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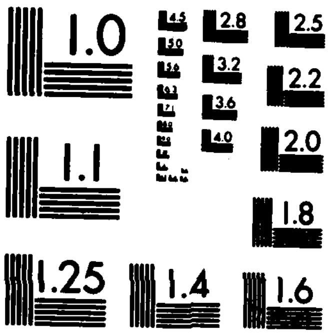

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NOTES ON *FINITE COOPERATIVE GAMES

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ABSTRACT

The following set of notes is extracted from lectures given by the author on the subject of NonStandard Games at U.C.L.A.'s Logic Colloquium in February 1980, Stanford University's Department of Operations Research in June of 1980, and the Second Victoria International Symposium on Non-Standard Analysis held at the University of Victoria in British Columbia in June 1980. The discussion is for the most part informal but nonetheless covers several aspects of cooperative games of the N-person variety, and provides an introduction to NonStandard Analysis in some detail. Reference is also made to the principal results of the author's Thesis, <u>A NonStandard Theory of Games</u>, written at Harvard in 1979 under the direction of Professors Kenneth J. Arrow and Hilary Putnam.

I. REMARKS ON THE THEORY OF GAMES

The Theory of Games is a discipline that seeks to mathematize into a theory, alternative forms of rational behavior on the part of persons whose interests are nonidentical.

The domains of application for the Theory of Games consist primarily in the subject areas of Mathematical Economics, Political Science, and Military Strategic Assessment.

By a game one typically refers to a situation in the above areas, with specific rules for the players. A play of a game refers to a particular occurrence of a game.

The oldest known theorem in the Theory of Games considered as a mathematical discipline is attributed to Zermelo and was given at the Fifth International Congress of Mathematicians in 1912: "On An Application of Set Theory to the Theory of Chess."

▶Theorem: For the Game of Chess, either white can force a win, or Black can force a win, or both sides can at least draw.

Pf: Von Neumann/Morgenstern, Ch. III.14, pp.98-128 by induction on the length of admissible moves.

The game of Chess is a two person Zero-Sum Game. Before we characterize such a game however, we shall consider the concept of a strategy. By a strategy one means a complete specification of a plan for a given play of the game with respect to a given player of the game, which takes into account the contingencies of the moves of the opponent.

For example, in the game of Tic-Tac-Toe, the first player has at most 5 moves per any given play of the game, and at each move there are at most 9 alternatives. But the first player has at least 10 times more than 45 = 5x9 strategies, for a strategy is a complete specification of play. One sees that the number of possible strategies for just the first two moves of the first player is already at 504, i.e., $(9\times(8\cdot 7)) = 504$

Move 1 ⟶ Move 2

A_{21} . . . . A_{28}

9 Alternatives

7 Alternatives 7 Alternatives

at Move 1 there are 9 alternatives, at Move 2 for each of the 8 possible responses there are at least 7 alternatives, etc. The number of possible strategies for Chess becomes astronomical.

One can obtain a useful abstraction of the type of game we have been considering as follows:

$$\Gamma(\mathbb{N}, \{h_j\}_{j\in\mathbb{N}}, \{S_j\}_{j\in\mathbb{N}})$$

and

(i) $h_j : (\prod_{j\in\mathbb{N}} S_j) \longrightarrow \{-1,0,1\}$

(ii) $\sum_{j\in\mathbb{N}} h_j = 0$

where $\mathbb{N}$ is the set of players

$\{S_j\}_{j\in\mathbb{N}}$ is the collection of strategies for each player

$\{h_j\}_{j\in\mathbb{N}}$ is the collection of payoff evaluations for the players.

To state the principal result concerning the solution of such games, one requires the notion of <u>an equilibrium point of Γ</u>.

Definition: An equilibrium point of Γ is an $\tilde{s} \in (\prod_{j \in \mathbb{N}} S_j)$ such that for each $j \in \mathbb{N}$ and any strategy $t_j \in S_j$, $h_j(\tilde{s}|t_j) \leq h_j(\tilde{s})$, where $\tilde{s}|t_j$ is identical to $\tilde{s}$ with $\tilde{s}_j$ replaced by t_j.

The principal result for such games was given by J. Nash, <u>Annals of Mathematics</u>, 1951, for the general case of $||\mathbb{N}|| \geq 2$ and finite:

▶Theorem (Nash): Any game of the form $\Gamma(\mathbb{N}, \{h_j\}_{j \in \mathbb{N}}, \{S_j\}_{j \in \mathbb{N}})$ where $\mathbb{N}$ is finite and each S_j a compact convex subset of metric space X, and each $h_j : S_j \longrightarrow \mathbb{R}$ continuous, has an equilibrium point.

For the case of $\mathbb{N} = \{1,2\}$, Γ has an equilibrium point if, and only if

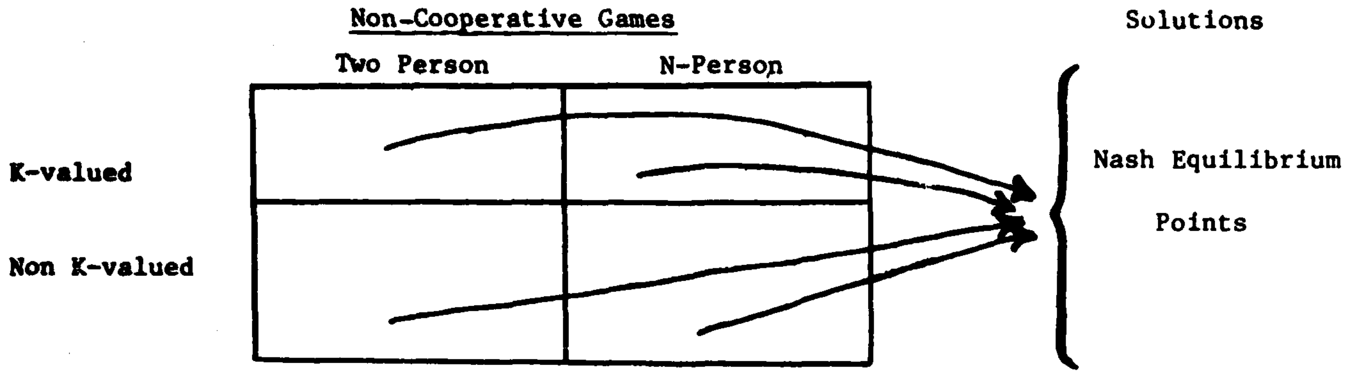
$$\exists \tilde{s} \in (\prod_{j \in \mathbb{N}} S_j) \text{ and } \exists h : (\prod_{j \in \mathbb{N}} S_j) \longrightarrow \mathbb{R}$$

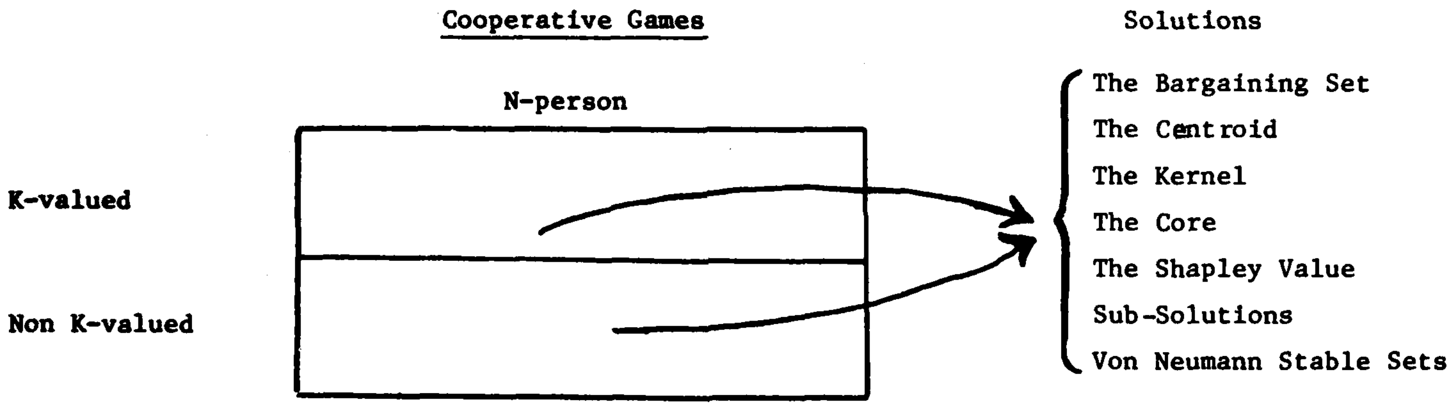
such that

$$\max_{s_1} \min_{s_2} h_1 = h(\tilde{s}) = \min_{s_2} \max_{s_1} h_2$$

The so-called Min-Max Theorem of Von Neumann.

Game Theory can be classified effectively into the following subdivisions of the diagrammes:





The topic of NonStandard Games that will be of concern to us is within the domain of cooperative N-person games. The construction of a *Finite cooperative game, which we define subsequently, is designed to permit the continuous representation of principal solution concepts by means of deriving nonatomic measures from the NonStandard domain. We will treat, in brief fashion, in the next section those aspects of N-person cooperative games that will be needed in our subsequent discussion.

II. N-PERSON COOPERATIVE GAMES

A finite cooperative game in the classic Von Neumann/Morgenstern sense is a pair, $\Gamma(\mathbb{N},v)$, for $\mathbb{N}$ a finite set $\{1, \ldots, n\}$ and $v : \mathcal{P}(\mathbb{N}) \longrightarrow \mathbb{R}_+$ such that

(1) $v(\{i\}) = 0$

(2) $v(\emptyset) = 0$

(3) $v(\mathbb{N}) < \infty$

One frequently assumes an additional property of superadditivity:

$$v(S \cup T) \geq v(S) + v(T) \text{ if } (T \cap S) = \emptyset$$

for $S, T \in \mathcal{P}(\mathbb{N})$

We will ignore coalition structures and assume players are aligned in the grand coalition $\mathbb{N}$. An <u>individually rational payoff configuration</u> is a pair, $(\chi, \mathbb{N})$ for $\mathbb{N}$ the coalition structure and $\chi : \mathbb{N} \longrightarrow \mathbb{R}_+$ such that:

(1) $\chi(i) \geq v(\{i\})$

(2) $\sum_{i \in \mathbb{N}} \chi(i) = v(\mathbb{N})$

Although there are six major solution concepts for games of the form $\Gamma(\mathbb{N},v)$, namely, the Core, the Nucleolus, the Von Neumann/Morgenstern Solution, the Shapley value, the Kernel, and the Bargaining Set, we will be primarily concerned with only two in what follows, the Bargaining Set and the Kernel, the latter having more emphasis.

Allow $T_{\ell K} = \{S\epsilon\mathcal{P}(\mathbb{N}) : \ell\epsilon S \;.\wedge.\; K\notin S\}$. For an i.r.p.c. $(\chi,\mathbb{N})$ an objection of i against j is a pair (y,S) for $y\epsilon(\mathbb{R}_+)^{|S|}$, $S\epsilon T_{ij}$ such that:

$$y(K) \geqq \chi(K) \quad \text{for } K\epsilon S$$

$$y(i) > \chi(i)$$

$$\sum_{K\epsilon S} y(K) \leqq v(S)$$

A counter-objection to (y,S) is a pair (Z,D) for $Z\epsilon(\mathbb{R}_+)^{|D|}$, $D\epsilon T_{ji}$ such that:

$$Z(t) \geqq \chi(t) \quad \text{for } t\epsilon D - S$$

$$Z(t) \geqq y(t) \quad \text{for } t\epsilon\ S\cap D$$

A justified objection is one for which there is no counter-objection.

The Bargaining Set $M_1^i(\Gamma)$ is the set of i.r.p.c.'s in $(\chi,\mathbb{N})$ such that no justified object can be made.

Define next, for $S\epsilon\mathcal{P}(\mathbb{N})$ the excess of the coalition S with respect to the i.r.p.c. $(\chi,\mathbb{N})$ as $e(S,\chi) = \left[v(S) - \sum_{j\epsilon S}\chi(j)\right]$ Then for $i,j\epsilon\mathbb{N}$ let $S_{ij}(\chi) = \sup_{S\epsilon T_{ij}} (e(S,\chi))$.

A player i is said to outweigh a player j with respect to $(\chi,\mathbb{N})$ if:

$$S_{ij}(\chi) > S_{ji}(\chi) \text{ and } \chi(j) > 0.$$

Outweighing is an indication of leverage. Note that if $\chi(j) = 0$, j can "play alone" since $v(\{j\}) = 0$; counter-objections can be trivially obtained in such a case.

We say that an i.r.p.c., $(\chi, \mathbb{N})$, is <u>balanced</u> if there is no pair of players $i, j \varepsilon \mathbb{N}$ such that <u>i outweighs j</u>. This condition is satisfied when:

$$[S_{ij}(\chi) - S_{ji}(\chi)]\ \chi(j) \geq 0$$

The Kernel, $K(\Gamma)$ is the set of all balanced i.r.p.c.'s.

An example of a Bargaining Set solution:

Allow $\mathbb{N} = \{1,2,3\}$ and define

$$v : 2^{\mathbb{N}} \longrightarrow \{1, \tfrac{1}{2}, 0\}$$

as: $v(\mathbb{N}) = v(\{1,2\}) = v(\{1,3\}) = 1$

$v(\{2,3\}) = \tfrac{1}{2}$ and $v(\{j\}) = 0$ $j=1,2,3$.

An i.r.p.c. for the above game is then such that

$$\chi_j \geq v(\{j\}) = 0 \qquad j = 1,2,3$$

$$\sum_{j \varepsilon \mathbb{N}} \chi_j = 1$$

Now, if 1 objects to 3, then he can offer to 2 the following:

$$(v(\{1,2\}) - \sum_{j=1}^{2} \chi_j) = 1 - (\chi_1 + \chi_2)$$

To counter 1's objection, 3 can offer to 2 the following:

$$(v(\{2,3\}) - \sum_{j=2}^{3} \chi_j) = \tfrac{1}{2} - (\chi_2 + \chi_3)$$

Then for $\chi \varepsilon\ M_1^i$, we must have

$$1 - \chi_1 - \chi_2 \leq \tfrac{1}{2} - \chi_2 - \chi_3$$

Since 1 cannot have a justified objection against 3, 3 must be able to offer at least as much to 2 as 1 can. Hence, the inequality.

One argues in an analogous fashion that if 1 objects to 2, he can offer to 3 the following:

$$(v(\{1,3\}) - \sum_{j=1}^{3} \chi_j) = 1 - (\chi_1 + \chi_3)$$

To counter 1's objection, 2 can offer to 3 the following:

$$(v(\{2,3\}) - \sum_{j=2}^{3} \chi_j) = \tfrac{1}{2} - (\chi_2 + \chi_3)$$

If then $\chi \varepsilon M_1^i$, we must have as before that

$$1 - \chi_1 - \chi_3 \leq \tfrac{1}{2} - \chi_2 - \chi_3$$

as it would not be permitted that 1's objection against 2 be justified.

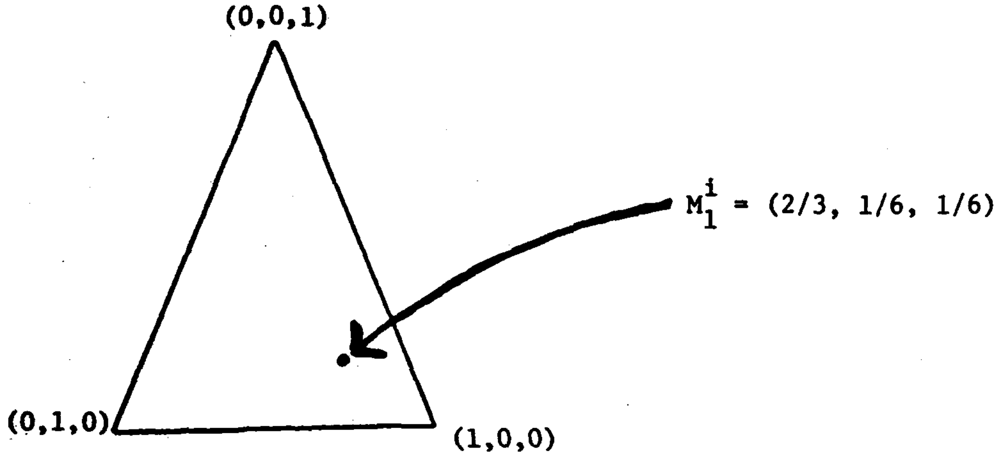
Continuing in a symmetric fashion, we see that for $\chi \varepsilon M_1^i$ in the game given above, then the following set of equations holds:

$$1 - \chi_1 - \chi_2 = \tfrac{1}{2} - \chi_2 - \chi_3$$

$$1 - \chi_1 - \chi_3 = \tfrac{1}{2} - \chi_2 - \chi_3$$

$$\chi_1 + \chi_2 + \chi_3 = 1$$

which gives the solution of the following diagramme



One notes that payoff allocations in this instance somehow indicate that $v(S)$ for $1 \varepsilon S$ is such that $v(S) > v(T)$ for $1 \notin T$, S, $T \varepsilon 2^{IN}$. This simple observation has been developed and extended to market games of competitive economics by J. Geanokoplos in "The Bargaining Set and NonStandard Analysis," Harvard Discussion Paper, 1978. Geanokoplos shows that a "version" of the Bargaining Set solution coincides with the Core in the NonStandard Exchange framework of Loeb and Brown* by means of coalitional contributions of non-negligible sets of players. We will not discuss Geanokoplos' Bargaining Set, as the analytical framework employed there to define his Bargaining Set differs from our own, in that it is specific to market games.

The following set of theorems are well-known results in the literature and serve to indicate the interrelationships between the major solution concepts:

▶Th.I: $(\forall IN)\ M_1^i(\Gamma) \neq \emptyset$ (Peleg)

▶Th.II: $(\forall IN)\ K(\Gamma) \neq \emptyset$ (Maschler & Peleg)

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*Donald Brown and Peter Loeb, "The Values of NonStandard Exchange Economies," <u>Israel Journal of Mathematics</u>, Vol.25, 1976.

▶Th.III: $(\forall \mathbb{N})\ K(\Gamma) \subsetneqq M_1^i(\Gamma)$ (Davis & Maschler)

▶Th.IV: $(\forall \mathbb{N})\ \mathcal{C}(\Gamma) \subseteq M_1^i(\Gamma)$

for $\mathcal{C}(\Gamma) = \{\chi \varepsilon (\chi, \mathbb{N}) : \sup_{S \varepsilon (\mathbb{N})} e(S, \chi) \leq 0\}$, the Core.

▶Th.V: $(\forall \mathbb{N})$ If $\mathcal{C}(\Gamma) \neq \emptyset$ then

$K(\Gamma) \cap \mathcal{C}(\Gamma) \neq \emptyset$ (Davis & Maschler)

▶Th.VI: $N(\Gamma) \neq \emptyset$ (Schmeidler)

▶Th.VII: $N(\Gamma) \subseteq K(\Gamma) \subseteq M_1^i(\Gamma)$ (Schmeidler, Davis & Maschler)

▶Th.VIII: If $\mathcal{C}(\Gamma) \neq \emptyset$, then $N(\Gamma) \subseteq \mathcal{C}(\Gamma)$ (Schmeidler)

III. *FINITE COOPERATIVE GAMES

1. Why a NonStandard Theory of Games?

(1) By the features weakly saturated enlargements that we employ, the theory is preserving of all standard results of finite cooperative games. Such enlargements are conservative.

(2) It is a less coarse semantic framework for many intuitive mathematical concepts that are difficult to be consistently formulated in standard mathematics.

(3) The elegance and power of NonStandard techniques can be obtained within weakened frameworks of the assumptions of set theory. In particular, weakly saturated enlargements can be obtained without the Axiom of Choice. We have shown elsewhere that this feature has implications for the issues of measurability that arise in nonatomic, noncooperative exchange. The latter can be linked directly to the nonatomic representation theorem.

(4) Weakened solution concepts can be obtained by external imbeddings of NonStandard Games. The space of solutions is thereby enlarged nontrivially by the *Finite context.

2. Preliminary Concepts on Filters

Df.2.1.: Allow B to be an algebra of sets derived from a subfield of an arbitrary set Y. Assume that $\mathfrak{F} \subseteq B$ and $\emptyset \notin \mathfrak{F}$. Then

(a) $\mathfrak{F}$ has the finite intersection property if $\left[\bigcap_{K \leq n} b_K\right] \neq \emptyset$ for $n < \omega = |\sim|_0$ and each $b_K \varepsilon \mathfrak{F}$.

(b) $\mathfrak{J}$ is a filter of B if for $a,b \in \mathfrak{J}$

(i) $(a \cap b) \in \mathfrak{J}$ for $a,b \in \mathfrak{J}$

(ii) $(a \subseteq b) \Longrightarrow b \in \mathfrak{J}$ for $a \in \mathfrak{J}$, $b \in B$

A filter $\mathfrak{J}$ is said to be proper if $\mathfrak{J} \neq B$, or alternatively $\emptyset \notin \mathfrak{J}$. The improper filter of B is merely B.

(c) Let F denote the family of filters on B. Then the filter generated by $\mathfrak{J}$ is the smallest filter containing $\mathfrak{J}$ in F, which we denote as $\langle\mathfrak{J}\rangle$. Alternatively, $\langle\mathfrak{J}\rangle = \bigcap_j \mathfrak{J}_j$ such that $\mathfrak{J} \subseteq \mathfrak{J}_j \in F$.

Additionally,

(c)(i) $\langle\mathfrak{J}\rangle = \{b \in B : (\exists n<\omega)(\exists \{b\}_{K=1}^{n} \subset \mathfrak{J}) : \bigcap_K b_K \subseteq b\}$

(c)(ii) $\langle\mathfrak{J}\rangle$ is proper if an only if $\mathfrak{J}$ has the finite intersection property.

Define the relation $\gtrsim$ on pairs in F as:

$$\mathfrak{J}_1 \gtrsim \mathfrak{J}_2 \text{ if } b \in \mathfrak{J}_2 \Longrightarrow b \in \mathfrak{J}_1.$$

In which case we say that $\mathfrak{J}_1$ is finer than $\mathfrak{J}_2$. One then sees that $\langle\mathfrak{J}\rangle$ is the least fine of the ordered set $\langle F(\mathfrak{J}), \gtrsim\rangle$, for $F(\mathfrak{J})$ the subfamily of F whose members include $\mathfrak{J}$.

Df.2.2: Allow B to be an algebra of sets and consider the ordered class $\langle\tilde{F}, \gtrsim\rangle$ where $\tilde{F}$ is the collection of proper filters of B.

An ultrafilter of B, UI, is a most fine element of $\langle\tilde{F}, \gtrsim\rangle$, that is, UI is a proper filter and is not properly contained in any other proper filter.

An important fact is:

Theorem: Allow B to be an algebra of sets and allow $\mathfrak{J} \subseteq B$. If $\mathfrak{J}$ has the finite intersection property, then there is an ultrafilter UI of B such that UI $\gtrsim \mathfrak{J}$.

In particular the theorem states that any proper filter can be extended to an ultrafilter with respect to an algebra B.

A further discussion and proof of the theorem can be found in the book by Comfort and Negrepontis, The Theory of Ultrafilters, Springer-Verlag, 1974.

The theorem, also known as the Ultrafilter Theorem is due to Tarski. As the proof is simple and straightforward, we give the details.

Lemma: Per premiss of the theorem, there is a filter, D, on B containing $\mathfrak{F}$.

Pf: Recall from (c)(ii) that per premiss of the theorem $\mathfrak{F}$ is proper and thus $\emptyset \notin \mathfrak{F}$. Then for some filter D, if $\{\chi_j\}_{j=1}^{K} \subseteq \mathfrak{F}$, then $(\bigwedge_{j=1}^{K} \chi_j) \varepsilon D$ and thus if $\mathfrak{F}$ were in D, $\bigcap_{j=1}^{K} \chi_j \neq \emptyset$. We show the converse to establish the Lemma.

If $\mathfrak{F}$ is proper, allow

$$\mathfrak{F}(+) = \{y \varepsilon B : \exists \{\chi_j\}_{j=1}^{K} \subseteq \mathfrak{F} : (\bigcap_{j=1}^{K} \chi_j) \subseteq y\}$$

Obviously $\mathfrak{F} \subseteq \mathfrak{F}(+)$. We claim that $\mathfrak{F}(+)$ is a filter. Clearly, by definition for $\chi, y \varepsilon \mathfrak{F}(+)$ for which $\chi \leq y, y \varepsilon \mathfrak{F}(+)$. Suppose $y, y' \varepsilon \mathfrak{F}(+)$. Then for $K, \ell \varepsilon \mathbb{N}$ $\exists \{\chi_j\}_{j=1}^{K} \subseteq \mathfrak{F}$ and $\exists \{\chi_i'\}_{j=1}^{\ell} \subseteq \mathfrak{F}$ for which $(\bigcap_{j=1}^{K} \chi_j) \subseteq y$ and $(\bigcap_{i=1}^{\ell} \chi_i') \subseteq y'$ from whence, $\left((\bigcap_{j=1}^{K} \chi_j) \cap (\bigcap_{i=1}^{\ell} \chi_i') \right) \subseteq y \cap y'$. Thus $y \cap y' \varepsilon \mathfrak{F}(+)$.

Finally, since $\mathfrak{F}$ is proper and $\emptyset \notin \mathfrak{F}, \mathfrak{F} \subseteq \mathfrak{F}(+)$ means $\emptyset \notin \mathfrak{F}(+)$.*

Q.E.D.

*(One can show that $\mathfrak{F}(+) \equiv \langle \mathfrak{F} \rangle$ defined as before.)

Lemma: (Tarski 1930) $\langle \mathfrak{J} \rangle$ is contained in an ultrafilter.

Pf: Consider the ordered structure $\langle \tilde{F}(\mathfrak{J}), \gtrsim \rangle$ which can be viewed as a POSET under $\gtrsim$. Suppose $\mathcal{C}$ were a chain in $\langle \tilde{F}(\mathfrak{J}), \gtrsim \rangle$. We show that $\langle \tilde{F}(\mathfrak{J}), \gtrsim \rangle$ is inductive, i.e., every $\mathcal{C}$ has a bound in $\langle F(\mathfrak{J}), \gtrsim \rangle$. Let $C = \bigcap \mathcal{C}$. Then if $\chi, y \varepsilon C$, therefore some $\mathcal{C}_K, \mathcal{C}_\ell \varepsilon \mathcal{C}$, $\chi \varepsilon \mathcal{C}_K$ and $y \varepsilon \mathcal{C}_\ell$. $\mathcal{C}$ is a chain, however; thus either $\mathcal{C}_K \gtrsim \mathcal{C}_\ell$ or $\mathcal{C}_\ell \gtrsim \mathcal{C}_K$. Assume $\mathcal{C}_K \gtrsim \mathcal{C}_\ell$. Then $\chi, y \varepsilon \mathcal{C}_K$. $\mathcal{C}_K$ is a filter however; therefore $(\chi \cap y) \varepsilon \mathcal{C}_K \subseteq C$. If $Z \varepsilon \mathfrak{J}$ and $y \leq Z$ ipso facto $Z \varepsilon \mathcal{C}_K \subseteq C$. Since $\emptyset \notin \mathcal{C}_j$ for any j, $\mathcal{C}_j \varepsilon C$, $\emptyset \notin C$. Then C is a filter and $C \gtrsim \mathcal{C}$.

By Zorn's Lemma there exists UI such that UI is order maximal for $\langle F(\mathfrak{J}), \gtrsim \rangle$.

Q.E.D.

3. NonStandard Analysis

Allow $\mathbb{R}$ to denote the real number system. Let D be an algebra of sets and let UI be an ultrafilter on D.

If A and B are two mappings of D into $\mathbb{R}$, i.e., $A, B \varepsilon \mathbb{R}^D$, then one says that $A \equiv_{UI} B$ if and only if $\{n : n \varepsilon D \text{ and } A(n) = B(n)\} \varepsilon UI$. The relation $\equiv_{UI}$ is an equivalence relation.

Denote by ${}^*\mathbb{R}$ the set $\mathbb{R}^D/_{UI}$ of all equivalence classes. The equivalence class of a mapping $A: D \longrightarrow \mathbb{R}$, is denoted as a, so that $A \varepsilon a$. One can then define algebraic operations in ${}^*\mathbb{R}$ as follows:

$a \dotplus b = c$ if and only if $\exists A \varepsilon a\ \exists B \varepsilon b\ \exists C \varepsilon c$ such that $\{n : n \varepsilon D$ and $A(n) + B(n) = C(n)\} \varepsilon UI$.

Similar definitions obtain for the operations $ab = c$, $a - b = c$, and $a \leqq b$.

In this manner, with these definitions, ${}^*\mathbb{R}$ is a totally ordered field, and by way of the constant mappings in $\mathbb{R}^D$, $\mathbb{R} \subseteq {}^*\mathbb{R}$.

If $\mathbb{R} \neq {}^*\mathbb{R}$, then ${}^*\mathbb{R}$ is said to enlarge $\mathbb{R}$ and ${}^*\mathbb{R}$ is therefore non-archimedian. This will occur if the ultrafilter UI on D is $\aleph_1$-adequate. The details of the construction and an in depth exposition of NonStandard Analysis can be found in the recent treatise by Luxemburg and Stroyan, Introduction to the Theory of Infinitessimals, Academic Press, 1976.

If the filter on D is at least $\aleph_1$-adequate and thus ${}^*\mathbb{R}$ enlarges $\mathbb{R}$, from the non-archimedian character of ${}^*\mathbb{R}$ (owing to the categorical nature of $\mathbb{R}$ as a continuously ordered dense archimedian field), infinite integers must exist in ${}^*\mathbb{N}$, i.e., $\exists \omega \in {}^*\mathbb{N} - \mathbb{N}$ such that $\forall n \in \mathbb{N}\, (\omega > n)$. The reciprocals of such integers, which exist by way of ${}^*\mathbb{R}$ being a field, are therefore such that $\forall n \in \mathbb{N}\, (1/\omega < 1/n)$ and are termed the infinitesimals.

There are three forms of NonStandard Numbers:

(A) The infinitesimals, M_1

$$M_1 = \{{}^*\chi \epsilon {}^*\mathbb{R} : |{}^*\chi| < v \text{ for any } v \epsilon \mathbb{R}_+ - \{o\}\}$$

(B) The S-finite NonStandard Numbers, M_o

$$M_o = \{{}^*\chi \epsilon {}^*\mathbb{R} : |{}^*\chi| < v \text{ for some } v \epsilon \mathbb{R}_+ - \{o\}\}$$

(C) The infinite NonStandard Numbers, ${}^*\mathbb{R}-M_o$

$${}^*\mathbb{R}-M_o = \{{}^*\chi \epsilon {}^*\mathbb{R} : |{}^*\chi| > v \text{ for all } v \epsilon \mathbb{R}_+ - \{o\}\}$$

Further terms and nomenclature are contained in Lewis, A NonStandard Theory of Games, Ch. I.

If the ultrafilter $\mathcal{U}$ on D is N_1-adequate, then we say that the enlargement of $\mathbb{R}$, $^*\mathbb{R}$ is weakly saturated.

Alternatively characterized, $^*\mathbb{R}$ is weakly saturated if for a relation $0 : \mathbb{R} \times \mathbb{R} \longrightarrow \{0,1\}$, for $\{a_j\}^n_{j\varepsilon 1} \subseteq \mathrm{Dom}(0)$ for some arbitrary $n \varepsilon \mathbb{N}$. If $\exists b(n) \varepsilon \mathrm{Rng}(0)$ such that 0 is concurrent, i.e.,

$$\text{if } (\bigwedge_{j=1}^{n} (a_j 0 b(n)) = 1$$

$$\text{then } \exists\, ^*b \varepsilon\, ^*\mathbb{R} \text{ such that}$$

$$(\bigwedge_{j\varepsilon \mathbb{N}} a_j\, ^*0\, ^*b) = 1$$

$^*\mathbb{R}$ is weakly saturated if every concurrent relation 0 in $\mathbb{R}$ is bounded in $^*\mathbb{R}$.

From this characterization of weak saturation, the following simple result can be expressed:

Theorem: If $^*\mathbb{R}$ is weakly saturated and if $\omega \varepsilon\, ^*\mathbb{N} - \mathbb{N}$, then ω is Non-Standard.

Pf: $^*\mathbb{R}$ is weakly saturated and the following relation is concurrent:

$$R_\Phi = \{(\chi, y) : \vdash \Phi(\chi, y)\}$$

where, $\forall \chi, y \varepsilon \mathbb{N}\ \Phi(\chi, y) = \chi < y$

Any linear ordering on $\mathbb{R}$ is concurrent, in fact.

By the properties of the enlargement therefore,

$\forall \chi \varepsilon \mathbb{N}\ \exists \mathcal{C}_\Phi \varepsilon\, ^*\mathbb{R} [^*\vdash \Phi(\chi, \mathcal{C}_\Phi)]$.

If $\mathcal{C}_\Phi \varepsilon \mathbb{N}$, then

$\forall \chi \varepsilon \mathbb{N}\ \exists n \varepsilon \mathbb{N} [\chi < n]$

which cannot be of $\mathbb{R}$ because it is archimedian. Then $\mathcal{C}_{\Phi}$ must be in ${}^*\mathbb{N}$ which gives the result.

Q.E.D.

A further result that can be established within the framework of an N_1-saturated enlargement is that the variety of set $[o,\omega]$ viewed as an initial segment of ${}^*\mathbb{N}$ with $\omega\epsilon{}^*\mathbb{N}$ with $\omega\epsilon{}^*\mathbb{N} - \mathbb{N}$ has an exceedingly large cardinality. In fact $||[o,\omega]|| > 2^{|\sim|_o}$ which has lent support to our intuitions that the NonStandard framework is appropriate for effectively modelling continuous games. The result is given in Lewis, <u>A NonStandard Theory of Games</u>, Ch.I, and is due primarily to results found in E. Zakon's fundamental paper: "Remarks on the NonStandard Real Axis" in <u>Application of Model Theory to Algebra, Analysis, and Probability</u>, W.A.J. Luxemburg, editor, Holt Rhinehart & Winston, 1969.

The approach we employ is to model an extension of the classical cooperative games of Von Neumann/Morgenstern on a *Finite Set $[o,\omega] \subseteq {}^*\mathbb{N}$, and then to demonstrate that such a model has a straightforward nonatomic representation in the standard domain. The following results are contained in Lewis, <u>A NonStandard Theory of Games</u>, Chs. I and II.

4. Principal Results on *Finite Games

Existence Theorem: For a *Finite game ${}^*\Gamma = \langle \mathbb{F}, A(\mathbb{F}), {}^*v\rangle$ when $\mathbb{F} = [o,\omega] \subseteq {}^*\mathbb{N}$ for $\omega\epsilon{}^*\mathbb{N} - \mathbb{N}$, $A(\mathbb{F})$ = the algebra of internal subsets of $\mathbb{F}$ taken as coalitions, ${}^*v : A(\mathbb{F}) \longrightarrow {}^*\mathbb{R}_+$ internally with Q-bound, i.e., $||{}^*v(\mathbb{F})|| < K$ for some $K\epsilon{}^*\mathbb{N}$ such that S-superadditivity obtains as:

${}^*v(S\cup T) \gtrsim {}^*v(S) + {}^*v(T)$ if $S,T\varepsilon A(\mathbb{F})$ and $S\cap T = \emptyset$. The set of payoff configurations $(\chi,\mathbb{F})$ (which is external) when $\chi\varepsilon(\chi,\mathbb{F})$ if and only if the following are satisfied:

(a) $\chi : \mathbb{F} \longrightarrow {}^*\mathbb{R}_+$ is internal

(b) $\left[\sum_{j\varepsilon\mathbb{F}} \chi(j) = {}^*v(\mathbb{F})\right]_{\text{Mod } M_1}$

for $\chi(j)\varepsilon M_o^+$ and $j\varepsilon\mathbb{F}$

(c) $\chi(j) \gtrsim 0$

then will contain some $\tilde{\chi}$ for which $\left[{}^*S_{ij}(\tilde{\chi}) = {}^*S_{ji}(\tilde{\chi})\right]_{\text{Mod } M_1}$ a.e. in $\mathbb{F}$.

▶Nonatomic Representation Theorem: Any *Finite cooperative game ${}^*\Gamma = \langle\mathbb{F},\ A(\mathbb{F}),\ {}^*v\rangle$, when viewed as a construction a nonstandard *Finite measure ${}^*\Phi = \langle\mathbb{F},\ A(\mathbb{F}),\ \mu_{\mathbb{F}}\rangle$ for $\mu_{\mathbb{F}}$ *Finite additive, has a nonatomic (standard) representation $\Psi({}^*\Gamma)$ on $\Psi = \langle\chi(\mathbb{F}),\ \chi(A(\mathbb{F})),\ m\rangle$ for $\chi(A(\mathbb{F}))$ the σ-algebra of internal subsets of $A(\mathbb{F})$ by way of the construction of P. Loeb, "Conversion from NonStandard to Standard Measure Spaces," TRANS. A.M.S., Vol. 211, 1975. Alternatively phrased, for the measure $\mu_{\mathbb{F}}(S) = \dfrac{\|\mathbb{F}\cap S\|}{\|\mathbb{F}\|}$, $S\varepsilon A(\mathbb{F})$, the payoffs $(\chi,\mathbb{F})$ are shown to be $\mu_{\mathbb{F}}$-measurable such that their standard parts: $st(\tilde{\chi})$, $\tilde{\chi}\varepsilon(\chi,\mathbb{F})$ are m-measurable in Ψ. Provided that $\dfrac{1}{\|\mathbb{F}\|} \sum_{j\varepsilon\mathbb{F}} \tilde{\chi}(j)$ is S-bounded, i.e.,

$$\left\| \frac{1}{\|\mathbb{F}\|} \sum_{j\varepsilon\mathbb{F}} \tilde{\chi}(j) \right\| < n\varepsilon\mathbb{N}$$

, one can show that $\tilde{\chi}\varepsilon QK({}^*\Gamma)$ if and only if $st(\tilde{\chi})\varepsilon QK(\Psi({}^*\Gamma))$.

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