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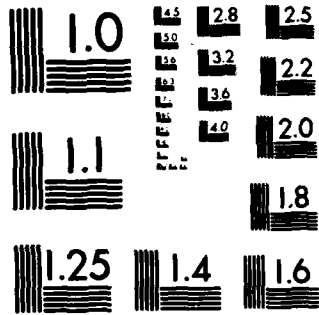
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APPLICATIONS OF NATURAL CONSTRAINTS
IN CRITICAL POINT THEORY TO BOUNDARY
VALUE PROBLEMS ON DOMAINS WITH
ROTATION SYMMETRY

E. W. C. van Groesen

**Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705**

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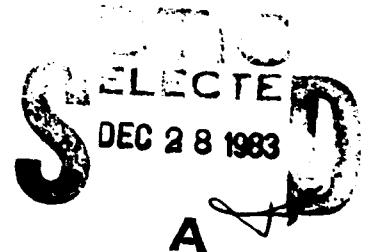
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APPLICATIONS OF NATURAL CONSTRAINTS IN CRITICAL POINT THEORY
TO BOUNDARY VALUE PROBLEMS ON DOMAINS WITH ROTATION SYMMETRY

E. W. C. van Groesen*

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Sub 2 ABSTRACT

In this paper a nonlinear Dirichlet problem for the Laplace operator is considered on a disc in \mathbb{R}^2 . It is shown that if the nonlinearity, which may explicitly depend on the radial variable, is odd and superlinear at infinity, there exist infinitely many non-radial solutions. If the nonlinearity is odd and sublinear at infinity, and satisfies certain conditions at zero, a finite number of radial and non-radial solutions will be found. This number is given by the number of radial, respectively non-radial, eigenvalues that are crossed by the nonlinearity. In any case, as a consequence of the oddness of the nonlinearity, these solutions inherit the nodal line structure of the eigenfunctions corresponding to the eigenvalues that are crossed.

The results are obtained by using natural constraints in a variational approach of the problem.

AMS (MOS) Subject Classification: 35J20, 35B10, 58E05, 58E30, 34C10.

Key Words: nonlinear Dirichlet problem, variational methods, natural constraints, periodic solutions

Work Unit Number 1 (Applied Analysis)

*Mathematical Institute, Catholic University of Nijmegen, The Netherlands

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SIGNIFICANCE AND EXPLANATION

For the nonlinear Dirichlet problem on the unit disc $D \subset \mathbb{R}^2$:

$$(*) \quad \begin{cases} -\Delta u = g(r,u) & \text{in } D = \{(r,\theta) : r \in [0,1], \theta \in (-\pi,\pi)\} \\ u(1,\theta) = 0 & \text{for } -\pi < \theta < \pi \end{cases}$$

where g is a given function which is odd in u , known multiplicity results for solutions of (*) provide not much information because of the invariance of solutions for rotations in \mathbb{R}^2 . Therefore we distinguish between radial (i.e. θ -independent) and non-radial (explicitly on θ dependent) solutions. The non-radial solutions we will look for are odd in θ . Thinking of the functions on D as being defined for all θ , and 2π -periodic in θ , we shall also consider "superharmonic" solutions, i.e. solutions with period $2\pi/2, 2\pi/3, \dots$ in θ . The sets of superharmonic functions are natural constraints for the functional of which (*) is the Euler-Lagrange equation, i.e. the critical points of this functional provide solutions of (*) even in case it is restricted to these subsets. In that way we show, under the sole condition that g grows faster than any linear function at infinity (the superlinear case), that there exist infinitely many non-radial solutions of (*). If g is bounded above by a specific linear function at infinity (the sublinear case), and if g satisfies certain conditions at $u = 0$, a finite number of radial and non-radial solutions will be found. This number is given by the number of radial, respectively non-radial, eigenvalues that are crossed by the nonlinearity $u \mapsto g(\cdot, u)$ as u runs from 0 to ∞ . In any case, the solutions inherit the nodal line structure of the eigenfunctions which correspond to the eigenvalues that are crossed.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

APPLICATIONS OF NATURAL CONSTRAINTS IN CRITICAL POINT THEORY
TO BOUNDARY VALUE PROBLEMS ON DOMAINS WITH ROTATION SYMMETRY

E. W. C. van Groesen*

1. INTRODUCTION AND RESULTS.

Consider for a smooth, bounded domain $\Omega \subset \mathbb{R}^n$, and a given function $g \in C^0(\Omega \times \mathbb{R}, \mathbb{R})$, with $u \in (0,1)$, the BVP

$$(1) \quad \begin{cases} -\Delta u = g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Writing

$$G(x,u) := \int_0^u g(x,s) ds,$$

it is well known that under suitable growth conditions on g classical solutions of (1) are in an one-to-one correspondence with the critical points of the functional ψ , given by

$$(2) \quad \psi(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \int_{\Omega} G(\cdot, u),$$

on the usual Sobolev space $\dot{H}^1(\Omega, \mathbb{R})$ of L_2 -functions which have generalized derivatives in L_2 and which vanish on the boundary. For the norm we shall take

$$\|u\| := \left\{ \int_{\Omega} |\nabla u|^2 \right\}^{1/2}, \quad u \in \dot{H}^1(\Omega, \mathbb{R}).$$

Using critical point theory, the aim is to get information about the number and properties of solutions of (1).

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In the following we shall write $\lambda_1 < \lambda_2 < \lambda_3 < \dots$ for the successive (distinct) eigenvalues of the linear eigenvalue problem associated with (1):

$$(3) \quad \begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega . \end{cases}$$

Let us first recall a classical result. Consider the following conditions:

(H₀) For every $x \in \Omega$, the function $u \mapsto G(x,u)$ is an even function.

(H₁) The function G is subquadratic at infinity in the following sense:

$$\limsup_{|u| \rightarrow \infty} \frac{G(x,u)}{|u|^2} < \frac{1}{2} \lambda_1 \quad \text{for all } x \in \Omega .$$

(H₂) _{γ} There exists a number $\gamma > 0$ such that

$$\liminf_{|u| \rightarrow 0} \frac{G(x,u)}{|u|^2} > \frac{1}{2} \gamma \quad \text{for all } x \in \Omega .$$

THEOREM A (Clark [1])

Suppose that G satisfies conditions (H₀), (H₁) and, for some $\gamma > 0$, (H₂) _{γ} .

Then the BVP (1) has at least j (pairs) of distinct solutions, where j is the number of linearly independent eigenfunctions belonging to eigenvalues not larger than γ .

The proof of this result is an application of Ljusternik-Schnirelmann theory to the even functional ψ to show that ψ has at least j pairs of distinct critical points. This theorem is the best result (concerning the number of solutions) one can generally expect.

Now suppose that Ω is the unit disc D in \mathbb{R}^2 . (For simplicity we shall restrict ourselves to \mathbb{R}^2 , but the same ideas can be used for more general sets $\Omega \subset \mathbb{R}^n$, $n > 2$, which are rotationally symmetric about one rotation axis).

Using polar coordinates (r,θ) on D , $D = \{(r,\theta) : 0 < r < 1, -\pi < \theta < \pi\}$, assume, moreover, that the function g depends only on r and u : $g = g(r,u)$.

In that case, the result stated in Theorem A is no longer very informative any more. Indeed, with $u(r,\theta)$ any solution of

$$(4) \quad \begin{cases} -\frac{1}{r} (ru_r)_r - \frac{1}{r^2} u_{\theta\theta} = g(r,u) & \text{in } D \\ u = 0 & \text{on } \partial D, \end{cases}$$

the function $R_{\theta_0} u$ defined by $R_{\theta_0} u(r, \theta) = u(r, \theta + \theta_0)$ is for any θ_0 also a solution of (4). Hence, the existence of one solution u for which $u_{\theta_0} \neq 0$, immediately implies the existence of infinitely many solutions.

Calling two functions u_1 and u_2 geometrically distinct (as in [2]) if $R_{\theta} u_1 \neq u_2$ for all $\theta \in (-\pi, \pi]$, one thus needs statements about the existence of geometrically distinct solutions of (4).

Some other obvious terminology that will be used in the sequel: functions (solutions) which do not depend on θ will be called radial functions (solutions), and the others non-radial. An eigenvalue λ_k of (3) will be called radial or non-radial depending on whether the corresponding eigenfunctions are radial or not. This makes sense, since the eigenvalues and corresponding eigenfunctions (modulo an arbitrary rotation R_{θ}) are given explicitly by

$$(5) \quad u_m^0 = j_{0,m}^2, \quad J_0(j_{0,m}r), \quad m \in \mathbb{N}$$

and for $k \in \mathbb{N}$

$$(6) \quad u_m^k = j_{k,m}^2, \quad J_k(j_{k,m}r) \sin k\theta, \quad m \in \mathbb{N},$$

where J_k , $k \in \mathbb{N} \cup \{0\}$ are Bessel functions of the first kind and $j_{k,m}$ denote the m -th strictly positive zero of J_k . Grouped together in this way, $\{u_m^0\}_{m \in \mathbb{N}}$ are the radial, and $\{u_m^k\}_{k, m \in \mathbb{N}}$ are the non-radial eigenvalues. (It is well known that all zero's of distinct Bessel-functions are different (see Watson [3, p. 485]). Hence, modulo rotations of the eigenfunctions, each eigenvalue is simple).

For subquadratic, convex (instead of even) functions $G = G(u)$, Costa and Willem [4] proved the existence of at least $j - k$ non-radial, geometrically distinct solutions of (4) if G satisfies

$$\frac{1}{2} \lambda_k < \liminf_{|u| \rightarrow 0} \frac{G(u)}{|u|^2} < \frac{1}{2} \lambda_j,$$

where λ_k, λ_j are such that all eigenvalues in $[\lambda_k, \lambda_j]$ are non-radial.

Struwe [5], requiring G to be superquadratic at infinity, i.e.

(H₃) There exist numbers $\mu > 2$ and $R > 0$ such that

$$G'(\cdot, u)u > \mu G(\cdot, u) \text{ for all } u, |u| > R;$$

moreover, g satisfies the usual Sobolev growth condition at infinity:

$$|g(\cdot, u)| < a \exp(\beta(u)),$$

for some number a and some function β , with $\beta(u)|u|^2 \rightarrow 0$ as $|u| \rightarrow \infty$

obtained, without convexity or evenness assumptions, the existence of infinitely many distinct radial solutions of (4) by, essentially, reducing the problem to a one-dimensional problem and exploiting the nodal structure of solutions to distinguish between them.

In this paper we shall prove:

THEOREM B.

Suppose that the function G satisfies condition (H₀) and (H₁), and condition

(H₂) _{γ} for some $\gamma > \lambda_j$, $j \in \mathbb{N}$.

Then problem (4) has at least j geometrically distinct (pairs) of solutions; in fact, (i) at least k radial solutions and (ii) at least l non-radial solutions, where k and l are, respectively, the number of radial and non-radial eigenvalues not larger than λ_j .

The proof of this result is an application of Ljusternik-Schnirelmann theory to the functional ψ restricted to the set of radial functions (for (i)), and to the functional ψ restricted to the set of functions which are odd in θ for (ii). These restricted sets are natural constraints for the original couple $(\psi, H^1(D, \mathbb{R}))$ in the sense as defined in [6] (see section 2). More specific information about the non-radial solutions can be obtained using the idea of naturally embedded sets ([6], see section 2). Roughly speaking, let for $k \in \mathbb{N}$, $H_k^0(D)$ denote the set of functions which are defined on D as the restriction of functions $u(r, \theta)$, $(r, \theta) \in [0, 1] \times \mathbb{R}$ which are odd and $2\pi/k$ -periodic in θ and which satisfy the boundary condition $u(1, \theta) = 0$ for all $\theta \in \mathbb{R}$. In particular, if $u \in H_k^0(D)$, then

$$u(r, \theta) = -u(r, -\theta) = u(r, \theta + \frac{2\pi}{k}) \text{ for all } r \in [0, 1], \theta \in \mathbb{R}.$$

and u has $2k$ "nodal lines":

$$u(r, \frac{\pi}{k} \cdot n) = 0 \text{ for } r \in [0,1], n \in \mathbb{Z}, -k < n < k.$$

These sets $H_k^0(D)$ turn out to be natural constraints for the problem at hand. Since $H_{mk}^0(D) \subset H_k^0(D)$ for all $m \in \mathbb{N}$, we shall say that $u \in H_k^0(D)$ has minimal period $2\pi/k$ if $u \notin H_{mk}^0(D)$ for any $m \in \mathbb{N}$, $m > 1$.

THEOREM B'

With the same conditions as in Theorem B, we have additionally:

(a) For each $k \in \mathbb{N}$ for which $\mu_1^k < \gamma$, there exists at least one non-radial solution of (4) which is a solution of the following minimization problem:

$$(7) \quad \inf\{\psi(u) : u \in H_k^0(D)\}.$$

Moreover, any solution of this minimization problem has minimal period $2\pi/k$.

(b) For every couple $(k,m) \in \mathbb{N} \times \mathbb{N}$ for which $\mu_m^k < \gamma$, there exist at least m distinct solutions of (4) with period $2\pi/k$.

For superquadratic functions G we shall prove the following result which is complementary to known results for radial solutions in that case (cf. Struwe [5]).

THEOREM C.

Suppose that the function G satisfies conditions (H_0) and (H_3) and that, moreover, the function $u \mapsto g(\cdot, u)$ is locally Lipschitz continuous.

Then there exist a number $k_0 \in \mathbb{N}$ and a sequence $\{u_k\}$, $k \in \mathbb{N}$, $k > k_0$, of geometrically distinct non-radial solutions, for which the minimal period in θ , to be denoted by τ_k , satisfies $\tau_k < 2\pi/k$ for all $k \in \mathbb{N}$.

The proof of this theorem is an application of the Mountain Pass Lemma (Ambrosetti and Rabinowitz [9]) to the restriction of the functional ψ to the naturally embedded sets $H_k^0(D)$ for each $k \in \mathbb{N}$ sufficiently large.

Under slightly more stringent conditions on g we can obtain solutions u_k as in theorem C which have minimal period precisely $2\pi/k$ and which are characterized as the

solutions of a specific minimization problem. To that end we need the following assumption:

(H₄) The function $u \mapsto g(\cdot, u)$ is C^1 , and, with g' the derivative of g with respect to u , the following conditions are satisfied:

- (i) $G(\cdot, u) > 0$ for all $u \neq 0$,
- (ii) $g'(\cdot, u) \cdot u^2 - g(\cdot, u) \cdot u > 0$ for all $u \neq 0$.

THEOREM D.

Suppose that G satisfies (H₀), (H₃) and (H₄). If $g'(\cdot, 0) = 0$, let $k_0 := 1$, and if $g'(\cdot, 0) \neq 0$ let $k_0 \in \mathbb{N}$ be such that $g'(\cdot, 0) < u_{k_0}^1$.

Then there exists a sequence $\{u_k\}$, $k \in \mathbb{N}$, $k > k_0$, of geometrically distinct non-radial solutions for which the minimal period in θ , τ_k , is given by $\tau_k = 2\pi/k$.

Furthermore, for $k > k_0$, u_k is a solution of the following minimization problem

$$(8) \quad \inf\{\psi(u) : u \in M_k\},$$

where M_k is a natural constraint, given by

$$M_k := \{u \in H_k^0(D) \setminus \{0\} : \int_D |\nabla u|^2 = \int_D g(\cdot, u)u\}.$$

REMARK. As we shall see from the proof, the minimization problem (8) is an explicit formulation of a mini-max expression for the functional ψ on $H_k^0(D)$. In fact, for $k > k_0$, and with $S_k^0(D) := \{v \in H_k^0(D) : \|v\| = 1\}$ the unit sphere in $H_k^0(D)$, for any $v \in S_k^0(D)$ condition (H₄) implies that the function $R_k \ni \rho \mapsto \psi(\rho v)$ has a unique critical point $\hat{\rho} = \hat{\rho}(v)$, at which point this function is maximal. Moreover, there exists $\rho_0 > 0$ such that $\hat{\rho}(v) > \rho_0$ for all $v \in S_k^0(D)$, which implies that M_k is given by $M_k = \{\hat{\rho}(v)v : v \in S_k^0(D)\}$ and that (8) is an explicit formulation of the following mini-max problem for the functional ψ :

$$(9) \quad \inf_{v \in S_k^0(D)} \max_{\rho > 0} \psi(\rho v).$$

In section 2 we shall recall for the readers convenience the concepts of natural constraint and naturally embedded sets. The non-radial solutions for the problem at hand

will be considered in section 3, and the radial solutions in section 4. In that last section also an example is given of the use of "forced" natural constraints to provide a specific characterization of the node of one of the radial solutions.

After the completion of this manuscript I learned from Paul H. Rabinowitz that he had obtained at the same time the multiplicity result of Theorem C in a completely different way. I thank him for drawing my attention to the references [10]-[12] related to proposition 8, and David G. Costa for several conversations.

2. DEFINITIONS.

In [6] the notion of natural constraint was introduced. It generalizes related concepts that have been used by several authors (see e.g. Berger [7, Chapter 6], Nehari [8] and other references in [6]).

For the definition we use the following notation. For a set E and a C^1 -functional ψ on E , we denote by $S(\psi, E)$ the set of critical points of ψ on E .

DEFINITION 1.

A subset \tilde{E} of E will be called a natural constraint for the couple (ψ, E) if $S(\psi, \tilde{E}) \subset S(\psi, E)$, i.e. if any critical point of the functional ψ restricted to the set \tilde{E} is also a critical point of ψ on E .

In [6], a restricted kind of natural constraints, namely naturally embedded sets, has been found useful in the study of periodic solutions of Hamiltonian systems. For the problem at hand, this same concept will provide rather detailed information on the nodal structure of the non-radial solutions.

DEFINITION 2.

A mapping $\phi : E \rightarrow \tilde{E} \subset E$ will be called a natural embedding for the couple (ψ, E) , and \tilde{E} defined by $\tilde{E} := \phi E$ will be called a naturally embedded set for (ψ, E) if:

- (i) the functional $\tilde{\psi}$ defined by $\tilde{\psi} := \psi \circ \phi : E \rightarrow \mathbb{R}$ belongs to $C^1(E, \mathbb{R})$, and
- (ii) for every $x \in S(\tilde{\psi}, E)$ it holds that $\phi x \in S(\psi, E)$.

It must be noted that by restricting to a subset \tilde{E} and considering $S(\psi, \tilde{E})$ instead of $S(\psi, E)$, one may not find all the elements of $S(\psi, E)$. But, as is clear from the results stated in the foregoing theorems, this possible loss of information is greatly compensated in Theorems B and C, and provides more specific properties of certain solutions in Theorems B' and D.

3. NON-RADIAL SOLUTIONS.

For the problem at hand, with ψ given by (2), standard regularity results for (4) imply

$$S(\psi, H(D)) \subset E := \{u : u \in C^2(D) \cap C_0^0(\bar{D})\},$$

where $H(D) \equiv \dot{H}^1(D, \mathbb{R})$.

In this section we shall look for non-radial solutions of (4), by investigating critical points of ψ that are odd in θ (the only radial solution which is odd in θ is the trivial one). In that way we get rid of the rotational symmetry of the problem.

To that end we essentially solve the equation on a semi-disc, say the upper half, with Dirichlet boundary conditions:

$$(3.1) \quad \begin{cases} -(ru_r)_r - \frac{1}{r} u_{\theta\theta} = g(r, u) & \text{in } S = \{(r, \theta) : r \in (0, 1), \theta \in (0, \pi)\} \\ u(r, 0) = u(r, \pi) = 0 & \text{for } r \in [0, 1] \\ u(1, \theta) = 0 & \text{for } \theta \in [0, \pi], \end{cases}$$

and continue the solution in an odd way to the lower-half of the disc. We denote this continuation by C : if v is defined on S , then Cv is defined by

$$Cv(r, \theta) := \begin{cases} v(r, \theta) & \text{for } 0 < \theta < \pi \\ -v(r, -\theta) & \text{for } -\pi < \theta < 0. \end{cases}$$

For solutions of (3.1) it turns out that this continuation is C^2 , therefore mapping classical solutions of (3.1) onto classical solutions of (4).

With $H(S) := \dot{H}^1(S, \mathbb{R})$, define the following sets of odd functions

$$H^0(D) := \{Cv : v \in H(S)\}$$

and

$$E^0 = \{u \in E : u \text{ is odd in } \theta\}.$$

Furthermore, let ϕ be the functional ψ restricted to functions defined on S :

$$(3.2) \quad \phi(v) := \frac{1}{2} \int_S |\nabla v|^2 - \int_S G(\cdot, v)$$

LEMMA 3.

- (i) $S(\psi, H^0(D)) \equiv C(S(\phi, H(S))) \subset \mathbb{E}^0$
(ii) $S(\psi, H^0(D)) \subset S(\psi, H(D))$, i.e. $H^0(D)$ is a natural constraint for $(\psi, H(D))$.
(iii) $\psi(Cv) = 2\psi(v)$ for any $v \in H(S)$.

Proof:

Suppose $u \in S(\psi, H^0(D))$. Then u satisfies the equation

$$-Au - g(r, u) = f(r, \theta) \text{ in } D,$$

for some function f which is even in θ . Since u , and consequently Au and $g(r, u)$, are odd in θ , the left hand side is an odd function of θ , from which it follows that f must vanish identically. Consequently, u satisfies the differential equation in D , and also the homogeneous boundary conditions on ∂D . Therefore, u is a classical solution of (4), which proves part (ii) of the lemma. Moreover, since $u \in C^2(D)$, we have in particular $u(r, 0) = u(r, \pi) = 0$ for $r \in [0, 1]$. Hence $u = Cv$, with $v \in H(S)$ a solution of (3.1). Since any solution of (3.1) is a critical point of ϕ on $H(S)$, it follows $S(\psi, H^0(D)) \subset C(S(\phi, H(S)))$. On the other hand, let $v \in S(\phi, H(S))$. Then v is a classical solution of (2.1), and $Cv \in C^1(D)$. In the lower-half disc Cv satisfies also the differential equation, and from this it readily follows that $Cv \in C^2(D)$. Hence $Cv \in \mathbb{E}^0$ is a classical solution of (4), which shows $C(S(\phi, H(S))) \subset S(\psi, H(D))$. Together with the reversed inclusion obtained above, this proves (i). Part (iii) is an immediate consequence of the definition of the functional ϕ and the continuation mapping C . ■

With the foregoing lemma, part (ii) of Theorem B is an immediate consequence of the following result.

PROPOSITION 4

Suppose that the function G satisfies the conditions of Theorem B. Then $S(\phi, H(S))$ consists of at least l distinct (pairs) of elements.

Proof. Since the functional ϕ and the set $H(S)$ are invariant for the action of the group $\mathbb{Z}_2 = \{id, -id\}$, standard Ljusternik-Schnirelmann theory may be applied with the genus of symmetric, compact subsets of $H(S) \setminus \{0\}$ as index theory.

To that end, first observe that because of condition (H_1) , the functional ϕ satisfies the Palais-Smale condition on $H(S)$ (cf. e.g. Clark [1]). Moreover, again because of condition (H_1) , it can be shown that ϕ is coercive on $H(S)$, i.e. $\phi(v) \rightarrow \infty$ as $|v| \rightarrow \infty$, so that, in particular, ϕ is bounded from below on $H(S)$.

Furthermore, let $u^{(m)}$, $1 < m < l$, be the non-radial, odd eigenfunctions of (3) with eigenvalue not larger than λ_j . Denote by $v^{(m)}$, $1 < m < l$, the restriction of $u^{(m)}$ to the upper-half disc S . Then $v^{(m)} \in H(S)$. Consider for $\rho > 0$ the set

$$\Sigma^\rho := \left\{ v = \sum_{m=1}^l \alpha_m v^{(m)} : \alpha_m \in \mathbb{R} \text{ for } 1 < m < l; |v| = \rho \right\}.$$

This set has genus l , and Ljusternik-Schnirelmann theory gives the existence of at least l distinct pairs of critical points for ϕ if $\phi(\Sigma^\rho) < 0$ for some $\rho > 0$.

In order to estimate $\phi(\Sigma^\rho)$, note that $v^{(m)}$, $1 < m < l$ are the eigenfunctions of problem (3) on the domain S for which the eigenvalues are not larger than λ_j . As a consequence, for any $v \in \Sigma^\rho$ we have

$$\frac{1}{2} \int_D |\nabla v|^2 < \frac{1}{2} \lambda_j \int_D v^2.$$

On the other hand, condition $(H_2)_\gamma$ implies

$$G(r, v) > \frac{1}{2} \gamma v^2 \text{ for } |v| \text{ sufficiently small.}$$

Thus it follows that

$$\int_S G(r, v) > \frac{1}{2} \gamma \int_S v^2 \text{ for all } v \in \Sigma^\rho, \rho \text{ sufficiently small.}$$

Since $\gamma > \frac{1}{2} \lambda_j$, these two inequalities imply that $\phi(\Sigma^\rho) < 0$ for ρ sufficiently small. This completes the proof. \square

In order to investigate the nodal structure with respect to the angle variable θ in more detail, we introduce naturally embedded sets H_k^0 of H^0 .

To that end, from now on we will think of the function $u \in H^0(D)$ as being defined for all $\theta \in \mathbb{R}$ and periodic in θ with period 2π . Defining a mapping θ_k for $k \in \mathbb{N}$ by:

$$\theta_k v(r, \theta) := v(r, k\theta),$$

the sets $H_k^0(D)$ will be defined by

$$H_k^0(D) := \Phi_k H^0(D) .$$

$H_k^0(D)$ can be interpreted as the set of odd, $2\pi/k$ periodic functions ("superharmonic functions"). The restriction of the functional ψ to $H_k^0(D)$ defines a functional ψ_k on $H^0(D)$ according to

$$\psi_k := \psi \circ \Phi_k .$$

Then $\psi_k \in C^1(H^0(D), \mathbb{R})$. It is not difficult to see that

$$\Phi_k S(\psi_k, H^0(D)) \subset S(\psi, H^0(D)) ,$$

which, by definition, means that $H_k^0(D)$ is a naturally embedded set for the couple $(\psi, H^0(D))$, with Φ_k the natural embedding. Therefore we shall investigate $S(\psi_k, H^0(D))$, as each element of this set gives rise to a solution of the original problem (4) which is odd in θ and $2\pi/k$ -periodic.

Because of lemma 3 (1), we can instead of $S(\psi_k, H^0(D))$ consider $S(\psi_k, H(S))$, where $\psi_k \in C^1(H(S))$ is defined by

$$(3.3) \quad \psi_k(v) := q_k(v) - \int_S G(\cdot, v) ,$$

with q_k the quadratic functional

$$q_k(v) = \int_0^1 \int_0^\pi \frac{1}{2} r v_r^2 dr d\theta + k^2 \int_0^1 \int_0^\pi \frac{1}{2} \frac{1}{r} v_{\theta\theta}^2 dr d\theta .$$

Hereafter we shall need the following relation between the eigenvalues and -functions of problem (3) and the eigenvalue problem on S associated with q_k

PROPOSITION 1.5

Let $k \in \mathbb{N}$. The solutions of the variational problem

$$(3.4) \quad \text{stat}\{q_k(v) : v \in H(S), \int_S v^2 = 1\}$$

(i.e. the eigenfunctions of the eigenvalue problem associated with q_k) are given, up to some normalization coefficient, by the functions

$$v_{km,s}(r,\theta) := J_{mk}(j_{mk,s}r)\sin m\theta, \quad m \in \mathbb{N}, s \in \mathbb{N},$$

and the corresponding eigenvalues are μ_s^{km} .

In particular, up to a normalization, the function $v_{k,1}$ is the solution of the minimization problem

$$\inf\{q_k(v) : v \in H(S), \int_S v^2 = 1\}.$$

Consequently,

$$q_k(v) > \frac{1}{2} \mu_1^k \int_S v^2 \quad \text{for all } v \in H(S).$$

Proof:

It is easily verified that if v is any solution of (3.3), then the function $\phi_k Cv$ is a solution of the eigenvalue problem (3) on D , with $\lambda = q_k(v)$, and conversely, if u is any eigenfunction of (3) on D which belongs to $H_k^0(D)$, i.e. $u = \phi_k Cv$ for some $v \in H(S)$, then v , normalized such that $\int_S v^2 = 1$, is a solution of (2.4) and $q_k(v)$ is the eigenvalue corresponding to u . As the functions $v_{km,s}$, defined above, are such that

$$J_{km}(j_{km,s}r)\sin km\theta = \phi_k Cv_{km,s}(r,\theta),$$

the conclusions of the proposition follow immediately. \square

With these preparations we can now easily give the

Proof of Theorem B'.

We shall prove the theorem by considering the functional ϕ_k on $H(S)$. In fact, the minimization problem

$$(3.5) \quad \inf\{\phi_k(v) : v \in H(S)\}$$

is equivalent to the minimization problem (7), in the sense that if $u \in H_k^0(D)$, with $u = \phi_k Cv$, $v \in H(S)$, then u is a solution of (7) iff v is a solution of (3.5). Just as for the functional ϕ , it may be verified that ϕ_k on $H(S)$ satisfies the Palais-Smale condition. Furthermore, since $\phi_k(v) > \phi(v)$ for all $v \in H(S)$ and ϕ is coercive, ϕ_k is coercive on $H(S)$. Consequently, the minimization problem (3.5) has a finite value and the infimum is attained at some point $v_1 \in H(S)$. To show that v_1 is nontrivial if G satisfies $(H_2)_\gamma$ with $\gamma > \mu_k^1$, note that, as in the proof of proposition 4 we have the

following estimate (using proposition 5 and the function $v_{k,1}$ defined there)

$$\begin{aligned} \phi_k(\rho v_{k,1}) &= \rho^2 \phi_k(v_{k,1}) - \int_S G(\cdot, \rho v_{k,1}) \\ &< \frac{1}{2} \rho^2 (\mu_k^1 - \gamma) \int_S v_{k,1}^2 < 0 \text{ for } |\rho| \text{ sufficiently small.} \end{aligned}$$

From this it readily follows that the solutions of (3.4) are nontrivial. With v any solution of (3.5), the function $\phi_k(v)$ is a solution of (4) and has period $2\pi/k$. The statement that any solution of the minimization problem (7) has $2\pi/k$ as its minimal period, follows from the fact that any solution of (3.5) is sign definite on S . To show this, note that with $v \in H(S)$ also the positive part $v^+(\cdot) := \max(0, v(\cdot))$ and the negative part $v^-(\cdot) := \min(0, v(\cdot))$ belong to $H(S)$ and $\phi_k(v) = \phi_k(v^+) + \phi_k(v^-)$. Hence, if v is a solution of (3.5), the assumption $v^+ \not\equiv 0 \not\equiv v^-$ implies that $\min(\phi_k(v^+), \phi_k(v^-))$ is less than the infimum defined in (3.5), contradicting the assumption that v is a solution of (3.5). This completes the proof of part (a).

For the proof of part (b) one uses Ljusternik-Schnirelmann theory for the functional ϕ_k on $H(S)$: since $\mu_m^k < \gamma$, it follows that $\phi_k(\Sigma^0) < 0$ for $\rho > 0$ sufficiently small, where

$$\Sigma^0 = \{v = \sum_{j=1}^m a_j v_{k,j} : a_j \in \mathbb{R}, 1 < j < m; |v| = \rho\}.$$

Since the genus of Σ^0 is m , the existence of m pairs of solutions $(\pm v_1, \dots, \pm v_m)$ follows, and $\phi_k(v_j)$, $1 < j < m$ are the corresponding solutions of (4), which have period $2\pi/k$.

REMARK. As we have seen, any solution v of the minimization problem (3.5) is sign-definite on S , which implied that the corresponding solution $\phi_k(v)$ of (4) has $2\pi/k$ as its minimal period. Without additional conditions on G it cannot be decided if the same is true for the other solutions found in part (b) of Theorem B'. However, under the additional condition that G satisfies

$$G(\cdot, u) < \frac{1}{2} \mu_{2k}^1 u^2 \text{ for all } u \in \mathbb{R},$$

it can be shown that all the critical points of ϕ_k on $H(S)$ give rise to solutions of (4) which have $2\pi/k$ as minimal period (cf. [6]).

Using the same ideas as before, Theorem C is a consequence of the following proposition.

PROPOSITION 6

Let G satisfy conditions (H_0) and (H_3) , and suppose that the function $g(r,u)$ is locally Lipschitz continuous in the variable u .

Then there exists a number $k_0 \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k > k_0$, the functional ϕ_k , defined in (3.3), has at least one nontrivial critical point v on $H(S)$. For such a critical point v , the function $\phi_k(v)$ is a solution of (4) with period $2\pi/k$.

Proof.

Since G satisfies (H_3) , the functional ϕ_k is neither bounded from below nor from above on $H(S)$. We shall use the well-known Mountain Pass Lemma (Ambrosetti and Rabinowitz [9]) to prove the existence of a critical point.

For the applicability of this lemma, the Palais-Smale condition has to be verified. For functions G which satisfy condition (H_3) , this is a standard result (see e.g. [9]). Next define a number C_k by

$$C_k := \inf \max \phi_k(v),$$

where "max" is taken over the points of a continuous path in $H(S)$ connecting two points v_0 and v_1 in $H(S)$, and "inf" is taken over all the paths with this property. Then the Mountain Pass Lemma states that C_k is a critical value of ϕ_k if

$$C_k > \max(\phi_k(v_0), \phi_k(v_1)).$$

We shall take $v_0 \equiv 0$ and for v_1 any point in $H(S)$ with sufficiently large norm, say $\|v_1\| > 1$, for which $\phi_k(v_1) < \phi_k(v_0) = 0$. The existence of points v_1 with this property is an easy consequence of condition (H_3) . Therefore it remains to show that C_k is strictly positive for all $k \in \mathbb{N}$ sufficiently large. To that end note that, since $G(\cdot, 0) = G'(0, 0) = 0$, and since g is locally Lipschitz, there exists constants $N > 0$ and $\delta > 0$ such that

$$(3.6) \quad G(\cdot, u) < \frac{1}{2} M u^2 \text{ for all } u \in \mathbb{R}, \text{ with } |u| < \delta.$$

With proposition 5, and the continuity of the embedding of $H(S)$ in $L_p(S)$ it readily follows that for some $\rho_0 < 1$ and all $\rho < \rho_0$:

$$(3.7) \quad \phi_k(u) > \frac{1}{2} (\mu_k^1 - M) \int_S u^2 \text{ for all } u \in H(S), \|u\| = \rho.$$

Since $\mu_k^1 \rightarrow \infty$ as $k \rightarrow \infty$, there exists $k_0 \in \mathbb{N}$ such that $\mu_k^1 - M > 0$ for all $k > k_0$. Consequently, $C_k > 0$ for all $k \in \mathbb{N}$ with $k > k_0$. The proof is complete once it is shown that $C_k > 0$ for $k > k_0$. Assume, on the contrary, that $C_k = 0$ for $k > k_0$. Then, for any $\rho < \rho_0$, there exists a sequence of functions $\{v_n\}$ for which $\phi_k(v_n) \rightarrow 0$ as $n \rightarrow \infty$, and $\|v_n\| = \rho$ for all $n \in \mathbb{N}$. This sequence has a subsequence, again to be denoted by v_n , that converges weakly in $H(S)$ to some element $\hat{v} \in H(S)$. Then v_n converges strongly to \hat{v} in $L_2(S)$. Since $\phi_k(v_n) \rightarrow 0$, it follows that $\int_S v_n^2 \rightarrow 0$, thus $\hat{v} = 0$. From this it follows that $\int_S G(\cdot, v_n) \rightarrow 0$ as $n \rightarrow \infty$. Since $\phi_k(v) = \frac{1}{2} \|v\|^2 + (k^2 - 1) \int_S \frac{1}{r^2} v_{\theta\theta} - \int_S G(\cdot, v)$, we have $\phi_k(v_n) > \frac{1}{2} \rho^2 - \int_S G(\cdot, v_n)$. This contradicts the assumption that $\phi_k(v_n) \rightarrow 0$. Hence $C_k > 0$ for $k > k_0$, and the proof is complete. ■

As in the foregoing, to prove Theorem D, we first observe that there is a one-to-one correspondence between the minimization problem (8) and the minimization problem

$$(3.8) \quad \inf\{\phi_k(v) : v \in N_k\},$$

where N_k is given by

$$(3.9) \quad N_k = \{v \in H(S) \setminus \{0\} : 2\alpha_k(v) = \int_S g(\cdot, v)v\}.$$

In fact, if v is any solution of (3.8) which is sign definite on S , then $\phi_k C v$ is a solution of (8) which has minimal period $2\pi/k$, and, conversely, if $u \in H_k^0(D)$, with $u = \phi_k C v$, $v \in H(S)$, is a solution of (8), then v is a solution of (3.8). Therefore, Theorem D and the remark following it, are consequences of the next result.

PROPOSITION 7.

Assume that G satisfies (H_0) , (H_3) and (H_4) , and let $k_0 \in \mathbb{N}$ be defined as in Theorem D. Then, for all $k > k_0$, the minimization problem (3.8) has at least one

solution v , which is sign definite on S and which satisfies the equation $\phi'_k(v) = 0$, i.e. N_k is a natural constraint.

Furthermore, for $k > k_0$, (3.8) is an explicit formulation of the mini-max problem

$$(3.10) \quad \inf_{\rho > 0} \{ \sup \phi_k(\rho w) : w \in H(S), |w| = 1 \} .$$

Proof.

First we show that for $k > k_0$ the set N_k is a smooth manifold, radially diffeomorphic to the unit sphere $S^1 := \{w \in H(S) : |w| = 1\}$. To that end, define for $w \in S^1$ the function $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ by

$$f(\rho) := \phi_k(\rho w) \text{ for } \rho > 0 .$$

Then f is a C^2 -function, $f(0) = 0$ since $G(\cdot, 0) = 0$, and $f(\rho) \rightarrow \infty$ as $\rho \rightarrow \infty$ because of condition (H_3) . If $\hat{\rho} = \hat{\rho}(w) > 0$ is any critical point of f , write $\hat{v} := \hat{\rho}(w)w$, and the equation $f'(\hat{\rho}) = 0$ reads

$$\frac{1}{\hat{\rho}} \left[2q_k(\hat{v}) - \int_S q(\cdot, \hat{v}) \hat{v} \right] = 0 ,$$

which implies that $\hat{v} \in N_k$. Conversely, for $v \in N_k$, $\hat{\rho} := |v|$ is a critical point of $f(\rho)$ for $w = \frac{1}{\hat{\rho}} v$.

At a critical point $\hat{\rho}$ of f , the second derivative is given by

$$f''(\hat{\rho}) = \frac{1}{\hat{\rho}^2} \left[2q_k(\hat{v}) - \int q'(\cdot, \hat{v}) \hat{v}^2 \right] .$$

Using $f'(\hat{\rho}) = 0$ it follows from condition (H_4) that $f''(\hat{\rho}) < 0$. In other words, at any positive critical point, the function is a strict (local) maximum, which implies by the global behaviour of f , that positive critical points of f are unique. Now we show that f has indeed a positive critical point if $k > k_0$. In fact, for any constant $M > 0$ such that $M > q'(\cdot, 0)$, it follows that there exists a $\delta > 0$ such that G satisfies (3.6). Hence, for some $\rho_0 > 0$ and all ρ , $0 < \rho < \rho_0$, inequality (3.7) holds. Consequently, with k_0 defined as in Theorem D, the function $f(\rho)$ is positive for $\rho \in (0, \rho_0)$, and thus, by the foregoing, for each $w \in S^1$ and $k > k_0$, f has a positive maximum at its unique critical point. This result provides the equivalence of the

problems (3.8) and (3.10). To show that N_k is a smooth manifold for $k > k_0$, it remains to show that 0 is an isolated point of $N_k \cup \{0\}$. This is immediate from the fact that for all $v \in H(S)$ with $\|v\|$ sufficiently small, $\int_S g(\cdot, v)v < M \int_S v^2$, whereas $2q_k(v) > \mu_k^{-1} \int_S v^2 > M \int_S v^2$ for $k > k_0$.

Next we shall show that N_k is a natural constraint. The multiplier rule states that a solution v of (3.8) satisfies the homogeneous boundary conditions and for some multiplier $\mu \in \mathbb{R}$ the equation

$$q'_k(v) - g(\cdot, v) = \mu [2q'_k(v) - g(\cdot, v) - g'(\cdot, v)v].$$

Multiplying this equation by v , and integrating over S , the fact that $v \in N_k$ and condition (N_4) readily imply that $\mu = 0$, which was to be proven.

To prove the existence of a solution of (3.8), note that the infimum in (3.8) is finite, and in fact positive, so that it suffices to verify that the functional ϕ_k on N_k satisfies the Palais-Smale condition. This verification proceeds along well-known lines so that we shall omit the proof. Finally, in much the same way as in the proof of Theorem B', it follows that any solution of (3.8) is sign definite on S . This completes the proof. ■

4. RADIAL SOLUTIONS.

In this section we shall briefly investigate radial solutions of equation (4), in particular prove part (i) of Theorem B. To that end consider the space of radial functions H :

$$H := \{u \in \overset{\circ}{H}^1(D, \mathbb{R}) : u(r, \theta) = u(r)\}.$$

It is easily verified that this set is a natural constraint for the couple $(\psi, H(D))$.

Proof of Theorem B, part (i).

The proof is a simple application of the Ljusternik-Schnirelmann theory to the \mathbb{Z}_2 -invariant set H and functional ψ , so we shall be brief.

Since the functional ψ is coercive on H because of condition (H_1) , the Palais-Smale condition is easily verified. From this it also follows that ψ is bounded from below on H . For all $\rho > 0$ the set Σ^ρ defined by

$$\Sigma^\rho := \left\{ u(\cdot) = \sum_{m=1}^k \alpha_m J_0(j_{0,m} \cdot) : \alpha_m \in \mathbb{R} \text{ for } 1 \leq m \leq k; |u| = \rho \right\}$$

has genus k . As a consequence of condition $(H_2)_\gamma$ it follows, as in proposition 4, that for all $v \in \Sigma^\rho$:

$$\psi(u) < \frac{1}{2} (u_k^0 - \gamma) \int_D u^2 \text{ for } \rho > 0 \text{ sufficiently small.}$$

Hence, if $\gamma > u_k^0$, the Ljusternik-Schnirelmann theory provides at least k distinct (pairs of) critical points of ψ on H , i.e. radial solutions of (4), which was to be proved. •

REMARK.

The restriction to radial solution of (4), essentially reduces the problem to a one-dimensional problem. For such problems one may obtain multiplicity results by considering subsets H_k of H which, roughly speaking, consist of functions which have precisely k simple zero's in $(0,1)$, and then looking for critical points of ψ on H_k . This idea has been exploited by Struwe [5] for the case that G is superquadratic (without evenness assumption), to get the result stated in the introduction.

A more constructive method to get results of this type has been used by Nehari [10] for a special class of superquadratic, even functions G . Fixing k nodes in $(0,1)$, appropriate solutions on the subintervals between the nodes are glued together to give a function on $[0,1]$. Only for a specific choice of the nodes, this function will be a solution of the original problem on $[0,1]$. In [10], these "critical" nodes are obtained as the minimal elements of a certain function of k variables which depends on the solutions of the subintervals. See also Hampel [12]. For a nonlinear eigenvalue problem, Rabinowitz [11] obtained the critical nodes as the fixed points of a continuous mapping from the set of ordered k -tuples into itself. For the problem at hand we shall give an example of this procedure of "forced" natural constraints for the simplest case of one interior node. From this it will be clear how the method can be generalized. (See also [12].

Proposition 8.

Assume that G satisfies (H_0) , (H_1) and $(H_2)_\gamma$ with $\gamma > \mu_2^0$. Moreover, suppose that g is differentiable, $g(r,u) = 0$ iff $u = 0$, and that g is convex for $u > 0$.

Then there exists at least one radial solution of (4) which has precisely one zero in the interval $(0,1)$.

Proof: Define for $\alpha \in (0,1)$ the sets $H(\alpha)$ by

$$H(\alpha) := \{u \in H : u(\alpha) = 0\}.$$

(This set is well defined, see [5, lemma 1]). We shall show that $\{H(\alpha)\}$, $0 < \alpha < 1$, is a forced natural constraint. Define the function $f : [0,1] \rightarrow \mathbb{R}$ as

$$f(\alpha) := m_1(0,\alpha) + m_2(\alpha,1),$$

where m_1 and m_2 are functions defined on $[0,1]$ by

$$(4.1) \quad m_1(0,\alpha) := \inf \left\{ \int_0^\alpha \left(\frac{1}{2} r v_x^2 - r G(r,v) \right) : v \in H(\alpha) \Big|_{[0,\alpha]} \right\} \text{ for } \alpha > 0,$$

$$m_1(0,0) := 0,$$

$$(4.2) \quad m_2(a,1) = \inf \left\{ \int_a^1 \left(\frac{1}{2} r v_r^2 - r G(r,w) \right) : v \in \hat{H}^1((a,1), \mathbb{R}) \right\} \text{ for } a < 1,$$

$$m_2(1,1) = 0$$

First observe that the infima in (4.1) and (4.2) are finite and are attained for some functions, say v and w , which are sign definite on the interval at which they are defined, i.e. v on $[0,a]$ and w on $[a,1]$. As a consequence of the convexity assumption for g , both v and w are unique solutions (apart from sign) of these minimisation problems. From this it follows that the functions m_1 and m_2 are continuously differentiable, with derivative

$$\frac{d}{da} m_1(0,a) = \left. \frac{1}{2} r v_r^2 - G(r,v) \right|_{r=a-0} = \frac{1}{2} a v_a^2(a-0),$$

$$\frac{d}{da} m_2(a,1) = \left. -\frac{1}{2} r w_r^2 + G(r,w) \right|_{r=a+0} = -\frac{1}{2} a w_a^2(a+0).$$

Hence f is differentiable on $(0,1)$ and if $\hat{a} \in (0,1)$ can be found such that

$\frac{df}{da}(\hat{a}) = 0$, then $v_a^2(a-0) = w_a^2(a+0)$, and the function

$$u(r) := \begin{cases} v(r) & \text{for } 0 < r < \hat{a} \\ w(r) & \text{for } \hat{a} < r < 1, \end{cases}$$

is a solution of the original problem on the interval $[0,1]$ and u has precisely one zero, namely \hat{a} .

We claim that at least one critical point $\hat{a} \in (0,1)$ of the function f can be found, namely any number \hat{a} for which f attains its maximum value. To show that, it suffices to prove that $f(a) < 0$ for all $a \in [0,1]$ and that f is not maximal at $a = 0$ or $a = 1$. To that end observe that there exist numbers $a_1 > 0$ and $a_2 < 1$ such that:

$m_1(0,a) = 0$ for $0 < a < a_1$, $m_1(0,a)$ is monotonically decreasing for $a > a_1$,

$m_2(a,1) = 0$ for $a_2 < a < 1$, $m_2(a,1)$ is monotonically increasing for $a < a_2$.

(a_1 and a_2 depend on the behaviour of g at $u = 0$).

Now let $a^* \in (0,1)$ denote the zero of the eigenfunction corresponding to the eigenvalue μ_2^0 . Then $a_1 < a^* < a_2$. Indeed, $a_1 < a^*$ because μ_2^0 is the lowest

eigenvalue of the eigenvalue problem associated with the interval $[0, a^*]$, and, as $\gamma > \mu_2^0$, it follows that $m_1(0, a^*) < 0$. In the same way it follows that $a^* < a_2$. From these observations it follows easily that the function f attains its (negative) maximum at some interior point $\hat{a} \in (0, 1)$, and the proof is complete. ■

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ABSTRACT (cont.)

number of radial and non-radial solutions will be found. This number is given by the number of radial, respectively non-radial, eigenvalues that are crossed by the nonlinearity. In any case, as a consequence of the oddness of the nonlinearity, these solutions inherit the nodal line structure of the eigenfunctions corresponding to the eigenvalues that are crossed.

The results are obtained by using natural constraints in a variational approach of the problem.

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