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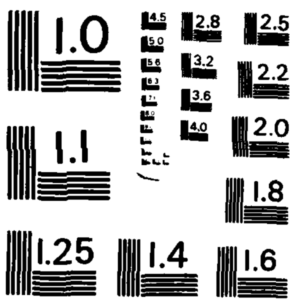

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Department of Statistics
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Chapel Hill, North Carolina

HARMONIZABLE STABLE PROCESSES ON GROUPS:
SPECTRAL, ERGODIC AND INTERPOLATION PROPERTIES

Aleksander Weron

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HARMONIZABLE STABLE PROCESSES ON GROUPS: SPECTRAL, ERGODIC AND INTERPOLATION PROPERTIES*

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Summary

This work extends to symmetric α-stable (SαS) processes, $1 < \alpha < 2$, which are Fourier transforms of independently scattered random measures on locally compact Abelian groups, some of the basic results known for processes with finite second moments and for Gaussian processes. Analytic conditions for subordination of left (right) stationarily related processes and a weak law of large numbers are obtained. The main results deal with the interpolation problem. Characterization of minimal and interpolable processes on discrete groups are derived. Also formulas for the interpolator and the corresponding interpolation error are given. This yields a solution of the interpolation problem for the considered class of stable processes in this general setting.

Keywords: Harmonizable stable process, minimal process, interpolation problem interpolator, weak law of large numbers.

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0. Introduction

Many features of the theory of Gaussian processes, in particular some path properties [10], linear estimation and system identification [4], nonparametric estimates for spectral density [11] as well as linear prediction [7], [5] have been shown to extend to appropriate classes of α-stable processes. The main difficulty is due to the fact that, while the linear space of a Gaussian process is a Hilbert space, the linear space of a stable process is an L_p space and its geometry is completely different.

With the aim of carrying over L^2-stationarity type arguments to the theory of α-stable processes, $1<\alpha<2$, Y. Hosoya [7], S. Cambanis and R. Soltani [5] have considered the class of harmonizable symmetric α-stable sequences and processes. This idea goes back to K. Urbanik [18], who studied first harmonizable processes with infinite second moments and their prediction, but under a more restrictive assumption that the processes admit independent prediction. It turns out that such an assumption which is useful in α-stable contexts in general, unfortunately is not satisfied for harmonizable α-stable processes. The main obstacle here is the lack of independent random variables in the linear span of the process, cf. [5], th. 3.3. However, the importance of harmonizable processes is that their theory can be penetrated by Fourier analysis type arguments.

The approach given in this paper follows the recent work of S. Cambanis and R. Soltani [5] with the extension to our general setting which is motivated as follows. In the development of the theory of L^2-stationary processes $(X_t)_{t\in T}$ a natural trend can be observed. First classical results derived for the processes with discrete or continuous time T ($T = \mathbb{Z}$ or $\mathbb{R}$) were extended to the case of random fields on $T = \mathbb{Z}^n$ or on $T = \mathbb{R}^n$, and next to the more general parameter sets such as groups or homogeneous spaces. This was motivated not only by theoretical aims, but also by some practical needs. Probably the simplest example is given by a class of processes considered in meteorology, where $T = S_3 \times \mathbb{Z}$, S_3 is the unit sphere in $\mathbb{R}^3$, see [14] and references therein. Having this in mind, it seems desirable to develop a theory for α-stable processes at once in such a general setup. This will permit inclusion also of the class of dyadic stationary processes, which for the L^2-case has been used recently for several purposes, mainly due to computational advantages of Walsh spectral analysis, see [12] and references therein.

The fact that the linear span of a SαS process can be considered as a semi-inner product space with respect to the covariation $[\cdot,\cdot]_\alpha$, introduced for complex SαS variables by S. Cambanis [3], will play a fundamental role in this paper. It should also be mentioned that the norm $\|\cdot\|_\alpha$ defined by this semi-inner product is equivalent to the usual pth norm, where $1 \le p < \alpha \le 2$ and the convergence in $\|\cdot\|_\alpha$ norm is equivalent to the convergence in probability.

The plan of the paper is as follows. In Section 1, we set up the basic notations and conventions, we present a general isomorphism lemma and we study the conditions for subordination of harmonizable SαS processes. The fact that the covariation is not linear in the second argument forces us to introduce a class of left (right) stationarily related processes. Th. 1.1 gives necessary and sufficient analytic conditions for subordination of left (right) stationarily related processes, which is an extension of A.N. Kolmogorov's [8] and L. Bruckner's [2] results from the symmetric $\alpha = 2$ case.

Section 2 is devoted to the study of ergodic properties. Th. 2.1 gives a law of large numbers for harmonizable SαS processes on second countable locally compact Abelian (LCA) groups. As a corollary Prop. 2.1 and remarks related to the Maruyama-Grenander characterization of metric transitivity are mentioned.

In Section 3 basic concepts and theorems related to interpolation are investigated. Using th. 1.1 on subordination, we derive a complete characterization of minimal harmonizable SαS processes on a discrete Abelian group, which is an extension of Kolmogorov's theorem. This result for $\alpha = 2$ reduces to known facts, cf. [2] and [9]. Recently, minimality of harmonizable SαS processes on the group of integers has been studied by M. Pourahmadi [13], under the restrictive assumption that the reciprocal of the spectral density exists a.s. As we prove in th. 3.1 any minimal process has such property and this restriction is not needed. Moreover, an interpolation problem is considered, when the values of the process (defined on a discrete Abelian group) on a compact subset are missing or cannot be observed. Th. 3.2 provides formulas for the interpolation error and interpolator related to this problem. Also a characterization of interpolable (exactly predictable) processes is derived. For $\alpha = 2$, these results were obtained first by A.M. Yaglom [20] for processes with discrete time and then successively extended to processes on groups cf. [2], [16], [19] and references therein.

Failure of the least-squares method of forecasting in economic time series was first explained by B. Mandelbrot, The variation of certain speculative prices, J. Business 36 (1963), 394-419 and 45 (1972), 542-543. He introduced a radically new approach based on α-stable processes to the problem of price variation. This additionally motivated our study.

1. Spectral domain analysis

A stochastic process $(X_t)_{t\in T}$ is called a *symmetric α-stable* (SαS) *process* if all the linear combinations $a_1X(t_1) + \ldots + a_nX(t_n)$ are SαS random variables, $1<\alpha\le 2$. In particular, if $\alpha = 2$, X is a Gaussian process. Recall that a complex random variable $X = X_1 + iX_2$ is SαS if X_1, X_2 are jointly SαS and its characteristic function is written with $t = t_1 + it_2$ as

$$E\exp\{iR(t\overline{X})\} = E\exp(i(t_1X_1 + t_2X_2))$$

$$= \exp\{-\int_{S_2} |t_1x_1 + t_2x_2|^{\alpha} d\Gamma_{X_1,X_2}(x_1,x_2)\} ,$$

where Γ_{X_1,X_2} is a symmetric measure on the unit sphere S_2 of $\mathbb{R}^2$.

When $X = X_1 + iX_2$ and $Y = Y_1 + iY_2$ are jointly SαS and $1<\alpha\le 2$, the *covariation* of X with Y is defined in [3] as

$$[X,Y]_\alpha = \int_{S_4} (x_1 + ix_2)(y_1 + iy_2)^{<\alpha-1>} d\Gamma_{X_1,X_2,Y_1,Y_2}(x_1,x_2,y_1,y_2) \tag{1.1}$$

where for a complex number z and $\beta > 0$ we use throughout the convention

$$z^{<\beta>} = |z|^{(\beta-1)} \cdot \overline{z} \tag{1.2}$$

where $\overline{z}$ is the complex conjugate of z.

Elementary, but useful properties of the function $z^{<\beta>}$ are listed in the following

LEMMA 1.1

(i) $|z|^\beta = z \cdot z^{<\beta-1>}$,

(ii) $|z^{<\beta>}| = |z|^{<\beta>}$,

(iii) *if* $z^{<\beta>} = v$, *then* $z = |v|^{(1-\beta)/\beta}\,\overline{v}$.

The covariation of jointly SαS random variables defined by formula (1.1) is not generally symmetric and unlike the covariance (to which it reduces in Gaussian case $\alpha = 2$) it is not linear in the second argument, but introduces on the linear space $\underline{S}$ of all SαS random variables a useful concept of a *semi-inner product*.

The basic properties of covariation are contained in

LEMMA 1.2 ([3])

(i) $[X_1 + X_2, Y]_\alpha = [X_1, Y]_\alpha + [X_2, Y]_\alpha$

(ii) $[aX, bY]_\alpha = ab^{<\alpha-1>}[X,Y]_\alpha$

(iii) $[X,Y]_\alpha = 0$ *if* X, Y *are independent,*

(iv) $[X, Y_1 + Y_2]_\alpha = [X,Y_1]_\alpha + [X,Y_2]_\alpha$, *if* Y_1, Y_2 *are independent.*

(v) $\|X\|_\alpha = [X,X]_\alpha^{1/\alpha}$ *is a norm on* $\underline{S}$ *equivalent to convergence in probability.*

In the real case the $\|\cdot\|_\alpha$-norm is related to the usual pth norm by $\|x\|_\alpha = C(p,\alpha)(E|x|^p)^{1/p}$, where $C(p,\alpha)$ is the constant depending only on α and p, $1\le p<\alpha\le 2$, see [4], p. 45. This is no longer valid for the complex case, however $\|\cdot\|_\alpha$ is equivalent to the pth norm, which is sufficient for our aims. Cf. also [3].

Let G be a *locally compact Abelian* (LCA) *group* and $\hat{G}$ the dual group of G. Then $\hat{G}$ is also a LCA group under the compact-open topology. Because of the duality between G and $\hat{G}$ we will denote the *characters* of G by $<g,\gamma>$, $g \in G$, $\gamma \in \hat{G}$. They have the following properties.

$$<g,\gamma><h,\gamma> = <g+h,\gamma> \tag{1.3}$$
$$|<g,\gamma>| = 1$$
$$<-g,\gamma> = \overline{<g,\gamma>} = <g,\gamma^{-1}> .$$

On any LCA group there exists a non-negative measure, finite on compact sets and positive on non-empty open sets, the so-called *Haar measure* of the group, which is translation invariant. We will denote usually the Haar measures on G and $\hat{G}$ by dg and $d\gamma$, respectively. But one exception will be given in Section 2. For more information see [15].

DEFINITION 1.1

A SαS process $(X_g)_{g\in G}$, $1<\alpha\le 2$ is said to be *harmonizable* if there exists an independently scattered SαS measure $Z(\cdot)$ on the Borel σ-field $B_{\hat{G}}$ of the dual group $\hat{G}$ such that

$$X_g = \int_{\hat{G}} \langle g,\gamma\rangle\, Z(d\gamma) \ , \quad g \in G \ ,$$

where the scalar valued measure $F(\Delta) = \|Z(\Delta)\|_\alpha^\alpha$ is finite. F is called the *control measure* of the process.

Some comments are in order. First we recall that a random measure $Z(\cdot)$ is *independently scattered* if

(i) for every sequence $E_1, E_2, \ldots$ of disjoint Borel sets

$$Z\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} Z(E_n) \ ,$$

where the series converges in probability,

(ii) for every sequence $E_1, E_2, \ldots, E_n$ of disjoint Borel sets the random variables $Z(E_1), Z(E_2), \ldots, Z(E_n)$ are independent.

In our case of SαS random variables from Lemma 1.2 follows that $Z(\cdot)$ is *orthogonally scattered* in the sense that

$$[Z(E_1),\ Z(E_2)]_\alpha = 0 \quad \text{whenever } E_1 \cap E_2 = \emptyset \ ,$$

and one may repeat the classical construction of the integral with respect to $Z(\cdot)$. Namely, if $f(\cdot)$ is a simple function of the form $f = \sum_{k=1}^n a_k 1_{E_k}$ then

$$\int_{\hat{G}} f(\gamma) Z(d\gamma) = \sum_{k=1}^{n} c_k Z(E_k)$$

and

$$\Big\|\int_{\hat{G}} f(\gamma) Z(d\gamma)\Big\|_\alpha^\alpha = \int_{\hat{G}} |f(\gamma)|^\alpha F(d\gamma) \ ,$$

where

$$F(E) = \|Z(E)\|_\alpha^\alpha$$

is the control measure

Next for an $f \in L^\alpha$ there exists a sequence of simple functions $f_n \to f$ with respect to $\|\cdot\|_\alpha$. If we put

$$\int_{\hat{G}} f(\gamma)Z(d\gamma) \equiv \lim_{n\to\infty} \int_{\hat{G}} f_n(\gamma)Z(d\gamma) \ ,$$

then this integral is well defined, does not depend on the choice of $\{f_n\}$ and defines linear isometry I: $L^\alpha(F)$ into $\underline{S}$.

Such processes for $G = \mathbb{Z}$ -- the integers has been introduced recently by Y. Hosoya [7] and for $G = \mathbb{R}$ -- the reals by S. Cambanis and R. Soltani [5]. Observe that in both cases the random measure $Z(\cdot)$ can be realized by means of a right continuous SαS process ξ_t with independent increments using the formula $Z((a,b]) = \xi_b - \xi_a$ for each $a \le b$.

The following lemma will be used later .

LEMMA 1.3

(i) *If* $p(\gamma), q(\gamma) \in L^\alpha(F)$, *then*

$$[\int_{\hat{G}} p(\gamma)Z(d\gamma), \int_{\hat{G}} q(\gamma)Z(d\gamma)]_\alpha = \int_{\hat{G}} p(\gamma)q(\gamma)^{<\alpha-1>}F(d\gamma) \ .$$

(ii) *Each harmonizable* SαS *process* $(X_g)_{g\in G}$ *is covariation stationary i.e.,*

$$[X_g, X_h]_\alpha = [X_{g-h}, X_0]_\alpha = [X_0, X_{g-h}]_\alpha = \int_{\hat{G}} \langle g-h,\gamma\rangle F(d\gamma) \ .$$

(iii) *There exists a preserving semi-inner product correspondence (an isometric isomorphism* I*) between the time domain* $L(X,G)$ $(= \overline{sp}\{X_g, g\in G\}$ *in* $\underline{S}$*) of the harmonizable* SαS *process* X_g *and the spectral domain of the process* $L^\alpha(F)$ *given by*

$$I\ p(\gamma) = \int_{\hat{G}} p(\gamma)Z(d\gamma) \ , \ p(\cdot) \in L^\alpha(F) \ .$$

Proof:

(i) It is enough to check this formula on the dense subset of simple functions in $L^\alpha(F)$. For this let $p(\gamma) = \sum_k a_k 1_{A_k}$ and $q(\gamma) = \sum_j b_j 1_{B_j}$. Then by Lemma 1.2

$$[\int_{\hat{G}} p(\gamma)Z(d\gamma), \int_{\hat{G}} q(\gamma)Z(d\gamma)]_\alpha = \sum_{k,j} a_k b_j^{<\alpha-1>}[Z(A_k), Z(B_j)]_\alpha$$

$$= \sum_{k,j} a_k b_j^{<\alpha-1>} F(A_k \cap B_j)$$

$$= \int_{\hat{G}} p(\gamma) q(\gamma)^{<\alpha-1>} F(d\gamma) .$$

The rest follows from the definition of the integral with respect to $Z(\cdot)$.

(ii) It is immediate from (i).

(iii) Observe that $I<g,\gamma> = \int_{\hat{G}} <g,\gamma> Z(d\gamma) = X_g$. By (i) I is an isometry which preserves a semi-inner product on the set of all characters onto $\{X_g, g \in G\}$. I can be extended to an isometry on the linear hulls of these sets and hence to an isometry on their closures. The closure of the latter set is $L(X,G)$ and the closure of the former is $L^\alpha(F)$. □

It is known, that in contrast with the Gaussian case, there are for $\alpha < 2$ covariation stationary SαS processes which are not harmonizable. The simplest example $X_g = A^{1/2} \cdot Y_g$, where A is $\alpha/2$-stable random variable independent from a stationary Gaussian process Y_g. For details, see [5], th. 3.4.

DEFINITION 1.2

A harmonizable SαS process $(Y_g)_{g \in G}$ is said to be obtained by a *linear transformation* (LT) from the harmonizable SαS process $(X_g)_{g \in G}$ if there exists a function $p(\gamma) \in L^\alpha(F_X)$ such that

$$Y_g = \int_{\hat{G}} <g,\gamma> p(\gamma) Z_X(d\gamma) ,$$

where Z_X is the random measure and F_X the control measure of the process $(X_g)_{g \in G}$.

The concept of *subordination* of stationary L^2-processes was introduced, studied and used in prediction of such processes by A.N. Kolmogorov (1941). The problem of finding analytic conditions for subordination in terms of the spectral measures leads us to study of linear transformations. We want to obtain necessary and sufficient conditions for subordination of harmonizable SαS processes. In a contrast with case $\alpha = 2$ the results are non-symmetric, and we need to consider *left (right) stationarily related processes* to a given harmonizable SαS process.

DEFINITION 1.3

A harmonizable SαS process $(Y_g)_{g\in G}$ is said to be *left (right) stationarily related* to the harmonizable SαS process $(X_g)_{g\in G}$ if there exists a finite measure $F_{YX}(F_{XY})$ such that $[Y_g, X_h]_\alpha = \int_{\hat{G}} \langle g-h,\gamma\rangle F_{YX}(d\gamma)$ (or $[X_g, Y_h]_\alpha = \int_{\hat{G}} \langle g-h,\gamma\rangle F_{XY}(d\gamma)$,

It is easy to observe that if $(Y_g)_{g\in G}$ is left (right) stationarily related to $(X_g)_{g\in G}$, then $[Y_g, X_h]_\alpha = [Y_0, X_{h-g}]_\alpha ([X_g, Y_h]_\alpha = [X_{g-h}, Y_0]_\alpha)$.

THEOREM 1.1

If a harmonizable SαS *process* $(Y_g)_{g\in G}$ *is left (right) stationarily related to the harmonizable* SαS *process* $(X_g)_{g\in G}$, *then the following conditions are equivalent:*

(i) *There exists a function* $p(\gamma) \in L^\alpha(F_X)$ *such that*

$$Y_g = \int_{\hat{G}} \langle g,\gamma\rangle p(\gamma) Z_X(d\gamma) \qquad g\in G .$$

(ii) *There exists a function* $p(\gamma) \in L^\alpha(F_X)$ *such that*

$$F_Y(\Delta) = \int_\Delta |p(\gamma)|^\alpha F_X(d\gamma)$$

and

$$F_{YX}(\Delta) = \int_\Delta p(\gamma) F_X(d\gamma) \qquad (\text{or } F_{XY}(\Delta) = \int_\Delta p(\gamma)^{\langle\alpha-1\rangle} F_X(d\gamma) ,$$

for all Borelian sets Δ *on* $\hat{G}$.

(iii) $(Y_g)_{g\in G}$ *is subordinate to* $(X_g)_{g\in G}$ *i.e.,* $L(Y:G) \subset L(X:G)$.

Proof:

Let us consider first the left-stationarily related process.

(i) $\Rightarrow$ (ii)

By Lemma 1.3 and (1.3) we have

$$[Y_g, X_h]_\alpha = [\int_{\hat{G}} \langle g,\gamma\rangle p(\gamma) Z_X(d\gamma), \int_{\hat{G}} \langle h,\gamma\rangle Z_X(d\gamma)]_\alpha$$

$$= \int_{\hat{G}} \langle g-h,\gamma\rangle p(\gamma) F_X(d\gamma).$$

Again by Lemma 1.3, Lemma 1.1 and (1.3) we have

$$[Y_g, Y_h]_\alpha = [\int_{\hat{G}} \langle g,\gamma\rangle p(\gamma) Z_X(d\gamma), \int_{\hat{G}} \langle h,\gamma\rangle p(\gamma) Z_X(d\gamma)]_\alpha$$

$$= \int_{\hat{G}} \langle g,\gamma\rangle p(\gamma) [\langle h,\gamma\rangle p(\gamma)]^{\langle\alpha-1\rangle} F_X(d\gamma)$$

$$= \int_{\hat{G}} \langle g,\gamma\rangle p(\gamma) \overline{\langle h,\gamma\rangle} |\langle h,\gamma\rangle|^{\alpha-2} \overline{p(\gamma)} |p(\gamma)|^{\alpha-2} F_X(d\gamma)$$

$$= \int_{\hat{G}} \langle g-h,\gamma\rangle |p(\gamma)|^\alpha F_X(d\gamma)$$

Since from the other side

$$[Y_g, Y_h]_\alpha = \int_{\hat{G}} \langle g-h,\gamma\rangle F_Y(d\gamma) \quad \text{and} \quad [Y_g, X_h]_\alpha = \int_{\hat{G}} \langle g-h,\gamma\rangle F_{YX}(d\gamma)$$

then by uniqueness of the Fourier transform (see [15], p. 17) we get (ii).

(ii) $\Rightarrow$ (iii)

Let $\eta \in L(Y;G)$, then there exists a function $f \in L^\alpha(F_Y)$ such that

$$\eta = \int_{\hat{G}} f(\gamma) Z_Y(d\gamma).$$

Observe that the function $f(\gamma)p(\gamma) \in L^\alpha(F_X)$, where $p(\gamma)$ is the function from condition (ii). Indeed,

$$\int_{\hat{G}} |f(\gamma)p(\gamma)|^\alpha F_X(d\gamma) = \int_{\hat{G}} |f(\gamma)|^\alpha F_Y(d\gamma) = \|\eta\|_\alpha^\alpha .$$

Consequently, the SαS random variable defined by

$$\xi = \int_{\hat{G}} f(\gamma) p(\gamma) Z_X(d\gamma)$$

is by Lemma 1.3 an element of $L(X;G)$. Now condition (ii) implies that

$$Z_Y(\Delta) = \int_\Delta p(\gamma) Z_X(\Delta) \quad \text{for any } \Delta$$

and consequently

$$\eta = \xi \in L(X;G) .$$

(iii) $\Rightarrow$ (i)

Assume $Y_g \in L(X;G)$ for all $g \in G$. Then for each $g \in G$ there exists a function $p(\gamma;g) \in L^{\alpha}(F_X)$ such that

$$Y_g = \int_{\hat{G}} p(\gamma;g) Z_X(d\gamma) .$$

Since $(Y_g)_{g\in G}$ is left-stationarily related to $(X_g)_{g\in G}$ we have

$$\int_{\hat{G}} p(\gamma;g)\langle -h,\gamma\rangle F_X(d\gamma) = [Y_g, X_h]_{\alpha} = [Y_0, X_{h-g}]_{\alpha} = \int_{\hat{G}} p(\gamma;0)\langle g-h,\gamma\rangle F_X(d\gamma).$$

Hence

$$\int_{\hat{G}} \langle -h,\gamma\rangle [p(\gamma;g) - p(\gamma;0)\langle q,\gamma\rangle] F_X(d\gamma) = 0 \qquad h \in G .$$

So by the uniqueness of the Fourier transform [15], p. 17, we have

(1.4) $$p(\gamma;g) = p(\gamma;0)\langle g,\gamma\rangle .$$

Hence letting $p(\gamma) = p(\gamma;0)$ we have that $(Y_g)_{g\in G}$ is a LT of $(X_g)_{g\in G}$ and we get (i).

The proof in the case of right-stationarily related processes is very similar and only the last implication needs some explanation.

(iii) $\Rightarrow$ (i)

Let $Y_g \in L(X;G)$ and $Y_g = \int_{\hat{G}} p(\gamma;g) Z_X(d\gamma)$, $p(\gamma;g) \in L^{\alpha}(F_X)$. Since $(Y_g)_{g\in G}$ is right-stationarily related to $(X_g)_{g\in G}$ we have

$$\int_{\hat{G}} \langle g,\gamma\rangle p(\gamma;h)^{\langle\alpha-1\rangle} F_X(d\gamma) = [X_g, Y_h]_{\alpha} = [X_{g-h}, Y_0]_{\alpha}$$

$$= \int_{\hat{G}} \langle g,\gamma\rangle\langle -h,\gamma\rangle p(\gamma;0)^{\langle\alpha-1\rangle} F_X(d\gamma) .$$

So again by the uniqueness of the Fourier transform we have

(1.5) $$p(\gamma;h)^{\langle\alpha-1\rangle} = \langle -h,\gamma\rangle p(\gamma;0)^{\langle\alpha-1\rangle} .$$

Observe that relation (1.5) is different from relation (1.4) obtained for left-stationarily related processes. Using Lemma 1.1 (iii) we may write (1.5) as

$$p(\gamma;h) = \overline{\langle -h,\gamma\rangle} |p(\gamma;0)^{\langle\alpha-1\rangle}|^{(2-\alpha)/(\alpha-1)} \overline{p(\gamma;0)^{\langle\alpha-1\rangle}} = \langle h,\gamma\rangle p_1(\gamma) \ ,$$

where $\quad p_1(\gamma) = |p(\gamma;0)^{\langle\alpha-1\rangle}|^{(2-\alpha)/(\alpha-1)} \overline{p(\gamma;0)^{\langle\alpha-1\rangle}}$

belongs to $L^\alpha(F_X)$ since

$$|p_1(\gamma)|^\alpha = |p(\gamma;0)^{\langle\alpha-1\rangle}|^{\alpha/(\alpha-1)} \in L^1(F_X)$$

from the definition of the bracket power function (cf. (1.2)). Thus $(Y_g)_{g\in G}$ is a LT of $(X_g)_{g\in G}$ and the proof is completed. □

2. Law of large numbers and metric transitivity

In this section we will assume that G is a second countable LCA group and the Haar measure on the Borel σ-field $\mathcal{B}_G$ will be denoted for convenience by $m(\cdot)$. It is known (see for example [6] and references therein) that G always possesses at least one sequence of subsets $\{K_n\}$ satisfying the following conditions

(2.1) K_n is a compact for each n,

(2.2) $m(K_n) > 0$ for sufficiently large n,

(2.3) If U is any symmetric, relatively compact, open neighborhood of 0, then

$$\lim_{n\to\infty} \frac{m(\{g \in G: g + U \subset K_n\})}{m(\{K_n + U\})} = 1 .$$

Such a sequence is called *regular*, and it has also the following useful property:

$$\lim_{n\to\infty} \frac{m\{(K_n + g)\Delta K_n\}}{m\{K_n\}} = 0 \quad \text{for each } g \in G . \tag{2.4}$$

The following result in connection with the Chebyshev inequality implies the weak law of large numbers for harmonizable stable processes on groups. Case $\alpha = 2$ reduces to the result of [6].

THEOREM 2.1

If $(X_g)_{g\in G}$ *is a harmonizable* SαS *process on* G, *then there exists a* SαS *random variable* Λ *such that for any regular sequence* $\{K_n\}$ *of subsets of* G

$$\lim_{n\to\infty} \frac{1}{m(K_n)} \int_{K_n} X_g(\omega)m(dg) = \Lambda(\omega)$$

in $L^p(\Omega,P)$ *for* $p < \alpha^*$, *where* $\alpha^* = \infty$ *if* $\alpha = 2$ *and* $\alpha^* = \alpha$ *if* $\alpha < 2$.

Proof:

By Fubini's theorem for random measures we have

$$(2.5)\qquad \frac{1}{m(K_n)}\int_{K_n} X_g\, m(dg) = \frac{1}{m(K_n)}\int_{K_n}\left\{\int_{\hat{G}} \langle g,\gamma\rangle Z(d\gamma)\right\} m(dg)$$

$$= \int_{\hat{G}}\left\{\frac{1}{m(K_n)}\int_{K_n} \langle g,\gamma\rangle m(dg)\right\} Z(d\gamma) \ .$$

Denote by $A_n(\gamma) = \frac{1}{m(K_n)}\int_{K_n} \langle g,\gamma\rangle m(dg)$, where $\{K_n\}$ is a fixed regular sequence of subsets in G. Then

$$(2.6)\qquad \lim_{n\to\infty} A_n(\gamma) = 1_{\{\hat{0}\}}(\gamma) \qquad \forall\, \gamma \in G,$$

where $\hat{0}$ is the zero element of the dual group $\hat{G}$ and 1_B stands for indicator function.

Indeed, formula (2.6) holds for $\gamma = \hat{0}$. So assume that $\gamma \neq \hat{0}$ and choose $g_0 \in G$ such that $\langle g_0,\gamma\rangle \neq 1$. From the following equality, which follows from the translation invariance of the Haar measure m,

$$\langle g_0,\gamma\rangle \int_{K_n} \langle g,\gamma\rangle m(dg) = \int_{K_n+g_0} \langle g,\gamma\rangle m(dg)$$

$$= \int_{K_n} \langle g,\gamma\rangle m(dg) + \int_{(K_n+g_0)\setminus K_n} \langle g,\gamma\rangle m(dg) - \int_{K_n\setminus(K_n+g_0)} \langle g,\gamma\rangle m(dg)$$

We conclude

$$A_n(\gamma)(\langle g_0,\gamma\rangle - 1) = \frac{1}{m(K_n)} \int_{(K_n+g_0)\Delta K_n} \langle g,\gamma\rangle m(dg) \ .$$

Thus by (2.4)

$$\lim_{n\to\infty} |A_n(\gamma)|\,|\langle g_0,\gamma\rangle - 1| \le \lim_{n\to\infty} \frac{m\{(K_n+g_0)\Delta K_n\}}{m(K_n)} = 0 \ ,$$

and we get (2.6). Since $|A_n(\gamma)| \le 1$ and $A_n(\gamma) \to 1_{\{\hat{0}\}}(\gamma)$ pointwise thus

$$\int_{\hat{G}} |A_n(\gamma) - 1_{\{\hat{0}\}}(\gamma)|^{\alpha} F(d\gamma) \to 0 \quad \text{as } n \to \infty \ ,$$

where $F(\Delta) = \|Z(\Delta)\|_{\alpha}^{\alpha}$. Hence by Lemma 1.3 (iii) we get

$$\| \int_{\hat{G}} \Lambda_n(\gamma) Z(d\gamma) - Z(\{\hat{0}\}) \|_\alpha^\alpha \to 0 \quad \text{as } n \to \infty ,$$

which by (2.5) shows that

$$\| \frac{1}{m(K_n)} \int_{K_n} X_g \, m(dg) - Z(\{\hat{0}\}) \|_\alpha^\alpha \to 0 \quad \text{as } n \to \infty.$$

Finally, using the fact that for SαS random variables the $\|\cdot\|_\alpha$-convergence is equivalent to $\|\cdot\|_p$-convergence for all $p < \alpha^*$ (see [3]) we conclude that there exists a SαS random variable $\Lambda(\omega) = Z(\{\hat{0}\})$ such that

$$\lim_{n\to\infty} \frac{1}{m(K_n)} \int_{K_n} X_g(\omega) m(dg) = \Lambda(\omega)$$

in $L^p(\Omega, \ )$ for $p < \alpha^*$. ||

Similarly as in Gaussian case the above result tells that time average of the process is a consistent estimate of the mean if and only if the control measure $F(\cdot)$ is continuous at $\hat{0}$ $F(\{\hat{0}\}) = \| Z(\{\hat{0}\}) \|_\alpha^\alpha = 0$. While this is useful to know, it is not general enough from the statistical point of view, since it only tells us something about a particular parameter of the process, the mean, and a particular estimate of it. To probe deeper into the consistency question, one must consider more general parameters. The question of consistent estimation leads us to study strictly stationary processes and their ergodic properties. But it is well known that ergodicity is equivalent to metric transitivity.

Assume now that a harmonizable SαS process $(X_g)_{g\in G}$ is strictly stationary. It is known [3] that this holds if and only if the random measure $Z(\cdot)$ of the process X_g is isotropic (or rotationally invariant) i.e. the distribution of $\{e^{i\phi} Z(A), A \in \mathcal{B}_{\hat{G}}\}$ does not depend on ϕ. Since X_g is strictly stationary the shift transformation $T_g : X_n \to X_{n+g}$ preserves distributions.

Recall that a strictly stationary process X_g is called metrically transitive if all shift invariant events have probability zero or one. This is equivalent to the fact that $T_g f = f$ for all $g \in G$ and $f \in L^1(dP_X)$ implies $f = \text{const.}$, where T_g is the shift transformation and P_X the canonical probability measure induced by the process X_g, cf. [1].

PROPOSITION 2.1

Let $(X_g)_{g \in G}$ *be a harmonizable* SαS *process on a second countable LCA group* G. *If* F *has no atoms, then for each* $\varepsilon > 0$ *and for each finite sequence* $g_1, g_2, \ldots, g_N \in G$ *there exists* $g \in G$ *such that*

$$\sum_{i=1}^{N} |[X_{(g+g_i)}, X_0]_\alpha|^2 < \varepsilon . \tag{2.7}$$

Proof:

Put $K(h) = [X_h, X_0]_\alpha$ for $h \in G$. Then

$$|K(h)|^2 = |[X_h, X_0]_\alpha|^2 = |\int_{\hat{G}} \langle h,\gamma\rangle F(d\gamma)|^2 = \int\int_{\hat{G}\times\hat{G}} \langle h,\gamma_1\rangle \overline{\langle h,\gamma_2\rangle}\, F(d\gamma_1)F(d\gamma_2)$$

$$= \iint_{\gamma_1 \neq \gamma_2} \langle h, \gamma_1\gamma_2^{-1}\rangle F(d\gamma_1)F(d\gamma_2) ,$$

since $F(\cdot)$ has no atoms and consequently the double integral over the set $\gamma_1 = \gamma_2$ is equal to zero. Now choosing any regular sequence $\{K_n\}$ of subsets in G, as in the proof of th. 2.1, it is seen that for any finite set $g_1, \ldots, g_N$

$$\frac{1}{m(K_n)} \int_{K_n} \sum_{i=1}^{N} |K(g+g_i)|^2 \, dm(g)$$

$$= \iint_{\gamma_1 \neq \gamma_2} \sum_{i=1}^{N} \left(\frac{1}{m(K_n)} \int_{K_n} \langle g + g_i, \gamma_1\gamma_2^{-1}\rangle dm(g)\right) F(d\gamma_1)F(d\gamma_2)$$

But (2.6), the translation invariance of the Haar measure m and the fact $\gamma_1 \neq \gamma_2$, implies that

$$\lim_{n\to\infty} \frac{1}{m(K_n)} \sum_{i=1}^{N} |K(g+g_i)|^2 dm(g) = 0 .$$

It follows that for each $\varepsilon > 0$ and each finite set $g_1, \ldots, g_N \in G$ there exists $g \in G$ such that $\sum_{i=1}^{N} |K(g+g_i)|^2 < \varepsilon$. □

Remark: For $\alpha = 2$ condition (2.7) and the assumption that $(X_g)_{g\in G}$ is strictly stationary imply that $(X_g)_{g\in G}$ is metric transitive. It is just an extension of the Maruyama-Grenander theorem, which says that a stationary Gaussian process is metric transtivie if and only if F has no atoms, cf. [1]. The proof of the necessary part is easily extendable to the case of (general, not necessarily SαS) harmonizable processes. However, we don't know whether (2.7) implies the metric transitivity nor any example of a harmonizable SαS process which is metric transitive.

3. Interpolation of harmonizable SαS processes

Extrapolation of harmonizable SαS processes on $\mathbb{Z}$ and $\mathbb{R}$ has been studied by Y. Hosoya [7] and by S. Cambanis and R. Soltani [5]. M. Pourahmadi in a recent paper [13] has formulated an interpolation problem on $\mathbb{Z}$ and has found an analog of Kolmogorov's minimality condition. However, his main result was obtained under more restrictive assumptions on the density of the control measure than originally by A.N. Kolmogorov (1941) for stationary L^2-processes. In this section basic concepts and theorems related to interpolation are investigated in the more general setting of harmonizable SαS processes on LCA groups.

Using Theorem 1.1 on subordination of right-stationarily related processes from Section 1 we are able to obtain an analog of Kolmogorov'w minimality theorem in full generality for SαS processes on discrete groups. Also the more general interpolation problem on discrete groups, when a finite number of the values of the process are missing, is studied. An analog of A.M. Yaglom's (1949) result is obtained (th. 3.2). This provides formulas for the interpolation error and the interpolator of a harmonizable SαS process, under some natural assumptions, which are, for example, satisfied by minimal processes. Note that the results and their proofs are more complicated when $1 < \alpha < 2$ as compared to the case of $\alpha = 2$, cf. [2]. Also it should be pointed out that all calculations depend here on the different fractional powers of the index α, which in the Gaussian case reduce to integer powers ± 1 or ± 2.

Let C be any proper non-empty compact subset of G. The *interpolation problem* arises if one wants to make linear predictions, if exactly X_g for $g \in G\setminus C$ are known. That is to say, we are looking for a *predictor* $\hat{X}_s$ of an unknown value X_s of the process basing on linear space of observations:

(1) $\hat{X}_s \in L(X;G\setminus C)$, $s \in C$

(2) $\|X_s - \hat{X}_s\|_\alpha^\alpha = \min_Y \|X_s - Y\|_\alpha^\alpha$,

where minimum is taken over all $Y \in L(X;G\backslash C)$. It is known, see [7] and [5] that $\hat{X}_s$ always exists and it is obtained by a metric projection of X_s in the strictly convex Banach space $L(X;G)$. Thus it is the best approximation of X_s in $L(X;G\backslash C)$.

For stationary L^2-processes there exists a general interpolation theory for processes on groups. However, the most interesting results are obtained for discrete groups only (see [19], [16]). Therefore we will consider the case of discrete groups here. Let us note only that in the general case, the class of trigonometric polynomials $\sum a_k\overline{\langle g_k,\gamma\rangle}$ arising in the next proposition and further, should be replaced by the class of functions on the dual group $\hat{G}$ which are Fourier transforms of functions $q(x)$ on G such that supp $q(x) \subset C$, $q(x) \in L^1(dg)$ and $q(x)$ is positive definite, cf. [19].

PROPOSITION 3.1

Let G be a discrete Abelian group and C a compact (hence finite) subset of G. Suppose the control measure F, of a harmonizable SαS process $(X_g)_{g\in G}$, is absolutely continuous with respect to the Haar measure $d\gamma$ and such that $dF/d\gamma > 0$ a.e. $d\gamma$. Then there exists a trigonometric polynomial $P_C(\gamma) = \sum_{g_k\in C} a_k\overline{\langle g_k,\gamma\rangle}$ such that

$$(3.1)\qquad \hat{X}_s = \int_{\hat{G}}[\langle s,\gamma\rangle - P_C(\gamma)^{\langle 1/(\alpha-1)\rangle}(dF/d\gamma)^{-(1/(\alpha-1))}]Z(d\gamma)\ ,\quad s\in C$$

and

$$(3.2)\qquad \|X_s - \hat{X}_s\|_\alpha^\alpha = \int_{\hat{G}}[|P_C(\gamma)|^\alpha/(dF/d\gamma)]^{(1/(\alpha-1))}d\gamma\ .$$

Proof:

Put $\phi(\gamma)$ for isomorph of $\hat{X}_s$ in $L^\alpha(F)$, which exists by Lemma 1.3. Then $\phi(\gamma)$, as a metric projection of $\langle s,\gamma\rangle$ onto subspace $L^\alpha(F;G\backslash C)$, satisfies the following James-orthogonality relation

$$\langle s,\gamma\rangle - \phi(\gamma) \perp_\alpha L^\alpha(F;G\backslash C)$$

which reads as follows

$$\int_{\hat{G}} \langle g,\gamma\rangle(\langle s,\gamma\rangle - \phi(\gamma))^{<\alpha-1>} F(d\gamma) = 0 \qquad \text{for } g \in G\setminus C\ .$$

Put $\int_{\hat{G}} \langle g_k,\gamma\rangle(\langle s,\gamma\rangle - \phi(\gamma))^{<\alpha-1>} F(d\gamma) = a_k$

for $g_1, g_2, \ldots, g_n \in C$. Consider two functions:

$$(\langle s,\gamma\rangle - \phi(\gamma))^{<\alpha-1>} dF/d\gamma$$

and

$$\sum_{g_k \in C} a_k \overline{\langle g_k,\gamma\rangle} \equiv P_C(\gamma)\ . \tag{3.3}$$

We see that both functions have the same Fourier coefficients, hence they coincide. Thus

$$(\langle s,\gamma\rangle - \phi(\gamma))^{<\alpha-1>} dF/d\gamma = P_C(\gamma)\,.$$

By Lemma 1.1 we have

$$\langle s,\gamma\rangle - \phi(\gamma) = |P_C(\gamma)|^{(2-\alpha)/(\alpha-1)} \overline{P_C(\gamma)}\ (dF/d\gamma)^{-(1/(\alpha-1))}$$

$$= P_C(\gamma)^{<1/(\alpha-1)>} (dF/d\gamma)^{-(1/(\alpha-1))}\ .$$

Hence by Lemma 1.3 (iii) we obtain formulas (3.1) and (3.2), and the proof is completed. □

Recall that a stochastic process is called *minimal* if for all $s \in G$, $X_s \notin L(X; G\setminus\{s\})$. Minimal processes exist only on discrete groups and their study is related to the simplest interpolation problem, when $C = \{s\}$ is a singleton, cf. [9].

THEOREM 3.1

Let $(X_g)_{g\in G}$ *be a discrete Abelian group and* $(X_g)_{g\in G}$ *a harmonizable* SαS *process such that the control measure* F *of the process is absolutely continuous with respect to the Haar measure* dγ. *Then* $(X_g)_{g\in G}$ *is minimal if and only if* $dF/d\gamma > 0$ *a.s.* $-d\gamma$ *and* $(dF/d\gamma)^{-(1/(\alpha-1))} \in L^1(d\gamma)$.

Proof:

Assume that $(X_g)_{g\in G}$ is minimal, i.e. $\|X_g - \hat{X}_g\|_\alpha^\alpha \neq 0$ for a fixed $g \in G$ (thus for all $g \in G$). Consider the decomposition

$$X_g = \hat{X}_g + Y_g \quad , \text{ where } Y_g = X_g - \hat{X}_g.$$

Moreover,

$$\hat{X}_g \in L(X;G\backslash\{g\}) \quad \text{and} \quad Y_g \perp_\alpha L(X;G\backslash\{g\}) \quad \text{i.e., } [X_h, Y_g]_\alpha = 0 \text{ for each } h \neq g.$$

Hence

$$(3.4) \qquad [X_h, Y_g]_\alpha = \begin{cases} 0 & \text{if } h \neq g \\ a = \|Y_g\|_\alpha^\alpha > 0 & \text{if } h = g \end{cases}$$

and consequently

$$[X_h, Y_g]_\alpha = [X_{h-g}, Y_0]_\alpha = a\int_{\hat{G}} \langle h-g, \gamma\rangle \, d\gamma \ .$$

Since $(Y_g)_{g\in G}$ is a harmonizable SαS process which is right-stationarily related to $(X_g)_{g\in G}$, then by Theorem 1.1 there exists a function $p(\gamma) \in L^\alpha(F_X)$ such that

$$Y_g = \int_{\hat{G}} \langle g, \gamma\rangle p(\gamma) Z_X(d\gamma)$$

and moreover

$$(3.5) \qquad F_{XY}(\Delta) = \int_\Delta p(\gamma)^{\langle\alpha-1\rangle} F_X(d\gamma).$$

From (3.4) it is seen that $[X_h, Y_g]_\alpha = a\int_{\hat{G}} \langle h-g, \gamma\rangle d\gamma$. But we also have

$$[X_h, Y_g]_\alpha = \int_{\hat{G}} \langle h-g, \gamma\rangle F_{XY}(d\gamma)$$

so that $F_{XY}(\Delta) = a\, d\gamma(\Delta)$, where $d\gamma$ stands for the normalized Haar measure* on $\hat{G}$. Clearly by (3.5) we have

$$(3.6) \qquad d\gamma(\Delta) = 1/a \int_\Delta p(\gamma)^{\langle\alpha-1\rangle} F_X(d\gamma)$$

Thus the derivative $d\gamma/dF_X = p(\gamma)^{\langle\alpha-1\rangle}/a$ is finite a.e. with respect to F_X and

*It is finite since the dual group of a discrete group is compact.

from (3.6) is finite a.e. with respect to the Haar measure $d\gamma$. Since by the assumption the control measure $F = F_X$ is absolutely continuous with respect to $d\gamma$, thus the above considerations show that $dF/d\gamma$ is positive a.e. with respect to $d\gamma$ and by Prop. 3.1 we conclude that

$$(dF/d\gamma)^{-(1/(\alpha-1))} \in L^1(d\gamma) \ .$$

Conversely, if g is fixed then by (3.2) there exists a non-zero $P_C(\gamma) = d\overline{\langle g,\gamma\rangle}$ such that

$$\|X_g - \hat{X}_g\|_\alpha^\alpha = |d|^\alpha \int_{\hat{G}} (dF/d\gamma)^{-(1/(\alpha-1))} d\gamma \neq 0$$

and consequently $(X_g)_{g\in G}$ is minimal. □

Remark: For $\alpha = 2$ this theorem reduces to the celebrated Kolmogorov's result on $G = \mathbb{Z}$. Recently it has been extended to SαS processes on $\mathbb{Z}$ in [13], but under additional assumption $(dF/d\gamma)^{-1}$ exists a.e. As it is easily seen from the proof this is an essential part of the theorem. The rest follows from Prop. 3.1. Case $\alpha = 2$ for any discrete Abelian group reduces to [2], th. 4.1 and [9], Cor. 4.8.

Observe that Prop. 3.1 has an existential character only. It was enough for obtaining Theorem 3.1, but it doesn't describe precisely the interpolation error or (the formula for) the interpolator. In the case $C = \{s\}$, however, it is easy to solve the problem completely. Indeed, we have by (3.6) and Lemma 1.1

$$\|X_g - \hat{X}_g\|_\alpha^\alpha = \|Y_g\|_\alpha^\alpha = \int_{\hat{G}} |p(\gamma)|^\alpha F(d\gamma) = \int_{\hat{G}} \left(\frac{a}{dF/d\gamma}\right)^{(\alpha/(\alpha-1))} (dF/d\gamma)\ d\gamma$$

$$= a^{(\alpha/(\alpha-1))} \int_{\hat{G}} (dF/d\gamma)^{-(1/(\alpha-1))} d\gamma$$

and, on the other hand, from (3.4)

$$\|X_g - \hat{X}_g\|_\alpha^\alpha = \|Y_g\|_\alpha^\alpha = a \ .$$

Thus we have that

$$a = a^{(\alpha/(\alpha-1))}\int_{\hat{G}}(dF/d\gamma)^{-(1/(\alpha-1))}d\gamma$$

and consequently we get

COROLLARY 3.1

If $(X_g)_{g\in G}$ *is a minimal harmonizable* SαS *process on a discrete group* G *with the control measure* F *absolutely continuous with respect to* $d\gamma$, *then*

$$\sigma \equiv \|X_g - \hat{X}_g\|_\alpha^\alpha = [\int_{\hat{G}}(dF/d\gamma)^{-(1/(\alpha-1))}d\gamma]^{1-\alpha}$$

and

$$\hat{X}_g = \int_{\hat{G}}[<g,\gamma> - (\frac{\sigma}{dF/d\gamma})^{(1/(\alpha-1))}]Z(d\gamma) \ .$$

Now we will return to a more general interpolation problem when $C = \{g_1, g_2, \ldots, g_n\}$ i.e., a finite number of the values of the process are missing or cannot be observed. For stationary L^2-processes this problem was first considered by A.N. Yaglom (1949) cf. [20]. See also [2], [16], [17], [19]. For SαS processes on $\mathbb{Z}$, see [13]. Our approach is based, similarly as in [17], on a duality relation for homogeneous functionals on a cone in linear space.

Let $\quad P = \{p(\gamma) \text{ on } \hat{G} \mid p(\gamma) = <s,\gamma> + \sum_{\substack{g\in C\\ g\neq s}} c_k<g_k,\gamma>\}$

For any $p\in P$ denote by C_p the following cone

(3.7) $\qquad C_p = \{\phi \in L^\alpha(F) \mid \int_{\hat{G}}\phi(\gamma)\overline{p(\gamma)}d\gamma \text{ exists}\},\ p\in P,$

where F is the control measure of the harmonizable SαS process. Let us introduce the following homogeneous functional $J(p)$ on C_p

(3.8) $\qquad J(p) = \inf_{\phi\in C_p}\{\int_{\hat{G}}|\phi(\gamma)|^\alpha F(d\gamma) \mid \int_{\hat{G}}\phi(\gamma)\overline{p(\gamma)}d\gamma \geq 1\}$

The following duality relation is a special case of a more general relation, which is frequently used in approximation theory in linear spaces. For an elementary proof see, for example, [17], p. 24.

For each $p \in P$ we have

$$\begin{aligned} J(p) &= \inf_{\phi \in C_p} \{\int_{\hat{G}} |\phi|^{\alpha} F(d\gamma) \mid \int_{\hat{G}} \phi \bar{p} d\gamma \geq 1\} \\ &= [\sup_{\phi \in C_p} \{\int_{\hat{G}} \phi \bar{p} d\gamma \mid \int_{\hat{G}} |\phi|^{\alpha} F(d\gamma) \leq 1\}]^{-1} \equiv S^{-1}(p). \end{aligned} \tag{3.9}$$

Now we may state the following main result.

THEOREM 3.2

Under the assumptions of Prop. 3.1 we have

$$\|X_s - \hat{X}_s\|_{\alpha}^{\alpha} = \max_{p \in P} (\int_{\hat{G}} [|p(\gamma)|^{\alpha}/(dF/d\gamma)]^{(1/(\alpha-1))} d\gamma)^{1-\alpha} . \tag{3.10}$$

If $p \in P$ *and fulfills condition (3.10) with* $\|X_g - \hat{X}_g\|_{\alpha}^{\alpha} = \sigma$, *then*

$$\hat{X}_s = \int_{\hat{G}} [<s,\gamma> - \sigma^{(1/(\alpha-1))} |p(\gamma)|^{((2-\alpha)/(\alpha-1))} \overline{p(\gamma)} (dF/d\gamma)^{-(1/(\alpha-1))}] Z(d\gamma). \tag{3.11}$$

Proof:

We shall split the proof for several steps.

Step one:

$$\|X_s - \hat{X}_s\|_{\alpha}^{\alpha} \geq \max_{p \in P} J(p) .$$

Pick $p \in P$ and let ϕ be the isomorph of $\hat{X}_s$ in $L^{\alpha}(F)$. Then we have

$$\begin{aligned} \|X_s - \hat{X}_s\|_{\alpha}^{\alpha} &= \int_{\hat{G}} |<s,\gamma> - \phi(\gamma)|^{\alpha} F(d\gamma) \\ &= \inf_{A,b} \int_{\hat{G}} |<s,\gamma> - \sum_{g_k \in A} b_k <g_k,\gamma>|^{\alpha} F(d\gamma), \end{aligned}$$

where infimum is taken over all finite subsets $A \subset G \backslash C$ and finite complex sequences $b = (b_k)$, $g_k \in A$. For brevity we shall use the symbol

$$\psi_{\Lambda,b}(\gamma) = \langle s,\gamma\rangle - \sum_{g_k\in\Lambda} b_k\langle g_k,\gamma\rangle$$

and thus we have

$$\|X_s - \hat{X}_s\|_\alpha^\alpha = \inf_{\Lambda,b} \int_{\hat{G}} |\psi_{\Lambda,b}(\gamma)|^\alpha F(d\gamma). \tag{3.13}$$

Since $\Lambda \cap C = \emptyset$ and the Haar measure $d\gamma$ is finite ($\hat{G}$ is compact as the dual group of the discrete group G) and consequently normalized, then for any $p \in P$

$$\int_{\hat{G}} \psi_{\Lambda,b}(\gamma)\overline{p(\gamma)}d\gamma = \int_{\hat{G}} |\langle s,\gamma\rangle|^2 d\gamma = 1 \ .$$

Thus $\psi_{\Lambda,b} \in C_p$ for all $p \in P$ and (3.13) and (3.8) imply that for all $p \in P$

$$\|X_s - \hat{X}_s\|_\alpha^\alpha \geq J(p) \ .$$

Step two:

$$\|X_s - \hat{X}_s\|_\alpha^\alpha \leq \max_{p\in P} J(p) \ .$$

If $\|X_s - \hat{X}_s\|_\alpha^\alpha = 0$ then nothing remains to be proved. So we may assume $\|X_s - \hat{X}_s\|_\alpha^\alpha = a > 0$. In an entirely analogous manner as in the proof of Prop. 3.1. cf. (3.3) we have

$$0 < a = \|X_s - \hat{X}_s\|_\alpha^\alpha = [X_s, X_s - \hat{X}_s]_\alpha = \int_{\hat{G}} \langle s,\gamma\rangle(\langle s,\gamma\rangle - \phi(\gamma))^{\langle\alpha-1\rangle} F(d\gamma)$$

$$= \int_{\hat{G}} \langle s,\gamma\rangle[(\langle s,\gamma\rangle - \phi(\gamma))^{\langle\alpha-1\rangle})dF/d\gamma]d\gamma = \int_{\hat{G}} \langle s,\gamma\rangle \ P_C(\gamma)d\gamma$$

$$= \int_{\hat{G}} \langle s,\gamma\rangle \sum_{g_k\in C} a_k\overline{\langle g_k,\gamma\rangle}\, d\gamma = a_s \int_{\hat{G}} |\langle s,\gamma\rangle|^2 d\gamma = a_s \ .$$

In the last equality for the integrals we use the fact that characters are orthonormal, when G is discrete, see [15]. If we put now

$$p_0(\gamma) = \langle s,\gamma\rangle + \sum_{\substack{g_k\in C \\ g_k\neq s}} \overline{a_k}/a\langle g_k,\gamma\rangle = 1/a\ \overline{P_C}(\gamma) \tag{3.14}$$

$$= 1/a\overline{(\langle s,\gamma\rangle - \phi(\gamma))^{\langle\alpha-1\rangle}}dF/d\gamma$$

Then $p_0(\gamma)$ is an element of P and has to be taken into account for a calculation of $\max J(p)$. Select $\psi \in C_{p_0}$ such that

$$|\int_{\hat{G}} \psi(\gamma)\overline{p_0(\gamma)}d\gamma| \geq 1 \ . \tag{3.15}$$

We will show now that

$$\int_{\hat{G}} |\langle s,\gamma\rangle - \phi(\gamma)|^{\alpha} F(d\gamma) \leq \int_{\hat{G}} |\psi(\gamma)|^{\alpha} F(d\gamma) \ . \tag{3.16}$$

For this split ψ in $L^{\alpha}(F)$ in such a way that

$$\psi(\gamma) = \delta(\langle s,\gamma\rangle - \phi(\gamma)) + \varepsilon\rho(\gamma) \ ,$$

where δ, ε are constants and

$$\langle s,\gamma\rangle - \phi(\gamma) \perp_{\alpha} \rho(\gamma) \quad \text{in } L^{\alpha}(F). \tag{3.17}$$

Thus by (3.14)

$$0 = \int_{\hat{G}} \rho(\gamma)(\langle s,\gamma\rangle - \phi(\gamma))^{\langle\alpha-1\rangle} dF/d\gamma \cdot d\gamma$$

$$= a\int_{\hat{G}} \rho(\gamma)\overline{p_0(\gamma)}d\gamma \ .$$

Consequently, (3.15) implies

$$|\delta|\,|\int_{\hat{G}} (\langle s,\gamma\rangle - \phi(\gamma))\overline{p_0(\gamma)}d\gamma| \geq 1 .$$

More specifically, note that by (3.14)

$$|\int_{\hat{G}} (\langle s,\gamma\rangle - \phi(\gamma))\overline{p_0(\gamma)}d\gamma| = \frac{1}{a}\int_{\hat{G}} |\langle s,\gamma\rangle - \phi(\gamma)|^{\alpha} F(d\gamma) = 1$$

and we conclude that $|\delta| \geq 1$. Moreover

$$\int_{\hat{G}} |\psi(\gamma)|^{\alpha} F(d\gamma) = \int_{\hat{G}} |\delta(\langle s,\gamma\rangle - \phi(\gamma)) + \varepsilon\rho(\gamma)|^{\alpha} F(d\gamma)$$

$$\geq |\delta| \int_{\hat{G}} |\langle s,\gamma\rangle - \phi(\gamma)|^{\alpha} F(d\gamma) \ ,$$

where the last inequality follows from (3.17) and the property of James orthogonality ($\|x + \varepsilon\rho\| \geq \|x\|$ for all $\varepsilon > 0$ if $x \perp_{\alpha} \rho$). Hence we get the desired inequality (3.16).

Finally, (3.16) and arbitrary choice of $\psi \in C_{p_0}$ satisfying (3.15) implies

$$\|x_s - \hat{x}_s\|_\alpha^\alpha \le J(p_0) \le \max_{p \in P} J(p)$$

and the second step of the proof is completed. Consequently we have

$$(3.18) \qquad \|x_s - \hat{x}_s\|_\alpha^\alpha = \max_{p \in P} J(p) .$$

Step three:

$$S(p) = \left(\int_{\hat{G}} [|p(\gamma)|^\alpha/(dF/d\gamma)]^{(1/(\alpha-1))} d\gamma\right)^{((\alpha-1)/\alpha)} ,$$

where $S(p)$ is defined in (3.9) and $p \in P$.

Take $\psi \in C_p$ with $\int_{\hat{G}} |\psi(\gamma)|^\alpha F(d\gamma) \le 1$. Then

$$\left|\int_{\hat{G}} \psi(\gamma)\overline{p(\gamma)}d\gamma\right| \le \int_{\hat{G}} |\psi(\gamma)(dF/d\gamma)^{1/\alpha}| \cdot |(dF/d\ )^{-1/\alpha}\overline{p(\gamma)}|d\gamma$$

by Hölder's inequality

$$\le \left(\int_{\hat{G}} |\psi(\gamma)|^\alpha (dF/d\gamma)d\gamma\right)^{1/\alpha}\left(\int_{\hat{G}} |\overline{p(\gamma)}(dF/d\gamma)^{-1/\alpha}|^{(\alpha/(\alpha-1))}\right)^{((\alpha-1)/\alpha)}$$

by the above choice of ψ

$$\le \left(\int_{\hat{G}} [|p(\gamma)|^\alpha/(dF/d\gamma)]^{(1/(\alpha-1))} d\gamma\right)^{((\alpha-1)/\alpha)} .$$

Consequently,

$$(3.19) \qquad S(p) = \sup_{\psi \in C_p} \left|\int_{\hat{G}} \psi(\gamma)\bar{p}(\gamma)d\gamma\right| \le \left(\int_{\hat{G}} [|p(\gamma)|^\alpha/(dF/d\gamma)]^{(1/(\alpha-1))} d\gamma\right)^{((\alpha-1)/\alpha)} .$$

To prove the converse, let's introduce the following two sequences of auxiliary functions

$$\xi_n(\gamma) = \max(dF/d\gamma(\cdot),\ 1/n) \qquad n = 1,2,\ldots$$

$$\psi_n(\gamma) = (c_n p(\gamma)^{<(1/(\alpha-1))>})/(\xi_n(\gamma)^{(1/(\alpha-1))}) \qquad n = 1,2,\ldots$$

where $p(\gamma) \in P$ and $c_n = \left(\int_{\hat{G}} [|p(\gamma)|^\alpha/\xi_n(\gamma)]^{(1/(\alpha-1))} d\gamma\right)^{-(1/\alpha)}$.

These definitions make sense in view of

$$|p(\gamma)|^{(\alpha/(\alpha-1))}/\xi_n^{(1/(\alpha-1))}(\gamma) < n^{(1/(\alpha-1))}|p(\gamma)|^{(\alpha/(\alpha-1))} \in L^1(d\gamma)$$

and

$$|c_n\, p(\gamma)^{<(1/(\alpha-1)>}/\xi_n(\gamma)^{(1/(\alpha-1))}| \le n^{(1/(\alpha-1))}c_n|p(\gamma)|^{(1/(\alpha-1))} \in L^1(d\gamma).$$

Observe first that

$$\int_{\hat{G}}|\psi_n(\gamma)|^\alpha F(d\gamma) = c_n^\alpha\int_{\hat{G}}|p(\gamma)|^{(\alpha/(\alpha-1))}/\xi_n^{(\alpha/(\alpha-1))}(\gamma)F(d\gamma)$$

$$\le c_n^\alpha\int_{\hat{G}}|p(\gamma)|^{(\alpha/(\alpha-1))}/(\xi_n^{(1/(\alpha-1))}(\gamma)\cdot\tfrac{dF}{d\gamma})\cdot\frac{dF}{d\gamma}\,d\gamma$$

$$= c_n^\alpha\int_{\hat{G}}|p(\gamma)|^{(\alpha/(\alpha-1))}/\xi_n(\gamma)^{(1/(\alpha-1))}d\gamma = c_n^\alpha\cdot c_n^{-\alpha} = 1$$

and

$$\int_{\hat{G}}\psi_n(\gamma)\overline{p(\gamma)}d\gamma = \int_{\hat{G}}c_n p(\gamma)^{<(1/(\alpha-1))>}/\xi_n(\gamma)^{(1/(\alpha-1))}\overline{p(\gamma)}d\gamma$$

$$= c_n\int_{\hat{G}}|p(\gamma)|^{(\alpha/(\alpha-1))}/\xi_n(\gamma)^{(1/(\alpha-1))}d\gamma = c_n\cdot c_n^{-\alpha} = c_n^{1-\alpha}.$$

Thus $\psi_n(\gamma)\in C_p$ and

$$S(p) = \sup_{\psi\in C_p}|\int_{\hat{G}}\psi(\gamma)\overline{p(\gamma)}d\gamma| \ge |\int_{\hat{G}}\psi_n(\gamma)\overline{p(\gamma)}d\gamma|$$

$$= c_n^{1-\alpha} = (\int_{\hat{G}}[|p(\gamma)|^\alpha/\xi_n(\gamma)]^{(1/(\alpha-1))})^{((\alpha-1)/\alpha)}.$$

Since $\lim_{n\to\infty}\xi_n(\gamma) = dF/d\gamma(\gamma)$ for all $\gamma\in\hat{G}$, thus the limit inequality together with (3.19) gives

$$S(p) = (\int_{\hat{G}}[|p(\gamma)|^\alpha/(dF/d\gamma)]^{(1/(\alpha-1))})^{((\alpha-1)/\alpha)}. \tag{3.20}$$

The final step:

To complete the proof of the theorem it only remains to observe that formula (3.10) is an immediate consequence of (3.18) and (3.20), and formula (3.11) follows from (3.1) and the fact that, similarly as in (3.14), $p_C(\gamma)\in P$ if $p_C(\gamma) = (1/\sigma)p(\gamma)$, where $p(\gamma)\in P$. □

Recall that a process is interpolable if it can be errorless predicted for missed values from a compact set, cf. [19]. The proof of the following result is an immediate consequence of Th. 3.2. For the case $\alpha = 2$, cf. [2] and [19].

COROLLARY 3.2

If $(X_g)_{g \in G}$ *is a harmonizable* SαS *process satisfying assumptions of Prop. 3.1, then it is interpolable if and only if* $|\,|p(\gamma)|^{\alpha}/(dF/d\gamma)\,|^{(1/(\alpha-1))} \notin L^1(d\gamma)$ *for any non-zero* $p(\gamma) \in \mathcal{P}$.

Remark: It is rather surprising that, unlike the case $\alpha = 2$, for $1 < \alpha < 2$ the Hellinger integral technique seems not to be useful in studying the interpolation problem. For $\alpha = 2$ the error space $N_C = \{X_g - \hat{X}_g,\ g \in C\}$ has an isometric description as a subspace of those complex valued measures μ which are Hellinger square integrable with respect to F and with the Fourier transforms $\hat{\mu}(g) = 0$ for $g \in C$, cf. [19]. This approach is not suitable for SαS processes because of the fact that the James-orthogonality used here is not a symmetric relation. Consequently, the above convenient description of the error space N_C is no longer valid.

4. Appendix

For the cases that occur most often in applications the characters are given in the following table.

Group G	Characters $\langle g,\gamma\rangle$, $g\in G$, $\gamma\in\hat{G}$	Dual Group $\hat{G}$
$\mathbb{R}$	$\exp ig\gamma$	$\mathbb{R}$
$\mathbb{R}^n$	$\exp[i\sum_{k=1}^n g_k\gamma_k]$	$\mathbb{R}^n$
$\mathbb{T}$	$\exp ig\gamma$	$\mathbb{Z}$
$\mathbb{T}^n$	$\exp[i\sum_{k=1}^n g_k\gamma_k]$	$\mathbb{Z}^n$
$\mathbb{Z}$	$\exp ig\gamma$	$\mathbb{T}$
$\mathbb{Z}^n$	$\exp[i\sum_{k=1}^n g_k\gamma_k]$	$\mathbb{T}^n$
$\mathbb{Z}_k$ $g=1,2,\ldots,k$	$\exp ig\gamma$	$\mathbb{Z}_k$ $\gamma=\frac{2\pi}{k},\frac{4\pi}{k},\ldots,2\pi$
$\mathbb{P}$	Walsh function $W(g,\gamma)$	$\mathbb{D}$

Where we use the following convention: $\mathbb{R}$ - the reals, $\mathbb{Z}$ - the integers, $\mathbb{T}$ - one-dimensional torus (circle), $\mathbb{Z}_k$ - cyclic group over k-object, $\mathbb{P}$ -dyadic group of non-negative integers with dyadic addition $\dot{+}$, and $\mathbb{D}$ - dyadic group of all sequences $\bar{x}=\{x_n\}$, where $x_n=0$ or $x_n=1$, $n=1,2,\ldots$ with the group operation defined by $\bar{z}=\bar{x}+\bar{y}$ if $\bar{x},\bar{y}\in\mathbb{D}$, where $z_n=x_n+y_n \pmod 2$. There is a topology for $\mathbb{D}$, based on the system of neighborhoods of $\bar{0}=(0,0,\ldots)$, with which $\mathbb{D}$ becomes a LCA group. To each $\bar{x}\in\mathbb{D}$ one may assign a real number $x=d(\bar{x})=\sum_{i=1}^{\infty}x_i2^{-i}$ in the interval $[0,1)$. The Walsh functions $\{W(n,x),\ n=0,1,\ldots,\ 0\le x<1\}$ are defined as follows:

(i) $W(0,x) = 1$, $0 < x < 1$

(ii) If n has the dyadic expansion $n = \sum_{i=0}^{\infty} x_i 2^i$, with $x_i = 0$ or $x_i = 1$, and $x_i = 0$ for $i > m_r$, then

$$W(n,x) = \prod_{i=1}^{r} \{R_{m_i}(x)\} ,$$

where $m_1, \ldots, m_r$ correspond to the coefficients $x_{m_i} = 1$ and where $\{R_k(x)\}$ are the Rademacher functions. For more details see [12], [15] and references therein.

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