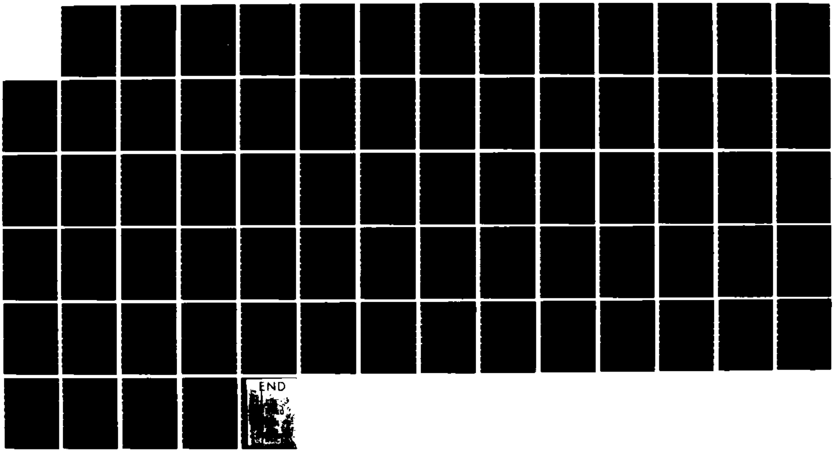
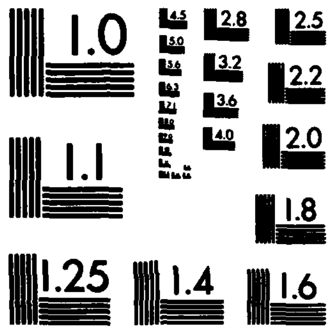
AD-A136 637 NEW METHOD OF ANALYZING VIBRATION OF CRACKED CYLINDRICAL SHELLS(U) CONNECTICUT UNIV STORRS DEPT OF MECHANICAL ENGINEERING R SOLECKI 26 DEC 83 1/1

UNCLASSIFIED UCONN-ME-82-83-ONR-1 N00014-81-K-0651 F/G 13/13 NL





MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

MECHANICAL ENGINEERING DEPARTMENT

NEW METHOD OF ANALYZING VIBRATION OF CRACKED CYLINDRICAL SHELLS

SCHOOL OF ENGINEERING
THE UNIVERSITY OF CONNECTICUT
STORRS, CONNECTICUT

NEW METHOD OF ANALYZING VIBRATION OF
CRACKED CYLINDRICAL SHELLS

by

Roman Solecki

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER ME-82-83-ONR-1	2. GOVT ACCESSION NO. AD-A136637	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) New Method of Analyzing Vibration of Cracked Cylindrical Shells		5. TYPE OF REPORT & PERIOD COVERED Final Report August 1, 1980-Sept. 30,1982
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Roman Solecki		8. CONTRACT OR GRANT NUMBER(s) U.S. Navy Contract N00014-81-K-0651
9. PERFORMING ORGANIZATION NAME AND ADDRESS Roman Solecki, University of Connecticut Department of Mechanical Engineering Storrs, Connecticut 06268		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Task No. 064-670
11. CONTROLLING OFFICE NAME AND ADDRESS Department of the Navy. Office of Naval Research Code 474, 800 N. Quincy Street, Arlington, VA 22217		12. REPORT DATE December 26, 1983
		13. NUMBER OF PAGES
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

This document has been approved for public release and role distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Parts of this report will be published in Engineering Fracture Mechanics.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

crack, cylindrical, plate, rectangular, shallow, shell, simply-supported, vibration

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The method of Fourier transformation of functions with lines of discontinuity and with "built-in" point singularities, is applied to analysis of vibration of cracked rectangular plates and of cracked cylindrical shells of rectangular planform. Application of the above method in conjunction with Green-Gauss theorem leads to an infinite system of linear algebraic equations. Its characteristic determinant when equated to zero provides with the frequency equation. Numerical examples include a plate with a parallel crack, a plate with a diagonal crack and a shell cracked at the apex.

PREFACE

This report covers a two year effort supported by the Department of the Navy, Office of Naval Research (ONR) under Contract Number N00014-81-K-0651. The Principal Investigator was Dr. Roman Solecki, the Project Monitor for ONR was Dr. Nicholas Basdekas.

The author wishes to thank Mr. Farshid Forouhar, a University of Connecticut graduate student, for his help in deriving and programming some of the equations and Mrs. Joanne Moore for typing the manuscript.

TABLE OF CONTENTS

ABSTRACT

A method is developed for calculating natural frequencies of cracked rectangular plates and of cracked cylindrical shells of rectangular planform. In either case the crack is rectilinear of arbitrary length and of arbitrary location. The analysis is based on finite Fourier transform method of discontinuous functions in the form suggested earlier by the author (Refs. 1,2,3,6,24). For the shell the problem is described by Donnell's equations. For either case the discontinuities of the displacement and of the slope across the crack are the unknowns of the problem. The unknown amplitudes of those discontinuities are determined by satis- fying the boundary conditions at the crack's edge (2 for the plate and 4 for the shell). This requires differentiation of Fourier series representing discontinuous functions and is achieved by using generalized Green-Gauss theorem. The square-root singularities of the bending moment (and also of the normal and shear stresses for the shell) are built into the solution. Ultimately the problem is reduced to an infinite system of linear algebraic equations. The method of reduction is applied to the characteristic determinant of the problem and numerical values of the lowest two frequencies are found, as examples, for (1) a simply supported square plate with symmetrically located crack, parallel to one edge for various d/a ratios (d-length of the

crack, a-length of the plate side); (2) for a simply supported square plate with a crack of fixed length, parallel to one edge at varying distances from it; (3) for a simply supported square plate with a diagonal crack of varying length; (4) for a simply supported cylindrical shell of a square planform with a crack of varying length located at the apex parallel to the straight boundary for various radius/edge length ratios.

I. INTRODUCTION

For the past two years, the author has carried out a research program for the Department of the Navy that was aimed at the analysis of natural vibration of cracked rectangular plates and of cracked shallow cylindrical shells of rectangular planform. The results of this investigation have been reported elsewhere,[1], [2],[3], here more details of derivation are added.

The report consists basically of two parts: in Part 1 the cracked plate is analyzed while the cracked shell is investigated in Part 2.

The studies of bending and vibration of finite cracked plates are limited to few papers. Bending of a cracked rectangular plate was first investigated by Keer and Sve in [4]. Their analysis was limited to such location of the crack that allowed reduction of the problem to a dual series equation. Hence the crack was confined to a position along the symmetry axis. Analogous method was applied by Stahl and Keer [5] for analysis of natural vibration and stability of rectangular plate and was bounded by similar limitations as encountered in [4]. In [6] Solecki attempted to remove existing restrictions by developing a method that would allow to study rectangular plates with arbitrarily located crack. He partially suceeded by developing a method based on the combination of finite Fourier transformation and of the generalized Green-Gauss theorem. A cracked rectangular plate was discussed as one of the examples. Numerical data were not however obtained partially because the singularity of the curvature at the tips was not explicitly isolated. The only other paper dealing with a similar topic was written by Ali and Atwani [7] who used a version of Rayleigh's method to study natural vibrations of rectangular plates with cutouts.

Numerous papers were devoted to analyses of vibrations of thin shells of various geometries (see, for example, Dym [8] and Kraus [9] for references). All reported studies however were applied to homogeneous (uncracked) shells. Apparently no investigation of vibration of cracked shells was done so far. Considerable mathematical difficulties are encountered when attempting to solve this problem formulated by a coupled system of partial differential equations describing functions with discontinuous derivatives.

Let us survey briefly, related studies on statics of cracked cylindrical shells noting that almost all of the analyses refer to infinite shells. The first attempt to determine deflection, stresses and stress intensity factor of a cracked shallow cylindrical shell is due to Folias [10] who reduced Marguerre-Reissner differential equations describing the problem to a system of dual integral equations. This in turn can be transformed to a set of coupled singular integral equations of the Cauchy type.

This technique of reducing a system of partial differential equations to a system of dual integral equations, and subsequently to a system of singular equations, seems to be the favorite of most authors of papers on statics of cracked shells.

A similar type of shell to that discussed in [10], was treated with similar mathematical devices by Copley and Sanders in [11]. Erdogan and Kibler [12] also studied shallow cylindrical shells using Marguerre-Reissner equations and reducing them to a system of singular integral equations. Their solutions however is valid, unlike Folias's solution [10], also for relatively long cracks. Keer and Watts [13] based their analysis of a complete cylindrical shell with a circumferential crack, on the equations of three-dimensional theory of elasticity. Sanders [14] used his equations to analyze an infinite cylindrical shell with a circumferential through-crack.

Few solutions exist that are based on 10th order shallow shell theories (Naghdi equations). These are also obtained by reducing the differential equations to singular integral equations via dual integral equations. Krenk [15] used this theory (which takes into account transverse shear deformation) to analyze cylindrical shell with an axial crack. Most recently, Delale and Erdogan [16] also investigated the effect of transverse shear on the behavior of circumferentially cracked cylindrical shell.

In [17] Delale analyzed, among others, also an axially or circumferentially cracked infinite cylindrical shell. An original approach often applicable to quite general geometries, partially based on Sanders paper [18], characterizes works of Simmonds and colleagues (see, for instance, [19], [20] and [21]). Finally, still another technique due to Lakshminararyana and Murthy [22] and [23] is valid for short arbitarily located cracks in infinite shallow shells. It is based on formulating the problem in elliptic coordinates. This allows the crack edges to become coordinate lines (as a limit of an elliptic cutout). This procedure although elegant and useful in establishing close form expressions for local quantities would be too cumbersome, even if applicable, to deal with global quantities, because it would require manipulating infinite series of Mathieu functions.

In the static case differential equations describing various shell theories are amenable, for certain geometries and special crack's locations, to a system of dual integral equations. These are in turn transformed into singular integral equations which are usually solved numerically. When inertia terms are included in the differential equations and when in addition the crack's location and its relative length are arbitrary, the above procedure encounters considerable difficulties or is even not applicable (in particular for finite geometries).

In part 1 of the present report a method is demonstrated for determining frequencies of steady-state vibration of a rectangqgular, simply supported, isotropic plate with an arbitrarily located crack (Fig. 1). It is based on application of finite Fourier sine transformation in conjunction with the generalized Green-Gauss Theorem. This method, as was shown in [6] and [24], eliminates considerable amount of tedious integration by parts.

After applying double finite Fourier transformation to the differential equation governing the problem, and after using the inversion theorem one obtains, as usual, a system of integral equations with respect to the unknown discontinuities of the deflection and of the slope across the crack. It is known [25], [1] that using as the unknowns higher order derivatives one improves the convergence of the resulting infinite series. Therefore using the condition that the bending moment at the crack equals zero one can replace the integral involving unknown discontinuity of the deflection by the discontinuity of the curvature in the direction normal to the crack. The unknown quantities are in turn expanded into Fourier series. Since the curvature is square-root singular at the tips therefore it is represented as a sum of a regular part (expanded into Fourier series) and of a singular part with known strengths of the singularities multiplied by unknown coefficients determined from the conditions of the problem. Finally, the unknown infinite sequences of Fourier coefficients are obtained by applying the conditions that the bending moment and the shear force across the crack are zero. Application of these conditions requires differentation of Fourier series of discontinuous functions. A novel approach is suggested in relation to this operation. Determining Fourier coefficients of derivatives constitutes in this case a major undertaking. This effor is reduced to a fraction by applying again the generalized Green-Gauss theorem. In Part 2 of the report Donnell equations are used to describe vibration of cylindrical shells. The same method as in Part 1 is applied again here. Its main feature consists in expanding displacement functions discontinuous inside the region of interest into infinite series and then proceeding according to the rules of integral transformations developed previously for discontinuous functions (application of integration by parts, Green-Gauss theorem, etc.). Differential equations are transformed as the result of these operations, into algebraic equations with respect to the transforms of the unknown displacements. Application of the inversion theorem allows to determine formally the displacements that still include at this stage certain integrals depending on unknown quantities. Specifically, the unknown functions appearing in the integrands are the quantities discontinuous across the crack (discontinuities of the first kind).

In addition, depending on the boundary conditions on the shell's circumference, other unknown integrals may appear. These include certain unknown boundary quantities. The objective of the next step is evaluation of the unknown functions in the integrands.

At this point it is very important to realize (1) which unknown functions are singular at the ends of their range of definition, and (2) what are the strengths of singularities.

Sufficient experience and information was gathered by various researchers to answer these questions. It is therefore possible to represent the unknown functions as the sum of a regular part in the form of infinite series with unknown Fourier coefficients, and of a singular part of known form multiplied by an unknown constant (the intensity factor).

Subsequently available boundary conditions at the edge of the crack are applied. At this stage particular care must be exercised when differentiating infinite series representing functions with "inside" discontinuities: they never can be differentiated term by term. The appropriate formulae for differentiation are derived applying again generalized Green-Gauss theorem. Once all these mathematical obstacles are overcome the final result is obtained in the form of an infinite system of linear algebraic equations. In the case of natural vibration the system is homogeneous and depending on the unknown natural frequencies. Thus the characteristic equation is in the form of an infinite determinant equated to zero. As usual, assuming that the system is regular, one applies the method of reduction obtaining frequencies with any desired accuracy. Adaptation of this procedure to higher order theory would not present any major difficulties.

II. Vibration of Cracked Rectangular Plates

1. Series Solution of the Differential Equation

The differential equation which governs the amplitude of flexural vibration w(x,y) of the plate shown in Fig. 1 is

$$D\,\nabla^2\nabla^2 w - \rho\omega^2 w = q(x,y) \qquad (1)$$

with the boundary conditions of simple support around the contour and the conditions of vanishing bending moment M_u and vanishing shear force Q_u at the crack's edge where u is the direction normal to the crack.

The solution of Eq. (1) is assumed in the form

$$w(x,y) = \frac{4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn}\sin\alpha_m x\,\sin\beta_n y \qquad (2)$$

In order of finding w_{mn} one applies finite Fourier sine transformation to eq. (1). As in [24] this is performed in conjunction with the generalized Green-Gauss theorem

$$\iint_S F_{,i}\,dS = \int F n_i\,d\Gamma + \int_{\bar{\Gamma}} F\bar{n}_i\,d\bar{\Gamma} \qquad (i=1,2,3) \qquad (3)$$

where Γ is the outer contour of the plate, $\bar{\Gamma}$ - its inner contour, n_i and $\bar{n}_i$ are the components of unit vectors normal to Γ and $\bar{\Gamma}$ respectively and where F is any tensor.

Application of this procedure and subsequent solution of the resulting system of algebraic equations with respect to w_{mn} yields the following expression:

$$w_{mn} = \frac{b^4}{\pi^4 \Delta_{mn}} \{ -\int_0^d [(\alpha_m{}^2 + \beta_n{}^2) \frac{\partial \phi_{(mn)}}{\partial u} - (1-\nu) \frac{\partial^3 \phi_{(mn)}}{\partial u \, \partial c^2}] \Delta w \, dc$$

$$+ \int_0^d [\alpha_m{}^2 + \beta_n{}^2) \phi_{(mn)} + (1-\nu) \frac{\partial^2 \phi_{(mn)}}{\partial c^2}] \Delta \frac{\partial w}{\partial u} \, dc$$

$$+ (\alpha_m{}^2 + \beta_n{}^2) \int_\Gamma w \frac{\partial \phi_{(mn)}}{\partial u} \, d\Gamma + \frac{1}{D} \int_\Gamma [H_2 - D(1-\nu) \frac{\partial^2 w}{\partial \Gamma^2}] \frac{\partial \phi_{(mn)}}{\partial u} \, d\Gamma + q_{mn} \} \tag{4}$$

where Δw and $\Delta(\partial w/\partial u)$ are discontinuities across the crack of the displacement and of normal slope respectively. Also: ν is the Poisson's ratio, q_{mn}-the transform of $q(x,y)$, H_2 is the bending moment,

$$\phi_{(mn)} = \sin\alpha_m(s_1 + c \cos \psi) \sin\beta_n(y_1 + c \sin \psi) \tag{5}$$

and

$$\Delta_{mn} = [(m\phi)^2 + n^2]^2 - \Omega^2 \tag{6}$$

where

$$\phi = b/a, \quad \Omega^2 = \frac{\rho \omega^2 b^4}{D \pi^4} \tag{7}$$

Since Γ is the contour of the plate therefore the last two integrals in (4) vanish for a simply supported plate. In order of improving convergence the approach of Fletcher and

Thorne [25] is applied, that is the unknown Δw is replaced by the unknown discontinuity of the curvature in the direction normal to the crack: $\Delta\, \partial^2 w/\partial u^2$. This is achieved by utilizing the condition that

$$\Delta \frac{\partial^2 w}{\partial u^2} + \nu \frac{\partial^2 w}{\partial c^2} = 0 \qquad (8)$$

Integrating the first integral in (4) by parts and using (8) one obtains the following relations

$$\int_0^d [(\alpha_m{}^2 + \beta_n{}^2) \frac{\partial \phi(mn)}{\partial u} - (1-\nu) \frac{\partial^3 \phi(mn)}{\partial u \partial c^2}] \Delta w \, dc$$

$$= -\frac{1}{\nu} \int_0^d \{ \iint [(\alpha_m{}^2 + \beta_n{}^2) \frac{\partial \phi(mn)}{\partial u} - (1-\nu) \frac{\partial^3 \phi(mn)}{\partial u \partial c^2}] \, dc \, dc\} \, \Delta \frac{\partial^2 w}{\partial u^2} \, dc \qquad (9)$$

The unknown slope discontinuity is now expanded into Fourier cosine series

$$\Delta \frac{\partial w}{\partial u} = \frac{2}{d} \sum_{k=0,2}^{\infty} \lambda_k U_k \cos \frac{k\pi c}{d}, \qquad c\varepsilon(0,d) \qquad (10)$$

The unknown discontinuity of the curvature has square-root singularities at the tips of the crack. It is therefore represented as a sum of a the unknown regular function which is expanded into Fourier sine series and of a singular function with square-root singularities at the end of the interval $c \epsilon (0,d)$ and adjusted in such a way that, in concert with the physical conditions of the problem, both Δw and $\Delta \partial w/\partial u$ vanish at the ends.

This leads to the following representation:

$$\Delta \frac{\partial^2 w}{\partial u^2} = \frac{2}{d} \sum_{k=1}^{\infty} W_k \sin \frac{k\pi c}{d} + \frac{3}{\pi\sqrt{d}} \sum_{k=1}^{\infty} \frac{W_k}{k} \left(-\frac{1}{\sqrt{c}} + \frac{1}{\sqrt{d-c}}\right)$$

$$+ \frac{2}{\pi\sqrt{d}} \sum_{k=1,3}^{\infty} \frac{W_k}{k} \left(\frac{1}{\sqrt{c}} - \frac{2}{\sqrt{d-c}}\right) \tag{11}$$

Presently eqs. (10) and (11) are used in (4) which is in turn substituted into eq. (2) leading to a formal solution of the problem which still includes unknown Fourier coefficients U_k and W_k.

2. Boundary Conditions and the Characteristic Equations

In order to determine two infinite sets of unknown Fourier coefficients, two boundary conditions available at the crack's edge are applied

$$\frac{\partial^2 w}{\partial u^2} + \nu \frac{\partial^2 w}{\partial c^2} = 0 \text{ at } x=x_1 + c\cos\psi,\ y=y_1 + c\sin\psi \tag{12}$$

$$\frac{\partial^3 w}{\partial u^3} + (2-\nu) \frac{\partial^3 w}{\partial u \partial c^2} = 0 \text{ at } x=x_1 + c\cos\psi,\ y=y_1 + c\sin\psi \tag{13}$$

It is seen that the series (2) must be differentiated and that this can not be done term-by-term because (2) represents a function with discontinuous second and third derivatives. As indicated in [1] proceeding in traditional way will result in proper formulae being devised but at a cost of considerable amount of cumbersome manipulations. The procedure devised in [1], based on Green-Gauss theorem is generalized here to include the use of straight line discontinuity arbitrarily located within the plate's area.

Let

$$\phi_{(1mn)} = \sin\alpha_m x \; \sin\beta_n y$$

$$\phi_{(2mn)} = \cos\alpha_m x \; \sin\beta_n y$$

$$\phi_{(3mn)} = \sin\alpha_m x \; \cos\beta_n y$$

$$\phi_{(4mn)} = \cos\alpha_m x \; \cos\beta_n y \tag{14}$$

Assume that we want to obtain the expression for $\partial w/\partial u$ where, as before, u is the direction normal to the crack.

Let $F \equiv w\ \phi_{(2mn)}$ be substituted into (3) for i = 1.

The following relation results

$$\iint_S \frac{\partial}{\partial x}\left(w\ \phi_{(2mn)}\right)dS = \int_{\bar{\Gamma}} w\ \phi_{(2mn)} u_x d\bar{\Gamma} \tag{15}$$

where the integral along Γ vanished because of the assumed boundary conditions.

Expanding the integral on the left hand side of eq. (15) and taking into account that

$$\iint_S \frac{\partial}{\partial x} \phi_{(2mn)} = -\alpha_m \phi_{(1mn)}$$

and that

$$\iint_S w\, \phi_{(1mn)}\, dS = w_{mn}$$

one obtains finally

$$\iint_S \frac{\partial w}{\partial x} \cos\alpha_m x \sin\beta_n y\, dxdy = \alpha_m w_{mn} + \int_{\bar{\Gamma}} w\, \phi_{(2mn)} u_x d\bar{\Gamma} \tag{16}$$

One can easily find now the formula for $\partial w / \partial x$:

$$\frac{\partial w}{\partial x} = \frac{2}{ab} \sum_{n=1}^{\infty} \int_{\bar{\Gamma}} w(\sin\beta_n y)_{\bar{\Gamma}}\, u_x d\bar{\Gamma} \sin\beta_n y$$

$$+ \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{\alpha_m w_{mn} + \int_{\bar{\Gamma}} w(\cos\alpha_m x \sin\beta_n y)_{\bar{\Gamma}}\, u_x d\bar{\Gamma} \cos\alpha_m x \sin\beta_n y \tag{17}$$

In a similar way, starting with the expression $F \equiv w\,\phi_{(3mn)}$ one obtains from

$$\iint_S \frac{\partial}{\partial y} (w\, \phi_{(3mn)})\, dS = \int_{\bar{\Gamma}} w\, \phi_{(3mn)}\, u_y\, d\bar{\Gamma} \tag{18}$$

the following formula

$$\frac{\partial w}{\partial y} = \frac{2}{ab} \sum_{m=1}^{\infty} \int_{\bar{\Gamma}} w(\sin\alpha_m x)_{\bar{\Gamma}}\ u_y\ d\bar{\Gamma}\ \sin\alpha_m x$$

$$+ \frac{4}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{\beta_n\ w_{mn} + \int_{\bar{\Gamma}} w(\sin\alpha_m x\ \cos\beta_n y)_{\bar{\Gamma}}\ u_y\ d\bar{\Gamma}\ \ \sin\alpha_m x\ \cos\beta_n y \tag{19}$$

Now using known expression for directional derivative

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\ \cos\psi + \frac{\partial w}{\partial y}\ \sin\psi \tag{20}$$

yields the desired result.

The same procedure was used to obtain higher order derivatives appearing in the boundary conditions (12) and (13). Resulting system of equations still depends on the local coordinate c (along the crack). This dependence is removed by multiplying the equations obtained from (12) and (13) by $\cos(\ell\pi c/d)$ or $\sin(\ell\pi c/d)$ respectively (where $\ell = 0,1,\ldots\infty$) and integrating the results with respect to $c\,\varepsilon(0,d)$.

This yields an infinite system of linear algebraic equations of the following form

$$\sum_{k=1}^{\infty} W_k\ c_{1k\ell} + \sum_{k=1,3}^{\infty} W_k\ c_{2k\ell} + \sum_{k=0}^{\infty} U_k\ c_{3k\ell} = 0, \quad \ell=0,1,\ldots\infty$$

$$\sum_{k=1}^{\infty} W_k\ c_{4k\ell} + \sum_{k=1,3}^{\infty} W_k\ c_{5k\ell} + \sum_{1=0}^{\infty} U_k\ c_{6k\ell} = 0, \quad \ell=1,\ldots\infty \tag{21}$$

where $U_o = -2 \sum_{k=1}^{\infty} U_k$ and where

$$
\begin{aligned}
c_{1k\ell} = \frac{\zeta}{\nu} \{ & \phi a_{5s} \sum_{m=1} m \, T_{4m0\ell} \, [\pi Z_{2m0k} - \frac{3}{2k} \sqrt{2\pi} \, E_{3m0}] \\
& + a_{5c} \sum_{n=1}^{\infty} n \, T_{40n\ell} \, [\pi Z_{10nk} - \frac{3\sqrt{2\pi}}{2k} E_{10n}]\} \\
& + \frac{\zeta}{\nu} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\Delta_{mn}} \{(X_{1mn} \, T_{3mn\ell} + X_{2mn} \, T_{4mn\ell})(-\pi Z_{1mnk} \\
& + \frac{3\sqrt{2\pi}}{2k} E_{1mn}) + (X_{3mn} \, T_{3mn\ell} + X_{4mn} \, T_{4mn\ell})(-\pi Z_{2mnk} \\
& + \frac{3\sqrt{2\pi}}{k} E_{3mn})\}
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
c_{2k\ell} = \frac{\zeta\sqrt{2\pi}}{k\nu} \{ & -\phi a_{5s} \sum_{m=1}^{\infty} m \, T_{4m0\ell} \, E_{4m0} - a_{5c} \sum_{n=1}^{\infty} n \, T_{40n\ell} \, E_{20n} \\
& + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\Delta_{mn}} [X_{1mn} \, E_{2mn} + X_{3mn} \, E_{4mn}) \, T_{3mn\ell} \\
& + (X_{2mn} \, E_{2mn} + X_{4mn} \, E_{4mn}) \, T_{4mn\ell}\}
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
c_{3k\ell} = \frac{\lambda_k}{\zeta\pi} \{ & a_{11} \sum_{m=1}^{\infty} T_{4m0k} \, T_{4mo\ell} + a_{11} \sum_{n=1}^{\infty} T_{40nk} \, T_{40n\ell} \\
& + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\Delta_{mn}} [(X_{5mn} \, T_{3mn\ell} + X_{6mn} \, T_{4mn\ell}) \, T_{3mnk} \\
& + (X_{7mn} \, T_{3mn\ell} + X_{8mn} \, T_{4mn\ell}) \, T_{4mnk}\} + \pi\zeta a_{11} \delta_{0\ell}
\end{aligned}
\tag{24}
$$

where $\delta_{0\ell}$ is Kronecker's delta, and $\lambda_k = 1 - \frac{1}{2}\delta_{0k}$
The coefficients c_{4kl}, c_{5kl}, c_{6kl} are obtained from c_{1kl}, c_{2kl} and c_{3kl} respectively by making the substitutions shown in Table 1. The expressions for the terms appearing above are given in the Appendix 1. The characteristic equation is obtained by equating to zero the infinite characteristic determinant of the system of linear algebraic equations (21). In practice the method of reduction is applied and the frequencies are determined from a finite characteristic equation.

3. Examples 1: A Plate with a Parallel Crack

As the first example consider a plate with a crack parallel to one edge (see Ref. [1] and Fig. 2). In the present case eq. 21 can be simplified because the double series appearing there (see Eqs. 27÷24) can be reduced to single series to yield the following system:

$$\sum_{k=1,3}^{\infty} W_k \left(C_{4pk} - \frac{1}{k^2} C_{5p}\right) - \sum_{k=2,4}^{\infty} U_k C_{2kp} = 0 \qquad p = 0,2,..\infty$$

$$\sum_{k=1,3}^{\infty} W_k \left(P_{kp} - \frac{1}{k^2} R_p\right) + \sum_{k=2,4}^{\infty} U_k C_{1pk} = 0 \qquad p = 1,3,..\infty \tag{25}$$

The dimensionless quantities appearing in eq. (25) are given below:

$$C_{1pk} = \frac{1}{a^2 p} \sum_{m=1,3}^{\infty} \frac{a_{pm}\, \bar{a}_{km}}{m^2} [(1-\nu)^2 m^4 - f^2] Z_{1m}(e,e) \qquad p = 1,3\ldots;k = 2,4\ldots$$

$$C_{2kp} = \frac{1}{a^2} \sum_{m=1,3}^{\infty} \frac{\bar{a}_{km}\, \bar{a}_{pm}}{m^2} \{[(1-\nu^2) m^4 - f^2] Z_{2m}(e,e)$$

$$+ (1-\nu)[Z_{3m}(0,0) - Z_{3m}(2e,0)]\} \qquad p = 0,2,\ldots;k = 2,4,\ldots$$

$$C_{4pk} = \frac{1}{a^2 k} \sum_{m=1,3}^{\infty} \frac{a_{pm}\, a_{km}}{m^2} [(1-\nu)^2 m^4 - f^2] Z_{1m}(e,e) \qquad p = 0,2,..;k = 1,3,\ldots$$

$$C_{5p} = \frac{\sqrt{2\phi}}{a\pi} \sum_{m=1,3}^{\infty} \frac{\bar{a}_{pm}}{m^2 \sqrt{m}} [\cos \frac{m\pi\phi}{2}\ C(\sqrt{m\pi\phi}) + \sin \frac{m\pi\phi}{2}\ S(\sqrt{m\pi\phi})]$$

$$[(1-\nu)^2 m^4 - f^2] Z_{1m}(e,e) \qquad p = 0,2,.. \qquad (26)$$

$$P_{kp} = \frac{1}{a^2 kp} \sum_{m=1,3}^{\infty} \frac{a_{pm}\, a_{km}}{m^2} \{- (1-\nu)m^2 + 2(1-\nu)m^2(f^2 - m^4) Z_{5m}(e,e)$$

$$+ [f^2 - (1-\nu^2)m^4] Z_{3m}(e,e)\} \qquad k,p = 1,3,\ldots$$

$$R_p = \frac{\sqrt{2\phi}}{a\pi p} \sum_{m=1,3}^{\infty} \frac{a_{pm}}{m^2\sqrt{m}} [\cos \frac{m\pi\phi}{2}\ C(\sqrt{m\pi\phi}) + \sin \frac{m\pi\phi}{2}\ S(\sqrt{m\pi\phi})]\{ - (1-\nu)m^2$$

$$+ 2(1-\nu)m^2(f^2 - m^4)\ Z_{5m}(e,e) + [f^2 - (1-\nu^2)m^4] Z_{3m}(e,e)\} \qquad p = 1,3,\ldots$$

where $\phi = d/a$, $\gamma = a/b$, and $e = e_1/b$

$$Z_{1m}(\xi_1,\xi_2) = \gamma \sum_{n=1}^{\infty} \frac{n \sin n\pi\xi_1 \cos n\pi\xi_2}{(\gamma^2 n^2 + m^2)^2 - f^2},$$

$$Z_{2m}(\xi_1,\xi_2) = \sum_{n=1}^{\infty} \frac{\sin n\pi\xi_1 \sin n\pi\xi_2}{(\gamma^2 n^2 + m^2)^2 - f^2},$$

$$(27)$$

$$Z_{3m}(\xi_1,\xi_2) = \gamma^2 \sum_{n=1}^{\infty} \frac{n^2 \cos n\pi\xi_1 \cos n\pi\xi_2}{(\gamma^2 n^2 + m^2)^2 - f^2},$$

$$Z_{5m}(\xi_1,\xi_2) = \sum_{n=1}^{\infty} \frac{\cos n\pi\xi_1 \cos n\pi\xi_2}{(\gamma^2 n^2 + m^2)^2 - f^2},$$

The characteristic equation is obtained by equating to zero the finite characteristic determinant of the system (25). In practice method of reduction is used to obtain the natural frequencies, that is the number of rows and columns is limited to a certain finite number.

For a square plate, $\gamma=1$, and the functions (27) become:

$$Z_{1m}(\xi_1,\xi_2) = \frac{\pi}{8f}\left\{\frac{C\sin[\pi(1-\xi_2+\xi_1)\sqrt{m^2-f}] + \sinh[\pi(1-\xi_1-\xi_2)\sqrt{m^2+f}}{\sinh\pi\sqrt{m^2-f}}\right.$$

$$\left. - \frac{C\sinh[\pi(1-\xi_2+\xi_1)\sqrt{m^2+f}] + \sinh[\pi(1-\xi_1-\xi_2)\sqrt{m^2+f}]}{\sinh\pi\sqrt{m^2+f}}\right.$$

$$\text{where } C = \begin{cases} 1 & \text{for } \xi_1 \geq \xi_2 \\ -1 & \text{for } \xi_1 \leq \xi_2 \end{cases}$$

$$Z_{2m}(\xi_1,\xi_2) = \frac{\pi}{8f}\left\{\frac{\cosh[\pi(1-|\xi_1-\xi_2|)\sqrt{m^2-f} - \cosh[\pi(1-\xi_1-\xi_2)\sqrt{m^2-f}]}{\sqrt{m^2-f}\sinh\pi\sqrt{m^2-f}}\right.$$

$$\left. - \frac{\cosh[\pi(1-|\xi_1-\xi_2|)\sqrt{m^2+f}] + \cosh[\pi(1-\xi_1-\xi_2)\sqrt{m^2+f}]}{\sqrt{m^2+f}\sinh\pi\sqrt{m^2+f}}\right\}$$

$$Z_{3m}(\xi_1,\xi_2) = \frac{\pi}{8f}\left\{-\sqrt{m^2-f}\;\frac{\cosh[\pi(1-|\xi_1-\xi_2|)\sqrt{m^2-f} + \cosh[\pi(1-\xi_1-\xi_2)\sqrt{m^2-f}]}{\sinh\pi\sqrt{m^2-f}}\right.$$

$$\left. + \sqrt{m^2+f}\;\frac{\cosh[\pi(1-|\xi_1-\xi_2|)\sqrt{m^2+f} + \cosh[\pi(1-\xi_1-\xi_2)\sqrt{m^2+f}]}{\sinh\pi\sqrt{m^2+f}}\right\}$$

$$Z_{5m}(\xi_1,\xi_2) = \frac{1}{4f}\left\{\frac{\pi\cosh[\pi(1-|\xi_1-\xi_2|)\sqrt{m^2-f}] + \pi\cosh[\pi(1-\xi_1-\xi_2)\sqrt{m^2-f}]}{2\sqrt{m^2-f}\;\sinh\pi\sqrt{m^2-f}}\right.$$

$$- \frac{\pi\cosh[\pi(1-|\xi_1-\xi_2|)\sqrt{m^2+f}] + \pi\cosh[\pi(1-\xi_1-\xi_2)\sqrt{m^2+f}]}{2\sqrt{m^2+f}\sinh\pi\sqrt{m^2+f}}$$

$$\left. - \frac{2f}{m^4-f^2}\right\} \tag{28}$$

The system of algebraic equations (25) was also put in more compact form (see [26]) by introducing the following definitions:

$$A_k = \begin{cases} W_k & \text{for } k=1,3,\ldots \\ A_k & \text{for } k=2,4,\ldots \end{cases}$$

$$C_{kp} = \begin{cases} C_{4pk} - \frac{1}{k^2} C_{5p} & \text{for } k=1,3,\ldots;\ p=0,2,\ldots \\ -C_{2kp} & \text{for } k=2,4,\ldots;\ p=0,2,\ldots \\ P_{kp} - \frac{1}{k^2} R_p & \text{for } k=1,3,\ldots;\ p=1,3,\ldots \\ C_{1pk} & \text{for } k=2,4,\ldots;\ p=1,3,\ldots \end{cases} \qquad (29)$$

Now the system of eqs. (25) takes the form

$$\sum_{k=1}^{\infty} A_k C_{kp} = 0 \qquad p=0,1,..\infty \qquad (30)$$

and the characteristic equation becomes

$$\det|C_{kp}| = 0 \qquad k=1,2,..\infty;\ p=0,1..\infty \qquad (31)$$

Characteristic equation (31) was first solved for the crack located on the symmetry line y = b/2 = a/2 (see fig. 2). This allowed for commparison with the existing results given in [5]. Table 2 presents the smallest values of the frequency factors f corresponding to symmetric-symmetric and symmetric-antisymmetric ("opened crack") vibrations and compares them with values calculated in [5]. It should be noted that the present results were obtained by considering a 20 X 20 matrix (in few cases a 30 X 30 matrix). For "opened-crack" vibrations the results were not sufficiently accurate for a crack extending almost to the edges.

When crack is not located on the symmetry line both modes are coupled. The frequency factors f for 2 lowest modes were calculated for d/a = 0.5 and for various relative distances of the crack from the edge ($0 < e_1/a < 0.5$). Results are given in table 3 and represented graphically in Fig. 3. It is interesting to note that the smallest value of the frequency factor corresponding to the second mode is obtained for a crack located approximately at $e\ /a_1$=0.25.

4. Example 2: A Diagonally Cracked Square Plate

The system of equations (21) is first put in a more convenient form (see [1])

$$\sum_{k=1}^{\infty} A_k\, C_{kp} = 0\ , \qquad p=0,1,\ldots\infty \tag{32}$$

where

$$A_{2k-1} \equiv W_k\ , \qquad A_{2k} \equiv U_k\ , \qquad k=1,2,\ldots\infty \tag{33}$$

and

$$C_{kp} = c_{1(2k-1)(p/2)} + c_{2(2k-1)(p/2)}, \quad k=1,5,9,\ldots; \quad p=0,2,\ldots$$

$$C_{kp} = c_{1(2k-1)(p/2)}, \quad k=3,7,11,\ldots; \quad p=0,2,\ldots$$

$$C_{kp} = c_{3(2k)(p/2)}, \quad k=2,3,\ldots; \quad p=0,2,\ldots$$

$$C_{kp} = c_{4(2k-1)(2p-1)/2} + c_{5(2k-1)(2p-1)/2}, \quad k=1,5,9,\ldots; \quad p=1,3,\ldots$$

$$C_{kp} = c_{4(2k-1)(2p-1)/2} + \quad k=3,7,11,\ldots; \quad p=1,3,\ldots$$

$$C_{kp} = c_{6(2k)(2p-1)/2}, \quad k=2,4,\ldots; \quad p=1,3$$

Characteristic equation for the system (32) becomes

$$\det | C_{kp} | = 0 \quad k=1,2,\ldots\infty; \quad p=0,1,\ldots\infty$$

The geometry of the plate and of the crack allows for modes which are symmetric or antisymmetric with respect to the diagonal.

For such modes the natural frequencies have been found for various relative lengths of the crack: $\zeta = d/a$. The results are presented in Fig. 4. When the crack extends to the corners the corresponding frequencies should equal these of a right triangular, equilateral plate simply supported along the legs and with the hypothenuse either free (symmetric vibration of the cracked plate) or "sliding" (antisymmetric vibration of the cracked plate). Unfortunately these data are not available for this type of supports of a triangular plate. The indirect verification of the results is obtained by noting that the frequencies of both modes seem to converge to the same value when ζ is increasing to $\sqrt{2}$. It was not possible to determine the exact value of this limit because the convergence of the series involved decreases when ζ approaches $\sqrt{2}$. This pheonomenon is similar to the one noted in ref. [24].

III. Vibration of Cracked Cylindrical Shells

1. Series solution of the differential equations

Consider a cracked shell shown in Fig. 5. It is assumed for simplification that the deflections, normal displacements, bending moments and shear stress vanish on the contour. Validity of Donnell Equations [8] is assumed:

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{1-\nu}{2}\frac{\partial^2}{\partial\eta^2}\right)u + \frac{\partial^2 v}{\partial\xi\partial\eta} + \nu\frac{\partial w}{\partial\xi} = -q_x R^2/C \qquad (1)$$

$$\frac{1+\nu}{2}\frac{\partial^2 u}{\partial\xi\partial\eta} + \left(\frac{1-\nu}{2}\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)v + \frac{\partial w}{\partial\eta} = -q_y R^2/C \qquad (2)$$

$$\nu\frac{\partial}{\partial\xi} + \frac{\partial v}{\partial\eta} + (1 + k^2\,\nabla^2\nabla^2)w = -q_n R^2/C \qquad (3)$$

where ν is Poisson's ration, k^2 is the thickness parameter, and C is the extensional rigidity

$$k^2 = \frac{1}{12}\left(\frac{h}{R}\right)^2, \qquad C = Eh/(1-\nu^2) \qquad (4)$$

where R is the radius of curvature, h is the thickness of the shell and where the body forces q_x, q_y, q_n also include inertia terms. The following Fourier tranforms of the displacement components u, v, w are defined:

$$u_{mn} = \int_0^a \int_0^b u(\xi,\eta)\, \phi_{mn}^{(1)}\, d\xi d\eta \tag{5}$$

$$v_{mn} = \int_0^a \int_0^b v(\xi,\eta)\, \phi_{mn}^{(2)}\, d\xi d\eta \tag{6}$$

$$w_{mn} = \int_0^a \int_0^b w(\xi,\eta)\, \phi_{mn}^{(3)}\, d\xi d\eta \tag{7}$$

where

$$\phi_{mn}^{(1)} \equiv \cos\alpha_m\xi \quad \sin\beta_n\eta \tag{8}$$

$$\phi_{mn}^{(2)} \equiv \sin\alpha_m\xi \quad \cos\beta_n\eta \tag{9}$$

$$\phi_{mn}^{(3)} \equiv \sin\alpha_m\xi \quad \sin\beta_n\eta \tag{10}$$

$$\alpha_m = m\pi/\bar{a}, \qquad \beta_n = n\pi/\bar{b}, \qquad \bar{a} = a/R, \qquad \bar{b} = b/R$$

The system of differential equations is subjected to finite Fourier transformations with kernels (8), (9) and (10) respectively.

The double integrals thus obtained depend on the derivatives of u, v, w. These integrals must be, as usual, integrated by parts in order to relate them to the transforms (5)÷(7). In view of the discontinuities present this standard procedure is extremely tedious. The method used in Part II of this report based on application of Green-Gauss theorem

$$\iint_S F_{,i}\, dS = \int_\Gamma F n_i\, d\Gamma + \int_{\bar{\Gamma}} F \bar{n}_i\, d\bar{\Gamma} \quad (i=1,2,3)$$

is very helpful in considerably reducing the amount of manipulations. Ultimately the following system of linear algebraic equations in u_{mn}, v_{mn}, w_{mn} is obtained, which includes integrals involving unknown discontinuities of the displacements and of the slope across the crack:

$$(\alpha_m + \frac{1+\nu}{2}\beta_n{}^2 - \frac{R^2}{C}\rho\omega^2)u_{mn} + \frac{1+\nu}{2}\alpha_m\beta_n\, v_{mn} - \nu\alpha_m\, w_{mn} = A_{mn}$$

$$\frac{1+\nu}{2}\alpha_m\beta_n\, u_{mn} + (\frac{1-\nu}{2}\alpha_m{}^2 + \beta_n{}^2 - \frac{R^2}{C}\rho\omega^2)v_{mn} - \beta_n\, w_{mn} = B_{mn}$$

$$-\nu\alpha_m\, u_{mn} - \beta_n\, v_{mn} + [k^2(\alpha_m{}^2 + \beta_n{}^2)^2 + 1 - \frac{R^2}{C}\rho\omega^2]w_{mn} = C_{mn}$$

$$m,n = 1,2,\ldots\infty \qquad (12)$$

The quantities A_{mn}, B_{mn}, C_{mn} appearing on the right hand sides represent certain linear combinations (see Apendix 2 for details) of the integrals of the unknown discontinuities of the displacements Δu, Δv, Δw and of the normal slope Δ $(\partial w/\partial u)$ across the crack. The system of algebraic equations (12) is solved with respect to u_{mn}, v_{mn}, w_{mn} and the results substituted into the inversion formulae:

$$u(\xi,\eta) = \frac{2}{a\bar{b}}\sum_{n=1}^{\infty} u_{on}\sin\beta_n\eta + \frac{4}{a\bar{b}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}\cos\alpha_m\xi\ \sin\beta_n y \qquad (13.1)$$

$$v(\xi,\eta) = \frac{2}{a\bar{b}}\sum_{m=1}^{\infty} v_{mo}\sin\alpha_m\xi + \frac{4}{a\bar{b}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v_{mn}\sin\alpha_m\xi\ \cos\beta_n y \qquad (13.2)$$

$$w(\xi,\eta) = \frac{4}{a\bar{b}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn}\sin\alpha_m\xi\ \sin\beta_n y \qquad (13.3)$$

In order to determine the integrals of the unknown functions the boundary conditions at the crack must be applied next.

2. Boundary Conditions and the Characteristic Equation

Four boundary conditions are available at the crack's edge to determine four unknown discontinuities. These are:

$$M_{nn} = 0, \qquad V_n = 0, \qquad \sigma_{nn} = 0, \qquad \sigma_{cn} = 0 \tag{14}$$

where M_{nn} is the bending moment in the direction normal to the crack, V_n is the reduced shear force at the edge, σ_{nn} and σ_{cn} are stress components normal and tangential to the crack respectively. Using standard expressions for cylindrical shells (see [8]) relating quantities (14) to the displacements u, v, w and the formulae for directional derivatives

$$\frac{\partial}{\partial n} = \frac{\partial \xi}{\partial n}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial n}\frac{\partial}{\partial \eta} \tag{15}$$

one obtains from (14) the following conditions at the edge of the crack:

$$(\sin^2\alpha + \nu\cos^2\alpha)\frac{\partial^2 w}{\partial \xi^2} + (\cos^2\alpha + \nu\sin^2\alpha)\frac{\partial^2 w}{\partial \eta^2} + 2(1-\nu)\sin\alpha\cos\alpha\frac{\partial^2 w}{\partial \xi \partial \eta} = 0 \tag{16.1}$$

$$\sin\alpha\,[1 + (1-\nu)\cos^2\alpha]\frac{\partial^3}{\partial \xi^3} - \cos\alpha\,[1 + (1-\nu)\sin^2\alpha]\frac{\partial^3 w}{\partial \eta^3}$$

$$+ \cos\alpha\,[(1-2\nu)\sin^2\alpha - (2-\nu)\cos^2\alpha]\frac{\partial^3 w}{\partial \xi^2 \partial \eta} - \sin\,[1-2\nu)\cos^2\alpha$$

$$- (2-\nu)\sin^2\alpha]\frac{\partial^3 w}{\partial \xi \partial \eta^2} = 0 \tag{16.2}$$

$$\left(\frac{\partial u}{\partial \xi} + \nu\frac{\partial v}{\partial \eta} + \nu w\right)\sin^2\alpha + \left(\nu\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} + w\right)\cos^2\alpha - \frac{2Gh}{C}\left(\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta}\right)\sin\alpha\cos\alpha = 0 \tag{16.3}$$

$$-\left(\frac{\partial u}{\partial \xi} + \nu\frac{\partial v}{\partial \eta} + \nu w\right)\sin\alpha\cos\alpha + \left(\nu\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} + w\right)\sin\alpha\cos\alpha$$

$$+ \frac{Gh}{C}\left(\frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \xi}\right)(\cos^2\alpha - \sin^2\alpha) = 0 \tag{16.4}$$

satisfy conditions (16). In order to derive proper differentiation formulae Green-Gauss theorem (11) is applied again in the way shown in Part I. Consider for instance the series (13.1) and the transform (5). In order to find $\partial u/\partial\xi$ Green-Gauss theorem is applied to the expression $u\,\phi^{3}_{mn}$ yielding

$$\iint\limits_S \frac{\partial u}{\partial \xi}\phi^{(3)}_{mn}\,dS = -\iint\limits_S u\frac{\partial \phi^{(3)}_{mn}}{\partial \xi}\,dS + \int\limits_{\bar\Gamma} u\,\phi^{(3)}_{mn}\,\bar u_\xi\,d\bar\Gamma \tag{17}$$

where $\bar\Gamma$ is the "contour" of the crack

Since $\phi^{(3)}_{mn,\xi} = \alpha_m\,\phi^{(1)}_{mn}$ therefore the above becomes

$$\iint\limits_S \frac{\partial u}{\partial \xi}\,\phi^{(3)}_{mn}\,dS = -\alpha_m\,u_{mn} + \int\limits_{\bar\Gamma} u\,\phi^{(3)}_{mn}\,\bar u_\xi\,d\bar\Gamma \tag{18}$$

The derivative of (13.1) must be of the following form:

$$\frac{\partial u}{\partial \xi} = \frac{4}{\bar a\bar b}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} g_{mn}\,\phi^{(3)}_{mn} \tag{19}$$

where g_{mn} are Fourier coefficients to be found. Substitution of (19) into (18) yields the desired formula

$$g_{mn} = -\alpha_m u_{mn} + \int_{\bar{\Gamma}} u \, \phi_{mn}^{(3)} \, \bar{u}_\xi \, d\bar{\Gamma} \tag{20}$$

Similar approach was used to derive expressions for other derivatives appearing in eqs. (16).

In order to derive conditional equations determining these quantities, boundary conditions (16) are applied by performing differentiation on eqs. 13 according to the rules explained in the previous section. After lengthuy manipulations this leads to an infinite system of linear algebraic equations of the following form

$$\sum_{i=1}^{\infty} (F_{1i\ell} U_i + F_{2i\ell} V_i + F_{3i\ell} W_i) + \sum_{i=2}^{\infty} F_{4i\ell} Z_i = 0 \qquad \ell=0,1,...$$

$$\sum_{i=1}^{\infty} (F_{5i\ell} V_i + F_{6i\ell} V_i + F_{7i\ell} W_i) + \sum_{i=2}^{\infty} F_{8i\ell} Z_i = 0 \qquad \ell=1,...$$

$$\sum_{i=1}^{\infty} (F_{9i\ell} V_i + F_{10i\ell} V_i + F_{11i\ell} W_i) + \sum_{i=2}^{\infty} F_{12i\ell} Z_i = 0 \qquad \ell=1,...$$

$$\sum_{i=1}^{\infty} (F_{13i\ell} V_i + F_{14i\ell} V_i + F_{15i\ell} W_i) + \sum_{i=2}^{\infty} F_{16i\ell} Z_i = 0 \qquad \ell=1,...$$

(21)

Coefficients $F_{1i\ell}, F_{2i\ell}, \ldots F_{16i\ell}$ are composed of double and single series with complicated general terms, some involving

Fresnel integrals (due to the presence of the square roots in the denominators of eqs. 31). Expressions leading to evaluation of these coefficiencies are given in Appendix 3 for the case when the tangential inertia terms are neglected. In order to obtain the frequencies of natural vibration of the cracked shell one should equate to zero the characteristic determinant of the system (26).

The quantities Δw, Δu, and Δv appearing under the integrals are not singular and can be therefore expanded into Fourier series. However, the fact that the bending moment and the stress are square-root singular at the tips of the crack makes these Fourier series very slowly convergent. To eliminate this difficulty the procedure adopted in Part I will be generalized here. To this end the singular quantities are chosen to be the primary unknows with "built-in" singularities:

$$\Delta \frac{\partial^2 w}{\partial n^2} = \frac{2}{d} \sum_{i=1}^{\infty} W_i \sin \frac{i\pi c}{d} + \frac{C_1}{\sqrt{c}} + \frac{C_2}{\sqrt{d-c}} \tag{22.1}$$

$$\Delta \frac{\partial u}{\partial n} = \frac{2}{d} \sum_{i=1}^{\infty} U_i \sin \frac{i\pi c}{d} + \frac{C_3}{\sqrt{c}} + \frac{C_4}{\sqrt{d-c}} \tag{22.2}$$

$$\Delta \frac{\partial v}{\partial n} = \frac{2}{d} \sum_{i=1}^{\infty} V_i \sin \frac{i\pi c}{d} + \frac{C_5}{\sqrt{c}} + \frac{C_6}{\sqrt{d-c}} \tag{22.3}$$

where the unknown constants $C_1,\ldots,C_6$ are determined using continuity conditions of the displacements at the tips of the crack and by applying integration by parts. Variable "c" is measured along the crack starting at one tip. Extensive but straightforward manipulations yield finally the following expressions:

$$C_1 = -\frac{3}{\pi\sqrt{d}} \sum_{i=1}^{\infty} \frac{W_i}{i} + \frac{2}{\pi\sqrt{d}} \sum_{i=1,3}^{\infty} \frac{W_i}{i}$$

$$C_2 = \frac{3}{\pi\sqrt{d}} \sum_{i=1}^{\infty} \frac{W_i}{i} - \frac{4}{\pi\sqrt{d}} \sum_{i=1,3}^{\infty} \frac{W_i}{i}$$

$$C_3 = \frac{1}{\pi\sqrt{d}} \sum_{i=1}^{\infty} \frac{(-1)^i U_i}{i} - \frac{2\gamma}{R\pi\nu} d^{3/2} \left[\sum_{i=1}^{\infty} \frac{W_i}{i} \left(\frac{2}{5} - \frac{(-1)^i}{2i^2\pi^2}\right) - \frac{3}{5} \sum_{i=1,3}^{\infty} \frac{W_i}{i} \right]$$

$$C_4 = -\frac{1}{\pi\sqrt{d}} \sum_{i=1}^{\infty} \frac{U_i}{i} + \frac{2\gamma}{R\pi\nu} d^{3/2} \left[\sum_{i=1}^{\infty} \frac{W_i}{i} \left(\frac{2}{5} - \frac{1}{2i^2\pi^2}\right) - \frac{8}{15} \sum_{i=1,3}^{\infty} \frac{W_i}{i} \right]$$

$$C_5 = \frac{1}{\pi\sqrt{d}} \sum_{i=1}^{\infty} \frac{(-1)^i V_i}{i} - \frac{2\lambda}{R\pi\nu} d^{3/2} \left[\sum_{i=1}^{\infty} \frac{W_i}{i} \left(\frac{2}{5} - \frac{(-1)^i}{2i^2\pi^2}\right) - \frac{3}{5} \sum_{i=1,3}^{\infty} \frac{W_i}{i} \right]$$

$$C_6 = -\frac{1}{\pi\sqrt{d}} \sum_{i=1}^{\infty} \frac{V_i}{i} + \frac{2\lambda}{R\pi\nu} d^{3/2} \left[\sum_{i=1}^{\infty} \frac{W_i}{i} \left(\frac{2}{5} - \frac{1}{2i^2\pi^2}\right) - \frac{8}{15} \sum_{i=1,3}^{\infty} \frac{W_i}{i} \right] \tag{23}$$

The only unknown function which is not singular is the slope discontinuity across the crack. It is represented by the following series:

$$\Delta \frac{\partial w}{\partial n} = \frac{1}{d} Z_o + \frac{2}{d} \sum_{i=1}^{\infty} Z_i \cos \frac{i\pi c}{d} \tag{24}$$

The condition that $\Delta\,(\partial w/\partial n)$ be zero at both tips leads to the following relationships:

$$Z_0 = -2 \sum_{i=2,4}^{\infty} Z_i \qquad (25.1)$$

$$Z_1 = - \sum_{i=3,5}^{\infty} Z_i \qquad (25.2)$$

The unknown functions Δw, $\Delta(\partial u/\partial n)$, $\Delta(\partial v/\partial n)$ appearing under the integral signs in the quantities A_{mn}, B_{mn}, C_{mn} (Eqs. 12) are replaced now by their representations (eq. 27.1÷27.3, 23). The discontinuities of the higher order derivatives of the displacements are eliminated by using the conditions (14) referred to the local coordinate system Ocn along the crack (c,n being directions respectively tangential and normal to the crack). Thus to eliminate for instance $\Delta\,\partial^3 w/\partial^3 n)$ we would use the condition.

$$\Delta \frac{\partial^3 w}{\partial n^3} = -\,(2-\nu)\,\Delta \frac{\partial^3 w}{\partial c^2 \partial n} \qquad (26)$$

and then eliminate the dependence on "c" by integration by parts.

Presently all the quantities appearing on the right hand sides of eqs. 12 (and, after inversion, in eqs. 13) depend solely on four infinite sequences of Fourier coefficients: U_i, V_i, W_i, and Z_i.

The frequencies have been calculated for the crack positioned at the apex, parallel to the straight edge of a shell with square planform for various d/a ratios and for three R/a ratios: R/a = 25, R/a = 2.5, and R/a = 1.0. (See: Figs. 6, 7, 8).

The first case is equivalent to an almost flat plate and was considered for the purpose of compassion. In the second and the third cases the effect of the curvature is more evident. Since no other data are available for these last c ases therefore the frequencies of the uncracked shell with identical geometry were evaluated first. For this end the characteristic equation for a shell with tangential inertia effects neglected has been solved

$$d_{mn} = \alpha_m^8 + 4\alpha_m^6\beta_n^2 + (6\beta_n^4 + \frac{1-\nu^2}{k^2} - \frac{p}{R^2})\alpha_m^4$$

$$+ (4\beta_n^6 - 2p\,\frac{\beta_n^2}{k^2})\alpha_m^2 + \beta_n^8 - \frac{p}{k^2}\,\beta_n^4 = 0 \qquad (27)$$

as well as the characteristic equation corresponding to the case when all inertial effects are included. It was found that the decrease of the curvature increases the frequencies and that neglecting tangential inertia slightly overestimates the magnitudes of frequencies as shown in Table3.

References

1. Solecki, R., "Bending Vibration of a Simply Supported Rectangular Plate with a Crack Parallel to One Edge". Engineering Fract. Mechanics, 18,6, 1983, pp. 1111-1118.

2. Solecki, R., "Bending Vibration of a Simply Supported Rectangular Plate with an Arbitrarily Located Crack", submitted for publication.

3. Solecki, R., "Vibration of a Cracked Cylindrical Shell of Rectangular Planform", submitted for publication.

4. L.M. Keer and C. Sve, On the Bending of Cracked Plates, Int. J. Sol. Structures, 6, 1545-1559 (1970).

5. B. Stahl and L. M. Keer, Vibration and Stability of Cracked Rectangular Plates, Int. J. Sol. Structures, 8, 69-92 (1972).

6. R. Solecki, Isotropic Finite Plates with Cracks, Proc. 12th Annual Meeting of the Society of Engineering Science, The University of Texas at Austin. Oct. 20-22, 1975, pp. 373-380.

7. R. Ali and S. J. Atwani, Prediction of Natural Frequencies of Vibration of Rectangular Plates with Rectangular Cutouts, Comp. and Structures, 12, 819-826 (1980).

8. Dym, C. L., "Introduction to the Theory of Shells", Pergamon Press, 1974.

9. Kraus, H., "Thin Elastic Shells", John Wiley, 1967.

10. Folias, E. S., "An Axial Crack in Pressurized Cylindrical Shell", Int. J. Fract. Mech., Vol. 1, 1965, pp. 104-113.

11. Copley, L. G., Sanders, J. L., Jr., "A Longitudinal Crack in a Cylindrical Shell Under Internal Pressure", Int. J. Fract. Mech., Vol. 5, 1969, pp. 117-131.

12. Erdogan, F., Kibler, J. J., "Cylindrical and Spherical Shells with Cracks", Int. J. Sol. Str., Vol. 12, 1967, pp. 229-237.

13. Keer, L. M., Watts, H. A., "Mixed Boundary Value Problems for a Cylindrical Shell", Int. J. Fract., Vol. 14, 1978, pp. 723-729.

14. Sanders. J. L., "Circumferential Through-Cracks in Cylindrical Shells Under Tension", ASME Journal of Applied Mechanics, Vol. 49, 1982, pp. 103-107.

15. Krenk, S., "Influence of Transverse Shear on an Axial Crack in a Cylindrical Shell", Int. J. Fract., Vol. 14, 1978, pp. 123-144.

16. Delale, F., Erdogan, F., "Transverse Shear Effect in a Circumferentially Cracked Cylindrical Shell", Quart. Appl. Math., Vol. 37, 1979, pp. 239-248.

17. Delale, F., "Cracked Shells Under Skew Symmetric Loading", Int. J. of Engin. Science, Vol. 20, 1982, pp. 325-348.

18. Sanders, J. L., Jr., "Cutouts in Shallow Shells", ASME Journal of Applied Mechanics, Vol. 37, 1970, pp. 374-383.

19. Simmonds, J. G., Bradley, M. R., "Stress-Intensity Factors for very Short cracks in Arbitrary Pressurized Shells", ASME Journal of Applied Mechanics, Vol. 43, 1976, pp. 657-662.

20. Latta, G.E., Simmonds, J. G., Bradley, M. R., "Analytical and Numberical Calculation of Stress-Intensity Factors in a Pressurized Cylindrical Shell with a Clamped Crack", ASME Journal of Applied Mechanics, Vol. 44, 1977, pp. 264-270.

21. Simmonds, J. G., Bradley, M. R., Nicholson, J. W., "Stress-Intensity Factors for Arbitrarily Oriented Cracks in Shallow Shells", ASME Journal of Applied Mechanics, Vol. 45, 1978, pp. 135-141.

22. Lakshminararyana, H. V., Murthy, M. V. V., "On Stresses Around an Arbitrary Oriented Crack in a Cylindrical Shell", Int. J. Fract., Vol. 12, 1976, pp. 547-566.

23. Lakshminarayana, H. V. Murthy, M. V. V., Srinath, L. S., "On an Analytical-Numerical Procedure for the Analysis of Cylindrical Shells with Arbitrarily Oriented Cracks", Int. J. of Fracture, Vol. 19, 1982, pp. 257-276.

24. Solecki, R., "Bending Vibration of Simply Supported Rectantular Plates with Internal Rigid Support", Int. J. Eng. Sci., Vol. 18, 1980, pp. 1309-1318.

25. H. J. Fletcher and C. J. Thorne, Bending of Thin Rectangular Plates. Proc. of the 2nd U.S. National Congress of Applied Mechanics, 389-406(1954).

26. L. W. Kantorowitsch and W. I.Krylow, Näherungs-methoden der Höherenanalysis. V.E.B. (1956).

Appendix 1

The following is the list of symbols appearing in equations (21) ÷ (24):

$$X_{2mn} = -2n(a_{5c}n^4 - a_{5c}\Omega^2 + a_{7c}\,\phi^2m^2n^2 + a_{6c}\,\phi^4m^4)$$

$$X_{3mn} = n(a_{1c}n^4 - a_{3c}\,\phi^2m^2n^2 - a_{2c}\,\phi^4m^4 + a_{4c}\Omega^2$$

$$X_{5mn} = a_{8s}\,\phi^4m^4 + a_9\,\phi^2m^2n^2 + a_{8c}\,n^4 - a_{10}\Omega^2$$

$$X_{6mn} = \phi mn\,(a_{10c}\phi^2m^2 + a_{10s}\,n^2)\ ,\quad X_{7mn} = \phi mn(a_{10s}\phi^2m^2 + a_{10c}n^2)$$

$$X_{8mn} = 2a_{11}(\phi^4m^4 + 2\nu\phi^2m^2n^2 + n^4 - \Omega^2$$

X_{1mn}, X_{4mn}, X_{7mn} are obtained from X_{3mn}, X_{2mn}, X_{6mn} respectively by replacing n by ϕm and a_{ic} by a_{is} and vice-versa

$$X_{9mn} = a_{15s}\,\phi^6m^6 + a_{16s}\,\phi^4m^4n^2 + a_{17s}\,\phi^2m^2n^4 - a_{19s}\,n^2(n^4-\Omega^2)$$
$$+ a_{18s}\,\phi^2m^2\Omega^2$$

$$X_{10mn} = \phi mn(a_{20s}\,\phi^4m^4 + a_{21s}\,\phi^2m^2n^2 + a_{22s}\,n^4 + a_{23s}\Omega^2)$$

$$X_{13mn} = \phi m(a_{24s}\,\phi^4m^4 + a_{25s}\,\phi^2m^2n^2 + a_{26s}n^4 + a_{27s}\Omega^2)$$

$$X_{16mn} = \phi m[a_{28c}n^4 + a_{29c}\,\phi^2m^2n^2 + 2a_{14c}\,(\phi^4m^4-\Omega^2)]$$

X_{11mn}, X_{12mn}, X_{15mn}, X_{14mn} are obtained from X_{9mn}, X_{10mn}, X_{13mn}, and X_{16mn} respectively by replacing n by ϕm and a_{ic} by a_{is} and vice-versa.

The symbols a_{ic} and a_{is} are listed below where for abbreviation

$$s \equiv \sin\psi, \qquad c \equiv \cos\psi$$

$$a_{1s} \equiv -(1-\nu^2)sc, \qquad a_{2s} = (1-\nu)[(1-3\nu)c^4 - 2(1-2\nu)c^2 + 1-\nu]s$$

$$a_{3s} = 2(1-\nu)(2-\nu)sc^2(1-2c^2), \qquad a_{4s} = [-1 + (2-\nu)c^4]s$$

$$a_{5s} = (1-\nu)sc^4, \quad a_{6s} = -\nu(1-\nu)s^3c^2$$

$$a_{7s} = (1-\nu)[-3 + 2\nu + (5-3\nu)sc^2, \quad a_{8s} = (1-\nu^2)c^4$$

$$a_{9s} = 2(1-\nu)[1-(3-\nu)c^2 + (3-\nu)c^4, \quad a_{10} = 1 -2(1-\nu)c^2 + 2(1+\nu)c^4$$

$$a_{10s} = -2(1-\nu)[1-(1-\nu)s^2]sc, \quad a_{11} = (1-\nu)s^2c^2$$

$$a_{12} = (1-\nu)sc^4, \quad a_{13s} = -s^4\,[s^4 - \nu s^2c^2 + (1+\nu)c^4]$$

$$a_{14s} = -s^4c(s^2-3c^2), \quad a_{15s} = (1+\nu)s^2c^4\,(1=4c^2)$$

$$a_{16s} = 2[-7 + 2\nu - 2(1+\nu)c^2]s^2c^4$$

$$a_{17s} = [-1 -\nu-2(1-5\nu)c^2 - (7+19\nu)c^4 + 4(1+\nu)c^6]s^2$$

$$a_{18s} = [\frac{1}{1-\nu} + 2c^2 - 2\nu c^4 + 4(1+\nu)c^6]s^2$$

$$a_{19s} = 2[1-(2+\nu)c^2 + 2(1+\nu)c^4]s^4$$

$$a_{20s} = [2(1+\nu)c^2 - 3(5+3\nu)c^4 + 8(1+\nu)c^6]sc$$

$$a_{21s} = 2[-1 -\nu -(1-9\nu)c^2 - (7+17\nu)c^4 + 8(1+\nu)c^6]sc$$

$$a_{22s} = [-5 + \nu + 12c^2 - 3(5+3\nu)c^4 + 8(1+\nu)c^6]sc$$

$$a_{23s} = [\frac{1-2\nu}{1-\nu} - 4(1+\nu)c^2 + 12(1+\nu)c^4 - 8(1+\nu)c^6]sc$$

$$a_{24s} = (-7+\nu+8c^2)sc^4, \quad a_{25s} = 2[-2 + \nu - 2(2+\nu)c^2 + 8c^4]sc^2$$

$$a_{26s} = [-1 + \nu + 2(1-2\nu)c^2 + 3(-3+\nu)c^4 + 8]s$$

$$a_{27s} = [\frac{1}{1-\nu} + 6c^4 - 8c^6]s, \quad a_{28s} = 2(6-\nu-4c^2)s^2c^3$$

$$a_{29s} = 2[-\nu + (7+3\nu)c^2 - 8c^4]s^2c$$

The quantities a_{Nc} are obtained from corresponding quantities a_{N_s} by replacing s by -c and c by -s.

Also:

$$T_{1mnk} = 2\int_0^d (\cos\alpha_m x \sin\beta_n y)_{\bar{\Gamma}} \sin\frac{k\pi c}{d}\, dc$$

$$T_{2mnk} = 2\int_0^d (\sin\alpha_m x \cos\beta_n y)_{\bar{\Gamma}} \sin\frac{k\pi c}{d}\, dc$$

$$T_{3mnk} = 2\int_0^d (\sin\alpha_m x \sin\beta_n y)_{\bar{\Gamma}} \cos\frac{k\pi c}{d}\, dc$$

$$T_{4mnk} = 2\int_0^d (\cos\alpha_m x \cos\beta_n y)_{\bar{\Gamma}} \cos\frac{k\pi c}{d}\, dc$$

$$t_{1mnk} = \int_0^d \sin\frac{v_{3mn}c}{d} \sin\frac{k\pi c}{d}\, dc, \quad t_{2mnk} = \int_0^d \sin\frac{v_{4mn}c}{d} \sin\frac{k\pi c}{d}\, dc$$

$$t_{3mnk} = \int_0^d \cos\frac{v_{3mn}c}{d} \sin\frac{k\pi c}{d} dc, \quad t_{4mnk} = \int_0^d \cos\frac{v_{4mn}c}{d} \sin\frac{k\pi c}{d}\, dc$$

All these integrals have been represented by closed form expressions. Also:

$$Z_{1mnk}, Z_{2mnk} = \frac{1}{v_{3mn}^2}(t_{3mnk} \sin v_{1mn} + t_{1mnk} v \cos v_{1mn})$$

$$\pm \frac{1}{v_{4mn}^2}(t_{4mnk} \sin v_{2mn} + t_{2mnk} \cos v_{2mn})$$

Where the upper sign applies to Z_{1mnk} and where

$$v_{1mn} = \alpha_m x_1 + \beta_n y_1, \qquad v_{2mn} = \alpha_m x_1 - \beta_n y_1$$

$$v_{3mn} = (\alpha_m \cos\psi + \beta_n \sin\psi)d, \; v_{4mn} = (\alpha_m \cos\psi - \beta_n \sin\psi)d$$

Finally

$$E_{1mn}, E_{3mn} = -(1/v_{3mn}^{5/2}) \{[\sin v_{1mn} - \sin(v_{1mn} + v_{3mn})]$$

$$C(\sqrt{v_{3mn}}) + [\cos v_{1mn} + \cos(v_{1mn} + v_{3mn})] S(\sqrt{v_{3mn}}$$

$$\pm (1/v_{4mn}^{5/2}) \{[\sin v_{2mn} - \sin(v_{2mn} + v_{4mn})] C(\sqrt{v_{4mn}})$$

$$+ [\cos v_{2mn} + \cos(v_{2mn} + v_{4mn})] S(\sqrt{v_{4mn}})\}$$

$$E_{2mn}, E_{4mn} = (1/v_{3mn}^{5/2}) [\sin v_{1mn} - 2\sin(v_{1mn} + v_{3mn})] C(\sqrt{v_{3mn}})$$

$$+ [\cos v_{1mn} + 2\cos(v_{1mn} + v_{3mn})] S(\sqrt{v_{3mn}})$$

$$\pm (1/v_{4mn}^{5/2}) [\sin v_{2mn} - 2\sin(v_{2mn} + v_{4mn})] C(\sqrt{v_{4mn}})$$

$$+ [\cos v_{2mn} + 2\cos(v_{2mn} + v_{4mn})] S(\sqrt{v_{4mn}})\}$$

where the upper signs refer to E_{1mn} and E_{2mn} and where

$$S(x) = \frac{1}{\sqrt{2\pi}} \int_0^{x^2} \frac{\sin t}{\sqrt{t}} dt \qquad C(x) = \frac{1}{\sqrt{2\pi}} \int_0^{x^2} \frac{\cos t}{\sqrt{t}} dt$$

are Fresnel's integrals.

Appendix 2

The right hand sides of eqs. (12), Part II are:

$$A_{mn} = R\,\frac{1-\nu}{2}\left[s\int_{P_1}^{P_2}\Delta u\,\phi^1_{(mn),\bar{c}}\,d\bar{c} - c\int_{P_1}^{P_2}\Delta u\,\phi^1_{(mn),n}\,d\bar{c}\right]$$

$$- R\nu c\int_{P_1}^{P_2}\Delta v\,\phi^1_{(mn),\bar{c}}\,d\bar{c} + Rs\int_{P_1}^{P_2}\Delta v\,\phi^1_{(mn),n}\,d\bar{c}$$

$$B_{mn} = -R\,\frac{1-\nu}{2}\left[c\int_{P_1}^{P_2}\Delta u\,\phi^2_{(mn)\bar{c}}\,d\bar{c} - s\int_{P_1}^{P_2}\Delta u\,\phi^2_{(mn),n}\,d\bar{c}\right.$$

$$-R\nu s\int_{P_1}^{P_2}\Delta v\,\phi^2_{(mn),\bar{c}}\,dc - Rc\int_{P_1}^{P_2}\Delta\bar{v}\,\phi^2_{(mn),n}\,d\bar{c}$$

$$C_{mn} = -k^2(\alpha_m^2+\beta_n^2)\int_{P_1}^{P_2}\Delta w\,\phi^3_{(mn),u}\,d\bar{c} + k^2R^2(1-\nu)\int_{P_1}^{P_2}\Delta w\,\phi^3_{(mn),u\bar{c}\bar{c}}\,d\bar{c}$$

$$+ Rk^2(\alpha_m^2+\beta_n^2)\int_{P_1}^{P_2}\Delta\frac{\partial w}{\partial n}\,\phi^3_{(mn)}\,d\bar{c} + k^2R^3(1-\nu)\int_{P_1}^{P_2}\Delta\frac{\partial w}{\partial n}\,\phi^3_{(mn),\bar{c}\bar{c}}\,d\bar{c}$$

$$+ (\nu s^2+c^2)\int_{P_1}^{P_2}\Delta v\,\phi^3_{(mn)}\,d\bar{c}$$

where P_1 and P_2 denote the tips of the crack.

<u>Appendix 3</u>: Symbols used in the series eq. (26)

In order to obtain the coefficients $F_{1i\ell},\ldots, F_{16i\ell}$ the conditional equations (26) are represented in the following form:

$$
\begin{aligned}
4 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{ & [A_1A_2A_3 + d_{mn}A_4]\overset{1}{W}_{mn} + [A_1A_2\bar{A}_3 + d_{mn}A_4]\overset{2}{W}_{mn} \\
& - R[A_1^2A_2 + d_{mn}(F_1s^2 + \bar{F}_1c^2)](W_{,n})^{3}_{mn} \\
& - 2R\, A_1A_2\, (1-\nu)\alpha_m\beta_n\, (W_{,n})^{4}_{mn}\, sc + A_1\, [A_7\overset{3}{\bar{U}}_{mn} + A_5\overset{3}{\bar{V}}_{mn} \\
& + A_8\overset{4}{\bar{U}}_{mn} + A_6\overset{4}{\bar{V}}_{mn}]\} \; \frac{(SSC)_{mn\ell}}{d_{mn}} \\
& + 8(1-\nu)sc \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{\beta_n[A_2A_3\alpha_m - d_{mn}s^3]\}\overset{1}{W}_{mn} + \alpha_m\, [A_2\bar{A}_3\beta_n \\
& + d_{mn}c^3]\overset{2}{W}_{mn} - R\, A_1A_2\alpha_m\beta_n\, (W_{,n})^{3}_{mn} - R[2(1-\nu)A_2\alpha_m^2\beta_n^2 \\
& + d_{mn}]\, (W_{,n})^{4}_{mn} sc + \alpha_m\beta_n\, [A_7\overset{3}{\bar{U}}_{mn} + A_5\overset{3}{\bar{V}}_{mn} \\
& + A_8\overset{4}{\bar{U}}_{mn} + A_6\overset{4}{\bar{V}}_{mn}\} \; \frac{(CCC)_{mn\ell}}{d_{mn}} \\
& + 2(1-\nu)sc\, \{2 \sum_{m=1}^{\infty} [\alpha_m\overset{1}{W}_m c^3 - R(W_{,n})^{2}_{m} sc]\, (CC)_{m\ell} \\
& + 2 \sum_{n=1}^{\infty} [-\beta_n\overset{1}{W}_n s^3 - R\, (W_{,n})^{2}_{n}\, sc]\, (CC)_{n\ell} - R\, (W_{,n})^{0}\, d\, \delta_{0\ell} sc\} \\
& = 0 \qquad\qquad \ell = 0,1,\ldots\infty
\end{aligned}
\qquad (A2\text{-}1)
$$

$$2 \sum_{n=1}^{\infty} \{[\beta_n^2 s^3(\bar{\gamma}c^2+\bar{\delta}s^2)W_n^1 - 2R[\beta_n(\bar{\gamma}-\bar{\delta})s^3c](W_{,n})_n^2 + R^2F_3 (W_{,nn})_n^1\} (SS)_{n\ell}$$

$$+ 4 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{[B_1A_2A_3 + d_{mn}B_2]W_{mn}^1 + [B_1A_2\bar{A}_3 + d_{mn}B_3]W_{mn}^2$$

$$+ R [-B_1A_1A_2 + d_{mn}B_4](W_{,n})_{mn}^3 + R [-2B_1A_2 (1-\nu)\alpha_m\beta_n sc$$

$$+ d_{mn}B_5] (W_{,n})_{mn}^4 + d_{mn}F_3R^2(W_{,nn})_{mn}^1 + B_1[A_7\bar{U}_{mn}^3 + A_5\bar{V}_{mn}^3$$

$$+ A_8\bar{U}_{mn}^4 + A_6\bar{V}_{mn}^4]\} \frac{(CSS)_{mn\ell}}{d_{mn}}$$

$$+ 4 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{[\bar{B}_1A_2A_3 + d_{mn} \bar{B}_3]W_{mn}^1 + [\bar{B}_1A_2\bar{A}_3 + d_{mn}\bar{B}_2]W_{mn}^2$$

$$+ R [-\bar{B}_1A_1A_2 + d_{mn}\bar{B}_4](W_{,n})_{mn}^3 + R [-2\bar{B}_1A_2(1-\nu)\alpha_m\beta_n sc$$

$$+ d_{mn}B_5](W_{,n})_{mn}^4 + d_{mn}\bar{F}_3R^2(W_{,nn})_{mn}^2 + \bar{B}_1[A_7\bar{U}_{mn}^3 + A_5\bar{V}_{mn}^3$$

$$+ A_8\bar{U}_{mn}^4 + A_6\bar{V}_{mn}^4]\} \frac{(SCS)_{mn\ell}}{d_{mn}}$$

$$+ 2 \sum_{m=1}^{\infty} \{[\alpha_m^2 c^3(\bar{\bar{\gamma}}s^2-\bar{\bar{\delta}}c^2) W_m^1 + 2R [\alpha_m(\bar{\bar{\gamma}}+\bar{\bar{\delta}})sc^3](W_{,n})_m^2$$

$$+ R^2\bar{F}_3(W_{,nn})_m^1 \}(SS)_{m\ell} = 0$$

$\ell=1,2,\ldots\infty$

(A2-2)

$$4 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{-[(1-\nu)(\alpha_m C_1 F_1 + \beta_n \bar{C}_1 \bar{F}_1)sc]\bar{U}^3_{mn} + [\tfrac{1-\nu}{2} \alpha_m \beta_n (C_2 F_1 -$$

$$- \bar{\bar{C}}_2 \bar{F}_1)(c^2-s^2)]\bar{U}^4_{mn} + [-\alpha_m F_1(\alpha_m C_3 F_1 + \beta_n C_4 \bar{F}_1 + C_5 F_1)$$

$$- \beta_n \bar{F}_1 (\beta_n \bar{C}_3 \bar{F}_1 + \alpha_m C_4 F_1)]\bar{V}^3_{mn} - [(1-\nu)\alpha_m \beta_n (C_2 F_1 - \bar{\bar{C}}_2 \bar{F}_1)sc]\bar{V}^4_{mn}$$

$$- [\alpha_m A_3 C_5 F_1 k^2 + \beta_n{}^2 A_3 C_6 \bar{F}_1 k^2]W^1_{mn} - [\beta_n^2 \bar{A}_3 C_6 \bar{F}_1 k^2$$

$$+ \alpha_m \bar{A}_3 C_5 F_1 k^2]\ W^2_{mn} + R\ A_1 k^2\ [\alpha_m C_5 F_1 + \beta_n^2 C_6 \bar{F}_1](W_{,n})^3_{mn}$$

$$+ 2(1-\nu)R\ k^2 \alpha_m \beta_n\ [\alpha_m C_5 F_1 + \beta_n C_6 \bar{F}_1](W_{,n})^4_{mn} sc$$

$$+ A_2 \bar{F}_1\ [A_3 W^1_{mn} + \bar{A}_3 W^2_{mn} - RA_1 (W_{,n})^3_{mn} - 2(1-\nu)R\alpha_m \beta_n\ sc(W_{,n})^4_{mn}]$$

$$+ \bar{F}_1\ [A_7\ \bar{U}^3_{mn} + A_5 \bar{V}^3_{mn} + A_8 \bar{U}^4_{mn} + A_6 \bar{V}^4_{mn}]$$

$$+ (1-\nu)d_{mn}\ (c^2-s^2)\ sc\ \bar{U}_{mn}{}^3 + d_{mn}\ F_2 \bar{V}_{mn}{}^3\} \ \frac{(SSS)_{mn\ell}}{d_{mn}}$$

$$- \frac{8Gh}{C} sc \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{[(1-\nu)(\beta_n C_1 + \alpha_m \bar{C}_1)sc]\bar{U}^3_{mn} + [\tfrac{1-\nu}{2} (\beta_n^2 C_2 -$$

$$- \alpha_m^2 \bar{\bar{C}}_2)(s^2-c^2)]\bar{U}^4_{mn} + [\beta_n(\alpha_m C_3 F_1 + \beta_n C_4 \bar{F}_1 + C_5 \bar{F}_1)$$

$$+ \alpha_m(\beta_n \bar{C}_3 \bar{F}_1 + \alpha_m C_4 F_1)]\bar{V}^3_{mn} + [(1-\nu)(\beta_n^2 C_2 - \alpha_m^2 C_2)sc]\bar{V}^4_{mn}$$

$$+ [\beta_n k^2\ A_3(C_5 + \alpha_m C_6)]W^1_{mn} + [\beta_n k^2 \bar{A}_3 (C_5 + \alpha_m C_6)]W^2_{mn}$$

$$- R\ A_1 k^2 \beta_n (C_5 + \alpha_m C_6)(W_{,n})^3_{mn} - 2(1-\nu)R\ k^2 \alpha_m \beta_n^2\ (C_5 + \alpha_m C_6)$$

$$sc(W_{,n})^4_{mn} + (c^2-s^2)d_{mn}\bar{U}^4_{mn} - 2\ sc\ d_{mn}\ \bar{V}^4_{mn}\} \ \frac{(CCS)_{mn\ell}}{d_{mn}}$$

$$- \frac{2Gh}{C}\ sc\ [(c^2-s^2)\bar{U}^2_0 - 2sc\ \bar{V}^2_o]\ \frac{[1-(-1)^l]d}{\pi\ell}$$

$$+ 2\sum_{m=1}^{\infty} [-c^2\bar{U}^2_m + sc\bar{V}^2_m]\ (CS)_{m\ell}$$

$$+ 2\sum_{m=1}^{\infty} [s^2\bar{U}^2_n + sc\ \bar{V}^2_n]\ (CS)_{n\ell} = 0 \qquad \ell=1,2,\ldots \qquad \text{(A2-3)}$$

$$- 2(1-\nu) \sin 2\alpha \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{- \frac{1-\nu}{2} \sin 2\alpha \, [D_1 + \frac{2}{1-\nu} d_{mn}] \bar{U}^3_{mn}$$

$$+ \frac{1-\nu}{2} \cos 2\alpha \, [D_2] \bar{U}^4_{mn} + [\alpha_m \bar{\bar{C}}_1 F_1 - A_5 + D_3 \bar{\bar{F}}_1 - d_{mn} \cos 2\alpha] \bar{V}^3_{mn}$$

$$- \frac{1-\nu}{2} \sin 2\alpha \, [\bar{D}_2] \bar{V}^4_{mn} + \frac{1-\nu^2}{2} k^2 \alpha_m^2 (\alpha_m^2 - \beta_n^2) \quad [-A_3 W^1_{mn} - \bar{\bar{A}}_3 W^2_{mn}$$

$$+ RA_1 (W_{,n})^3_{mn}] - 2Rk^2 \alpha_m \beta_n A_7 (W_{,n})^4_{mn} \frac{(SSS)_{mn\ell}}{d_{mn}}$$

$$+ \frac{4Gh}{C} \cos 2\alpha \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{- \frac{1-\nu}{2} \sin 2\alpha \, [D_2] \bar{U}^3_{mn} + \frac{1-\nu}{2} \cos 2\alpha \, [D_4 + \frac{2}{1-\nu} d_{mn}]$$

$$\bar{U}^4_{mn} + \alpha_m \beta_n [C_2 F_1 - \bar{C}_2 \bar{\bar{F}}_1] \bar{V}^3_{mn} - \frac{1-\nu}{2} \sin 2\alpha \, [D_4 + \frac{2}{1-\nu} d_{mn}] \bar{V}^4_{mn}$$

$$+ (1-\nu^2) k^2 \alpha_m^3 \beta_n [A_3 W^1_{mn} + \bar{\bar{A}}_3 W^2_{mn} - R A_1 (W_{,n})^3_{mn}$$

$$- (1-\nu) R \alpha_m \beta_n \sin 2\alpha (W_{,n})^4_{mn}] \frac{(CCS)_{mn}}{d_{mn}}$$

$$+ \frac{Gh}{C} \cos 2\alpha \, [\bar{U}_o{}^2 \cos 2\alpha - \bar{V}_o{}^2 \sin 2\alpha] \frac{[1-(-1)^\ell] d}{\pi \ell}$$

$$+ \frac{2Gh}{C} \cos 2\alpha \sum_{m=1}^{\infty} [-\bar{U}^2_m c^2 + \bar{V}_m{}^2 sc] (CS)_{m\ell}$$

$$+ \frac{2Gh}{C} \cos 2\alpha \sum_{n=1}^{\infty} [\bar{U}^2_n s^2 + \bar{\bar{V}}^2_n sc] (CS)_{n\ell} = 0$$

$$\ell = 1, 2, \ldots \infty \qquad \text{(A2-4)}$$

The symbols appearing in the equations (A2-1 - A2-4) are listed below:

$s = \sin\alpha \qquad c = \cos\alpha$

$F_1 = s^2+\nu c^2$

$\bar{F}_1 = c^2+\nu s^2$

$F_2 = 1 - 2(1-\nu)s^2c^2$

$\bar{\gamma} = [1 + (1-\nu)c^2]s$

$\bar{\bar{\gamma}} = [1 + (1-\nu)s^2]c$

$\bar{\delta} = [(2\nu-1)c^2 + (2-\nu)s^2]s$

$\bar{\bar{\delta}} = [(1-2\nu)s^2 + (\nu-2)c^2]c$

$F_3 = -s(\bar{\gamma}s^2+\bar{\delta}c^2)$

$\bar{F}_3 = c(-\bar{\bar{\gamma}}c^2+\bar{\bar{\delta}}s^2)$

$A_1 = F_1\alpha_m^2 + \bar{F}_1\beta_n^2$

$A_2 = \frac{1-\nu}{2} \; (\alpha_m^2 + \beta_n^2)^2k^2$

$A_3 = -\alpha_m s \; [\alpha_m^2+ \beta_n^2+(1-\nu)(\alpha_m^2c^2+\beta_n^2s^2)-2(1-\nu)\beta_n^2c^2]$

$\bar{A}_3 = \beta_n c[\alpha_m^2+\beta_n^2+(1-\nu)(\alpha_m^2c^2+\beta_n^2s^2)-2(1-\nu)\alpha_m^2s^2]$

$A_4 = \alpha_m s[-F_1(1+c^2)+\bar{F}_1c^2]$

$\bar{A}_4 = -\beta_n c[-\bar{F}_1(1+s^2)+F_1s^2]$

$A_5 = \frac{(1-\nu)(1-\nu^2)}{2} \; \alpha_m^2(\alpha_m^2c^2+\beta_n^2s^2)$

$A_6 = -(1-\nu)(1-\nu^2)\alpha_m^3\beta_n sc$

$$A_7 = \frac{(1-\nu)(1-\nu^2)}{2}\,\alpha_m^2(\alpha_m^2+\beta_n^2)sc$$

$$A_8 = -\,\frac{(1-\nu)(1-\nu^2)}{2}\,\alpha_m^3\beta_n(s^2-c^2)$$

$$B_1 = -\alpha_m^2(\alpha_m^2\bar{\gamma}+\beta_n^2\bar{\delta})$$

$$\bar{B}_1 = \beta_n(\beta_n^2\bar{\bar{\gamma}}-\alpha_m^2\bar{\bar{\delta}})$$

$$B_2 = [\alpha_m^2(1+c^2+c^4)+\beta_n^2s^2c^2]\bar{\gamma}s+(-\alpha_m^2c^4+\beta_n^2s^4)\bar{\delta}s$$

$$\bar{B}_2 = [\beta_n^2(1+s^2+s^4)+\alpha_m^2s^2c^2]\bar{\bar{\gamma}}c+(\beta_n^2s^4-\alpha_m^2c^4)\bar{\bar{\delta}}c$$

$$B_3 = [(1+2c^2)\bar{\gamma}s^2-(1+s^2-2s^4)\bar{\delta}]\alpha_m\beta_nc$$

$$\bar{B}_3 = [(1+2s^2)\bar{\bar{\gamma}}c^2+(1+c^2-2c^4)\bar{\bar{\delta}}]\alpha_m\beta_ns$$

$$B_4 = [(1+2c^2)\bar{\gamma}s^2+(1-2s^2)\bar{\delta}c^2]\alpha_m$$

$$\bar{B}_4 = [-(1+2s^2)\bar{\bar{\gamma}}c^2+(1-2c^2)\bar{\bar{\delta}}s^2]\beta_n$$

$$B_5 = 2(-\bar{\gamma}+\bar{\delta})\beta_ns^3c$$

$$\bar{B}_5 = 2(\bar{\bar{\gamma}}+\bar{\bar{\delta}})\alpha_msc^3$$

$$C_1 = [(\frac{1-\nu}{2}\,\alpha_m^6+\frac{5-\nu}{2}\,\alpha_m^4\beta_n^2+\frac{7+\nu}{2}\,\alpha_m^2\beta_n^4+\frac{3+\nu}{2}\,\beta_n^6)k^2+(1+\nu-p)\frac{1-\nu}{2}\,\alpha_m^2-\frac{3+\nu}{2}\,p\beta_n^2]\alpha_m$$

$$\bar{C}_1 = -[(\frac{1-\nu}{2}\,\beta_n^6+\frac{5-\nu}{2}\,\beta_n^4\alpha_m^2+\frac{7+\nu}{2}\,\beta_n^2\alpha_m^4+\frac{3+\nu}{2}\,\alpha_m^6)k^2-p\frac{1-\nu}{2}\,\beta_n^2-\frac{3+\nu}{2}\,p\alpha_m^2+(1+\nu)\frac{1-\nu}{2}\,\alpha_m^2]\beta_n$$

$$C_2 = \{[\nu\alpha_m^6 + (2\nu-1)\alpha_m^4\beta_n^2 + (\nu-2)\alpha_m^2\beta_n^4 - \beta_n^6]k^2 - p(\nu\alpha_m^2-\beta_n^2]\}$$

$$\bar{C}_2 = -\{[\nu\beta_n^6 + (2\nu-1)\beta_n^4\alpha_m^2 + (\nu-2)\beta_n^2\alpha_m^4 - \alpha_m^6]k^2 - p(\nu\beta_n^2-\alpha_m^2)\}$$

$$\bar{\bar{C}}_2 = \bar{C}_2 + (1-\nu^2)\alpha_m^2$$

$$C_3 = -[\frac{1-\nu}{2}\alpha_m^6 + (2-\nu)\alpha_m^4\beta_n^2 + \frac{5-\nu}{2}\alpha_m^2\beta_n^4 + \beta_n^6]k^2 + \frac{1-\nu}{2}(p-1)\alpha_m^2 + p\beta_n^2$$

$$\bar{C}_3 = -[\frac{1-\nu}{2}\beta_n^6 + (2-\nu)\beta_n^4\alpha_m^2 + \frac{5-\nu}{2}\beta_n^2\alpha_m^4 + \alpha_m^6]k^2 + \frac{1-\nu}{2}(p)\beta_n^2 + p\alpha_m^2$$

$$- \frac{\nu}{2}(1-\nu)\alpha_m^2$$

$$C_4 = \frac{1+\nu}{2}\ [k^2(\alpha_m^2+\beta_n^2)^2 + \frac{1-\nu}{1+\nu} - p]\alpha_m\beta_n$$

$$C_7 = -(1-\nu)k^2\alpha_m(\nu\alpha_m^2-\beta_n^2) = -2C_5k^2$$

$$C_6 = \alpha_m^2 - \nu\frac{1+\nu}{2}\alpha_m^2 + \frac{1-\nu}{2}\beta_n^2$$

$$C_5 = \frac{1-\nu}{2}\alpha_m(\nu\alpha_m^2-\beta_n^2)$$

$$D_1 = [\frac{1-\nu}{2}\alpha_m^8 + 4\alpha_m^6\beta_n^2 + (7+\nu)\alpha_m^4\beta_n^2 + 4\alpha_m^2\beta_n^6 + \frac{1-\nu}{2}\beta_n^8]k^2$$

$$+ [2(1+\nu)-p]\frac{1-\nu}{2}\alpha_m^4 - \frac{1}{2}(1+\nu + 2\nu^2 + 3p + \nu p)\alpha_m^2\beta_n^2 - \frac{1-\nu}{2}p\beta_n^4$$

$$D_2 = [\alpha_m^6 + \alpha_m^4\beta_n^2 - \alpha_m^2\beta_n^4 - \beta_n^6)k^2 - p(\alpha_m^2-\beta_n^2)](1+\nu)\alpha_m\beta_n$$

$$\bar{D}_2 = [\alpha_m^6 + \alpha_m^4\beta_n^2 - \alpha_m^2\beta_n^4 - \beta_n^6)k^2 - p(\alpha_m^2-\beta_n^2) + 2(1-\nu)\alpha_m^2](1+\nu)\alpha_m\beta_n$$

$$\bar{\bar{C}}_1 = [(\frac{1-\nu}{2}\alpha_m^6 + \frac{5-\nu}{2}\alpha_m^4\beta_n^2 + \frac{7+\nu}{2}\alpha_m^2\beta_n^4 + \frac{3+\nu}{2}\beta_n^6)k^2 + (1+p)\frac{1-\nu}{2}\alpha_m^2$$

$$- \frac{3+\nu}{2}p\beta_n^2 + \frac{1-\nu}{2}\beta_n^2]\alpha_m$$

$$D_3 = -[\frac{1-\nu}{2}\beta_n^6 + \frac{5-\nu}{2}\beta_n^4\alpha_m^2 + \frac{7+\nu}{2}\beta_n^2\alpha_m^4 + \frac{3+\nu}{2}\alpha_m^6]k^2\beta_n^2$$

$$+ \frac{1}{2}(3p - \nu + p\nu - \nu^2)\alpha_m^2\beta_n^2 + \frac{1-\nu}{2}p\beta_n^4 - \nu\frac{1-\nu}{2}\alpha_m^4$$

$$D_4 = [\alpha_m^8 + 2(1-\nu)\alpha_m^6\beta_n^2 + 2(1-2\nu)\alpha_m^4\beta_n^4 + 2(1-\nu)\alpha_m^2\beta_n^6 + \beta_n^8]k^2$$

$$+ (1-\nu^2-p)\alpha_m^4 + 2\nu p\alpha_m^2\beta_n^2 - p\beta_n^4$$

$$W_{mn}^1 = \frac{R^2}{\nu}\frac{1}{(\alpha_m^2c^2-\beta_n^2s^2)^2}[(\alpha_m^2c^2 + \beta_n^2s^2)(W_{,nn})_{mn}^1 - 2\alpha_m\beta_n sc\,(W_{,nn})_{mn}^2]$$

$$W_{mn}^2 = \frac{R^2}{\nu}\frac{1}{(\alpha_m^2c^2-\beta_n^2s^2)^2}[(\alpha_m^2c^2 + \beta_n^2s^2)(W_{,nn})_{mn}^2 - 2\alpha_m\beta_n sc(W_{,nn})_{mn}^1]$$

$$(W_{,nn})_{mn}^k = \frac{2}{d}\sum_{i=1}^{\infty} W_i \int_0^d \sin\frac{i\pi\bar{c}}{d}\,\phi_{(mn)}^k\,d\bar{c} + \frac{1}{\pi\sqrt{d}}\sum_{i=1}^{\infty}\frac{W_i}{i}\int_0^d [-\frac{3}{\sqrt{\bar{c}}} + \frac{3}{\sqrt{d-\bar{c}}}]$$

$$\phi_{(mn)}^k\,d\bar{c} + \frac{1}{\pi\sqrt{d}}\sum_{i=1,3}^{\infty}\frac{W_i}{i}\int_0^d [\frac{2}{\sqrt{\bar{c}}} - \frac{4}{\sqrt{d-\bar{c}}}]\,\phi_{(mn)}^k d\bar{c}$$

$$(W_{,n})_{mn}^k = \frac{2}{d}\sum_{i=0}^{\infty}\lambda_i Z_i \int_0^d \cos\frac{i\pi\bar{c}}{d}\,\phi_{(mn)}^k d\bar{c}$$

$$\bar{U}^3_{mn} = \frac{1}{\nu(\beta_n^2 s^2 - \alpha_m^2 c^2)} [-R\beta_n s \int_b^d \Delta \frac{\partial v}{\partial n} \phi^2_{(mn)}\, d\bar{c} + R\alpha_m c \int_o^d \Delta \frac{\partial v}{\partial n} \phi^1_{(mn)} d\bar{c}$$

$$+ \lambda(-\beta_n s\, W^2_{mn} + \alpha_m c\, W^1_{mn})]$$

$$\bar{U}^4_{mn} = \frac{1}{\nu(\beta_n^2 s^2 - {}_m^2 c^2)} [R\beta_n s \int_o^d \Delta \frac{\partial v}{\partial n} \phi^2_{(mn)} d\bar{c} + \lambda(\beta_n s\, W^1_{mn} - \alpha_m c\, W^2_{mn})]$$

$$\bar{V}^3_{mn} = \frac{1}{\beta_n^2 s^2 - \alpha_m^2 c^2} [-R\beta_n s \int_o^d \Delta \frac{\partial u}{\partial n} \phi^2_{(mn)} d\bar{c} + R\alpha_m c \int_o^d \Delta \frac{\partial u}{\partial n} \phi^1_{(mn)} d\bar{c}$$

$$+ \gamma(-\beta_n s\, W^2_{mn} + \alpha_m c\, W^1_{mn})]$$

$$\bar{V}^4_{mn} = \frac{1}{\beta_n^2 s^2 - \alpha_m^2 c^2} [R\beta_n s \int_o^d \Delta \frac{\partial u}{\partial n} \phi^1_{(mn)} d\bar{c} - R\alpha_m c \int_o^d \Delta \frac{\partial u}{\partial n} \phi^2_{(mn)} d\bar{c}$$

$$+ \gamma(\beta_n s\, W^1_{mn} - \alpha_m c\, W^2_{mn})]$$

Also: $\bar{U}^2_m = \bar{U}^4_{mo}$, $\bar{U}^2_n = \bar{U}^4_{on}$, $\bar{V}^4_{on}$, $\bar{V}^2_m = \bar{V}^4_{mo}$, $\bar{V}^2_n = \bar{V}^4_{on}$

$\bar{U}^2_o = \bar{U}^4_{oo}$, $\bar{V}^2_o = \bar{V}^4_{oo}$

$(W_{,nn})^1_n \equiv (W_{,nn})^1_{on} \quad (W_{,nn})^1_m \equiv (W_{,nn})^2_{mo}$

$W^1_m \equiv W^2_{mo}$, $W^1_n \equiv W^1_{on}$, $(W_{,n})^2_m \equiv (W_{,n})^4_{mo}$, $(W_{,n})^2_n \equiv (W_{,n})^4_{on}$

$(W_{,n})^o_n = Z_o$

$$(SSC)_{mn\ell} = \int_0^d \sin\alpha_m(\xi_1 + \frac{\bar{c}}{R}\,c)\,\sin\beta_n(\eta_1 + \frac{\bar{c}}{R}\,s)\,\cos\frac{\ell\pi\bar{c}}{d}\,d\bar{c}$$

$$(CCC)_{mn\ell} = \int_0^d \cos\alpha_m(\xi_1 + \frac{\bar{c}}{R}\,c)\,\cos\beta_n(\eta_1 + \frac{\bar{c}}{R}\,s)\,\cos\frac{\ell\pi\bar{c}}{d}\,d\bar{c}$$

$$(SSS)_{mn\ell} = \int_0^d \sin\alpha_m(\xi_1 + \frac{\bar{c}}{R}\,c)\,\sin\beta_n\,(\eta_1 + \frac{\bar{c}}{R}\,s)\,\sin\frac{\ell\pi\bar{c}}{d}\,d\bar{c}$$

$$(CCS)_{mn\ell} = \int_0^d \cos\alpha_m\,(\xi_1 + \frac{\bar{c}}{R}\,c)\,\cos\beta_n\,(\eta_1 + \frac{\bar{c}}{R}\,s)\,\sin\frac{\ell\pi\bar{c}}{d}\,d\bar{c}$$

$$(CSS)_{mn\ell} = \int_0^d \cos\alpha_m\,(\xi_1 + \frac{\bar{c}}{R}\,c)\,\sin\beta_n\,(\eta_1 + \frac{\bar{c}}{R}\;s)\,\sin\frac{\ell\pi\bar{c}}{d}\,d\bar{c}$$

$$(SCS)_{mn\ell} = \int_0^d \sin\alpha_m\,(\xi_1 + \frac{\bar{c}}{R}\,c)\,\cos\beta_n\,(\eta_1 + \frac{\bar{c}}{R})s\,\sin\frac{\ell\pi\bar{c}}{d}\,d\bar{c}$$

$$(CC)_{m\ell} \equiv (CCC)_{mo\ell}, \qquad (CC)_{n\ell} = (CCC)_{on\ell}$$

$$(CS)_{m\ell} = (CCS)_{mo\ell}, \qquad (CS)_{n\ell} = (CCS)_{on\ell}$$

$$(SS)_{m\ell} = (SCS)_{mo\ell}, \qquad (SS)_{n\ell} = (CSS)_{on\ell}$$

Table 1

Evaluation of terms appearing in c_{4kl}, c_{5kl}, c_{6kl}

Quantity	X_{1mn}	X_{2mn}	X_{3mn}	X_{4mn}	X_{5mn}	X_{6mn}	X_{7mn}	X_{8mn}
Substitution	X_{9mn}	X_{12mn}	X_{10mn}	X_{11mn}	X_{13mn}	X_{15mn}	X_{14mn}	X_{16mn}
Quantity	$T_{3mn\ell}$	$T_{4mn\ell}$	a_{5S}	a_{5c}	$T_{4m0\ell}$	$T_{4on\ell}$	$a_{11}T_{4m0k}$	$a_{11}T_{4onk}$
Substitution	$T_{1mn\ell}$	$T_{2mn\ell}$	$-a_{13c}$	$-a_{13S}$	$\phi mT_{1m0\ell}$	$nT_{10n\ell}$	$a_{14s}T_{4m0k}$	$a_{14c}T_{40nk}$

Table 2

Comparison of the Fundamental Frequency Factors with the Results Obtained in [5].

Symmetric-Symmetric				Symmetric-Antisymetric		
d/a		Ref. [5]	Comments		Ref. [5]	Comments
.1	1.988	1.988		4.999	4.998	Inaccurate
.2	1.955	1.956		4.983	4.982	
.3	1.906	1.908		4.912	4.912	
.4	1.849	1.852		4.721	4.724	
.5	1.792	1.794	30 X 30	4.354	4.360	
.6	1.738	1.742	30 X 30	3.846	3.848	
.7	1.693	1.696	30 X 30	3.298	3.304	
.8	1.660	1.662	30 X 30	2.807	2.814	
.9	1.640	1.642			2.289	Inaccurate
1.0	1.635	1.634			1.634	Inaccurate

Table 3

Influence of the Relative Location (e_1/a) of the Crack on the Frequency Factors f for Lowest Two Modes (d/a = 0.5).

e_1/a	f_1	f_2	Comments
.01	1.947	4.605	Inaccurate
.05	1.924	4.384	
.10	1.900	4.250	
.15	1.877	4.179	All Results
.20	1.856	4.154	From
.25	1.835	4.166	20 X 20 Matrix
.30	1.818	4.203	
.35	1.804	4.253	
.40	1.794	4.303	
.45	1.787	4.338	
.50	1.785	4.351	

Table 4

Effect of curvature and of tangential inertia on the two lowest frequencies of vibration of an uncracked shell of square planform

	Tangent. inertia not included		Tangential inertia included	
R/a	f_1	f_2	f_1	f_2
∞	2.000	5.000	2.000	5.000
25	2.003	5.000	2.003	5.001
2.5	2.269		2.256	
1.0	3.343	5.112	3.227	5.042

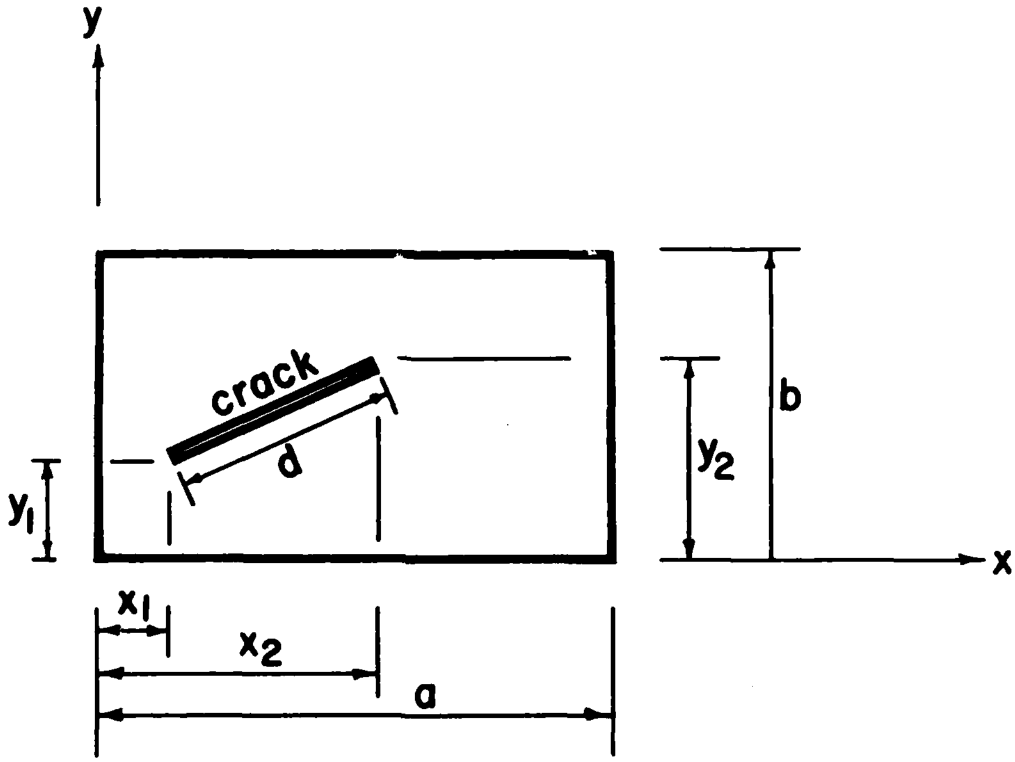


Figure I.

Plate with diagonally located crack

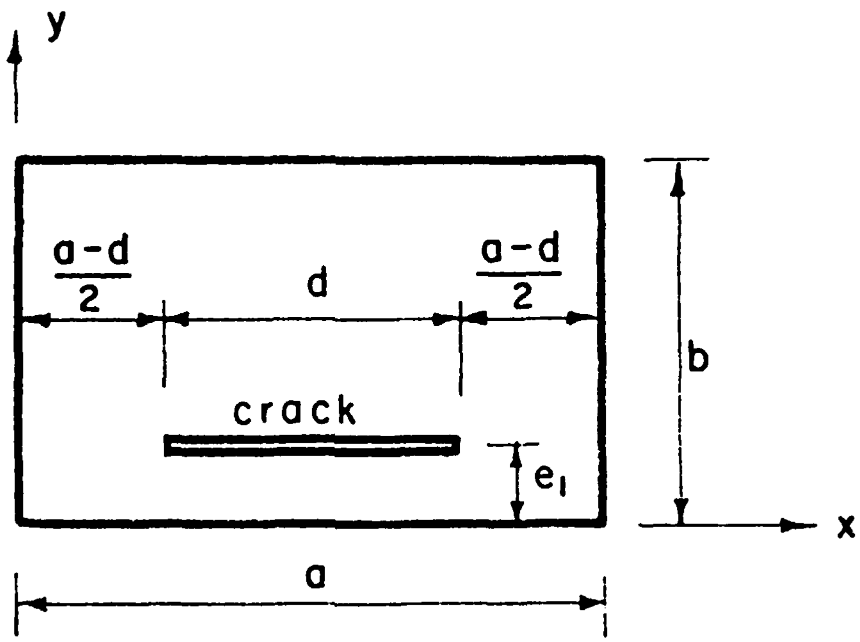


Figure 2.
Plate with a parallel crack

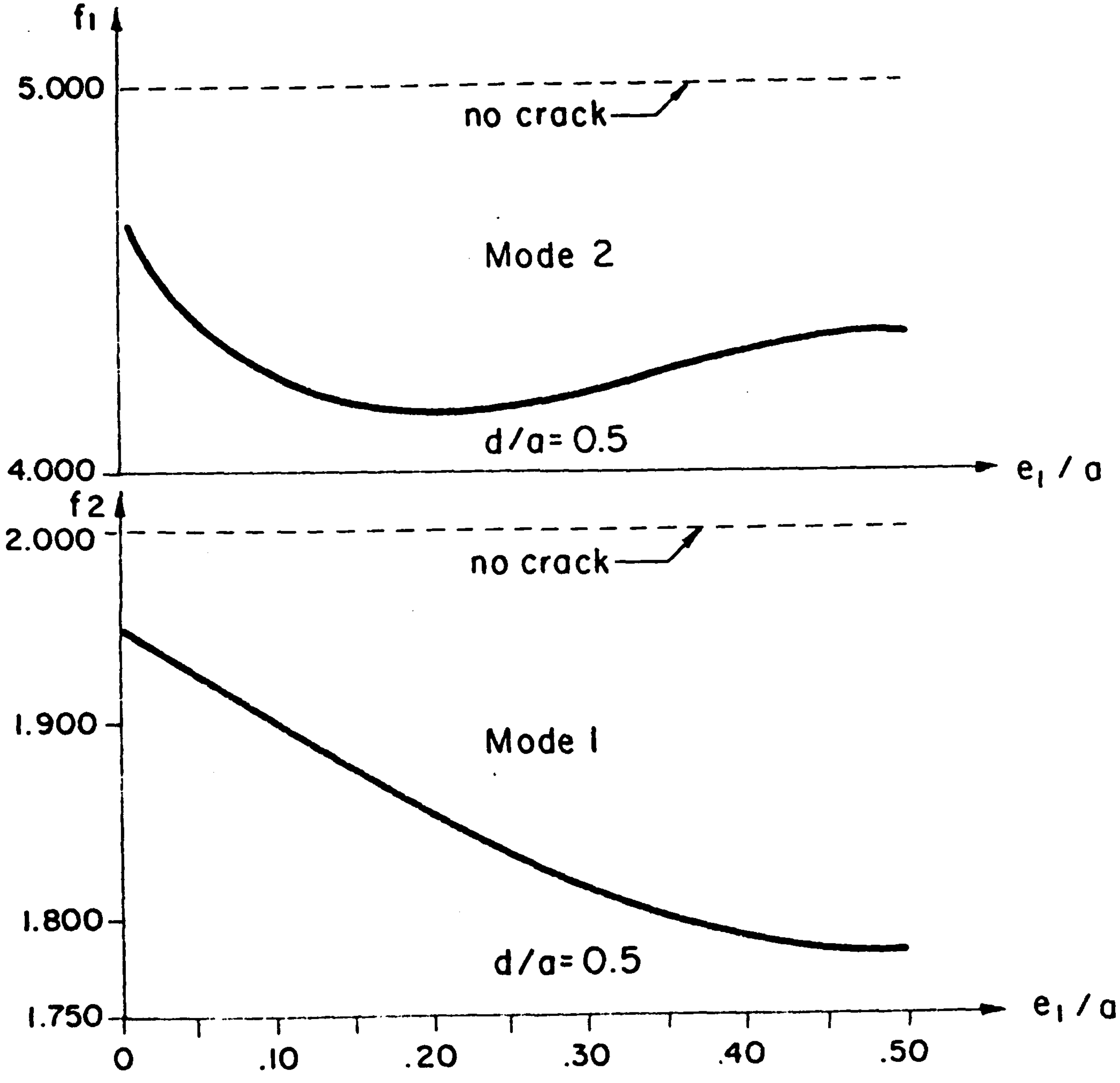


Figure 3.
Frequency coefficients
for a square plate with a parallel crack

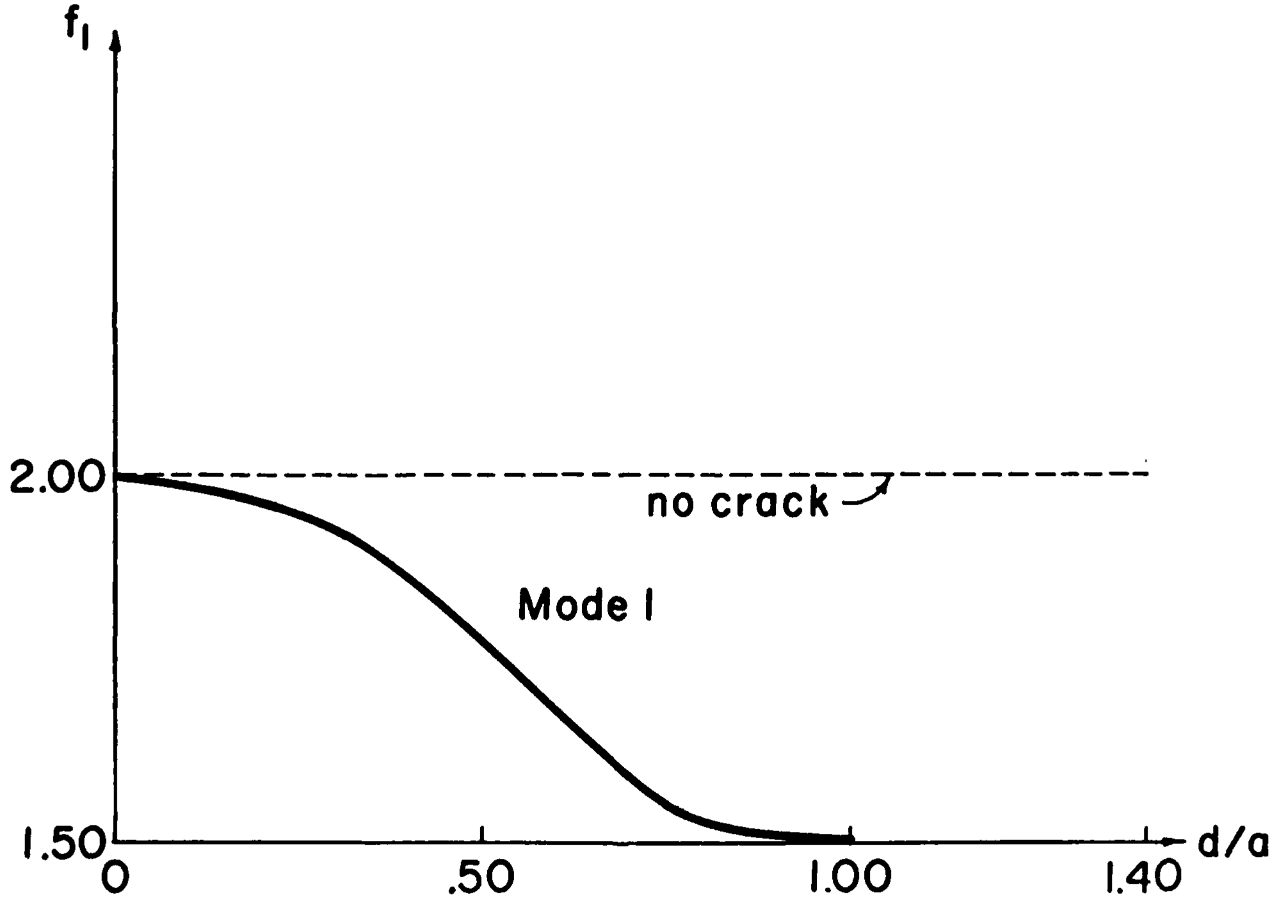


Figure 4.

Frequency coefficients for a square plate with a diagonal crack

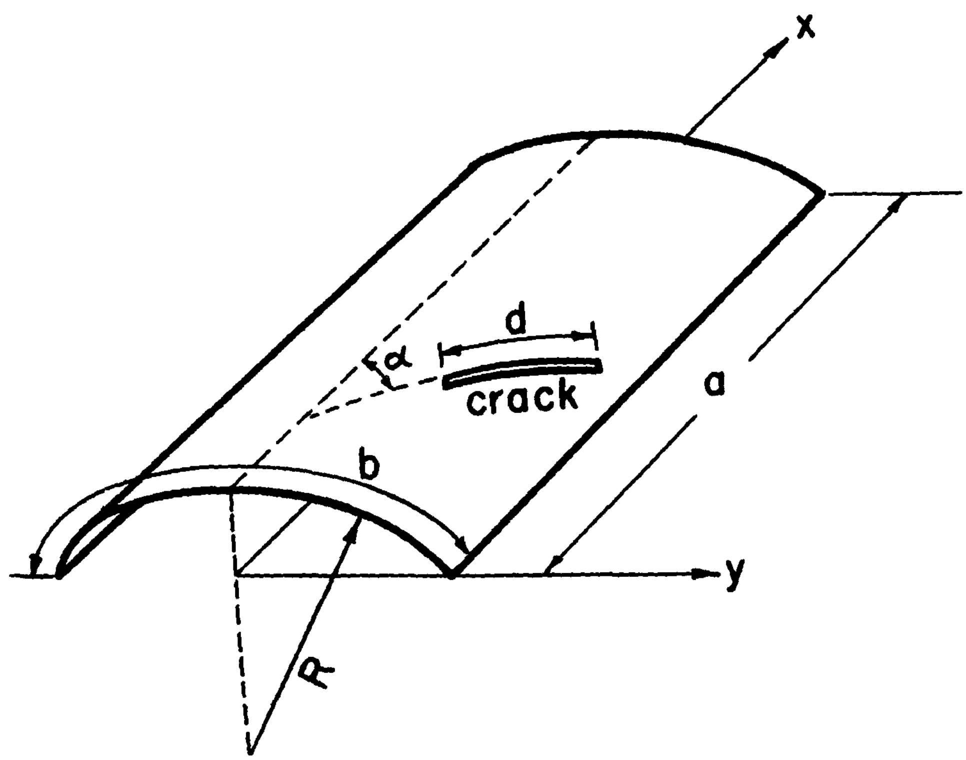


Figure 5
Geometry of the shell

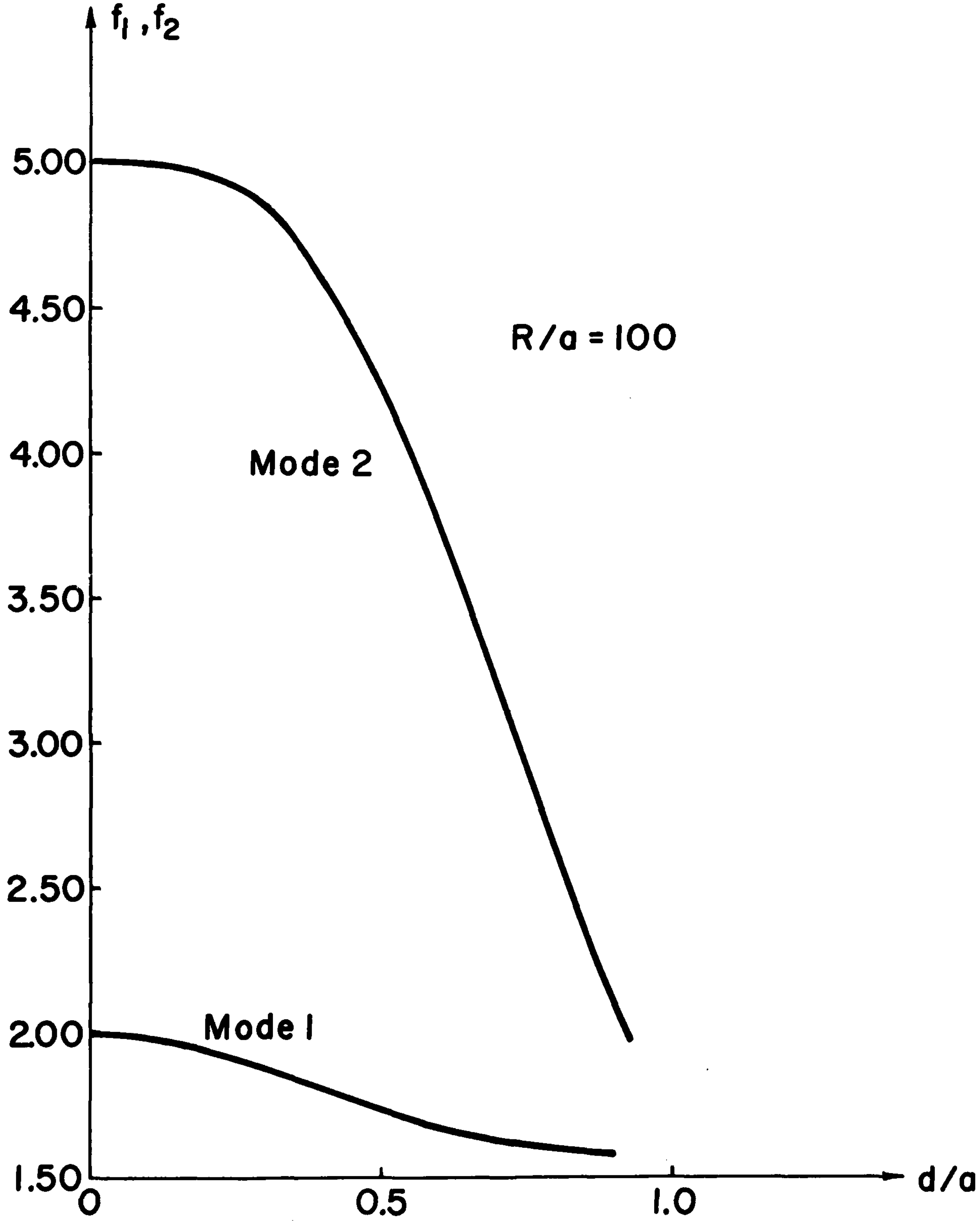


Figure 6.

Frequency coefficients for a shell cracked at the apex

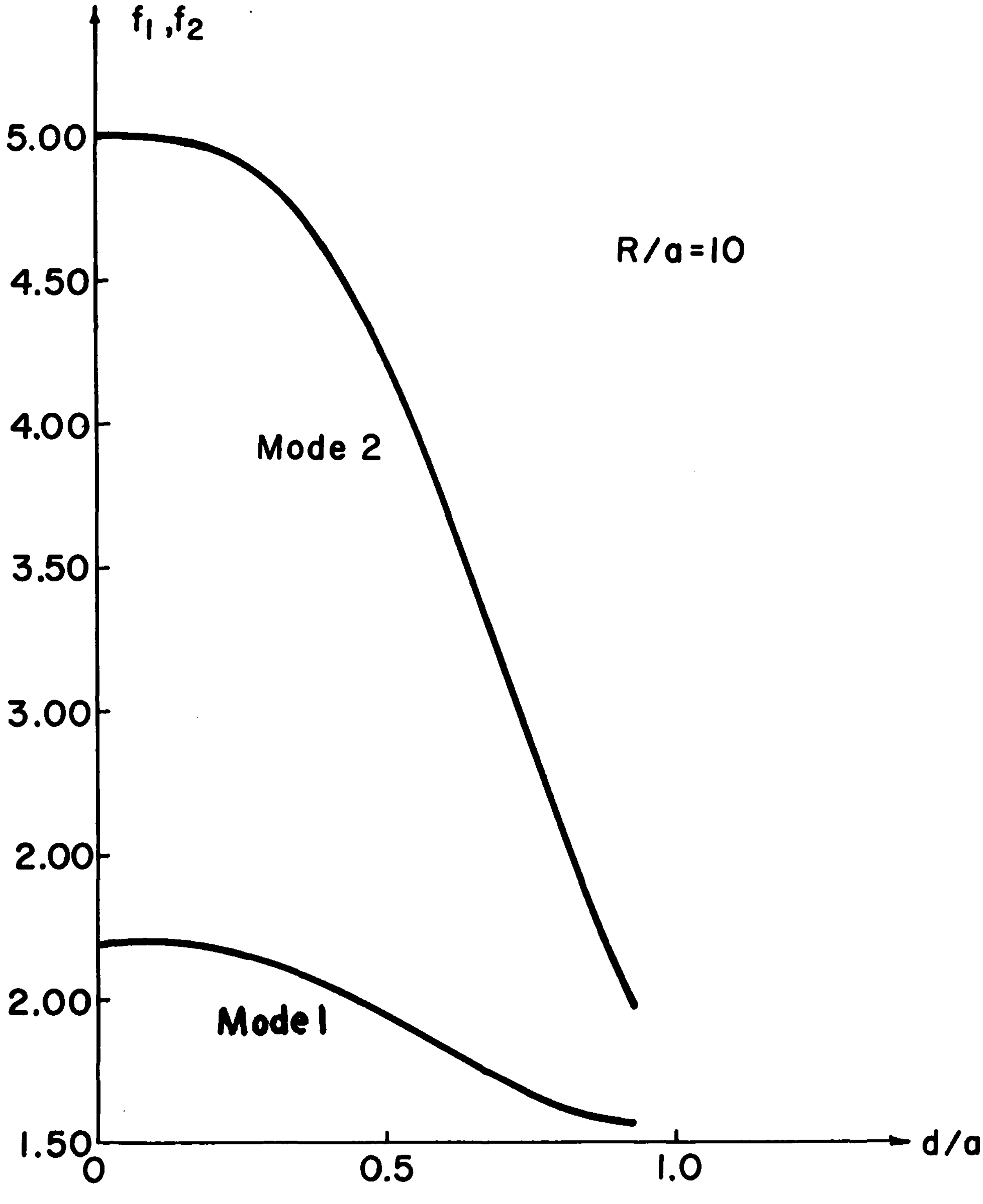


Figure 7

Frequency coefficients for a shell cracked at the apex

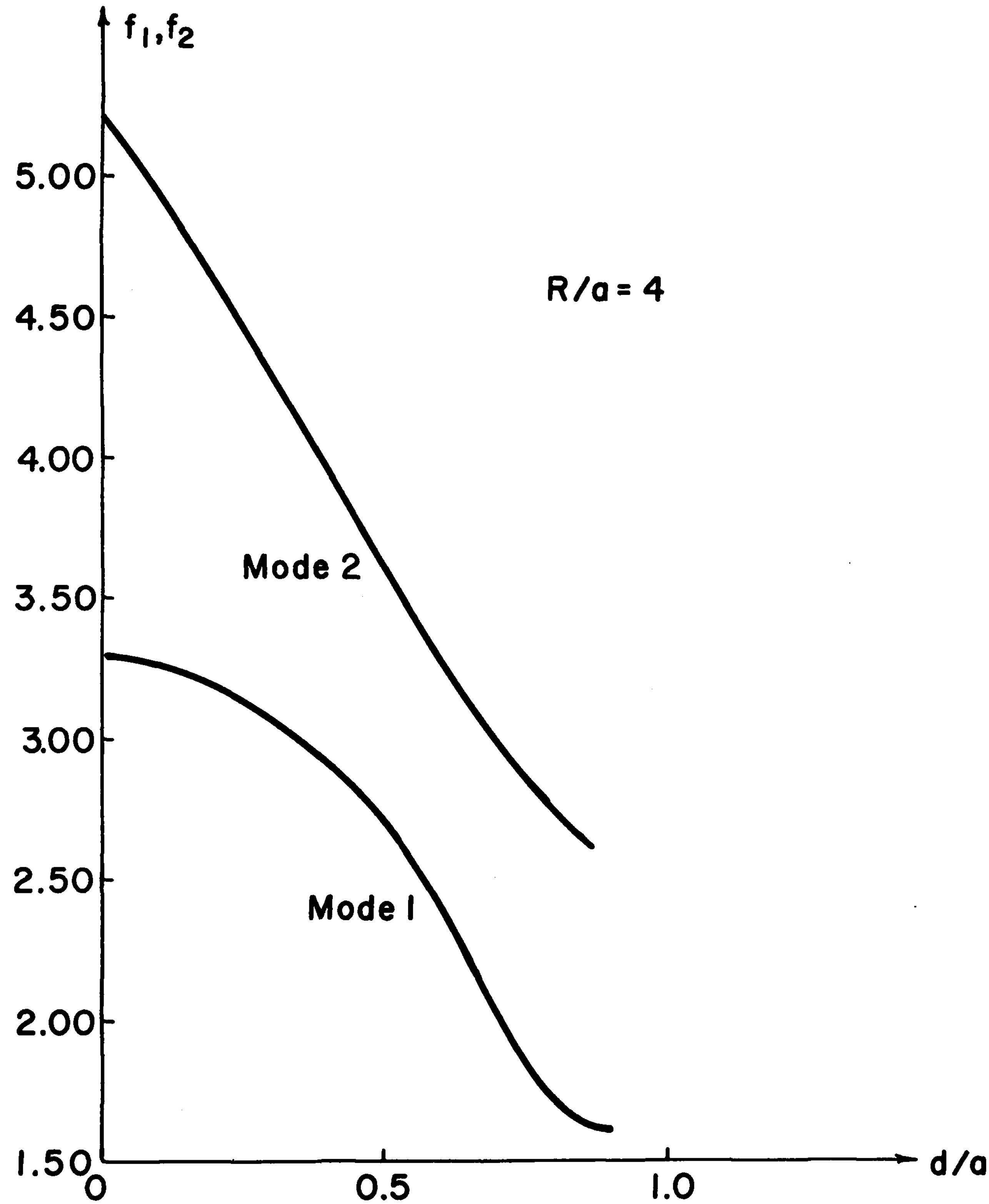


Figure 8

Frequency coefficients for a shell cracked at the apex

FILM

2 -8

DTI