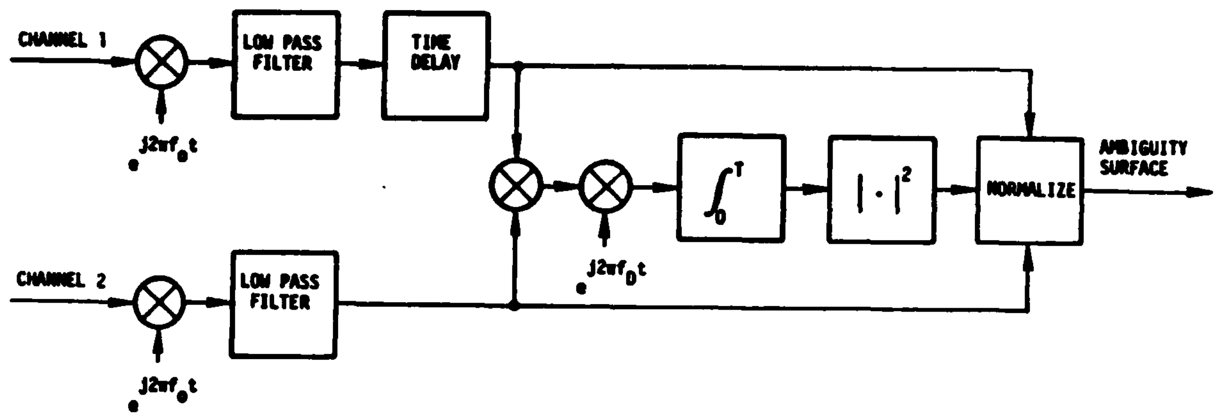

Figure 1-1. Schematic for Generating Ambiguity Surfaces with a Narrowband Correlation Algorithm

Since the ambiguity surface is usually computed digitally, the ambiguity surface is quantized into cells of width $\Delta\tau$ seconds in the delay dimension and Δf_D Hz in the Doppler shift dimension, where $\Delta\tau \leq 1/2W_p$ and $\Delta f_D \leq 1/T$. The actual structure and statistics are affected by power level fluctuations and processing parameters. The accuracy with which the time delay and Doppler shift can be estimated and the ability to detect a signal is in turn affected by the statistics and structure of the surface.

There are many types of fluctuation conditions which depend on the rate at which the fluctuation processes can vary. The types of fluctuations are bounded by very slow fluctuation conditions and very rapid fluctuation conditions. Very slow fluctuation conditions occur when the signal and noise powers are unknown but remain constant throughout the observation interval. On the other hand, very rapid fluctuation conditions occur when the power fluctuations from sample to sample are so large that successive samples may be considered independent. Then there is the whole range of fluctuation conditions between the above two extremes. The signal model used to study the effects of fluctuation is presented in Chapter 2. The detection performance is described in Chapters 3 and 4 for slow and rapid fluctuations, respectively. The results are summarized in Chapter 5.

2. SIGNAL MODEL

The signal model used to analyze the effects of power fluctuations on the sample magnitude-squared correlation coefficient (MSCC) is described. The model is used to analyze the effects of slow and rapid power fluctuations. The fluctuation model is discussed in Section 2.1. The probability law governing fluctuations and the resulting statistics are developed in Section 2.2.

2.1 Fluctuation Model

There are many types of fluctuation conditions which depend on the rate at which the fluctuation processes can vary. The types of fluctuations are bounded by very slow fluctuation conditions and very rapid fluctuation conditions. Very slow fluctuation conditions occur when the signal and noise powers are unknown but remain constant throughout the observation interval. On the other hand, very rapid fluctuation conditions occur when the power fluctuations from sample to sample are so large that successive samples may be considered independent. Then there is the whole range of fluctuation conditions between the above two extremes. The procedures used to analyze the effects of fluctuations will depend on the type of fluctuation condition.

The fluctuation model is a generalization of the zero mean complex Gaussian signal model that is used to analyze the statistics of the sample MSCC (refs. 1-5). The effects of fluctuations can be included by modeling signals as a compared process (refs. 6, 7). In this case, signals are modeled as

$$s(t) = \sqrt{p(t)}\, x(t) \tag{2.1}$$

where $x(t)$ is a zero mean, unit-variance, complex, stationary Gaussian process independent of $p(t)$; $p(t)$ is a non-negative random process called the power process. Slow fluctuation conditions exist when the correlation time of $p(t)$

is much larger than the correlation time of $x(t)$, while rapid fluctuation conditions exist when the correlation time of $p(t)$ is much smaller than the correlation time of $x(t)$. Nonfluctuation conditions exist when $p(t)$ is a known constant in which case $s(t)$ is a Gaussian process with variance p.

Let $Z(\ell)$ be a two-dimensional zero mean complex random column vector with elements $z_1(\ell)$ and $z_2(\ell)$ representing samples from channels 1 and 2 at time ℓT_S for $\ell = 1,2,\ldots,N_T$. T_S is the sampling interval, and $T = N_T T_S$ is the observation interval. The cross-covariance matrix of $Z(\ell)$ is defined as:

$$R_z(\ell,k) \;=\; E\{Z(\ell)\, Z'(k)\} \tag{2.2}$$

where $E\{\cdot\}$ denotes statistical expectation and ' is the complex conjugate of the transpose. Let $Z(\ell)$ contain spatially uncorrelated noise under the H_o hypothesis and contain correlated signal plus spatially uncorrelated noise under the H_1 hypothesis. Then

$$Z(\ell) \;=\; \begin{cases} \sqrt{N(\ell)}\; Y(\ell) & , H_o \\ \sqrt{S(\ell)}\; X(\ell) + \sqrt{N(\ell)}\; Y(\ell) & , H_1 \end{cases} \tag{2.3}$$

where $S(\ell)$ and $N(\ell)$ are the independent two-dimensional power vectors for signal and noise, respectively: $X(\ell)$ is a two-dimensional, unit-variance, zero-mean, complex Gaussian random vector with $\rho_s e^{j\theta s}$ the correlation coefficient between $x_1(\ell)$ and $x_2(\ell)$; and $Y(\ell)$ is a two-dimensional unit-variance, zero-mean, complex Gaussian random vector with independent components.

The sample MSCC can be computed from the sample auto-correlation matrix. The two-dimensional positive definite Hermetian sample auto-correlation matrix is

$$A \;=\; \frac{1}{N_T}\sum_{\ell=1}^{N_T} Z(\ell)\, Z'(\ell) \quad . \tag{2.4}$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12}^* & a_{22} \end{bmatrix} . \tag{2.5}$$

The sample MSCC is the sample magnitude-squared cross-correlation coefficient between $z_1(\ell)$ and $z_2(\ell)$ and is given by

$$\rho^2 = \frac{|a_{12}|^2}{a_{11}a_{22}} . \tag{2.6}$$

The PDF of ρ^2 can be derived from the PDF of A by (1) performing the change of variables indicated in Eq. (2.6) and (2) integrating out the auxiliary variables a_{11}, a_{22}, and the phase angle of a_{12}.

The cross-covariance matrix of $Z(\ell)$ is sample independent when the power processes are stationary. In this case,

$$R_Z(\ell,k) = \begin{cases} R_N & , H_o \\ R_S+R_N & , H_1 \end{cases} \tag{2.7}$$

where R_S and R_N are the cross-covariance matrices of the signal and noise vectors, respectively. Combining Eqs. (2.2) and (2.7), we have

$$R_N = \begin{bmatrix} \bar{N}_1 & 0 \\ 0 & \bar{N}_2 \end{bmatrix} \tag{2.8a}$$

and

$$R_S = \begin{bmatrix} \bar{S}_1 & E\{\sqrt{S_1(\ell)S_2(\ell)}\}\rho_s e^{j\theta s} \\ E\{\sqrt{S_1(\ell)S_2(\ell)}\}\rho_s e^{-j\theta s} & \bar{S}_2 \end{bmatrix} \tag{2.8b}$$

where

$$E\{S(\ell)\} = \bar{S} = \begin{bmatrix} \bar{S}_1 \\ \bar{S}_2 \end{bmatrix} \tag{2.8c}$$

$$E\{N(\ell)\} = \bar{N} = \begin{bmatrix} \bar{N}_1 \\ \bar{N}_2 \end{bmatrix} \quad . \tag{2.8d}$$

Therefore, according to Eqs. (2.7) and (2.8),

$$R_o = R_N = \begin{bmatrix} \bar{N}_1 & 0 \\ 0 & N_2 \end{bmatrix} \tag{2.9a}$$

$$R_1 = R_S + R_N$$

$$= \begin{bmatrix} \bar{S}_1 + \bar{N}_1 & E\{\sqrt{S_1(\ell)S_2(\ell)}\}\rho_s e^{j\theta s} \\ E\{\sqrt{S_1(\ell)S_2(\ell)}\}\rho_s e^{-j\theta s} & \bar{S}_2 + \bar{N}_2 \end{bmatrix} \tag{2.9b}$$

$$= \begin{bmatrix} \bar{P}_1 & \sqrt{\bar{P}_1\bar{P}_2}\ \rho_T e^{j\theta} \\ \sqrt{\bar{P}_1\bar{P}_2}\ \rho_T e^{-j\theta} & \bar{P}_2 \end{bmatrix} \tag{2.9c}$$

where $\bar{P}_k$ is the average power in channek k, ρ_T is the true correlation coefficient between the channels, and θ is the phase of the true correlation between the channels. The true MSCC is defined as:

$$\rho_T^2 = \frac{\left[\ E\{\sqrt{S_1(\ell)S_2(\ell)}\}\ \right]^2 \rho_s^2}{(\bar{S}_1+\bar{N}_1)(\bar{S}_2+\bar{N}_2)} \tag{2.10}$$

In the absence of fluctuation, $\bar{S}_k = S_k$, $\bar{N}_k = N_k$, and

$$\rho_T^2 = \frac{SNR_1 \; SNR_2 \; \rho_s^2}{(SNR_1+1)(SNR_2+1)} \quad . \tag{2.11}$$

If $S_1(\ell)$ and $S_2(\ell)$ are independent,

$$E\{\sqrt{S_1(\ell)S_2(\ell)}\} = E\{\sqrt{S_1(\ell)}\} \, E\{\sqrt{S_2(\ell)}\} \tag{2.12}$$

and

$$\rho_T^2 = \frac{\left(E\{\sqrt{S_1(\ell)}\}\right)^2 \left(E\{\sqrt{S_2(\ell)}\}\right)^2 \rho_s^2}{(\bar{S}_1+\bar{N}_1)(\bar{S}_2+\bar{N}_2)} \quad . \tag{2.13}$$

2.2 Fluctuation Statistics

The two-dimensional power vectors, $S(\ell)$ and $N(\ell)$, for signal and noise are modeled as two-dimensional Gamma random vectors. This appears to be a reasonable statistical model because it is a generalization of the distribution of the observed single-site fluctuation processes (refs. 6-11). It will be assumed that the power processes in channel 1 is independent of the power process in channel 2 for both signal and noise.

The PDF of the signal power process in channel k is

$$f(S_k) = \begin{cases} \dfrac{(S_k)^{MS_k-1}}{(\bar{S}_k/MS_k)^{MS_k} \, \Gamma(MS_k)} \, e^{-(MS_k S_k / \bar{S}_k)} & , \; S_k \geq 0 \\ 0 & , \; S_k < 0 \end{cases} \tag{2.13}$$

where MS_k is the signal degrees of freedom in channel k and $\bar{S}_k$ is the mean signal power in channel k. Similarly,

$$f(N_k) = \begin{cases} \dfrac{(N_k)^{MN_k-1}}{(\bar{N}_k/MN_k)^{MN_k}\Gamma(MN_k)} e^{-(MN_k N_k / \bar{N}_k)} & , N_k \geq 0 \\ 0 & , N_k < 0 \end{cases} \tag{2.14}$$

where MN_k is the noise degrees of freedom in channel k and $\bar{N}_k$ is the mean noise power in channel k. According to Eq. (A.3) of Appendix A, the mean signal power and variance of the signal power is

$$\begin{aligned} M_{S_k} &= \bar{S}_k \\ \sigma^2_{S_k} &= (\bar{S}_k)^2/MS_k \end{aligned} \tag{2.15}$$

and for the noise, we have

$$\begin{aligned} M_{N_k} &= \bar{N}_k \\ \sigma^2_{N_k} &= (\bar{N}_k)^2/MN_k \end{aligned} \tag{2.16}$$

Therefore, the mean powers are unbiased and the variances vanish with increasing degrees of freedom.

Finally, the true MSCC is biased in fluctuation conditions. A larger SNR is needed under fluctuation conditions to achieve the same ρ_T^2 as for no fluctuations. Substitute Eqs. (2.13), (2.14), and (A.2) into Eq. (2.10). Then,

$$\rho_T^2 = \left[\frac{\Gamma(MS_1 + 1/2)\ \Gamma(MS_2 + 1/2)}{\sqrt{MS_1 MS_2}\ \Gamma(MS_1)\ \Gamma(MS_2)}\right]^2 \frac{SNR_1 SNR_2}{(SNR_1+1)(SNR_2+1)}\ \rho_s^2 \tag{2.17}$$

where $SNR_k = \bar{S}_k/\bar{N}_k$. By comparing Eqs. (2.11) and (2.17), the bias factor is

$$\mathrm{BIAS} = \left[\frac{\Gamma(MS_1 + 1/2)\,\Gamma(MS_2 + 1/2)}{\sqrt{MS_1 MS_2}\,\Gamma(MS_1)\,\Gamma(MS_2)}\right]^2 . \qquad (2.18)$$

Eq. (2.18) is plotted in Figure 2.1 for $MS_1 = MS_2 = MS$. It is compared to ρ_T^2 for no bias. It can be seen that the BIAS can require significant increases in SNR.

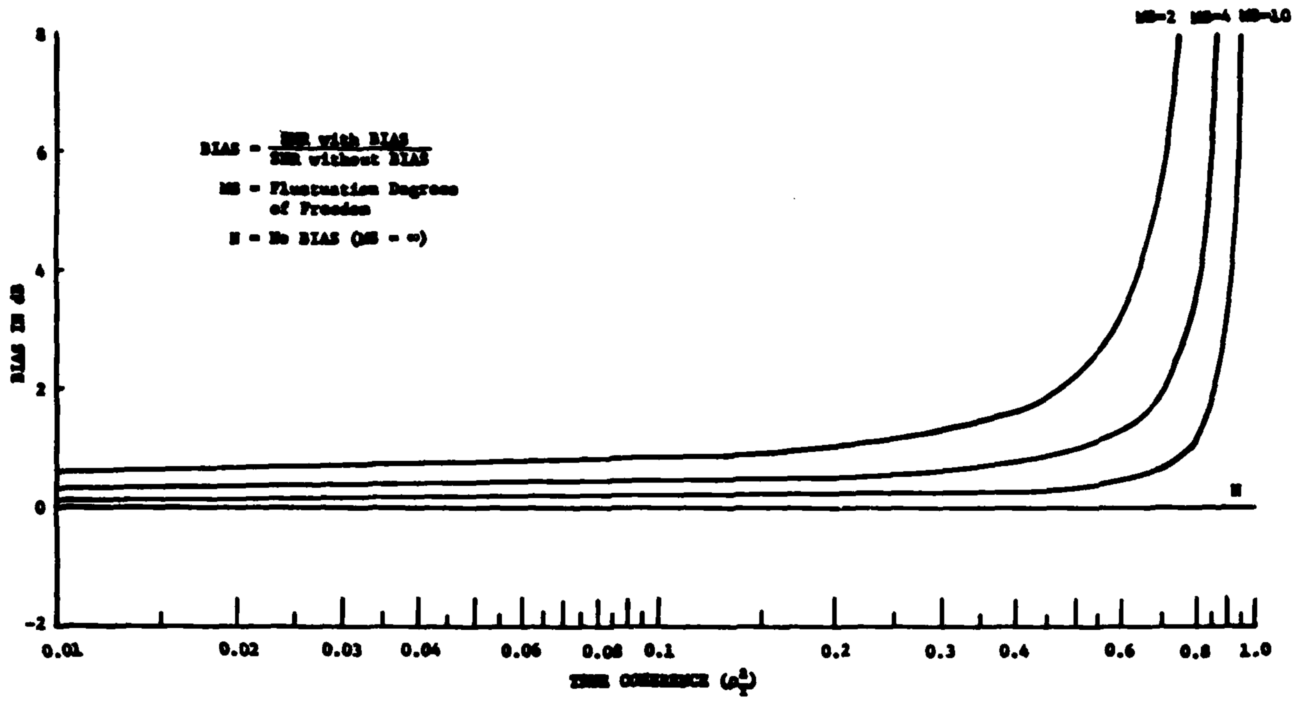

Figure 2.1. Bias Effects

3. SLOW FLUCTUATION

Slow fluctuation occurs when the correlation times of the signal and noise power processes are much larger than the observation interval. In this case, the power processes becomes an unknown constant but unknown. Therefore, the slow fluctuation case becomes the case of constant but unknown signal and noise power levels. The cumulative distribution function (CDF) of the sample magnitude-squared correlation coefficient (MSCC) is derived in Section 3.1. The detection performance of the sample MSCC is presented in Section 3.2. The results are summarized, and the implications discussed in Section 3.3.

3.1 Cumulative Distribution Function

The cumulative distribution function (CDF) of the sample MSCC for known SNR's is

$$F(\rho_t^2|\rho_1,\rho_2,\rho_s^2,N_T) = \rho_t^2(1-\rho_1\rho_2\rho_s^2)^{N_T} \cdot \sum_{\ell=0}^{N_T-2} (1-\rho_t^2)^{\ell}\,{}_2F_1(N_T,\ell-1;1;\rho_1\rho_2\rho_s^2) \tag{3.1}$$

where ρ_t^2 is the threshold, ρ_s is the correlation coefficient of the signal components, N_T is the degrees of freedom, ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ is the hypergeometric function, and

$$\rho_k = \frac{SNR_k}{SNR_k+1} \tag{3.2}$$

is the ratio of the SNR in channel k (ref. 1). For slow fluctuations, the SNR's are unknown constants. Therefore, the CDF of the sample MSCC for slow fluctuations becomes

$$F(\rho_t^2|\rho_s^2,N_T) = \int_0^1\!\!\int F(\rho_t^2|\rho_1,\rho_2,\rho_s^2,N_T)f(\rho_1,\rho_2)\, d\rho_1 d\rho_2 \tag{3.3}$$

where $f(\rho_1,\rho_2)$ is the joint probability density function (PDF) of the ρ_k's.

It is reasonable to assume that ρ_1 and ρ_2 are independent because the acoustic propagation conditions to the two receivers are different. Therefore, according to the fluctuation model discussed in Chapter 2,

$$f(\rho_k) = \begin{cases} \dfrac{\Gamma(MS_k + MN_k)}{\Gamma(MS_k)\,\Gamma(MN_k)} \, (\alpha_k)^{MS_k} \, \dfrac{\rho_k^{MS_k - 1}(1-\rho_k)^{MN_k - 1}}{\left(1+(\alpha_k - 1)\rho_k\right)^{NS_k + MN_k}} & , 0 \geq \rho_k \leq 1 \\ 0 & , \text{otherwise} \end{cases} \qquad (3.4a)$$

where

$$\alpha_k = \frac{MS_k}{MN_k} \frac{\bar{N}_k}{\bar{S}_k} \qquad (3.4b)$$

MS_k is the signal fluctuation degrees of freedom in channel k, MN_k is the noise fluctuation degrees of freedom in channel k, $\bar{S}_k$ is the mean signal power in channel k, and $\bar{N}_k$ is the mean noise power in channel k. The CDF of the sample MSCC is obtainable by substituting Eq. (3.4) into Eq. (3.3) and evaluating the integral.

Eq. (3.1) becomes, upon expanding the hypergeometric function,

$$F(\rho_t^2 | \rho_1, \rho_2, \rho_s^2, N_T) = \rho_t^2 \sum_{\ell=0}^{N-2} (1-\rho_t^2)^k \sum_{p=0}^{\infty} \frac{(N_T)_p (\ell+1)_p \rho_s^{2p}}{(p!)^2} \, t(p|\rho_1, \rho_2) \qquad (3.5a)$$

where $(x)_n = \Gamma(n+n)/\Gamma(x)$ is Pochammer's symbol,

$$t(p|\rho_1, \rho_2) = (\rho_1 \rho_2)^p (1-\rho_1 \rho_2 \rho_s^2)^{N_T} \qquad (3.5b)$$

Define

$$t(p) = \int_0^1\!\!\int t(p|\rho_1, \rho_2) f(\rho_1) f(\rho_2) d\rho_1 d\rho_2 \quad . \qquad (3.6)$$

Now

$$t(p|\rho_2) = \int_0^1 t(p|\rho_1,\rho_2)f(\rho_1)d\rho_1$$

$$= \frac{\Gamma(MS_1+MN_1)}{\Gamma(MS_1)\Gamma(MN_1)}\alpha_1^{MS_1}\rho_2^p \int_0^1 \frac{\rho_1^{MS_1+p-1}(1-\rho_1\rho_2\rho_s^2)^{N_T}(1-\rho_1)^{MN_1-1}}{(1-(1-\alpha_1)\rho_1)^{MS_1+MN_1}}d\rho_1$$

$$= \alpha_1^{MS_1}\sum_{q=0}^{N_T}\frac{(-N_T)_q(MS_1)_{p+q}}{(MS_1+MN_2)_{p+q}q!}\rho_s^{2q}\rho_2^{p+q} \cdot$$

$$_2F_1(MS_1+MN_1,\ MS_1+p+q;\ MS_1+MN_1+p+q;\ 1-\alpha_1) \tag{3.7}$$

according to Eq. (3.211) and Eq. (9.180.1) of reference 12.

$$t(p) = \int_0^1 t(p|\rho_2)f(\rho_2)d\rho_2$$

$$= \frac{\Gamma(MS_2+MN_2)}{\Gamma(MS_2)\Gamma(MN_2)}\alpha^{MS_1}\alpha_2^{MS_2}\sum_{q=0}^{N_T}\frac{(-N_T)_q(MS_1)_{p+q}\rho_s^{2q}}{(MS_1+MN_1)_{p+1}q!} \cdot$$

$$_2F_1(MS_1+MN_1,\ MS_1+p+q;\ MS_1+MN_1+p+q;\ 1-\alpha_1) \cdot$$

$$\int_0^1 \frac{\rho_2^{MS_2+p+q-1}(1-\rho_2)^{MN_2-1}}{(1-(1-\alpha_2)\rho_2)^{MS_2+MN_2}}d\rho_2 \qquad \text{(cont.)}$$

$$= \alpha_1^{MS_1} \alpha_2^{MS_2} \sum_{q=0}^{N_T} \frac{(-N_T)_q (MS_1)_{p+q} (MS_2)_{p+q} \, \rho_s^{2q}}{(MS_1+MN_1)_{p+q} (MS_2+MN_2)_{p+q} \, q!} \cdot$$

$${}_2F_1(MS_1+MN_1,\ MS_1+p+q;\ MS_1+MN_1+p+q;\ 1-\alpha_1) \cdot$$

$${}_2F_1(MS_2+MN_2,\ MS_2+p+q;\ MS_2+MN_2+p+q;\ 1-\alpha_2) \tag{3.8}$$

according to Eq. (3.197.3) of reference 12.

Under the H_1 hypothesis, the CDF of the sample MSCC is

$$F(\rho_t^2 | N_T)_1 = F(\rho_t^2 | \rho_s^2,\ N_t)$$

$$= \rho_t^2 \alpha_1^{MS_1} \sigma_o^{MS_2} \sum_{\ell=0}^{N_T-2} (1-\rho_t^2)^k \cdot$$

$$\sum_{q=0}^{N_T} \sum_{p=0}^{\infty} \frac{(N_T)_p (-N_T)_q (\ell+1)_p (MS_1)_{p+q} (MS_2)_{p+q}}{(MS_1+MN_1)_{p+q} (MS_2+MN_2)_{p+q} \, q! \, (p!)^2} (\rho_s^2)^{p+q} \cdot$$

$${}_2F_1(MS_1+MN_1,\ MS_1+p+q;\ MS_1+MN_1+p+q;\ 1-\alpha_1) \cdot$$

$${}_2F_1(MS_2+MN_2,\ MS_2+p+q;\ MS_2+MN_2+p+q;\ 1-\alpha_2) \tag{3.9a}$$

where

$$\alpha_k = \frac{MS_k}{MN_k} \frac{\bar{N}_k}{\bar{S}_k} \quad . \tag{3.9b}$$

Under the H_o hypothesis, $\rho_s^2 = 0$, and the CDF becomes

$$
\begin{aligned}
F(\rho_t^2 | N_T)_o &= F(\rho_t^2 | 0, N_T) \\
&= \rho_t^2 \sum_{q=0}^{N_T-2} (1-\rho_g^2)^k \\
&= 1 - (1-\rho_t^2)^{N_T-1} \quad . \qquad (3.10)
\end{aligned}
$$

Eq. (3.10) is the same as the equation for the CDF of the sample MSCC under H_o for known noise powers, Eq. (4.8) of reference 3. This is not surprising because the powers normalize out of the sample MSCC whenever the noise powers are constant, event if they are unknown Eqs. (2.4-2.6).

3.2 Detection Performance

The detection performance is quantified by the probability of false alarm (P_{FA}) and the probability of detection (P_D). These are defined as

$$
P_D = 1 - F(\rho_t^2 | N_T)_1 \qquad (3.11)
$$

and

$$
P_{FA} = 1 - F(\rho_t^2 | N_T)_0 = (1 - \rho_t^2)^{N_T-1} \qquad (3.12)
$$

These equations are evaluated for equal channel conditions where

$$
MS = MS_1 = MS_2 \qquad (3.13a)
$$

$$
MN = MN_1 = MN_2 \qquad (3.13b)
$$

$$
\overline{SNR} = \overline{SNR}_1 = \overline{SNR}_2 \qquad (3.13c)
$$

$$
\overline{SNR}_k = \overline{S}_k / \overline{N}_k \quad . \qquad (3.13d)
$$

The performance is quantified by (1) solving Eq. (3.12) for the threshold, ρ_t^2, for a specified P_{FA}, and (2) numerically solving Eq. (3.11) for the $\overline{SNR}$ required for the specified P_D, N_T, and ρ_s^2.

The performance is plotted as a function of N_T in Figure 3.1. It is immediately apparent that (1) uncertainty in signal and noise powers can require large increases in SNR to achieve the same performance in the absence of fluctuation, and (2) fluctuation effects decrease as the uncertainty decreases (i.e., MS and MN increase).

The performance is plotted as a function of P_D for various N_T's in Figures 3.2-3.4. It is apparent that fluctuation effects decrease as MS and MN increase. SNR is more sensitive to signal power fluctuations than to noise power fluctuations because the required SNR is larger for NS = 2, M = 10, than for MS = 10, MN = 2. This effect is larger for larger P_D's (> 0.4). The SNR sensitivity to signal fluctuations follows from the fact that the P_{FA} threshold is independent of noise power fluctuations.

3.3 Discussion

The cumulative density function (CDF) of the sample MSCC was derived for slow fluctuations. The CDF of the sample MSCC is independent of noise power fluctuations under the H_0 hypothesis.

It is observed that the SNR required to achieve the desired operating point decreased as the fluctuations decreased. The SNR is more sensitive to signal power fluctuations than to noise power fluctuations because the P_{FA} threshold is independent of noise power fluctuations. Slow fluctuations can require a 4-6 dB increased SNR over the SNR required to achieve comparable performance in the absence of fluctuation.

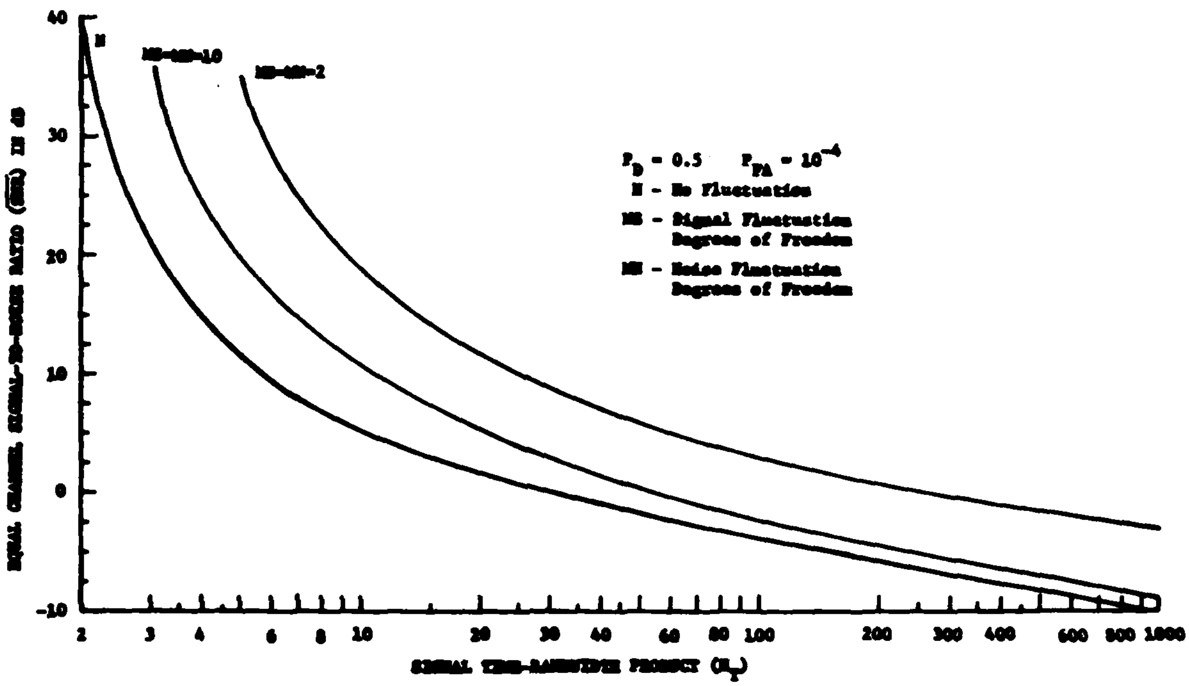

Figure 3-1. Slow Fluctuation Performance

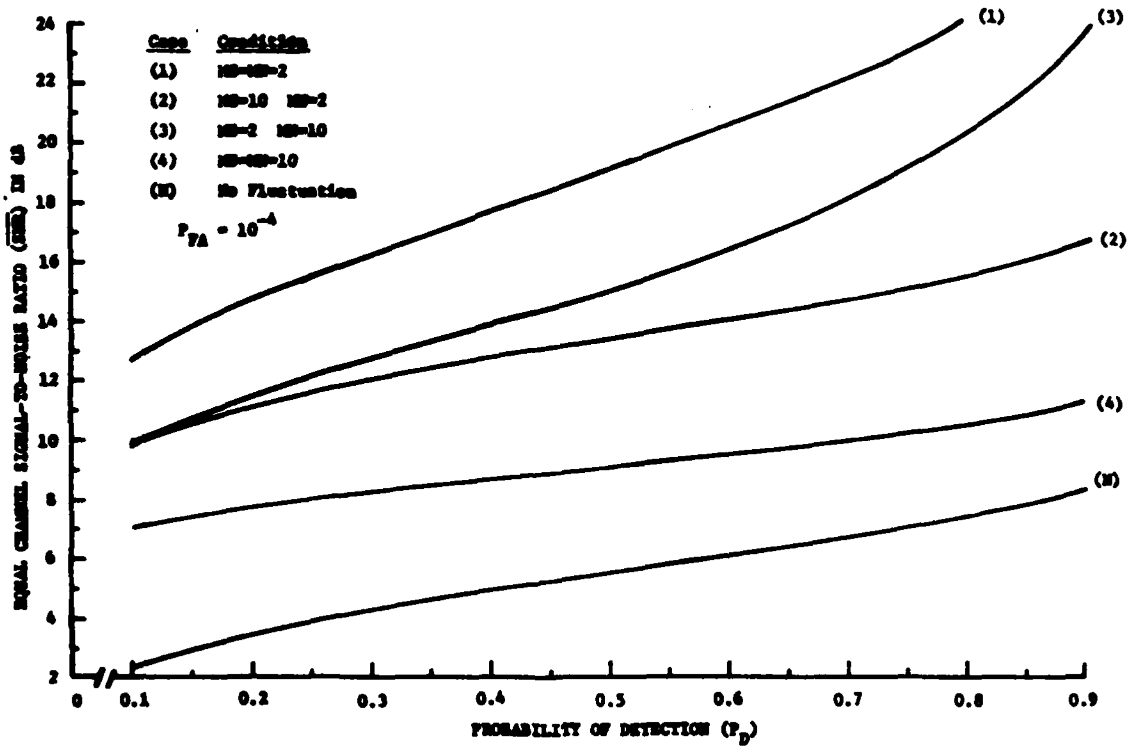

Figure 3-2. Slow Fluctuation Performance for N_T = 10

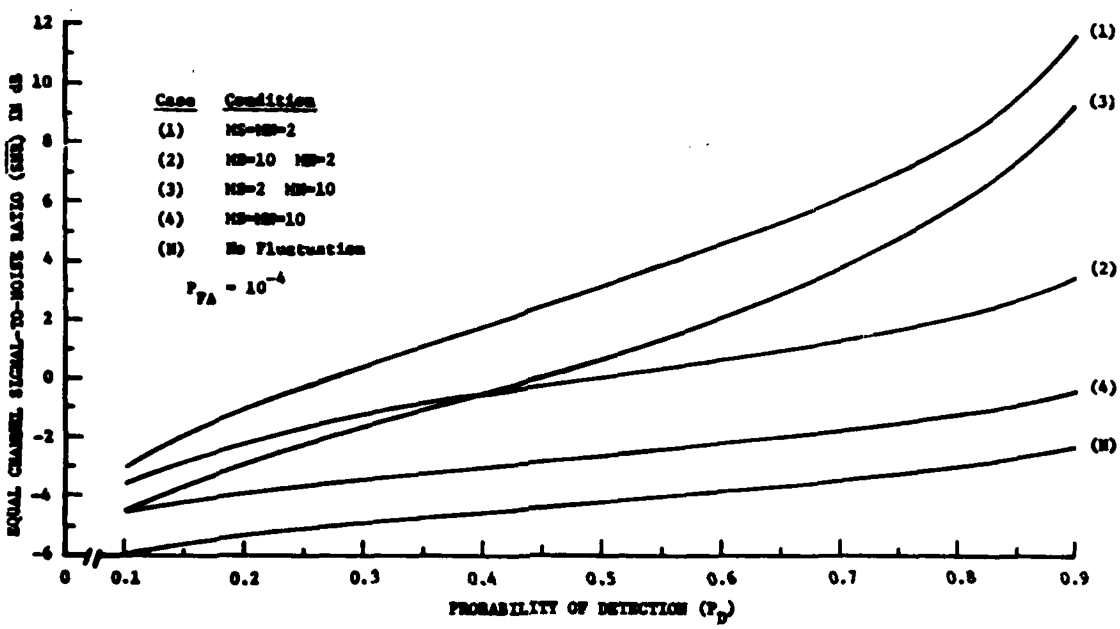

Figure 3-3. Slow Fluctuation Performance for N_T = 100

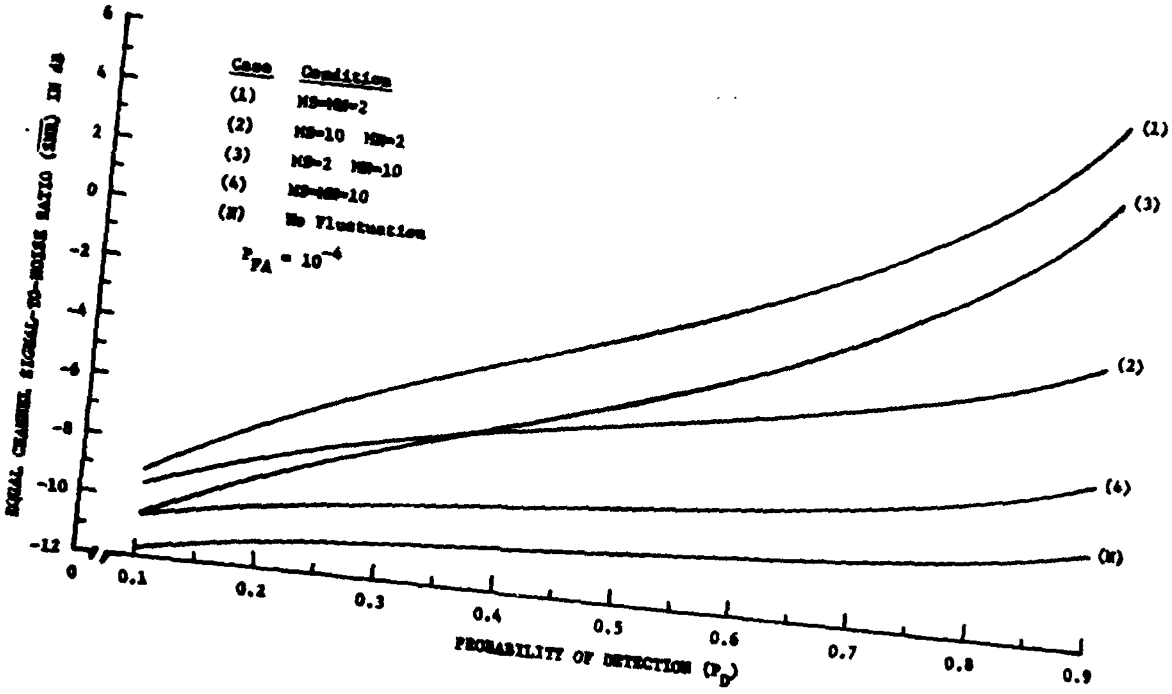

Figure 3-4. Slow Fluctuation Performance for N_T = 1000

4. RAPID FLUCTUATION

Rapid fluctuation occurs when the power level fluctuations from sample to same are so large that successive samples may be considered independent. The cumulative distribution function (CDF) of the sample magnitude-squared correlation coefficient (MSCC) is derived in Section 4.1, and the detection performance of the sample MSCC is presented in Section 4.2. The results are summarized and the implications discussed in Section 4.3.

4.1 CDF of the Sample MSCC

The CDF of the sample MSCC for rapid fluctuation is difficult to derive because the probability density function (PDF) of the observation, Z_ℓ defined in Eq. (2.3), is unknown for Gamma distributed fluctuations. This problem is overcome by using an Edgeworth series approximation to the PDF of Z_ℓ (Appendix B). The only way to incorporte the Edgeworth series into the derivatives of the CDF of the sample MSCC is to use an Edgeworth series approximation to the PDF of the sample auto-covariance matrix (Chapter 2). The Edgeworth series approximation to the CDF of the sample MSCC for rapid fluctuation is derived in Appendix C. The CDF of the sample MSCC with signal present is

$$G(\rho_t^2|N_T)_1 = F(\rho_t^2|N_T)_1 + N_T\tilde{F}(\rho_t^2|N_T)_1 \tag{4.1}$$

where

$$F(\rho_t^2|N_T)_1 = (1-\rho_t^2)^{N_T} \sum_{k=0}^{N_T-2} (1-\rho_t^2)^k {}_2F_1(N_T,k+1;1;\rho_t^2\rho_T^2) \tag{4.2}$$

is the CDF of ρ^2 for no fluctuation (ref. 1); $\rho_t^2 \in (0,1)$ is the threshold; and $\tilde{F}(\rho_t^2|N_T)_1$ is the Edgeworth series correction factor.

Since $G(\rho_t^2|N_T)$ and $F(\rho_t^2|N_T)$ are CDFs, $\tilde{F}(1|N_T)_1 = 0$ because $G(1|N_T)_1 = F(1|N_T)_1 = 1$. The correction factor is

$$
\begin{aligned}
\tilde{F}(\rho_t^2|N_T)_o \;=\; & PR_1 F(\rho_t^2|0,N_T-2,N_T,N_T,1) \\
& + PR_2 F(\rho_t^2|0,N_T-2,N_T,N_T+1,1) \\
& + PR_3 F(\rho_t^2|0,N_T-2,N_T,N_T+2,1) \\
& + PR_4 F(\rho_t^2|0,N_T-2,N_T+1,N_T+1,1) \\
& + PR_5 F(\rho_t^2|0,N_T-1,N_T+1,N_T+1,1) \\
& + PR_6 F(\rho_t^2|0,N_T-1,N_T+1,N_T+2,1) \\
& + PR_7 F(\rho_t^2|0,N_T,N_T+2,N_T+2,1) \\
& + PR_8 F(\rho_t^2|1,N_T-2,N_T+1,N_T+1,1) \\
& + PR_9 F(\rho_t^2|1,N_T-2,N_T+1,N_T+1,2) \\
& + PR_{10} F(\rho_t^2|1,N_T-2,N_T+1,N_T+2,2) \\
& + PR_{11} F(\rho_t^2|1,N_T-2,N_T+2,N_T+2,2) \\
& + PR_{12} F(\rho_t^2|2,N_T-2,N_T+2,N_T+2,3)
\end{aligned} \tag{4.3}
$$

where

$$F(\rho_t^2|\alpha,\beta,\theta,\phi,\gamma) =$$

$$\frac{\rho_t^{2(\alpha+1)}}{(\beta+1)_{\alpha+1}} \sum_{k=0}^{\beta} (k+1)_\alpha (1-\rho_t^2)^k \, {}_3F_2(\theta,\phi,\alpha+k+1;\gamma,\alpha+\beta+2;\rho_t^2\rho_T^2) \tag{4.4a}$$

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} \tag{4.4b}$$

is Pochammer's symbol, where the PR's are:

$$PR_1 = CN_7 \frac{(N_t-1)^2}{N_T+1} (1-\rho_t^2)^{N_T-2} \tag{4.5a}$$

$$PR_2 = CN_5 \frac{(N_T-1)^2}{N_T+1} (1-\rho_T^2)^{N_T-1} - 4CN_7 \frac{N_T(N_T-1)}{N_T+1} (1-\rho_T^2)^{N_T-2} \tag{4.5b}$$

$$PR_3 = \left(CN_1(1-\rho_T^2)^{N_T} - CN_5(1-\rho_T^2)^{N_T-1} + 2CN_7(1-\rho_T^2)^{N_T-2}\right)(N_T-1) \tag{4.5c}$$

$$PR_4 = \left(CN_4(1-\rho_T^2)^{N_T} - CN_5(1-\rho_T^2)^{N_T-1} + 2CN_7(1-\rho_T^2)^{N_T-2}\right) \frac{N_T(N_T-1)}{N_T+1} \tag{4.5d}$$

$$PR_5 = (-CN_5 + 2CN_6) \frac{N_T^2}{N_T+1}(1-\rho_T^2)^{N_T-1} + 2CN_7N_T(1-\rho_T^2)^{N_T-1} \tag{4.5e}$$

$$PR_6 = (-2CN_1 + CN_3 - 2CN_4)\, N_T(1-\rho_T^2)^{N_T} + \left(CN_5(3-\rho_T^2) - 4CN_6 - 4CN_7\right) N_T(1-\rho_T^2)^{N_T-1} \tag{4.5f}$$

$$PR_7 = \left(CN_1 + 2CN_2 - CN_3 + CN_4(1+\rho_T^2) - CN_5 + 2CN_6 + CN_7\right)(N_T+1)(1-\rho_T^2)^{N_T} \tag{4.5g}$$

$$PR_8 = \left(CN_1(1-\rho_T^2)^{N_T} - 2CN_6(1-\rho_t^2)^{N_T-1} + 2CN_7\rho_T^2(1-\rho_T^2)^{N_T-2}\right) \frac{N_T(N_T-1)}{N_T+1} \tag{4.5h}$$

$$PR_9 = \left(- 2CN_6(N_T-1)(1-\rho_T^2)^{N_T-1} + 4CN_7N_T\rho_T^2(1-\rho_T^2)^{N_T-2}\right) \frac{N_T(N_T-1)}{N_T+1} \tag{4.5i}$$

$$PR_{10} = \left(- 2CN_3(1-\rho_T^2)^{N_T} + (2CN_5\rho_T^2 + CN_6)(1-\rho_T^2)^{N_T-1} - 8CN_7\rho_T^2(1-\rho_T^2)^{N_T-2}\right) N_T(N_T-1) \tag{4.5j}$$

$$PR_{11} = -\left(- (4CN_2 - CN_3 + 2CN_4\rho_T^2)(1-\rho_T^2)^{N_T} + (2CN_5\rho_T^2 + 2CN_6(1-2\rho_T^2) - 4CN_7\rho_T^2)(1-\rho_T^2)^{N_T-1}\right) \cdot N_T(N_T+1) \tag{4.5k}$$

$$PR_{12} = \left(CN_2(1-\rho_T^2)^{N_T} - CN_6\rho_T^2)1-\rho_T^2)^{N_T-1} + CN_7\rho_T^2(1-\rho_T^2)^{N_T-2}\right)N_T(N_T+1)(N_T-1) \tag{4.5l}$$

where the CN's (Appendix D) are

$$CN_1 = \frac{SNR_1^2 + MS_1/MN_1}{MS_1(SNR_1+1)^2} + \frac{SNR_2^2 + NS_2/MN_2}{MS_2(SNR_2+1)^2} \tag{4.6a}$$

$$CN_2 = \frac{\rho_T^4}{BIAS}\left(1 - BIAS^{1/2}\right) \tag{4.6b}$$

$$CN_3 = 2\rho_T^2\left(\frac{SNR_1}{MS_1(SNR_1+1)} + \frac{SNR_2}{MS_2(SNR_2+1)}\right) \tag{4.6c}$$

$$CN_4 = \frac{\rho_T^2}{BIAS}(1 - BIAS) \tag{4.6d}$$

$$CN_5 = CN_3 - 2CN_1 - 2CN_4 \tag{4.6e}$$

$$CN_6 = 2CN_2 - \frac{CN_3}{2} + CN_4\rho_T^2 \tag{4.6f}$$

$$CN_7 = CN_1 + 2CN_2 - CN_3 + CN_4(1+\rho_T^2) \quad . \tag{4.6g}$$

and where

$$\rho_T^2 = BIAS \frac{SNR_1 SNR_2}{(SNR_1+1)(SNR_2+1)} \rho_s^2 \tag{4.7a}$$

$$BIAS = \left[\frac{\Gamma(MS_1 + 1/2)\,\Gamma(MS_2 + 1/2)}{\sqrt{MS_1 MS_2}\,\Gamma(MS_1)\,\Gamma(MS_2)}\right]^2 \tag{4.7b}$$

$$SNR_k = \bar{S}_k/\bar{N}_k \quad , \text{ and} \tag{4.7c}$$

ρ_s is the correlation coefficient between the signal components.

Under the H_o hypothesis, $\rho_T^2 = SNR_k = 0$. Then the CDF of the sample MSCC becomes

$$G(\rho_t^2|N_T)_o = F(\rho_t^2|N_T)_o + N_T\tilde{F}(\rho_t^2|N_T)_o \tag{4.8a}$$

where

$$F(\rho_t^2|N_T)_o = 1 - (1-\rho_t^2)^{N_T-1} \quad , \qquad (4.8b)$$

$$\tilde{F}(\rho_t^2|N_T)_o = CN_1 \left[\frac{4N_T^2 + 2N_T-6}{(N_T+1)(N_T-1)} F(\rho_t^2|N_T)_o - \frac{(8N_T + 10)}{N_T + 1} F(\rho_t^2|N_T+1)_o + 4F(\rho_t^2|N_T+2)_o \right] \quad , \text{ and} \qquad (4.8c)$$

$$CN_1 = \frac{1}{MN_1} + \frac{1}{MN_2} \quad . \qquad (4.8d)$$

4.2 Detection Performance

The detection performance is quantified by the probability of false alarm (P_{FA}) and the probability of detection (P_D). These are defined as:

$$P_D = 1 - G(\rho_t^2|N_T)_1 \qquad (4.9)$$

and

$$P_{FA} = 1 - G(\rho_t^2|N_T)_0 \qquad (4.10)$$

where $G(\rho_t^2|N_T)_1$ and $G(\rho_t^2|N_T)_0$ are defined in Eqs. (4.1) - (4.8). These equations are evaluated for equal channel conditions where

$$MS = MS_1 = MS_2 \qquad (4.11a)$$

$$MN = MN_1 = MN_2 \qquad (4.11b)$$

$$\overline{SNR} = \overline{SNR}_1 = \overline{SNR}_2 \quad . \qquad (4.13c)$$

The equation expressing the relationship between the P_{FA}, ρ_t^2, and N_T for no fluctuation is well known (ref. 1). It is

$$P_{FA} = (1 - \rho_t^2)^{N_T-1} \tag{4.13}$$

By comparing Eq. (4.13) to Eq. (4.10), it is apparent that rapid fluctuation affects the P_{FA} threshold, ρ_t^2. One of the attractions of using the sample MSCC for detection is that ρ_t^2 is independent of the noise properties in the absence of fluctuations. However, this property does not hold when rapid fluctuations are present in the noise. The rapid fluctuation thresholds for a specified P_{FA} and N_T are plotted in Figure 4.1 for various fluctuation parameters. The ρ_t^2 are computed by numerically solving Eq. (4.10) for specified P_{FA}, N_T, and MN. It is seen that rapid fluctuation has the largest effect on ρ_t^2 for $6 \leq N_T \leq 500$. It is also apparent that the influence of rapid fluctuation decreases with increasing fluctuation degrees of freedom (MN) because the variance of the fluctuation process decreases as MN increases, Eg. (2.16).

The performance is plotted as a function of N_T in Figure 4.2. It is apparent that (1) rapid fluctuation can require large increases in SNR with respect to the SNR in the absence of fluctuation, (2) fluctuation effects decrease as MS and MN increase, and (3) fluctuation effects decrease as N_T increases. This means that the performance becomes somewhat insensitive to rapid fluctuation for large N_T.

The performance is plotted as a function of P_D for various N_T's in Figures 4.3 through 4.5. It is apparent tht fluctuation effects decrease as MS and MN increase. SNR is more sensitive to noise power fluctuations than to signal power fluctuations because the required SNR is larger for MS = 10, MN = 2, than for MS = 2, MN = 10. This sensitivity follows from the fact that the P_{FA} threshold is dependent on the noise power fluctuations. It is also seen that the effects of rapid fluctuation can be reduced by increasing N_T.

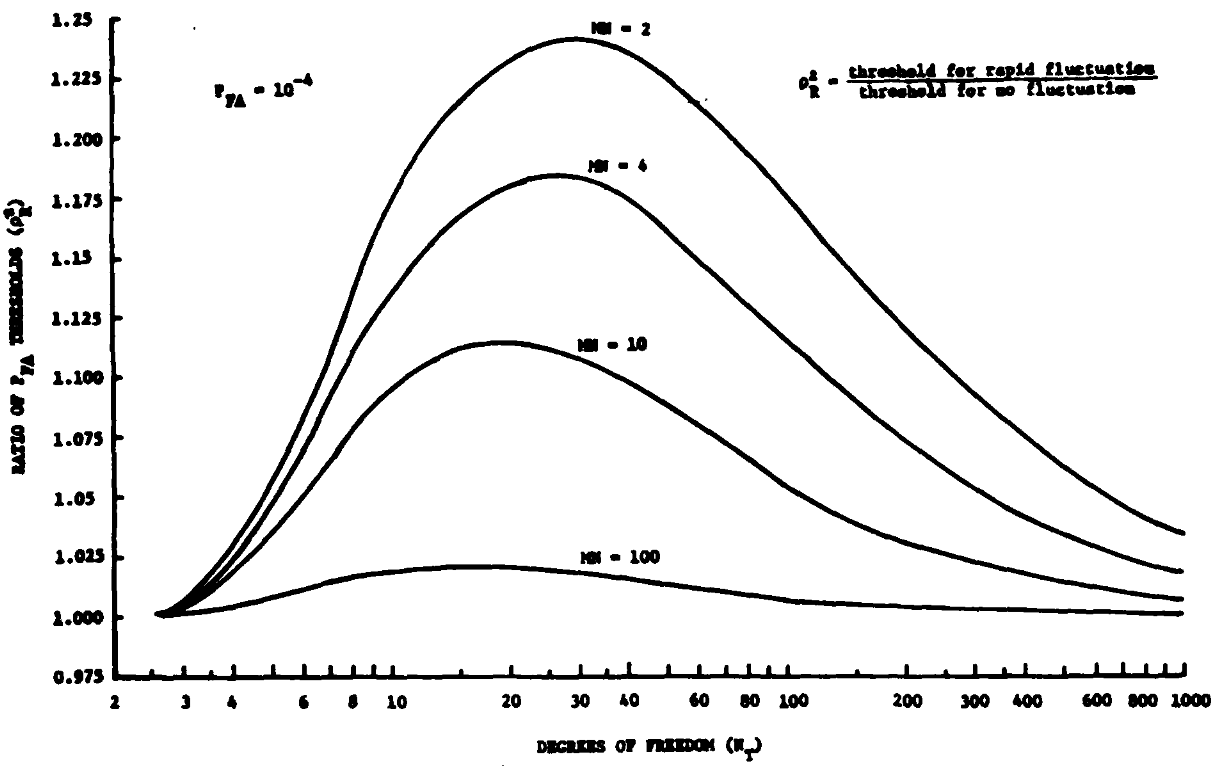

Figure 4-1. False Alarm Thresholds

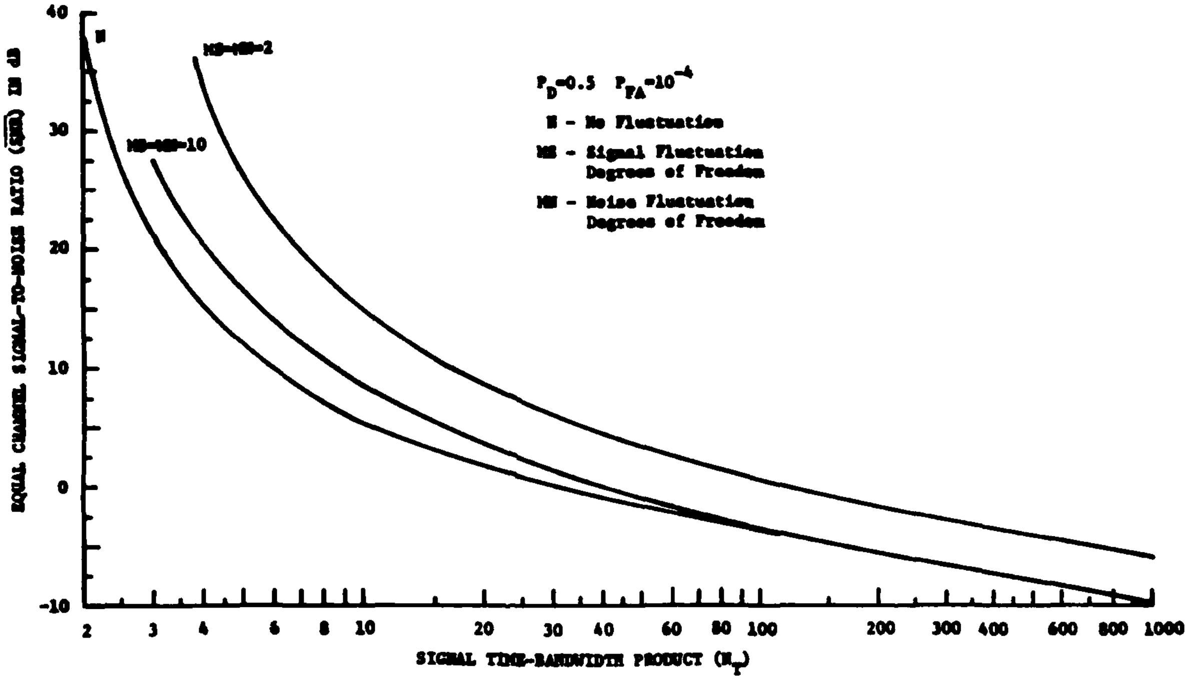

Figure 4-2. Rapid Fluctuation Performance

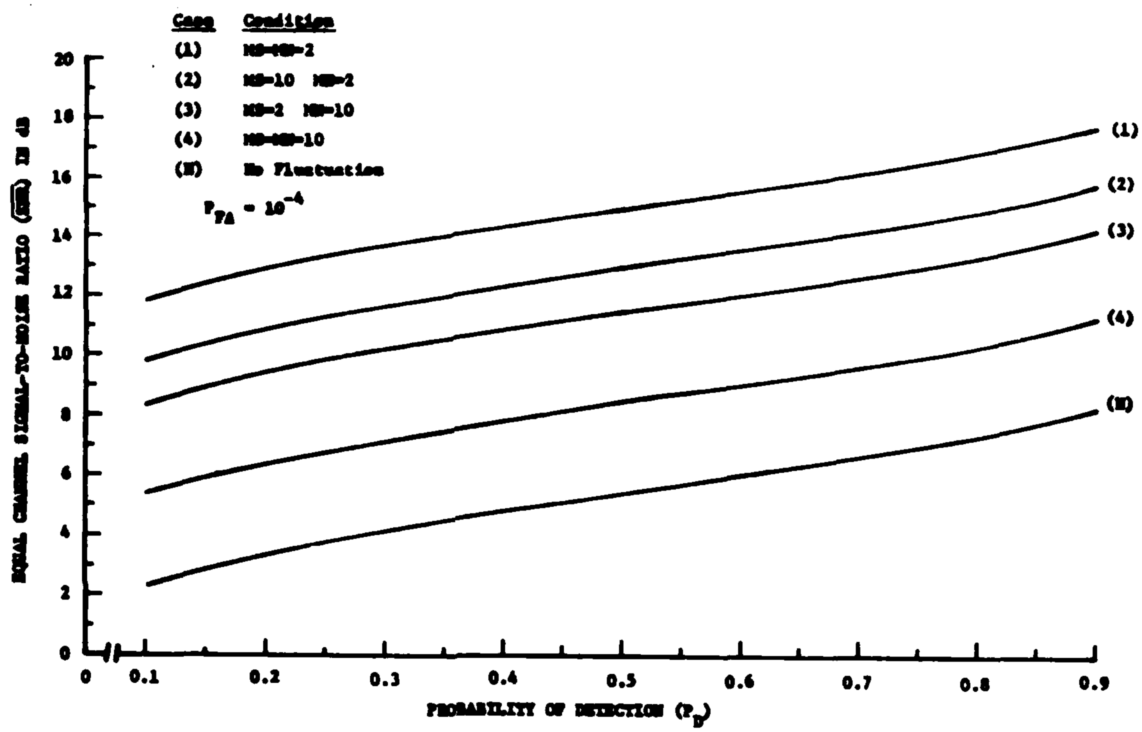

Figure 4-3. Rapid Fluctuation Performance for $N_T = 10$

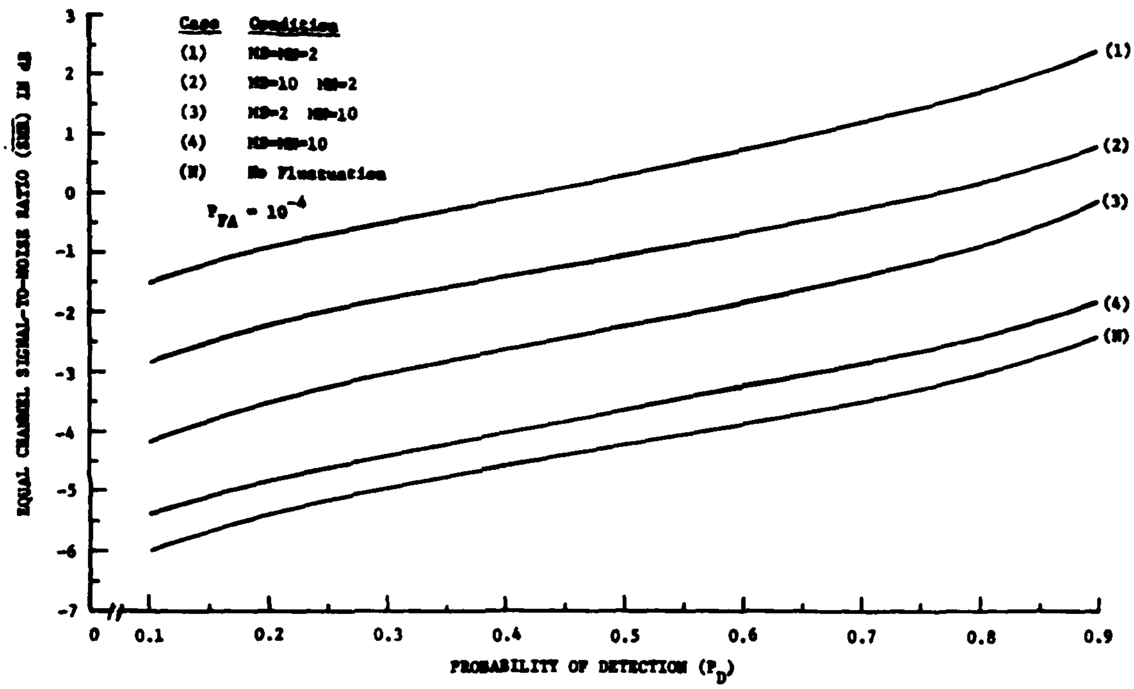

Figure 4-4. Rapid Fluctuation Performance for N_T = 100

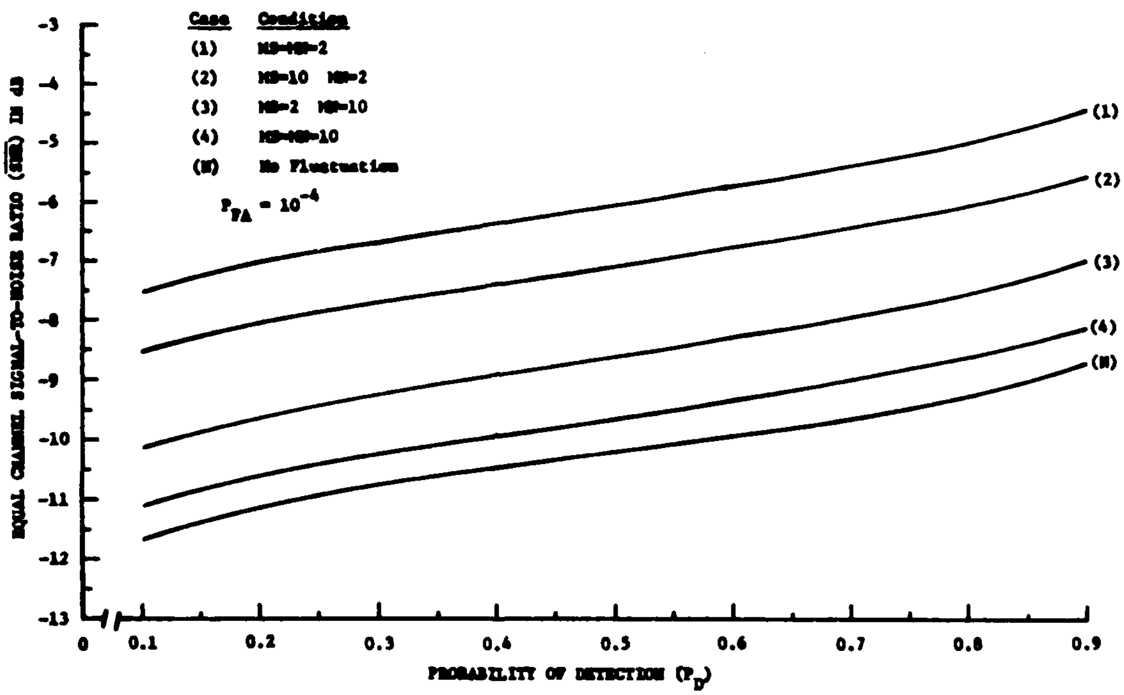

Figure 4-5. Rapid Fluctuation Performance for N_T = 1000

4.3 Discussion

The cumulative density function (CDF) of the sample MSCC was derived for rapid fluctuation. Due to the mathematical difficulties inherent in the derivation, the CDF was approximated with an Edgeworth series.

The CDF of the sample MSCC under H_0 is dependent on the noise power fluctuation parameters. Consequently, the sample MSCC loses some of its attractiveness as a detector because the P_{FA} threshold is dependent on the fluctuation parameters. This is in contrast to the CDF of the sample MSCC in the absence of fluctuations where the CDF is only dependent on the noise degrees of freedom.

It is observed that the SNR required to achieve the desired operating point decreased as the fluctuation decreased (i.e., MS and MN increased). The SNR is more sensitive to noise power fluctuations than to signal power fluctuations because the P_{FA} threshold is affected by the noise power fluctuation. Rapid fluctuations can require a 3-4 dB increase in SNR over the SNR required to achieve the comparable performance in the absence of fluctuations.

5. CONCLUSIONS

A detailed analysis of the detection performance of the sample magnitude-squared correlation coefficient (MSCC) in the presence of fluctuations has been presented. Fluctuations can be characterized as slow or rapid, or as anything in between. The fluctuation process samples are completely correlated over the observation interval for slow fluctuation, while the fluctuation process samples are completely uncorrelated over the observation interval for rapid fluctuations. These two bounds on fluctuations can be studied analytically. Simulation is required to study fluctuation processes with correlation times that lie between the bounds.

It is concluded that fluctuations require a 3-4 dB increase in SNR for rapid fluctuations and a 4-6 dB increase in SNR for slow fluctuations over the SNR required to achieve comparable peformance in the absence of fluctuations. In all fluctuation cases, the required SNR decreases as the fluctuation becomes "less" random (i.e., the variance decreases). However, the effects of slow fluctuation are basically independent of the signal time-bandwidth product (N_T), while the effects of rapid fluctuation can be reduced by increasing N_T.

The P_{FA} threshold (ρ_t^2) is independent of the noise fluctuation process for slow fluctuations, but it is dependent on the fluctuation process for rapid fluctuation. It can be concluded that ρ_t^2 is dependent on the fluctuation process for all correlation times except for slow fluctuation. The dependency of ρ_t^2 on the noise fluctuation process decreases as the correlation time of the fluctuation process increases.

The SNR for rapid fluctuations is more sensitive to noise power fluctuations than to signal power fluctuations for all fluctuation processes with correlation times less than the observation interval. This is caused by the fact that (1) ρ_t^2 is dependent on the noise fluctuation process and (2) some of the noise dependency is accounted for in selecting ρ_t^2. On the other hand, the SNR for slow fluctuations is more sensitive to signal power fluctuations than to noise power fluctuations.

References

1. G.C. Carter, Estimation of the Magnitude-Squared Coherence Function (Spectrum), Report No. 7R4343, Naval Underwater Sytems Center, New London, CT, May 1972.

2. R.D. Truebloo c and D.L. Alzrach, A Comparison of Incoherent and Coherent Test Statistics for Magnitude-Squared Coherence (U), Report No. TN1571, Naval Ocean Systems Center, San Diego, CA, July 1975 (Secret).

3. G.C. Carter, "Receiver Operating Characteristics for a Linearly Thresholded Coherence Estimation Detector," IEEE Trans. Acoust., Speech, Signal Processing, vol. 25, pp. 90-92, February 1977.

4. J. LaPointe, Jr., Ambiguity Surface Statistics, Final Report SV8007-1, ATAC, Sunnyvale, CA, September 1981.

5. J. LaPointe, Jr., "The Impact of Signal Overcontainment on Cross-Correlation Detection Performance," Proc. 1982 Internat'l. Conf. on Acoustics, Speech and Signal Processing.

6. M. Moll and D. Goodman, "Effects of Input Power Fluctuation on Passive Sonar Receiver Performance," Bolt Beranek and Newman, Inc., Report No. 4262, September 1980.

7. M. Moll, "Prediction of Passive Sonar Detection Performance in Environments with Acoustics Fluctuations," Bolt, Beranek and Newman, Inc., Report No. 3656, November 1978.

8. T. Arase and E.M. Arase, "Deep-Sea Ambient Noise Sttistics," J. Acoust. Soc. Am., vol. 44, no. 6, December 1968, pp. 1679-1684.

9. J.C. Heine and J.R. Nitsche, "The Effects of Fluctuating Signals and Noise on Detection Performance," presented at the 94th Meeting Acoustical Society of America, December 1977.

10. G.V. Frisk, "Intensity Statistics for Long-Range Acoustic Propagation in the Ocean," J. Acoust. Soc. Am., vol. 64, no. 1, July 1978, pp. 257-259.

11. W.J. Jobst and S.L. Adams, "Statistical Analysis of Ambient Noise," J. Acoust. Soc. Am., vol. 62, no. 1, July 1977, pp. 63-71.

12. I.S. Gradshteyn and I.M. Ryzhik, Tables of Integrals, Sines and Products, Academic Press, New York, 1965.

Appendix A

GAMMA DISTRIBUTION

The normalized Gamma probability density function (PDF) is the standard Gamma PDF normalized so that the mean is independent of the degrees of freedom. The normalized Gamma PDF is

$$f(x) = \begin{cases} \dfrac{x^{M-1} e^{-Mx/x_o}}{(x_o/M)^M \Gamma(M)} , & x \geq 0 \\ \\ o , & x < 0 \end{cases} \qquad (A.1)$$

where M is the degrees of freedom and x_o is the mean. The αth moment of x is

$$M_\alpha = E(x^\alpha)$$

$$= \frac{1}{(x_o/M)^M \Gamma(M)} \int_0^\infty x^{M+\alpha-1} e^{-Mx/x_o} dx$$

$$= \frac{\Gamma(M+\alpha)}{M^\alpha \Gamma(M)} x_o^\alpha \qquad (A.2)$$

Therefore, the mean and variance are:

$$\bar{x} = M_1 = x_o \qquad (A.3a)$$

$$\sigma_x^2 = M_2 - M_1^2 = x_o^2/M \quad . \qquad (A.3b)$$

Note that the variance vanishes as $M \to \infty$.

Let x and y be two independent Gamma distributed random variables with degrees of freedom M_x and M_y, respectively, and mean x_o and y_o, respectively. Define the random variable

$$z = x/y \quad . \qquad (A.4)$$

What is the PDF of z? Define the auxiliary variable w = y. Then the Jacobian of transformation from (x,y) to (w,z) is

$$J(x,y) \;=\; \begin{vmatrix} 1/y \;, & -\,x/y^2 \\ 0 \;, & 1 \end{vmatrix} \;=\; 1/y \;=\; 1/w \quad .$$

Then,

$$f(z,w) \;=\; wf_{xy}(zw,w)$$

and

$$f(z) \;=\; \int_0^{\infty} f(z,w)\,dw \;=\; \int_0^{\infty} wf_{xy}/xw,w)\,dw \quad . \tag{A.5}$$

Therefore,

$$f(z) \;=\; \frac{z^{M_x-1}}{\left(x_o/M_x\right)^{M_x}\,\Gamma(M_x)\,\left(x_o/M_y\right)^{M_y}\,\Gamma(M_y)} \cdot$$

$$\int_0^{\infty} w^{M_x+M_y-1}\, e^{-(M_x z/x_o \,+\, M_y/y_o)w}\,dw$$

$$= \begin{cases} \dfrac{\Gamma(M_x+M_y)}{\Gamma(M_x)\Gamma(M_y)}\,\alpha^{M_x}\,\dfrac{z^{M_x-1}}{(1+\alpha z)^{M_x+M_y}} \;, & z \geq 0 \\[2ex] 0 \;, & z < 0 \end{cases} \tag{A.6a}$$

where

$$\alpha \;=\; \frac{M_x y_o}{M_y x_o} \quad . \tag{A.6b}$$

The mean and variance of z are:

$$M_z = \frac{M_y}{M_y-1}\frac{x_o}{y_o} \qquad \text{for } M_y > 1 \tag{A.7a}$$

$$\sigma_z^2 = \frac{M_x+M_y-1}{M_x(M_y-2)}(M_z)^2 \qquad \text{for } M_y > 2 \quad . \tag{A.7b}$$

Define the random variable

$$\rho = \frac{z}{z+1} \quad . \tag{A.8}$$

What is the PDF of ρ? It is easily shown that

$$f(\rho) = \frac{f_z\left(\frac{\rho}{1-\rho}\right)}{(1-\rho)^2} \quad . \tag{A.9}$$

Substitue Eq. (A.6a) into Eq. (A.9). Then,

$$f(\rho) = \begin{cases} \dfrac{\Gamma(M_x+M_y)}{\Gamma(M_x)\,\Gamma(M_y)}\,\alpha^{M_x}\,\dfrac{\rho^{M_x-1}(1-\rho)^{M_y-1}}{\left[1+(\alpha-1)\rho\right]^{M_x+M_y}} & , \quad 0\le\rho\le1 \\ 0 & , \quad \text{otherwise} \end{cases} \tag{A.10}$$

The β'th moment of ρ is

$$\begin{aligned} M_\rho(\beta) &= E(\rho^\beta) \\ &= \frac{\Gamma(M_x+M_y)\,\Gamma(M_x+\beta)}{\Gamma(M_x)\,\Gamma(M_x+M_y+\beta)}\,{}_2F_1(M_x+M_y,\ M_x+\beta;\ M_x+M_y+\beta;\ 1-\alpha) \end{aligned} \tag{A.11}$$

according to Eq. 3.197.3 of Reference A1.

References

A1 I.S. Gradshteyn and I.M. Ryzhik, Tables of Integrals, Sines and Products (New York: Academic Press, 1965).

Appendix B

EDGEWORTH SERIES FOR COMPLEX SPHERICALLY INVARIATE RANDOM PROCESSES

The derivation of the Edgeworth Series for complex, spherically invariant random processes is presented in this appendix. A random process is spherically invariant if and only if it is a zero-mean Gaussian process that is multiplied by an independent random variable (Ref. B.1). The derivation of the Edgeworth Series for a specific type of spherically invariant random process is obtained by (1) computing the moment generating function, (2) computing the cumulant generating function from the moment generating function, and (3) finally identifying terms.

B.1 Process Description

Let $Z = (z_1, z_2)^T$ be a two-dimensional, complex zero compound process described as

$$z_k = \sqrt{S_k}\, x_k + \sqrt{N_k}\, y_k \quad , \tag{B.1}$$

where S_k and N_k are independent, non-negative random variables called power processes, which are also independent of x_k and y_k; x_k and y_k are independent, zero mean complex Gaussian random variables with unit variance; ρ_s is the correlation coefficient of x_1 and x_2; and T indicates transpose. Given S_k and N_k, Z is a two-dimensional, zero mean, complex Gaussian random variable with covariance matrix

$$R = \begin{bmatrix} S_1+N_1 & \sqrt{S_1 S_2}\, \rho_s \\ \sqrt{S_1 S_2}\, \rho_s^* & S_2+N_2 \end{bmatrix} \tag{B.2a}$$

$$= \begin{bmatrix} \rho_1 & \sqrt{S_1 S_2}\, \rho_s \\ \sqrt{S_1 S_2}\, \rho_s^* & \rho_2 \end{bmatrix} \tag{B.2b}$$

where * indicates complex conjugate. Therefore, the probability density function (PDF) of Z, given the power process, is

$$f(Z|S_1,S_2,M_1,M_2) = \frac{1}{\Pi^2|R|^{1/2}} e^{-Z'R^{-1}Z} \tag{B.3}$$

where ' indicates complex conjugate of the transpose. Finally, the PDF of z is

$$f(Z) = E\{f(Z|S_1,S_2,N_1,N_2)\}_{S_1,S_2,N_1,N_2} \tag{B.4}$$

where $E\{\cdot\}_{S_1,S_2,N_1,N_2}$ is the expectation over S_1,S_2,N_1,N_2.

B.2 Moment Generating Function

The moment generating function, given the power processes, is

$$M_z(\phi|S_1,S_2,N_1,N_2) = e^{\phi' R\phi} \tag{B.5a}$$

where

$$\phi = (\phi_1,\phi_2)^T \quad . \tag{B.5b}$$

Expand Eq. (B.5a):

$$\begin{aligned} M_z(\phi|S_1,S_2,N_1,N_2) &= \exp\left\{r_1|\phi_1|^2 + \rho_2|\phi_2|^2 + 2\sqrt{S_1S_2}\,\mathrm{Re}(\rho_s\phi_1^*\phi_2)\right\} \\ &= \sum_{\ell=0}^{\infty}\frac{1}{\ell!}\sum_{k=0}^{\ell}\sum_{q=0}^{\ell-k}\sum_{p=0}^{k} C_{\ell,k,q,p}\,\cdot \\ &\qquad \phi_1^{k-p+q}\phi_1^{*\,\ell-q-p}\phi_2^{\ell-k-q+p}\phi_2^{*\,-q+p} \end{aligned} \tag{B.6a}$$

where

$$C_{\ell,k,q,p} = \binom{\ell}{k}\binom{\ell-k}{q}\binom{k}{p}(S_1S_2)^{\frac{\ell-k}{2}}\, p_1^{k-p}\, p_2^{p}\, \rho_s^{\ell-k-q}\, \rho_s^{*q} \qquad \text{(B.6b)}$$

and $\binom{\ell}{k}$ is the binomial coefficient. From the discussion in section B.1, we know that

$$M_z(\phi) = E\{M_z(\phi|S_1,S_2,N_1,N_2)\}_{S_1,S_2,N_1,N_2}$$

$$= \sum_{\ell=0}^{\infty}\frac{P_\ell(\phi)}{\ell!} \qquad \text{(B.7a)}$$

where

$$P_\ell(\phi) = \sum_{k=0}^{\ell}\sum_{q=0}^{\ell-k}\sum_{p=0}^{k}\overline{C_{\ell,k,q,p}}\,\phi_1^{k-p+q}\phi_1^{*\ell-q-p}\phi_2^{\ell-k-q+p}\phi_2^{*q+p} \qquad \text{(B.7b)}$$

$$\overline{C_{\ell,k,q,p}} = \binom{\ell}{k}\binom{\ell-k}{q}\binom{k}{p}m_1(k-p,\ell-k)m_2(p,\ell-k)\rho_s^{\ell-k-q}\rho_s^{*q} \qquad \text{(B.7c)}$$

$$m_k(\alpha,\beta) = E\left(r_1^{\alpha}S_1^{\beta/2}\right) \quad . \qquad \text{(B.7d)}$$

B.3 Cumulant Generating Function

The cumulant generating function is

$$K_z(\phi) = \ln\left(M_z(\phi)\right)$$

$$= \ln\left(1 + \tilde{M}_z(\phi)\right)$$

$$= \sum_{u=1}^{\infty}(-1)^{u+1}\frac{\left(\tilde{M}_z(\phi)\right)^u}{u!} \qquad \text{(B.8)}$$

where $M_z(0) = 0$. In light of the discussion in section (B.2), $K_z(0)$ can be expanded in two ways. First,

$$K_z(\phi) = \sum_{r=1}^{\infty} \theta_r(\phi) \tag{B.8a}$$

where

$$\theta_r(\phi) = \sum_{u+v+x+y=r} \frac{\lambda_{uvxy}}{u!v!x!y!}(\phi_1)^u(\phi_1^*)^v(\phi_2)^x(\phi_2^*)^y \tag{B.8b}$$

is a polynomial of rth order, and

$$\lambda_{uvxy} = \frac{\partial^{u+v+x+y}}{\partial\phi_1^u \partial\phi_1^{*v} \partial\phi_2^x \partial\phi_2^{*y}} K_z(\phi)\bigg|_{\phi=0} \quad . \tag{B.8c}$$

The second way is to substitute Eq. (B.7) into Eq. (B.8):

$$K_z(\phi) = \sum_{u=1}^{\infty} \frac{(-1)^{u+1}\left[\sum_{\ell=1}^{\infty} P_\ell(\phi)\right]^u}{u} \tag{B.9}$$

Only terms up to r=4 in Eq. (B.8) will be considered because of the complexity of the problem. By expanding Eqs. (B.8) and (B.9) and identifying terms, it follows that Q's are related to the P's in the following manner:

$$\theta_1(\phi) = \theta_3(\phi) = 0 \tag{B.10a}$$

$$\theta_2(\phi) = P_1(\phi) \tag{B.10b}$$

$$\theta_4(\phi) = P_2(\phi) - \frac{P_1^2(\phi)}{2} \quad . \tag{B.10c}$$

Also, the λ's are related to the $\bar{C}$'s according to the following:

<u>$\theta_2(\phi)\lambda$'s</u> :

$$\lambda_{2000} = \lambda_{0200} = \lambda_{0020} = \lambda_{0002} = 0$$

$$\lambda_{1010} = \lambda_{0101} = 0$$

$$\lambda_{1100} = \bar{C}_{1100} \tag{B.11}$$

$$\lambda_{0011} = \bar{C}_{1101}$$

$$\lambda_{0110} = \bar{C}_{1000}$$

$$\lambda_{1001} = \bar{C}_{1010}$$

<u>$\theta_4(\phi)\lambda$'s</u> :

$$\lambda_{2200} = 4\bar{C}_{2200} - 2\bar{C}^2_{1100}$$

$$\lambda_{0022} = 4\bar{C}_{2202} - 2\bar{C}^2_{1101}$$

$$\lambda_{0220} = 4\bar{C}_{2000} - 2\bar{C}^2_{1000}$$

$$\lambda_{2002} = 4\bar{C}_{2020} - 2\bar{C}^2_{1010}$$

$$\lambda_{1210} = 2\bar{C}_{2100} - 2\bar{C}_{1000}\bar{C}_{1100} \tag{B.12}$$

$$\lambda_{0121} = 2\bar{C}_{2101} - 2\bar{C}_{1000}\bar{C}_{1101}$$

$$\lambda_{2101} = 2\bar{C}_{2110} - 2\bar{C}_{1010}\bar{C}_{1100}$$

$$\lambda_{1012} = 2\bar{C}_{2111} - 2\bar{C}_{1010}\bar{C}_{1101}$$

(cont.)

$$\lambda_{1111} = \bar{c}_{2010}\bar{c}_{2201} - \bar{c}_{1100}\bar{c}_{1101} - \bar{c}_{1000}\bar{c}_{1010}$$

and all other λ's = 0.

Substitute Eq. (B.7) into Eqs. (B.11) and (B.12). Then,

$$\begin{aligned}
\lambda_{1100} &= m_1(1,0) = E\{r_1\} \\
\lambda_{0011} &= m_2(1,0) = E\{r_2\} \\
\lambda_{0110} &= m_1(0,1)m_2(0,1)\rho_s = E\{\sqrt{S_1}\}\, E\{\sqrt{S_2}\}\rho_s \\
\lambda_{1001} &= \lambda^*_{0110} \\
\lambda_{2200} &= 2\left(m_1(2,0) - m_1(1,0)^2\right) \\
\lambda_{0022} &= 2\left(m_2(2,0) - m_2(1,0)^2\right) \\
\lambda_{0220} &= 2\left(m_1(0,2)m_2(0,2) - m_1(0,1)^2 m_2(0,1)^2\right)\rho_s^2 \\
\lambda_{2002} &= \lambda^*_{0220} \\
\lambda_{1210} &= 2\left(m_1(1,1)m_2(0,1) - m_1(1,0)m_1(0,1)m_2(0,1)\right)\rho_s \\
\lambda_{2101} &= \lambda^*_{1210} \\
\lambda_{0121} &= 2\left(m_1(0,1)m_2(1,1) - m_2(1,0)m_1(0,1)m_2(0,1)\right)\rho_s \\
\lambda_{1012} &= \lambda^*_{0121} \\
\lambda_{1111} &= \left(m_1(0,2)m_2(0,2) - m_1(0,1)^2 m_2(0,1)^2\right)|\rho_s|^2
\end{aligned} \qquad \text{(B.13)}$$

Therefore,

$$
\begin{aligned}
K_z(\phi) &= \theta_1(\phi) + \theta_2(\phi) \\
&= \lambda_{1100}|\phi_1|^2 + \lambda_{0011}|\phi_2|^2 + \lambda_{0110}\phi_1^*\phi_2 + \lambda_{0110}^*\phi_1\phi_2^* \\
&\quad + \frac{\lambda_{2200}}{4}|\phi_1|^4 + \frac{\lambda_{0022}}{4}|\phi_2|^4 \\
&\quad + \frac{\lambda_{0220}}{4}\phi_1^{*2}\phi_2^2 + \frac{\lambda_{0220}^*}{4}\phi_1^2\phi_2^{*2} \\
&\quad + \frac{\lambda_{1210}}{2}\phi_1\phi_1^{*2}\phi_2 + \frac{\lambda_{1210}^*}{2}\phi_1^2\phi_1^*\phi_2 \\
&\quad + \frac{\lambda_{0121}}{2}\phi_1^*\phi_2^2\phi_2^* + \frac{\lambda_{0121}^*}{2}\phi_1\phi_2\phi_2^{*2} \\
&\quad + \lambda_{1111}|\phi_1|^2|\phi_2|^2 \quad .
\end{aligned} \tag{B.14}
$$

B.4 Edgeworth Series

Let

$$
\tilde{R} = \begin{bmatrix} \lambda_{1100} & \lambda_{0110} \\ \lambda_{0110}^* & \lambda_{0011} \end{bmatrix} \tag{B.15}
$$

$$
= \begin{bmatrix} r_{11} & r_{12} \\ r_{12}^* & r_{22} \end{bmatrix} \quad .
$$

Then,

$$
M_z(\phi) = e^{K_z(\phi)} \quad . \tag{B.16}
$$

Substitute Eq. (B.14) into Eq. (B.16):

$$M_z(\phi) = e^{\theta_2(\phi)} e^{\theta_4(\phi)} = e^{\theta_2(\phi)} \sum_{\ell=0}^{\infty} \frac{\theta_4(\phi)^\ell}{\ell!} \quad . \tag{B.17}$$

Using only the first two terms of the series expansion for the exponential,

$$\begin{aligned} M_z(\phi) &\simeq \left(1 + \theta_4(\phi)\right) e^{\theta_2(\phi)} \\ &= \Bigg[1 + \frac{\lambda_{2200}}{4}|\phi_1|^4 + \frac{\lambda_{0022}}{4}|\phi_2|^4 \\ &\quad + \frac{\lambda_{0220}}{4}\phi_1^{*2}\phi_2^2 + \frac{\lambda_{0220}^*}{4}\phi_1^2\phi_2^{*2} \\ &\quad + \frac{\lambda_{1210}}{2}\phi_1\phi_1^{*2}\phi_2 + \frac{\lambda_{1210}^*}{2}\phi_1^2\phi_1^*\phi_2 \\ &\quad + \frac{\lambda_{0121}}{2}\phi_1^*\phi_2^2\phi_2^* + \frac{\lambda_{0121}^*}{2}\phi_1\phi_2\phi_2^{*2} \\ &\quad + \lambda_{1111}|\phi_1|^2|\phi_2|^2\Bigg] e^{\phi' R\phi} \quad . \end{aligned} \tag{B.18}$$

It is easily shown that

$$|\phi_1|^4 e^{\phi'\tilde{R}\phi} = \frac{\partial^2}{\partial r_{11}^2} e^{\phi'\tilde{R}\phi} \tag{B.19a}$$

$$|\phi_2|^4 e^{\phi'\tilde{R}\phi} = \frac{\partial^2}{\partial r_{22}^2} e^{\phi'\tilde{R}\phi} \tag{B.19b}$$

$$\phi_1^{*2}\phi_2^2 e^{\phi'\tilde{R}\phi} = \frac{\partial^2}{\partial r_{12}^2} e^{\phi'\tilde{R}\phi} \tag{B.19c}$$

$$\phi_1^2\phi_2^{*2} e^{\phi'\tilde{R}\phi} = \frac{\partial^2}{2\partial_{12}^{*2}} e^{\phi'\tilde{R}\phi} \tag{B.19d}$$

$$\phi_1^* \phi_1^2 \phi_2 e^{\phi' \tilde{R} \phi} = \frac{\partial^2}{\partial r_{11} \partial r_{12}} e^{\phi' \tilde{R} \phi} \tag{B.19e}$$

$$\phi_1^2 \phi_1^* \phi_2^* e^{\phi' \tilde{R} \phi} = \frac{\partial^2}{\partial r_{11} \partial r_{12}^*} e^{\phi' \tilde{R} \phi} \tag{B.19f}$$

$$\phi_1^* \phi_2^2 \phi_2^* e^{\phi' \tilde{R} \phi} = \frac{\partial^2}{\partial r_{22} \partial r_{12}} e^{\phi' \tilde{R} \phi} \tag{B.19g}$$

$$\phi_1 \phi_2 \phi_2^{*2} e^{\phi' \tilde{R} \phi} = \frac{\partial^2}{\partial r_{22} \partial r_{12}^*} e^{\phi' \tilde{R} \phi} \tag{B.19h}$$

$$|\phi_1|^2 |\phi_2|^2 e^{\phi' \tilde{R} \phi} = \frac{\partial^2}{\partial r_{11} \partial r_{22}} e^{\phi' \tilde{R} \phi} \quad . \tag{B.19i}$$

Substitute Eq. (B.19) into Eq. (B.18). Then,

$$\begin{aligned} M_z(\phi) \simeq \Bigg[1 &+ \frac{\lambda_{2200}}{4} \frac{\partial^2}{\partial r_{11}^2} + \frac{\lambda_{0022}}{4} \frac{\partial^2}{\partial r_{12}^2} \\ &+ \frac{\lambda_{0220}}{4} \frac{\partial^2}{\partial r_{12}^2} + \frac{\lambda_{0220}^*}{4} \frac{\partial 2}{\partial r_{12}^{*2}} \\ &+ \frac{\lambda_{1210}}{2} \frac{\partial^2}{\partial r_{11} \partial r_{12}} + \frac{\lambda_{1210}^*}{2} \frac{\partial^2}{\partial r_{11} \partial r_{12}^*} \\ &+ \frac{\lambda_{0121}}{2} \frac{\partial^2}{\partial r_{22} \partial r_{12}} + \frac{\lambda_{0121}^*}{2} \frac{\partial^2}{\partial r_{22} \partial r_{12}^*} \\ &+ \lambda_{1111} \frac{\partial^2}{\partial r_{11} \partial r_{22}} \Bigg] e^{\phi' R \phi} \quad . \end{aligned} \tag{B.20}$$

The probability density function of z is obtained by taking the inverse Fourier transform of $M_z(0)$. Taking the inverse Fourier transform of Eq. (B.20), we have

$$
\begin{aligned}
g(Z) = \Bigg[1 &+ \frac{\lambda_{2200}}{4} \frac{\partial^2}{\partial r_{11}^2} + \frac{\lambda_{2200}}{4} \frac{\partial^2}{\partial r_{22}^2} \\
&+ \frac{\lambda_{0220}}{4} \frac{\partial^2}{\partial r_{12}^{*2}} + \frac{\lambda^*_{0220}}{4} \frac{\partial^2}{\partial r_{12}^{*2}} \\
&+ \frac{\lambda_{1210}}{2} \frac{\partial^2}{\partial r_{11} \partial r_{12}} + \frac{\lambda^*_{1210}}{2} \frac{\partial^2}{\partial r_{11} \partial r^*_{12}} \\
&+ \frac{\lambda_{0121}}{2} \frac{\partial^2}{\partial 4_{22} \partial r_{12}} + \frac{\lambda^*_{0121}}{2} \frac{\partial^2}{\partial r_{22} \partial r^*_{12}} \\
&+ \lambda_{1111} \frac{\partial^2}{\partial r_{11} \partial r_{22}} \Bigg] f(Z)
\end{aligned}
\qquad (B.21)
$$

where

$$
f(Z) = \frac{1}{\Pi^2 |\tilde{R}|^{1/2}} e^{-Z'\tilde{R}^{-1}Z} \quad . \qquad (B.22)
$$

References

B.1 G.L. Wise and M.C. Gallagher, "On Spherically Invariant Random Processes," IEEE Trans. Inform. Theory, Vol. IT-24, January, 1978, pp. 118-120.

Appendix C

EDGEWORTH SERIES FOR THE CUMULATIVE DENSITY FUNCTION OF THE MSCC FOR A SPHERICALLY INVARIANT PROCESS

The derivation of the probability and cumulative density functions (PDF and CDF, respectively) of the sample magnitude-squared correlation coefficient (MSCC) is difficult for non-Gaussian signals. The Edgeworth series for the PDF of the sample MSCC will be developed for signals that are spherically invariant. The approach used is to derive the PDF of the sample auto-covariance matrix of the observations and then make a chnage of variables to obtain the PDF of the sample MSCC. The processes involved in the derivation are described in section C.1. The derivation of the characteristic functions and PDF of the sample auto-covariance matrix is presented in sections C.2 and C.3, respectively. The PDF of the sample MSCC is obtained from the PDF of the sample auto-covariance matrix in section C.4. Finally, the CDF is obtained in section C.5.

C.1 Approach

Let Z_ℓ be a two-dimensional zero mean complex random column vector with elements $z_1(\ell)$ and $z_2(\ell)$ representing samples from channels 1 and 2 at time ℓT_S for $\ell = 1,2,\ldots,N_T$. T_S is the sampling interval, and $T = N_T T_S$ is the observation interval. The cross-covariance matrix of $Z(\ell)$ is defined as:

$$R_z(\ell,k) = E\{Z(\ell)\, Z'(k)\} \tag{C.1}$$

where $E\{\cdot\}$ denotes statistical expectation and ' is the complex conjugate of the transpose. Let

$$Z_\ell = \sqrt{S(\ell)}\, X_\ell + \sqrt{N(\ell)}\, Y_\ell \tag{C.2}$$

where $S(\ell)$ and $N(\ell)$ are the independent two-dimensional power vectors for signal and noise, respectively; $X(\ell)$ is a two-dimensional unit-variance, zero-mean, complex Gaussian random vector with $\rho_R e^{j\theta_s}$ as the correlation coefficient between $x_1(\ell)$ and $x_2(\ell)$; and $Y(\ell)$ is a two-dimensional unit-variance, zero-mean, complex Gaussian random vector with independent components.

Given S_k and N_k, Z_ℓ is a two-dimensional, zero mean, complex Gaussian random variable with covariance matrix

$$R = \begin{bmatrix} S_1 + N_1 & \sqrt{S_1 S_2}\, \rho_s \\ \sqrt{S_1 S_2}\, \rho_s^* & S_2 + N_2 \end{bmatrix}$$

$$= \begin{bmatrix} r_1 & \sqrt{S_1 S_2}\, \rho_s \\ \sqrt{S_1 S_2}\, \rho_s^* & r_2 \end{bmatrix}$$

where * indicates complex conjugate.

The exact form of the PDF of Z_ℓ is unknown. However, the Edgeworth series form of the PDF is known (see Appendix B). The PDF of Z_ℓ is

$$g(Z_\ell) = (1 + P)\, f(Z_\ell) \tag{C.3}$$

where

$$P = \frac{C_1}{4} \frac{\partial^2}{\partial r_{11}^2} + \frac{C_2}{4} \frac{\partial^2}{\partial r_{22}^2}$$
$$+ \frac{C_3}{4}\left[\rho_s^2 \frac{\partial^2}{\partial r_{12}^2} + \rho_s^{*2} \frac{\partial^2}{\partial r_{12}^{*2}}\right]$$
$$+ \frac{C_4}{2}\left[\rho_s \frac{\partial^2}{\partial r_{11}\partial r_{12}} + \rho_s^* \frac{\partial^2}{\partial r_{11}\partial r_{12}^*}\right]$$

(cont.)

$$+\frac{C_5}{2}\left[\rho_s \frac{\partial^2}{\partial r_{22}\partial r_{12}} + \rho_s^* \frac{\partial^2}{\partial r_{22}\partial r_{12}^*}\right]$$

$$+ C_6|\rho_s|^2 \frac{\partial^2}{\partial r_{11}\partial r_{22}} \tag{C.4}$$

$$C_1 = 2\left(m_1(2,0) - m_1(1,0)^2\right) \tag{C.5a}$$

$$C_2 = 2\left(m_2(2,0) - m_2(1,0)^2\right) \tag{C.5b}$$

$$C_3 = 2\left(m_1(0,2)\ m_2(0,2) - m_1(0,1)^2\ m_2(0,1)^2\right) \tag{C.5c}$$

$$C_4 = 2\left(m_1(1,1)\ m_2(0,1) - m_1(1,0)\ m_1(0,1)\ m_2(0,1)\right) \tag{C.5d}$$

$$C_5 = 2(m_1(0,1)\ m_2(1,1) - m_2(1,0)\ m_1(0,1)\ m_2(0,1)) \tag{C.5e}$$

$$C_6 = m_1(0,2)\ m_2(0,2) - m_1(0,1)^2\ m_2(0,1)^2 \tag{C.5f}$$

$$m_k(\alpha,\beta) = E\{r_k^\alpha\ S_k^{\beta/2}\} \tag{C.6}$$

$$f(Z_\ell) = \frac{1}{\Pi^2|\tilde{R}|^{1/2}}\ e^{-Z_\ell^\dagger \tilde{R} Z_\ell} \tag{C.7}$$

$$R = \begin{bmatrix} E\{r_1\} & E\{\sqrt{S_1S_2}\}\ \rho_s \\ E\{\sqrt{S_1S_2}\}\ \rho_s^* & E\{r_2\} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{12}^* & r_{22} \end{bmatrix} \tag{C.8}$$

The sample MSCC can be computed from the sample auto-correlation matrix. The two-dimensional positive definite Hermetian sample auto-correlation matrix is

$$A = \sum_{\ell=1}^{N_T} A_\ell \tag{C.9}$$

where

$$A_\ell = \frac{Z_\ell Z_\ell^\dagger}{N_T} \quad . \tag{C.10}$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12}^* & a_{22} \end{bmatrix} \quad . \tag{C.11}$$

The sample MSCC is the sample magnitude-squared cross-correlation coefficient between $z_1(\ell)$ and $z_2(\ell)$ and is given by

$$\rho^2 = \frac{|a_{12}|^2}{a_{11}a_{22}} \quad . \tag{C.12}$$

The PDF of ρ^2 can be derived from the PDF of A by (1) performing the change of variables indicated in Eq. (C.6) and (2) integrating out the auxiliary variables A_{11}, A_{22}, and the phase angle of a_{12}.

C.2 Characteristic Function of A

The characteristic function of A_ℓ, using the Edgeworth form of the PDF of Z_ℓ, is

$$\begin{aligned} M_{A_\ell}(\Phi) &= E\left\{e^{jTR(\Phi A_\ell/N_T)}\right\} \\ &= E\left\{e^{Z_\ell^\dagger(\Phi/N_T)Z_\ell}\right\} \\ &= \frac{(1 + P)}{\left|I - j\tilde{R}\frac{\Phi}{N_T}\right|} \end{aligned} \tag{C.13}$$

where $\left|I - j\tilde{R}\frac{\Phi}{N_T}\right|^{-1}$ is the characteristic function of A_ℓ assuming the PDF of Z_ℓ is $f(Z_\ell)$ as defined in Eqs. (C.7) through (C.8),

$$\Phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi^*_{12} & \phi_{22} \end{bmatrix} .$$

Substitute Eq. (C.4) into Eq. (C.13) and carry out the indicated partial differentiation. After some tedious algebra,

$$M_{A_\ell}(\Phi) = D^{-1}\left(1 + \frac{Q}{D^2}\right) \tag{C.14a}$$

where

$$D = \left| I - j\,\tilde{R}\,\frac{\Phi}{N_T} \right| \tag{C.14b}$$

$$Q = Q_1 + Q_2 + C_3 \tag{C.14c}$$

$$\begin{aligned} Q_1 = \Bigg\{ & \frac{C_1}{2} r_{22}^2 + \frac{C_2}{2} r_{11}^2 \\ & + \frac{C_3}{2}\left(\rho_s^2 r_{12}^{*2} + \rho_s^{*2} r_{12}^2\right) \\ & - C_4\left(\rho_s r^*_{12} r_{22} + \rho^*_s r_{12} r_{22}\right) \\ & - C_5\left(\rho_s r^*_{12} r_{11} + \rho^*_s r_{12} r_{22}\right) \\ & + C_6 |\rho_s|^2 \left(r_{11} r_{22} + |r_{12}|^2\right) \Bigg\} \frac{|\Phi|^2}{N_T^4} \end{aligned} \tag{C.14d}$$

$$\begin{aligned} Q_2 = \Bigg\{ & - C_1 r_{22} \phi_{11} - C_2 r_{11} \phi_{22} \\ & + C_3\left(\rho_s^2 r^*_{12} \phi^*_{12} + \rho_s^{*2} r_{12} \phi_{12}\right) \\ & + C_4\left(\rho_s r^*_{12} \phi_{11} + \rho^*_s r_{12} \phi_{11} - (\rho_s \phi^*_{12} r_{22} + \rho^*_s \phi_{12} r_{22})\right) \end{aligned}$$

(cont.)

$$+ C_5\left(\rho_s r^*_1 \phi_{22} + \rho^*_2 r_{12}\phi_{11} - (\rho_s \phi^*_{12} r_{11} + \rho^*_s \phi_{12} r_{11})\right)$$

$$\left. - C_6 |\rho_s|^2 (r_{11}\phi_{11} + r_{22}\phi_{22} - r_{12}\phi^*_{12} - r^*_{12}\phi_{12}) \right\} \frac{|\phi|}{N_T^3} \qquad \text{(C.14e)}$$

$$Q_3 = \left\{ \frac{C_1}{2}\phi_{11}^2 + \frac{C_2}{2}\phi_{22}^2 \right.$$

$$+ C_3(\rho_s^2 \phi^{*2}_{12} + \rho^{*2}_s \phi^2_{12})$$

$$+ C_4(\rho_s \phi^*_{12}\phi_{11} + \rho^*_s \phi_{12}\phi_{11})$$

$$+ C_5(\rho_s \phi^*_{12}\phi_{22} + \rho^*_s \phi_{12}\phi_{22})$$

$$\left. + C_6 |\rho_s|^2 (\phi_{11}\phi_{22} + |\phi_{12}|^2) \right\} \frac{1}{N_T^2} \qquad \text{(C.14f)}$$

Since the A_ℓ's are independent,

$$M_A(\phi) = \left[M_{A_\ell}(\phi) \right]^{N_T} \qquad \text{(C.15)}$$

Substitute Eq. (C.14a) into Eq. (C.15):

$$M_A(\phi) = D^{-N_T}\left[1 + Q/D^2\right]^{N_T}$$

$$\approx D^{-N_T}\left[1 + N_T Q/D^2\right]$$

$$= D^{-N_T} + N_T Q D^{-(N_T+2)} \qquad \text{(C.16)}$$

C.3 Probability Density Function of A

The PDF of A is obtained by taking the inverse Fourier transform of $M_A(\phi)$ given in Eqs. (C.14) - (C.16). The inverse Fourier transform of D^{-N} is the complex Wishart

$$f(A|N_T) = C(N_T)\ V(A|N_T) \qquad \text{(C.17a)}$$

where

$$V(A|N_T) = |A|^{N_T} e^{-N_T TR(A\tilde{S})} \tag{C.17b}$$

$$C(N_T) = \frac{N^{2N_T} |\tilde{S}|^{N_T}}{\Pi\Gamma(N_T)\Gamma(N_T - 1)} \tag{C.17c}$$

$$\tilde{S} = \tilde{R}^{-1} = \begin{bmatrix} s_{11} & s_{12} \\ s^*_{12} & s_{22} \end{bmatrix} \tag{C.17d}$$

which was derived by Goodman (ref. C.1). Correspondingly, the PDF represented by the characteristic function $D^{-(N+2)}$ is $f(A|N_T+2)$.

By making use of the fact that the inverse Fourier transform of $\phi_{\ell k}D^{-N_T}$ can be obtained by $(\partial^{\alpha}/\partial a^{\alpha}_{\ell k})f(A|N_T)$, the PDF of A can be obtained from $M_A(\phi)$ by taking the proper partial derivatives with respect to the $a_{\ell k}$ of $f(A|N_T+2)$. Therefore, substitute Eqs. (C.17) into Eqs. (C.1d) - (C.16) and perform the indicated partial differentiation. After much tedious algebra,

$$g(A) = f(A|N_T) + N_T U(A|N_T) \tag{C.18a}$$

where U is the correction term of $f(A|N_T)$ which is the PDF of A for Z_ℓ complex Gaussian.

$$U(A|N_T) = U_1(A|N_T) + U_2(A|N_T) + U_3(A|N_T) \tag{C.18b}$$

$$N_T^4 U_1(A|N_T) = \left\{ \frac{C_1}{2} r_{22}^2 + \frac{C_2}{2} r_{11}^2 + \frac{C_3}{2}(\rho_s^2 r_{12}^{*2} + \rho_s^{*2} r_{12}^2) \right.$$

$$- (\rho_s r^*_{12} + \rho^*_s r_{12})(C_4 r_{22} + C_5 r_{11})$$

$$\left. + C_6 |\rho_s|^2 (r_{11} r_{22} + |r_{12}|^2) \right\} \cdot$$

$$\left\{ \left| N_T(N_T - 1) - 2N_T TR(AS) + TR(AS)^2 \right| \cdot \right.$$

(cont.)

$$C(N_T + 2)\ N_T(N_T - 1)V(A|N_T)$$

$$+ \left[2(N_T + 1)\ |S| - 2|S|\ TR(AS)\right] \cdot$$

$$C(N_T + 2)\ N_TV(A|A_T + 1)$$

$$+ |S|^2C(N_T + 2)\ V(A|N_T + 2)\Big\} \qquad (C.18c)$$

$$N_T^3U_2(A|N_T) = -C_1r_{22}\Big\{\left[(N_T - 1)a_{22} - a_{22}TR(AS)\right] \cdot$$

$$C(N_T + 2)\ N_T(N_T - 1)\ V(A|N_T)$$

$$- \left[N_Ts_{11} - a_{22}|S| - s_{11}TR(AS)\right] \cdot$$

$$C(N_T + 2)\ N_T\ V(A|N_T + 1)$$

$$- |S|s_{11}C(N_T + 2)\ V(A|N_T + 2)\Big\}$$

$$- C_2r_{11}\Big\{\left[(N_T - 1)a_{11} - a_{11}TR(AS)\right] \cdot$$

$$C(N_T + 2)\ N_{T(N_T} - 1)\ V(A|N_T)$$

$$- \left[N_Ts_{22} - a_{11}|S| - s_{22}TR(AS)\right] \cdot$$

$$C(N_T + 2)\ N_TV(A|N_T + 1)$$

$$- |S|s_{22}\ C(N_T + 2)\ V(A|N_T + 2)\Big\}$$

$$+ C_4(\rho_s r_{12}^* + \rho_s^* r_{12})$$

$$\Big\{\left[(N_T - 1)a_{22} - a_{22}^2TR(AS)\right] \cdot$$

(cont.)

$$C(N_T + 2)N\,N_T(N_T - 1)\,V(A|N_T)$$

$$- \left| N_T s_{11} - a_{22}|S| - s_{11}TR(AS) \right| \cdot$$

$$C(N_T + 2)\,N_T V(A|N_T + 1)$$

$$- |S| s_{11} C(N_T + 2)\,V(A|N_T + 2) \Big\}$$

$$+ C_5(\rho_s r_{12}^* + \rho_s^* r_{12})$$

$$\Big\{ \left| (N_T - 1)a_{11} - a_{11}TR(AS) \right| \cdot$$

$$C(N_T + 2)\,N_T(N_T - 1)\,V(A|N_T)$$

$$- \left| N_T s_{22} - a_{11}|S| - s_{22}TR(AS) \right| \cdot$$

$$C(N_T + 2)\,N_T V(A|N_T + 1)$$

$$- |S| s_{22}\,C(N_T + 2)\,V(A|N_T + 2) \Big\}$$

$$- (C_4 r_{22} + C_5 r_{11})$$

$$\Big\{ \Big| a_{11}s_{11}(\rho_s a_{12}^* + \rho_s^*\,a_{12})$$

$$+ a_{22}s_{22}(\rho_s a_{12}^* + \rho_s^* a_{12})$$

$$+ |a_{12}|^2(\rho_s s_{12}^* + \rho_s^* s_{12})$$

$$+ (\rho_s a_{12}^{*2} s_{12} + \rho_s^* a_{12} s_{12}^*)$$

$$- (N_T - 1)(\rho_s a_{12}^* + \rho_s^* a_{12}) \Big| \cdot$$

$$C(N_T + 2)\,N_T(N_T - 1)\,V(A|N_T)$$

(cont.)

$$+ \Big| a_{11}s_{11}(\rho_s s_{12}^* + \rho_2^* s_{12})$$

$$+ a_{22}s_{22}(\rho_s s_{12}^* + \rho_s^* s_{12})$$

$$+ |s_{12}|^2(\rho_s a_{12}^* + \rho_s^* a_{12})$$

$$+ (\rho_s s_{12}^{*2} a_{12} + \rho_s^* s_{12}^2 a_{12}^*)$$

$$- N_T(\rho_s s_{12}^* + \rho_s^* s_{12})$$

$$- |S|(\rho_s a_{12}^* + \rho_s a_{12}) \Big| \cdot$$

$$C(N_T + 2)\ N_T V(A|N_T + 1)$$

$$- \Big| |S|(\rho_s s_{12}^* + \rho_s^* s_{12}) \Big|$$

$$C(N_T + 2)\ V(A|N_T + 2) \Big\}$$

$$- C_6|\rho_s|^2 \Big\{ \Big| (N_T - 1)$$

$$(a_{11}r_{22} + a_{22}r_{11} + r_{12}a_{12}^* + r_{12}^* a_{12})$$

$$- (a_{11}^2 s_{11}r_{22} + a_{22}^2 s_{22}r_{11})$$

$$- a_{11}a_{22}(s_{11}r_{11} + s_{22}r_{22})$$

$$- (a_{11}r_{22} + a_{22}r_{11})(a_{12}s_{12}^* + a_{12}^* s_{12})$$

$$- (a_{11}s_{11} + a_{22}s_{22})(r_{12}a_{12}^* + r_{12}^* a_{12})$$

$$- |a_{12}|^2(r_{12}s_{11} + r_{12}^* s_{12})$$

$$- (r_{12}a_{12}^{*2} s_{12} + r_{12}^* a_{12}^2 s_{12}^*) \Big| \cdot$$

(cont.)

$$C(N_T + 2)\ N_T(N_T - 1)\ V(A|N_T)$$

$$+ \Big| - N_T(r_{11}s_{11} + r_{22}s_{22} + r_{12}s_{12}^* + r_{12}^*s_{12})$$

$$+ (s_{11}s_{22} + |S|)(a_{11}r_{22} + a_{22}r_{22})$$

$$+ a_{11}s_{11}^2r_{11} + a_{22}s_{22}^2r_{22}$$

$$+ (r_{11}s_{11} + r_{22}s_{22})(a_{12}s_{12}^* + a_{12}^*s_{12})$$

$$+ (a_{11}s_{11} + a_{22}s_{22})(r_{12}s_{12}^* + r_{12}^*s_{12})$$

$$+ (r_{12}s_{12}^{*2}a_{12} + r_{12}^*s_{12}a_{12}^*)$$

$$+ (|s_{12}|^2 - |S|)(r_{12}a_{12}^* + r_{12}^*a_{12})\Big| \ \cdot$$

$$C(N_T + 2)\ N_T V(A|VN_T + 1)$$

$$- \Big| |S|(r_{11}s_{11} + r_{22}s_{22} + r_{12}s_{12}^* + r_{12}^*s_{12})\Big| \ \cdot$$

$$C(N_T + 2)\ V(A|N_T + 2)\Big\} \qquad \text{(C.18d)}$$

$$N_T^2U_3(A|N_T) = \frac{C_1}{2}\Big\{ a_{22}^2C(N_T + 2)\ N_T(N_T - 1)\ V(A|N_T)$$

$$-2a_{22}s_{11}C(N_T + 2)\ N_TV(A|N_T + 1)$$

$$+ s_{11}^2C(N_T + 2)\ V(A|N_T + 2)\Big\}$$

$$+ \frac{C_2}{2}\Big\{a_{11}^2C(N_T + 2)\ N_T(N_T - 1)\ V(A|N_T)$$

$$- 2a_{11}s_{22}C(N_T + 2)\ N_TV(A|N_T + 1)$$

$$+ s_{22}^2C(N_T + 2)V(A|N_T + 2)\Big\}$$

(cont.)

$$
\begin{aligned}
&+ \frac{C_3}{2} \Bigg\{ (\rho_s^2 a_{12}^{*2} + \rho_s^{*2} a_{12}^2) \cdot \\
&\qquad C(N_T + 2)\ N_T(N_T - 1)\ V(A|N_T) \\
&\quad + 2(\rho_s^2 a_{12}^* s_{12}^* + \rho_s^{*2} a_{12} s_{12}) \cdot \\
&\qquad C(N_T + 2)\ N_T V(A|N_T + 1) \\
&\quad + (\rho_s^2 s_{12}^{*2} + \rho_s^{*2} s_{12}^2) \cdot \\
&\qquad C(N_T + 2)\ V(A|N_T + 2) \Bigg\} \\
&+ \Bigg\{ - (\rho_s a_{12}^* + \rho_s^* a_{12})(C_4 a_{22} + C_5 a_{11}) \cdot \\
&\qquad C(N_T + 2)\ N_T(N_T - 1)\ V(A|N_T) \\
&\quad + \Big[(\rho_s a_{12}^* + \rho_s^* a_{12})(C_4 s_{11} + C_5 s_{22}) \\
&\quad - (\rho_s s_{12}^* + \rho_s^* s_{12})(C_4 a_{22} + C_5 a_{11}) \Big] \cdot \\
&\qquad C(N_T + 2)\ N_T V(A|N_T + 1) \\
&\quad + (\rho_s s_{12}^* + \rho_s^* s_{12})(C_4 s_{11} + C_5 s_{22}) \cdot \\
&\qquad C(N_T + 2)\ V(A|N_T + 2) \Bigg\} \\
&+ C_6 |\rho_s|^2 \Bigg\{ (a_{11} a_{22} + |a_{12}|^2) \cdot \\
&\qquad C(N_T + 2)\ N_T(N_T - 1)\ V(A|N_T) \\
&\quad - (a_{11} s_{11} + a_{22} s_{22} - a_{12} s_{12}^* - a_{12}^* s_{12}) \cdot \\
&\qquad C(N_T + 2)\ N_T V(A|N_T + 1)
\end{aligned}
$$

(cont.)

$$+ (s_{11}s_{22} + |s_{12}|^2) \cdot$$

$$\left. C(N_T + 2)\ V(A|N_T + 2) \right\} \qquad \text{(C.18e)}$$

C.4 Probability Density Function of the Sample MSCC

The PDF of the sample MSCC, ρ^2, is obtained from the PDF of A by (1) performing the change of variables indicated in Eq. (C.12) and (2) integrating out the auxiliary variables a_{11}, a_{22}, and the phase angle of a_{12}. Performing this operation on Eq. (C.18a), we have

$$g(\rho^2) = f(\rho^2|N_T) + N_T\tilde{f}(\rho^2|N_T) \qquad \text{(C.19)}$$

where

$$f(\rho^2|N_T) = (N_T - 1)(1 - \rho_T^2)^{N_T}(1 - \rho^2)^{N_T-2}$$

$$_2F_1(N_T, N_T; 1; \rho^2\rho_T^2) \qquad \text{(C.19a)}$$

is the PDF of ρ^2 for no fluctuations and Gaussian signals (Ref. C.2); $\tilde{f}(\rho^2|N_T)$ is the correction term resulting from the Edgeworth series; and

$$\rho_T^2 = \frac{|r_{12}|^2}{r_{11}r_{22}} = \frac{(E\{\sqrt{S_1S_2}\})^2|\rho_s|^2}{(E\{r_1\})^2(E\{r_2\})^2} \quad . \qquad \text{(C.19b)}$$

Since $f(\rho^2|N_T)$ and $g(\rho^2|N_T)$ are PDF's, $\tilde{f}(\rho^2|N_T)$ must integrate to zero.

Perform the indicated change of variables on $U(A|N_T)$ in Eq. (C.18) to obtain $\tilde{f}(\rho^2|N_T)$. After much tedious integral evaluation and algebra,

$$\tilde{f}(\rho^2|N_T) = PR_1(1-\rho^2)^{N_T-2}\ {}_2F_1(N_T,N_T;1;\rho^2\rho_T^2)$$

$$+ PR_2(1-\rho^2)^{N_T-2}\ {}_2F_1(N_T,N_T+1;\ 1;\ \rho^2\rho_T^2)$$

$$+ PR_3(1-\rho^2)^{N_T-2}\ {}_2F_1(N_T,N_T+2;1;\rho^2\rho_T^2)$$

$$+ PR_4(1-\rho^2)^{N_T-2}\ {}_2F_1(N_T+1,N_T+1;1;\rho^2\rho_T^2)$$

$$+ PR_5(1-\rho^2)^{N_T-1}\ {}_2F_1(N_T+1,N_T+1;1;\rho^2\rho_T^2)$$

$$+ PR_6(1-\rho^2)^{N_T-1}\ {}_2F_1(N_T+1,N_T+2;1;\rho^2\rho_T^2)$$

$$+ PR_7(1-\rho^2)^{N_T}\ {}_2F_1(N_T+2,N_T+2;1;\rho^2\rho_T^2)$$

$$+ PR_8\rho^2(1-\rho^2)^{N_T-2}\ {}_2F_1(N_T+1,N_T+1;1;\rho^2\rho_T^2)$$

$$+ PR_9\rho^2(1-\rho^2)^{N_T-2}\ {}_2F_1(N_T+1,N_T+1;2;\rho^2\rho_T^2)$$

$$+ PR_{10}\rho^2(1-\rho^2)^{N_T-2}\ {}_2F_1(N_T+1,N_T+2;2;\rho^2\rho_T^2)$$

$$+ PR_{11}\rho^2(1-\rho^2)^{N_T-2}\ {}_2F_1(N_T+2,N_T+2;2;\rho^2\rho_T^2)$$

$$+ PR_{12}\rho^4(*1-\rho^2)^{N_T-2}\ {}_2F_1(N_T+2,N_T+2;3;\rho^2\rho_T^2) \qquad \text{(C.20)}$$

where

$$PR_1 = CN_7\,\frac{(N_T-1)^2}{N_T+1}\,(1-\rho_T^2)^{N_T-2} \qquad \text{(C.21a)}$$

$$PR_2 = CN_5\,\frac{(N_T-1)^2}{N_T+1}\,(1-\rho_T^2)^{N_T-1} - 4CN_7\,\frac{N_T(N_T-1)}{N_T+1}\,(1-\rho_T^2)^{N_T-2} \qquad \text{(C.21b)}$$

$$PR_3 = \left(CN_1(1-\rho_T^2)^{N_T} - CN_5(1-\rho_T^2)^{N_T-1}\right.$$

$$\left. + 2CN_7(1-\rho_T^2)^{N_T-2}\right)(N_T-1) \tag{C.21c}$$

$$PR_4 = \left[CN_4(1-\rho_T^2)^{N_T} - CN_5(1-\rho_T^2)^{N_T-1}\right.$$

$$\left. + 2CN_7(1-\rho_T^2)^{N_T-2}\right]\frac{N_T(N_T-1)}{N_T+1} \tag{C.21d}$$

$$PR_5 = \left(-CN_5 + 2CN_6\right)\frac{N_T2}{N_T+1}(1-\rho_T^2)^{N_T-1}$$

$$+ 2CN_7N_T(1-\rho_T^2)^{N_T-1} \tag{C.21e}$$

$$PR_6 = \left(-2CN_1 + CN_3 - 2CN_4\right)N_T(1-\rho_T^2)^{N_T}$$

$$+ \left(CN_5(3-\rho_T^2) - 4CN_6 - 4CN_7\right)N_T(1-\rho_T^2)^{N_T-1} \tag{C.21f}$$

$$PR_7 = \left(CN_1 + 2CN_2 - CN_3 + CN_4(1+\rho_T^2)\right.$$

$$\left. - CN_5 + 2CN_6 + CN_7\right)(N_T+1)(1-\rho_T^2)^{N_T} \tag{C.21g}$$

$$PR_8 = \left(CN_1(1-\rho_T^2)^{N_T} - 2CN_6(1-\rho_T^2)^{N_T-1}\right.$$

$$\left. + 2CN_7\rho_T^2(1-\rho_T^2)^{N_T-2}\right)\frac{N_T(N_T-1)}{N_T+1} \tag{C.21h}$$

$$PR_9 = \Big(- 2CN_6(N_T-1)(1-\rho_T^2)^{N_T-1} + 4CN_7N_T\rho_T^2)^{N_T-2}\Big) \frac{N_T(N_T-1)}{N_T+1} \qquad (C.21i)$$

$$PR_{10} = \Big(- 2CN_3(1-\rho_T^2)^{N_T} + (2CN_5\rho_T^2 + CN_6)(1 - \rho_T^2)^{N_T-1} - 8CN_7\rho_T^2(1-\rho_T^2)^{N_T-2}\Big) N_T(N_T-1) \qquad (C.21j)$$

$$PR_{11} = - \Big(- (4CN_2 - CN_3 + 2CN_4\rho_T^2)(1-\rho_T^2)^{N_T} + (2CN_5\rho_T^2 + 2CN_6(1-2\rho_T^2) - 4CN_7\rho_T^2)(1-\rho_T^2)^{N_T-1}\Big) \cdot N_T(N_T+1) \qquad (C.21k)$$

$$PR_{12} = \Big(CN_2(1-\rho_T^2)^{N_T} - CN_6\rho_T^2(1-\rho_T^2)^{N_T-1} + CN_7\rho_T^4(1-\rho_T^2)^{N_T-2}\Big) N_T(N_T+1)(N_T-1) \qquad (C.21l)$$

and

$$CN_1 = \frac{C_1}{2r_{11}^2} + \frac{C_2}{2r_{22}^2} \qquad (C.22a)$$

$$CN_2 = \frac{C_3}{2r_{11}r_{22}} |\rho_s|^2 \rho_T^2 \qquad (C.22b)$$

$$CN_3 = 2\left(\frac{C_4}{r_{11}^{3/2}\, r_{22}^{1/2}} + \frac{C_5}{r_{11}^{1/2}\, r_{22}^{3/2}}\right) |\rho_s|\ \rho_T \qquad (C.22c)$$

$$CN_4 = \frac{C_6}{r_{11}r_{22}} |\rho_s|^2 \qquad (C.22d)$$

$$CN_5 = -\frac{C_1}{r_{11}^2} - \frac{C_2}{r_{22}^2} + 2\left(\frac{C_4}{r_{11}^{3/2}\, r_{22}^{1/2}} + \frac{C_5}{r_{11}^{1/2}\, r_{22}^{3/2}}\right) |\rho_s|\ \rho_T$$

$$- 2\frac{C_6}{r_{11}r_{22}} |\rho_s|^2 \tag{C.22e}$$

$$CN_6 = \frac{C_3}{r_{11}r_{22}} |\rho_s|^2\rho_T^2 - \left(\frac{C_4}{r_{11}^{3/2}\, r_{22}^{1/2}} + \frac{C_5}{r_{11}^{1/2}\, r_{22}^{3/2}}\right) |\rho_s|\ \rho_T$$

$$+ \frac{C_6}{r_{11}r_{22}} |\rho_s|^2\ \rho_T^2 \tag{C.22f}$$

$$CN_7 = \frac{C_1}{2r_{11}^2} + \frac{C_2}{2r_{22}^2} + \frac{C_3}{r_{11}r_{22}} |\rho_s|^2\ \rho_T^2$$

$$- 2\left(\frac{C_4}{r_{11}^{3/2}\, r_{22}^{1/2}} + \frac{C_5}{r_{11}^{1/2}\, r_{22}^{3/2}}\right) |\rho_s|\ \rho_T$$

$$+ \frac{C_6}{r_{11}r_{22}} |\rho_s|^2\ (1 + \rho_T^2) \quad . \tag{C.22g}$$

C.5 Cumulative Density Function of the Sample MSCC

The CDF of the sample MSCC is

$$G(\rho_t^2|N_T) = \int_0^{\rho_t^2} G(\rho^2|N_T)\ d\rho^2 \tag{C.23}$$

where $0 \leq \rho_t^2 \leq 1$ is the threshold.

Substitute Eq. (C.19) into Eq. (C.23). Then,

$$G(\rho_t^2|N_T) = F(\rho_t^2|N_T) + N_T\tilde{F}(\rho_t^2|N_T) \tag{C.24}$$

where

$$F(\rho_t^2|N_T) = (1-\rho_T^2)^{N_T} \sum_{k=0}^{N_T-2} (1-\rho_t^2)^k {}_2F_1(N_T,k+1;1;\rho_t\rho_T^2) \qquad \text{(C.25)}$$

is the CDF of ρ^2 for no fluctuation (Ref. C.2); and

$$\tilde{F}(\rho_t^2|N_T) = \int_0^{\rho_t^2} \tilde{f}(\rho_t^2\,|N_T)\,dt \quad .$$

Define

$$f(\rho^2|\alpha,\beta,\theta,\phi,\gamma) = \rho^{2\alpha}(1-\rho^2)^\beta {}_2F_1(\theta,\phi;\gamma;\rho^2\rho_T^2) \qquad \text{(C.26a)}$$

and

$$F(\rho_t^2|\alpha,\beta,\theta,\phi,\gamma) = \int_0^{\rho_t^2} f(\rho^2|\alpha,\beta,\theta,\phi,\gamma)\,d\rho^2 \qquad \text{(C.26b)}$$

$\tilde{F}(\rho_t^2|N_T)$ has the same form as $\tilde{f}(\rho^2|N_T)$, Eq. (C.20), where the function $\tilde{f}(\rho^2|\alpha,\beta,\theta,\phi,\gamma)$ is replaced by $\tilde{F}(\rho_t^2|\alpha,\beta,\theta,\phi,\gamma)$.

Substitute Eq. (C.26a) into Eq. (C.26b) and expand the hypergeometric function. Then,

$$\tilde{F}(\rho_t^2|\alpha,\beta,\theta,\phi,\gamma) = \sum_{\ell=0}^{\infty} \frac{(\theta)_\ell(\phi)_\ell\rho_T^{2\ell}}{(\gamma)_\ell\cdot\ell!} \int_0^{\rho_t^2} \rho^{2(\alpha+\ell)}(1-\rho^2)^\beta\,d\rho^2 \qquad \text{(C.27)}$$

where

$$(?)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$$

is Pochammer's symbol. From reference (C.3),

$$\int_0^y x^{m-1}(1-x)^{n-1}\,dx = \frac{(n-1)!}{(m+n-1)!}y^m \sum_{k=0}^{n-1} \frac{(1-y)^k}{k!}(m+k-1)! \quad . \qquad \text{(C.28)}$$

Substitute Eq. (C.28) into Eq. (C.22). Then,

$$\tilde{F}(\rho_t^2|\alpha,\beta,\theta,\phi,\gamma) = \rho_t^{2(\alpha+1)} \sum_{\ell=0}^{\infty} \frac{(\theta)_\ell(\phi)_\ell}{(\gamma)_\ell \ell!} (\rho_t^2\rho_T^2)^\ell \sum_{k=0}^{\beta} \frac{(1-\rho_t^2)^k}{k!} \frac{(\alpha+k+\ell)!}{(\alpha+\beta+\ell+1)!}$$

$$= \frac{\rho_t^{2(\alpha+1)}}{(\beta+2)_\alpha} \sum_{k=0}^{\beta} \frac{(1-\rho_t^2)^k}{k!}(\alpha+k)! \sum_{\ell=0}^{\infty} \frac{(\theta)_\ell(\phi)_\ell(\alpha+k+1)_\ell}{(\gamma)_\ell \ell!(\alpha+\beta+2)_\ell} (\rho_t^2\rho_T^2)^\ell$$

$$= \frac{\rho_t^{2(\alpha+1)}}{(\beta+1)_{\alpha+1}} \sum_{k=0}^{\beta} (k+1)_\alpha (1-\rho_t^2)^k {}_3F_2(\theta,\phi,\alpha+k+1;\gamma,\alpha+\beta+2;\rho_t^2\rho_T^2) \qquad (C.29)$$

Therefore,

$$\begin{aligned}\tilde{F}(\rho_t^2|N_T) = {} & PR_1F(\rho_t^2|0,N_T-2,N_TN_T,1) + PR_2F(\rho_t^2|0,N_T-2,N_T,N_T+1,1) \\ & + PR_3F(\rho_t^2|0,N_T-2,N_T,N_T+2,1) + PR_4F(\rho_t^2|0,N_T-2,N_T+1;N_T+1,1) \\ & +PR_5F(\rho_t^2|0,N_T-1,N_T+1,N_T+1,1) + PR_6F(\rho_t^2|0,N_T-1,N_T+1,N_T+2,1) \\ & + PR_7F(\rho_t^2|0,N_T,N_T+2,N_T+2,1) + PR_8F(\rho_t^2|1,N_T-2,N_T+1,N_T+1,1) \\ & + PR_9F(\rho_t^2|1,N_T-2,N_T+1,N_T+1,2) \\ & + PR_{10}F(\rho_t^2|1,N_T-2,N_T+2,N_T+2,2) \\ & + PR_{11}F(\rho_t^2|1,N_T-2,N_T+2,N_T+2,2) \\ & + PR_{12}F(\rho_t^2|2,N_T-2,N_T+2,N_T+2,3) \qquad (C.30)\end{aligned}$$

where the PR's are defined in Eqs. (C.21) - (C.22). It is easily shown that $\tilde{F}(1|N_T) = 0$. Therefore, $G(1|N_T) = F(1|N_T) = 1$, which makes $G(\rho_t^2|N_T)$ a CDF.

References

C.1 N. R. Goodman, "Statistical Analysis Based on a Certain Multivariate Complex Gaussian Distribution (an Introduction)," Annals of Math. Stat., vol. 34, 1963, pp. 152-177.

C.2 G. C. Carter, Estimation of the Magnitude-Squared Coherence Function (Spectrum), Report No. 7R4343, Naval Underwater Systems Center, New London, Conn., May 1972.

C.3 R. A. Fisher, The General Sampling Distribution of the Multiple Correlation Coefficient," Proc. of the Royal Society, Series A, Vol. 121 (1928), pp. 654-673.

Appendix D

COEFFICIENT EVALUATION FOR THE CDF OF THE SAMPLE MSCC UNDER RAPID FLUCTUATION CONDITIONS

The expression for the Edgeworth series correction factor, $\tilde{F}(\rho t^2(N_T)$, to the cumulative probability function (CDF) of the sample MSCC contained seven constants (CN's) which depend on the statistical properties of the fluctuation model, Eqs. (C.5) and (C.22) of Appendix C. These constants will be evaluated for the statistical fluctuation model presented in Chapter 2. The important constants are:

$$C_1 = 2\left(m_1(2,0) - m_1(1,0)^2\right) \tag{D.1a}$$

$$C_2 = 2\left(m_2(2,0) - m_2(1,0)^2\right) \tag{D.1b}$$

$$C_3 = 2\left(m_1(0,2)m_2(0,2) - m_1(0,1)^2 m_2(0,1)^2\right) \tag{D.1c}$$

$$C_4 = 2\left(m_1(1,1)m_2(0,1) - m_1(1,0)m_1(0,1)m_2(0,1)\right) \tag{D.1d}$$

$$C_5 = 2\left(m_1(0,1)m_2(1,1) - m_2(1,0)m_1(0,1)m_2(0,1)\right) \tag{D.1e}$$

$$C_6 = m_1(0,2)m_2(0,1) - m_1(0,1)^2 m_2(0,1)^2 \tag{D.1f}$$

where

$$m_k(\alpha,\beta) = E\{r_k^{\alpha} S_k^{\beta/2}\} \tag{D.2}$$

and

$$r_k = S_k + N_k \quad . \tag{D.3}$$

Assume that S_k is Gamma distributed with mean $\bar{S}_k$ and MS_k degrees of freedom and N_k is also Gamma distributed with mean $\bar{N}_k$ and MN_k degrees of freedom as discussed in Chapter 2. Then, from Eq. (A.2) of Appendix A,

$$m_k(1,0) = \bar{S}_k + \bar{N}_k \tag{D.4a}$$

$$m_k(2,0) = E\{S_k^2 + 2S_kN_k + N_k^2\}$$

$$= (\bar{S}_k + \bar{N}_k)^2 + \frac{\bar{S}_k^{\,2}}{MS_k} + \frac{\bar{N}_k^{\,2}}{MN_k} \tag{D.4b}$$

$$m_k(0,1) = \frac{\Gamma(MS_k + 1/2)}{\sqrt{MS_k}\ \Gamma(MS_k)}\ \bar{S}_k^{\,1/2} \tag{D.4c}$$

$$m_k(0,2) = \bar{S}_k$$

$$m_k(1,1) = E\{(S_k+N_k)S_k^{1/2}\}$$

$$= \frac{\Gamma(MS_k + 1/2)\ \bar{S}_k^{\,1/2}}{MS_k^{1/2}\Gamma(MS_k)}\ \frac{MS_k + 1/2}{MS_k}\ \bar{S}_k + \bar{N}_k \tag{D.4d}$$

Substitute Eq. (D.4) into Eq. (D.1):

$$C_1 = 2\,\frac{\bar{S}_1^{\,2}}{MS_1} + \frac{\bar{N}_1^{\,2}}{MN_1} \tag{D.5a}$$

$$C_2 = 2\,\frac{\bar{S}_2^{\,2}}{MS_2} + \frac{\bar{N}_2^{\,2}}{MN_2} \tag{D.5b}$$

$$C_3 = 2\bar{S}_1\bar{S}_2\left[1 - \frac{\Gamma(MS_1 + 1/2)\ \Gamma(MS_2 + 1/2)}{(MS_1MS_2)^{1/2}\ \Gamma(MS_2)\ \Gamma(MS_2)}\right] \tag{D.5c}$$

$$C_4 = \frac{\Gamma(MS_1 + 1/2)\ \Gamma(MS_2 + 1/2)}{MS_1(MS_1MS_2)^{1/2}\ \Gamma(MS_1)\ \Gamma(MS_2)}\ \bar{S}_1^{3/2}\ \bar{S}_2^{1/2} \tag{D.5d}$$

$$C_5 = \frac{\Gamma(MS_1 + 1/2)\ \Gamma(MS_2 + 1/2)}{MS_2(MS_1MS_2)^{1/2}\ \Gamma(MS_1)\ \Gamma(MS_2)}\ \bar{S}_1^{1/2}\ \bar{S}_2^{3/2} \tag{D.5e}$$

$$C_6 = \bar{S}_1\bar{S}_2\left[1 - \left(\frac{\Gamma(MS_1 + 1/2)\ \Gamma(MS_2 + 1/2)}{(MS_2MS_2)^{1/2}\ \Gamma(MS_1)\ \Gamma(MS_2)}\right)^2\right] . \tag{D.5f}$$

According to Eqs. (2.17) and (2.18) of Chapter 2,

$$\rho_T^2 = \text{BIAS} \frac{\text{SNR}_1 \text{ SNR}_2}{(\text{SNR}_1 + 1)(\text{SNR}_2 + 1)} \rho_s^2 \tag{D.6a}$$

where

$$\text{BIAS} = \left[\frac{\Gamma(\text{MS}_1 + 1/2)\, \Gamma(\text{MS}_2 + 1/2)}{(\text{MS}_1 \text{MS}_2)^{1/2}\, \Gamma(\text{MS}_1)\, \Gamma(\text{MS}_2)} \right]^2 \tag{D.6b}$$

is the bias factor, and

$$\text{SNR}_k = \bar{S}_k / \bar{N}_k \quad . \tag{D.6c}$$

Substitute Eqs. (D.5) and (D.6) into Eq. (C.20) of Appendix C. The CN's become

$$\text{CN}_1 = \frac{\text{SNR}_1^2 + \text{MS}_1/\text{MN}_1}{\text{MS}_1(\text{SNR}_1 + 1)^2} + \frac{\text{SNR}_2^2 + \text{MS}_2/\text{MN}_2}{\text{MS}_2(\text{SNR}_2 + 1)^2} \tag{D.7a}$$

$$\text{CN}_2 = \frac{\rho_T^4}{\text{BIAS}} (1 - \text{BIAS}^{1/2}) \tag{D.7b}$$

$$\text{CN}_3 = 2\rho_T^2 \quad \frac{\text{SNR}_1}{\text{MS}_1(\text{SNR}_1 + 1)} + \frac{\text{SNR}_2}{\text{MS}_2(\text{SNR}_2 + 1)} \tag{D.7c}$$

$$\text{CN}_4 = \frac{\rho_T^2}{\text{BIAS}} (1 - \text{BIAS}) \tag{D.7d}$$

$$\text{CN}_5 = \text{CN}_3 - 2\text{CN}_1 - 2\text{CN}_4 \tag{D.7e}$$

$$\text{CN}_6 = 2\text{CN}_2 - \frac{\text{CN}_3}{2} + \text{CN}_4 \rho_T^2 \tag{D.7f}$$

$$\text{CN}_7 = \text{CN}_1 + 2\text{CN}_2 - \text{CN}_3 + \text{CN}_4(1 + \rho_T^2) \quad . \tag{D.7g}$$

If $|\rho_o|^2 = \rho_T^2 = SNR_k = 0$,

$$CN_1 = CN_7 = 1/MN_1 + 1/MN_2 \tag{D.8a}$$

$$CN_5 = -2CN_1 \tag{D.8b}$$

$$CN_2 = CN_3 = CN_4 = CN_6 = 0 \quad . \tag{D.8c}$$

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