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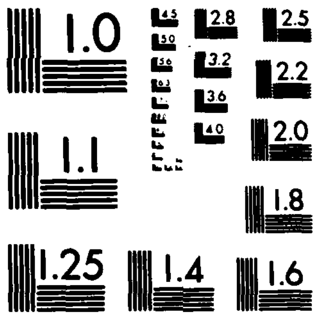

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STABLE CONVEX SETS OF MATRICES

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STABLE CONVEX SETS OF MATRICES

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ABSTRACT

A subset $\mathcal{A}$ of nxn complex valued matrices is stable if all powers of all matrices from the set $\mathcal{A}$ are uniformly bounded. We show that if $\mathcal{A}$ is a bounded convex circular set which spans a stable linear subspace of matrices, then $\mathcal{A}$ is stable if the spectral radius of any $A \in \mathcal{A}$ is bounded by 1.

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SIGNIFICANCE AND EXPLANATION

When solving partial differential equations numerically one often has to use iterations involving matrices restricted to a given set $\mathcal{A}$ of nxn complex valued matrices. It then follows that the iteration scheme is stable if and only if this set of matrices is stable, that is all powers of all matrices from the set $\mathcal{A}$ are uniformly bounded. Such sets were completely characterized by H. O. Kreiss. However, his criteria are hard to use.

In this paper we characterize in a very simple way stable sets of matrices $\mathcal{A}$, whenever $\mathcal{A}$ is closed, convex and circular. We show that if $\mathcal{A}$ span a stable subspace of matrices then $\mathcal{A}$ is stable if and only if for any matrix A in $\mathcal{A}$ the spectrum of A is contained in the unit disc $|z| \le 1$ in the complex plain. This paper extends the main result in MRC Technical Summary Report #2507.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

STABLE CONVEX SETS OF MATRICES

Shmuel Friedland

1. Introduction

Let $\mathcal{A}$ be a subset of $M_n(C)$, the set of $n \times n$ complex valued matrices. $\mathcal{A}$ is called stable if

$$|A^m| \le K, \; m = 0, 1, 2\ldots, \quad A \in \mathcal{A}. \qquad (1.1)$$

Here $|\cdot|$ is a vector norm on $M_n(C)$. The stable sets play an important role in the stability and the convergencve of certain numerical schemes for partial differential equations. See for example Richtmyer-Morton [4]. In 1962 Kreiss [2] characterized stable sets. In particular he showed that (1.1) is equivalent to

$$|(zI-A)^{-1}| \le c/(|z|-1), \text{ for all } |z| > 1, \quad A \in \mathcal{A} \qquad (1.2)$$

The serious draw-back of (1.2) is that it is difficult to verify in general. A natural approach is to replace (1.2) by a weaker condition assuming that $\mathcal{A}$ is of a certain type. See [4] for some examples. For $A \in M_n(C)$ let $\rho(A)$ denote the spectral radius of A. If $\mathcal{A}$ is a stable set then it is easy to show that

$$\rho(A) \le 1, \quad A \in \mathcal{A} \qquad (1.3)$$

Indeed, since on finite dimensional vector spaces all the norms are equivalent it follows

$$\rho(A) = \lim_{m \to \infty} |A^m|^{1/m}.$$

So (1.3) is implied by (1.1). Recently, Friedland-Zenger [1] showed that if $\mathcal{A}$ satisfies the assumptions

(i) $\mathcal{A}$ is a bounded convex closed set,

(ii) $\mathcal{A}$ is circular, i.e. $e^{i\theta} \mathcal{A} = \mathcal{A}$ for all real θ,

(iii) $\mathcal{A}$ contains an open set.

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Then $\mathcal{A}$ is stable if and only in (1.3) holds. The purpose of this paper is to replace condition (iii), or conditions (ii) and (iii), by some other conditions such that the stability of $\mathcal{A}$ is equivalent to (1.3). We now state our main results. To do that we need the following notions.

Definition 1 *Let* $\mathcal{L}$ *be a subspace of* $M_n(C)$ $(M_n(\mathbb{R}))$ - *the set of* $n \times n$ *complex (real) valued matrices*. *Denote*

$$\mathcal{L}(x) = \{ Ax,\ A \in \mathcal{L},\ x \in C^n(\mathbb{R}^n)\} \tag{1.4}$$

Then *is called stable if*

$$\dim \mathcal{L}(x) = k = \text{const.} \tag{1.5}$$

for all non-zero $x \in C^n(\mathbb{R}^n)$. *If* $k = n$ *then* $\mathcal{L}$ *is called maximally stable*.

Here by $C^n(\mathbb{R}^n)$ we denote the set of n-complex (real) valued column vectors.

Definition 2 *A subset* $\mathcal{A}$ *of* $M_n(\mathbb{R})$ *is called hyperbolic if each* $A \in \mathcal{A}$ *has only real eigenvalues*.

Recall that $\mathcal{A}$ is called balanced if $-\mathcal{A} = \mathcal{A}$. We now state our two main theorems.

Theorem 1 *Let* $\mathcal{A}$ *be a bounded convex closed circular set of complex valued matrices*. *Denote by* $\mathcal{L}$ *the complex subspace spanned by* $\mathcal{A}$. *Assume that* $\mathcal{L}$ *is stable and contains the identity matrix*. *Then* $\mathcal{A}$ *is stable if and only if* (1.3) *holds*.

Theorem 2. *Let* $\mathcal{A}$ *be a bounded convex closed balanced set of real valued matrices*. *Denote by* $\mathcal{L}$ *the real subspace spanned by* $\mathcal{A}$. *Assume that* $\mathcal{A}$ *is hyperbolic and* $\mathcal{L}$ *is stable and contains the identity matrix*. *Then* $\mathcal{A}$ *is stable if and only if* (1.3) *holds*.

2. *The complex case*.

As usual, let I be the identity matrix. In what follows we need the following simple result.

Lemma 1. *Let* $\mathcal{A} \subseteq M_n(C)$ *be a stable set*. *Put*

$\mathcal{B} = \{B,\ B = (1-\alpha)\, e^{i\theta} I + \alpha\, e^{i\rho}\ ,\ A \in \mathcal{A},\ 0 < \alpha < 1,\ \theta, \rho \in \mathbb{R}\}$.

Then B is a stable set.

Proof By enlarging K if necessary we may assume that $|I| \le K$. The triangle inequality and (1.1) imply

$$|B^m| \le \sum_{k=0}^{n} \binom{n}{k} \left| e^{ik\theta} e^{i(n-k)\rho} \alpha^k (1-\alpha)^{n-k} A^{n-k} \right|$$

$$\le K \sum_{k=0}^{n} \binom{n}{k} \alpha^k (1-\alpha)^{n-k} = K$$

The above lemma shows that when studying stable sets one always can assume that $I \in A$.

Let V be a finite dimensional vector space over $\mathbb{C}$ or $\mathbb{R}$. As usual denote by V^* the dual space of linear functionals on V. Let $\|\cdot\|$ be a norm on V. Then the dual norm $\|\cdot\|^*$ on V^* is given by

$$\|f\|^* = \sup_{\|v\| \le 1} |f(v)| \tag{2.1}$$

Since V is finite dimensional $\|\cdot\|^{**} = \|\cdot\|$ that is

$$\|v\| = \sup_{\|f\|^* \le 1} |f(v)|. \tag{2.2}$$

Also for any $0 \ne v \in V$ there exists $f \in V^*$ such that

$$1 = f(v) = \|f\|^* \|v\|. \tag{2.3}$$

As usual let $\|\cdot\|_2$ be the standard ℓ_2 norm on $\mathbb{C}^n$

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}, \quad x = (x_1, \ldots, x_n)^t.$$

Denote by $S = \{x, \|x\|_2 = 1\}$ the unit sphere of $\|\cdot\|_2$. For $A \in M_n(\mathbb{C})$ we denote by $\|A\|_2$ the induced operator norm $\|A\|_2 = \max_{\|x\|_2 \le 1} \|Ax\|_2$.

Proof of Theorem 1. Assuming (1.3) it is left to show the backward implication. As $I \in \mathcal{L}$ using Lemma 1 we may assume that $I \in \mathcal{A}$. Let

$$\mathcal{A}(x) = \{z, z = Ax, A \in \mathcal{A}\} \tag{2.4}$$

Assume that $x \in S$. Since $\mathcal{A}$ is a bounded closed convex circular set the set $\mathcal{A}(x)$ is also a bounded closed convex circular set. Clearly, the set $\mathcal{A}(x)$ span the subspace

$L(x)$. Thus $A(x)$ is a unit ball of some vector norm on $L(x)$. We denote this norm by $\|\cdot\|_x$. Since $I \in A$ we get that $x \in A(x)$. Next we note that x is on the boundary of $A(x)$. If not then there exists λ, $|\lambda| > 1$, such that $\lambda x \in A(x)$. That is $Ax = \lambda x$ for some $A \in A$. In particular $\rho(A) > 1$. This contradicts the hypothesis (1.3). So $\|x\|_x = 1$. Next we identify the dual space $L^*(x)$ with $L(x)$ by the equality

$$f(x) = (x,f) \tag{2.5}$$

where $(\cdot,\cdot)$ is the standard inner product in C^n. Let $\|\cdot\|_x^*$ be the dual norm on $L(x)$. According to (2.3) there exists $f_x \in L(x)$ such that

$$1 = (x,f_x) = \|f_x\|_x^* \tag{2.6}$$

Let

$$F = \{f,\ f = f_x,\ x \in S\} \tag{2.7}$$

We claim that F is bounded. Assume to the contrary that there exists a sequence $\{x_k\}$, $\|x_k\|_2 = 1$ such that $\|f_k\|_2 \to \infty$ $(f_k = f_{x_k})$. By taking a convergence subsequence if necessary we may assume that $x_k \to x$. Let $e_1,\ldots,e_m$ be an orthonormal basis in $V = L(x)$. So $e_i = B_i x$, $B_i \in L$, $i = 1, \ldots, m$. Put $e_{i,k} = B_i x_k$. Since $e_{i,k} \to e_i, i = 1,\ldots,m$ we deduce that $e_{i,k},\ldots,e_{m,k}$ are linearly independent for $k > N$. (Assume $N = 1$ for simplicity). As L is stable $e_{1,k},\ldots, e_{m,k}$ is a basis in $L(x_k)$. Let $g_{i,k},\ldots, g_{m,k}$ be a set of orthonormal vectors obtained from $e_{i,k},\ldots, e_{m,k}$ by the Gram-Schmidt process. Clearly we can choose $g_{1,k},\ldots, g_{m,k}$ such that $g_{i,k} \to e_i$. $i = 1, \ldots, m$. Let U_k be a unitary matrix with the property $U_k g_{i,k} = e_i$, $i = 1, \ldots, m$. Also we can choose U_k such that $U_k \to I$. That is

$$U_k\, L(x_k) \to L(x).$$

As the set A is bounded, the sets $A(x)$ are uniformly bounded for $x \in S$. So $B_k = U_k\, A(x_k) = (U_k - I)\, A(x_k) + A(x_k) \to A(x)$. As before B_k is a unit ball of a norm $\|\cdot\|_k$ on $L(x)$. It then follows that $U_k f_k$ lies in B_k^* - the unit ball of the dual norm $\|\cdot\|_k^*$. More precisely $\| U_k f_k \|_k^* = 1$. As $B_k \to A(x)$ we immediately deduce that $B_k^* \to A^*(x)$ - the unit ball of the dual norm $\|\cdot\|_x^*$. In particular B_k^* are uniformly bounded. Hence, the sequence $\|f_k\|_2 = \|U_k f_k\|_2$, $k = 1, 2, \ldots,$ is uniformly bounded which contradicts our assumption that $\|f_k\|_2 \to \infty$. The above contradiction proves the existence of C, such that

$$\|f_x\|_2 \leq C, \; x \in S. \tag{2.8}$$

To this end we show that A satisfies the Kreiss resolvent condition (1.2). Indeed, let $x \in S_2$ and $A \in \mathcal{A}$. Then

$$\|(zI-A)x\|_2 \geq \frac{|((zI-A)x,f_x)|}{C} = \frac{|z - (Ax,f_x)|}{C}$$

$$\geq \frac{|z| - |(Ax,f_x)|}{C} \geq \frac{|z| - \|Ax\|_x \|f\|_x^*}{C}$$

$$\geq \frac{|z| - 1}{C} = \frac{(|z| - 1)\|x\|_2}{C} .$$

Clearly, the above inequality holds for any x. Let $|z| > 1$. Choose $x = (zI-A)^{-1}y$ to get

$$\|(zI-A)^{-1}\|_2 \leq \frac{C}{|z| - 1}$$

Whence, according to the Kreiss resolvent theorem $\mathcal{A}$ is stable. The proof of the theorem is completed.

We now show that the assumption that L is stable can not be dropped. Let $|\cdot|$ be the max norm on $M_n(C)$

$$|A| = \max_{1 \leq i,j \leq n} |a_{ij}| \;, \; A = (a_{ij})_1^n \tag{2.9}$$

Denote by $UM_n(C)$ the set of all $n \times n$ upper triangular matrices. Clearly, the max norm is spectrally dominant on $UM_n(C)$ Let $\mathcal{A}$ be the restriction of the unit ball of the max norm to $L = UM_n(C)$. So $\mathcal{A}$ is a bounded convex circular set which satisfies the condition (1.3). Clearly $\mathcal{A}$ is not a stable set for $n \geq 2$. Indeed, take $A = (a_{ij})_1^n$, $a_{ij} = 1$ for $i \leq j$, $a_{ij} = 0$ for $i > j$. The matrix A is in $\mathcal{A}$ and the powers of A are unbounded. Clearly L is an unstable subspace, since

$$\dim L(e_i) = i,\ e_i = (\delta_{i1}, \ldots, \delta_{in})^t,\ i = 1, \ldots n.$$

In the next section we show that in certain sense the above L is special

3. Maximally stable subspaces.

Theorem 3 Let

$$2n-1 < k < n^2 \tag{3.1}$$

Then "most" of k dimensional subspaced of $M_n(C)$ are maximally stable.

The proof of the theorem follows from some basic notions and results in algebraic geometry. Also, the term "most" must be formulated in the language of algebraic geometry. We refer to Shafaverich [5] for the basic reference on the subject. In this context we restrict ourselves to projective spaces. Recall that the $n-1$ dimensional projective space P^{n-1} is the set $C^n - \{0\}$ where two collinear points ξ and η are identified. Let $\pi : C^n - \{0\} \to P^{n-1}$ be the above projection. If L is k (>1) dimensional subspace in C^n then $\pi(L)$ is called $k-1$ dimensional subspace in P^{n-1}. Vice versa, any $k-1$ dimensional subspace U of P^{n-1} is of the form $\pi(L)$ for some k dimensional subspace L in C^n. A set of points V in P^{n-1} is called an algebraic variety if $W = \Pi^{-1}(V)$ is given as zero set of a finite number of homogeneous polynominals

$$p_i(\zeta) = 0,\ i = 1, \ldots, m,\ \zeta = (z_1, \ldots, z_n) \in C^n \tag{3.2}$$

Here we assume that the above system of equations has a non-trivial solutions. It is well known, e.g. [5], that $W \bigcup_{i=1}^{N} W_i$ where each W_i is an irreducible homogeneous variety. If V is an irreducible variety in C^n then at "most" of it points it is a manifold. That is, there exists a subset $U_o \subset U$ such that U_o is a manifold and the closure of U_o is U. The dimension of U is defined to be the dimension of U_o and we denoted it by $\dim U$. Then the dimension of W $-\dim W$ is defined

$$\dim W = \max_{1 \le i \le N} \dim W_i .$$

The dimension of the algebraic variety V - dim V in P^{n-1} is given

$$\dim V = \dim W - 1.$$

The following theorem is a basic tool in theory of zero sets of homogeneous polynomials. See for example [5].

Theorem 4 *Let* U *and* W *be algebraic varieties in* P^m. *Assume that*

$$\dim U + \dim W \ge m. \tag{3.3}$$

Then U *and* W *intersect*.

Let $G_{m,k}$ the set of all k dimensional subspace of P^m. So any $U \in G_{m,k}$ is an algebraic variety of dimension k. Thus if $k \ge m - \dim W$ then any k dimensional subspace U intersect W. In fact this property can be used to define the dimension of W [5].

Theorem 5 *Let* W *be an algebraic variety in* P^m. *Then* d *is the dimension of* W *if and only if*

(i) *any subspace* U *of dimension* $m - d$ *intersets* W,

(ii) *there exists subspace* U *of dimension* $m - d - 1$ *which does not intersect* W.

In fact we claim that "most" subspaces in $G_{m,k}$ for $k < m-d$ do not intersect W. More precisely we claim

Lemma 2 *Let* W *be an algebraic variety in* P^m *of dimension* $d < m$. *Assume that* $k < m-d$ *and suppose that* $U \in G_{m,k}$ *intersects* W. *Then there exists an open neighborhood* $\mathcal{O}$ *of* U_o *in* $G_{m,k}$ *such that the set of all* U *in* $\mathcal{O}$ *which intersect* W *form an algebraic variety in* $\mathcal{O}$.

Proof We let $n = m+1$, $p = k+1$ and we will restrict ourselves to C^n rather than to P^{n-1}. Let $U_o = \Pi(V_o)$, V_o be a p dimensional vector space in C^n. By making a linear change of coordinates of necessary we may assume that V_o is spanned by the standard basis vectors $e_1, \ldots, e_p$. We then define $\mathcal{O}$ to be the set of all p dimensional subspaces V which have a basis of the form $f_1, \ldots, f_p$ such that

$$f_i = e_i + \sum_{j=p+1}^{n} f_{ij}\, e_j \;,\; i = 1, \ldots, p. \tag{3.4}$$

Here $e_1, \ldots, e_n$ is the standard basis of C^n. Clearly, the neighborhood of V_o consists of all subspaces V which have a basis $h_1, \ldots, h_p$ such that

$\|h_i - e_i\|_2 < \varepsilon,\ i = 1, \ldots, p,$

for a small positive ε. Let

$$h_i = \sum_{j=1}^{n} h_{ij} e_j,\ i = 1, \ldots, p,\ H = (h_{ij}),\ i = 1, \ldots, p,\ j = 1, \ldots, h.$$

Thus if ε small enough by using elementary now operations H is equivalent to $F = (f_{ij}),\ f_{ij} = \delta_{ij},\ i = 1, \ldots, p,\ j = 1, \ldots, n.$ Whence, any p - dimensional vector space V which is close to V_o has a basis of the form (3.4). Therefore, if V_1 and V_2 have basis of the form (3.4) then $V_1 = V_2$ if and only if the above bases are the same. We also claim that any p dimensional vector space V can be obtained as a limit of subspaces V_ℓ in $\mathcal{O}$. That is

$\lim_{\ell\to\infty} V_\ell = V,$

V_ℓ has a basis $h_1^{(\ell)}, \ldots, h_p^{(\ell)}$ of the form (3.4) such that

$$\lim_{\ell\to\infty} h_i^{(\ell)} = h_i$$

and $h_1, \ldots, h_p$ is a basis in V. Indeed a basis $h_1, \ldots, h_p$ in V is represented by pxn matrix H. The subspace V belongs to $\mathcal{O}$ if the pxp minor based on the first p columns is different from zero. Clearly, the set of such matrices H is dense in $M_{pn}(C)$. Next we claim that not all $V \in \mathcal{O}$ intersect $X = \Pi^{-1}(W)$. Otherwise all U in $\Pi(\mathcal{O})$ will intersect W. Since P^m is compact all U in the closure of $\Pi(\mathcal{O})$ will intersect W. That is any $U \in G_{m,k}$ will intersect W which contradicts Theorem 5. Let $V \in \mathcal{O}$. Then V intersects X if the system (3.2) together with

$$\xi = \sum_{j=1}^{p} t_j f_j \tag{3.5}$$

has a non-zero solution. Since (3.2) + (3.5) do not always have a non-trivial solution, the unknowns f_{ij}, $i = 1,\dots,p$, $j = p+1,\dots, n$ must satisfy some non-trivial algebraic conditions. See for example Shafarevich [5]. The proof of the lemma is completed. ∎

Consider next the space $M_n(C)$. We identify $M_n(C)$ with C^{n^2}. So P^{n^2-1} can be viewed as the set of one dimensional subspaces of $n\times n$ matrices. Let R_1 be the set of non-zero rank 1 matrices. We then claim

$$\Pi(R_1) \cong P^{n-1} \times P^{n-1} \tag{3.6}$$

Indeed if $0 \neq A$ is rank one matrix then $A = xy^t$, $x \neq 0$, $y \neq 0$.

Clearly x and y are determined up to a multiplication by scalar. So we have (3.6).

Recall that R_1 is given by the conditions that all 2×2 minors of $A \in M_n(C)$ are zero. So R_1 is an algebraic variety in $M_n(C)$ and (3.6) implies

$$\dim \Pi(R_1) = 2n-2 \tag{3.7}$$

Let L be a subspace of $M_n(C)$. Denote by L^1 the orthogonal subspace to L

$$L^1 = \{B,\ tr(AB) = 0 \ \text{ for } \ A \in L\}. \tag{3.8}$$

Clearly

$$\dim \Pi(L^1) = n^2-1 - \dim \Pi(L) \tag{3.9}$$

<u>Lemma 3</u>. <u>Let</u> L <u>be a subspace of</u> $M_n(C)$. <u>Then</u> L <u>is maximally stable if and only if</u> $\Pi(L^1)$ <u>does not intersect</u> $\Pi(R_1)$.

<u>Proof</u> Assume that L is not maximally stable. Then $\dim (x) < n$ for some $x \neq 0$. So there exists $0 \neq y \in C_n$ such that

$$y^t(Ax) = 0 \ \text{ for all } \ A \in L. \tag{3.10}$$

Put $B = xy^t$ and we get that $B \in L^1$. Vice versa if $0 \neq B = xy^t \in L^1$ then (3.10) holds and $\dim L(x) < n$.

<u>Proof of Theorem 3</u>. Let L be $k < n^2$ dimensional subspace of $M_n(C)$. Satisfying (3.1). Then

$$\dim \Pi\, (L^1) = n^2 - k - 1 < n^2 - 2n$$

So

$$\dim \Pi(\mathcal{L}^1) + \dim \Pi(R_1) < n^2-1.$$

Hence "most" of the subspaces $\Pi\ (\mathcal{L}^1)$ do not intersect $\Pi\ (R_1)$. Hence, according to Lemma 2, "most" of the subspaces $\mathcal{L}$ which satisfy (3.1) are maximally stable. ∎

In fact Theorem 3 is best possible in the following sense.

Theorem 6 *Let* $\mathcal{L}$ *be a* k *- dimensional subspace of* $M_n(C)$. *If*

$$k < 2n-1 \tag{3.11}$$

then $\mathcal{L}$ *is not maximally stable*.

Proof The above condition imply that $\dim \Pi(\mathcal{L}^1) + \dim \Pi(R_1) > n^2-1$. According to Theorem 4 $\Pi\ (\mathcal{L}^1)$ intersects $\Pi\ (R_1)$ and the result follows from Lemma 3.

Problem *Characterize an open set of all stable subspaces* $\mathcal{L}$ *of* $M_n(C)$ *which satisfy* (1.5) *for a fixed* $k < n$.

4.1. *The real case*

Let $\mathcal{A}$ be a set of matrices in $M_n(\mathbb{R})$ - the set of $n \times n$ real valued matrices. Clearly, the definition of the stability of $\mathcal{A}$ involves only real valued matrices. However, if we look on the Kreiss resolvent condition we see that it involves complex numbers and must be stated in the space $M_n(C)$ rather than in $M_n(\mathbb{R})$. This can be explained by the fact that the spectrum of a real valued matrix can be complex valued. Recall that $\mathcal{A}$ is called hyperbolic if the spectrum of A is real for any $A \in \mathcal{A}$. In that case the Kreiss resolvent condition can be stated in $M_n(\mathbb{R})$.

Theorem 7 *Let* $\mathcal{A}$ *be a hyperbolic set in* $M_n(\mathbb{R})$. *Then* $\mathcal{A}$ *is stable if and only if*

$$|(rI \pm A)^{-1}| < C/(r-1), \text{ for all } r > 1,\ A \in \mathcal{A}. \tag{4.1}$$

Proof Our proof is a modification of the proof of the Kreiss resolvent condition given in [4], p' 77-80. So we point out the necessary changes one should make. Since all the eigenvalues of $A \in \mathcal{A}$ are real there exists an orthogonal matrix Q such that $Q A Q^t$ is an real apper triangular matrix. See for example Marcus and Minc [3]. The rest of the proof is exactly as the proof given in [4] since the values of z used there turn out to be real values. So one uses the inequality (1.2) only for $z = \pm r$, i.e., one needs (4.1) to deduce that A is stable. #

<u>Proof of Theorem 2</u> The proof of this theorem is analogous to the proof of Theorem 1. So we point out the necessary changes. We first note that the norm $\| \cdot \|_x$ and its dual $\|\cdot\|_x^*$ are defined on real subspace $L(x)$ of R^n. Since $L(x)$ is stable as in the proof of Theorem 1 it follows that (2.8) holds. By choosing z to be real valued we deduce the inequality (1.2) for $|z| = r > 1$. Now Theorem 7 implies that A is stable.

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A subset $\mathcal{A}$ of $n \times n$ complex valued matrices is stable if all powers of all matrices from the set $\mathcal{A}$ are uniformly bounded. We show that if $\mathcal{A}$ is a bounded convex circular set which spans a stable linear subspace of matrices, then $\mathcal{A}$ is stable if the spectral radius of any $A \in \mathcal{A}$ is bounded by 1.

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