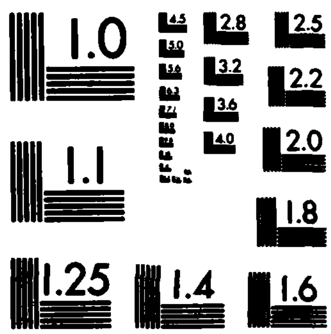
AD-A136 348 APPLICATIONS OF NATURAL CONSTRAINTS IN CRITICAL POINT THEORY TO PERIODIC..(U) WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER E W VAN GROESEN NOV 83 1/1

UNCLASSIFIED MRC-TSR-2593 DAAG29-80-C-0041 F/G 12/1 NL

END



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

A136348

MRC Technical Summary Report #2593

APPLICATIONS OF NATURAL CONSTRAINTS IN CRITICAL POINT THEORY TO PERIODIC SOLUTIONS OF NATURAL HAMILTONIAN SYSTEMS

E. W. C. van Groesen

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

November 1983

(Received September 22, 1983)

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

APPLICATIONS OF NATURAL CONSTRAINTS IN CRITICAL POINT THEORY TO PERIODIC SOLUTIONS OF NATURAL HAMILTONIAN SYSTEMS

E. W. C. van Groesen*

Technical Summary Report #2593

November 1983

ABSTRACT

This paper deals with periodic solutions of Hamiltonian systems of the form $-\ddot{x} = V'(x)$ with V a given function. Assuming V to be either a convex or an even function, and prescribing the period, existence results are obtained for the number of solutions in relation to the minimal period of these solutions, assuming superquadratic growth at infinity only, or subquadratic growth at infinity together with specific behaviour at the origin for V. By introducing natural constraints, these results are obtained by applying variational methods directly to the action functional.

AMS (MOS) Subject Classifications: 34C15, 34C25, 58E30.

Key Words: periodic solution, Hamiltonian system, variational methods, natural constraints.

Work Unit Number 1 - Applied Analysis

---

* Mathematical Institute, Catholic University of Nÿmegen, The Netherlands.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This research has been partially supported by the Niels Stensen Stichting, Amsterdam.

SIGNIFICANCE AND EXPLANATION

The system of equations $-\ddot{x} = V'(x)$, with V a given potential energy function, model the motion of a dynamical system. Prescribing the period T, the action functional associated with this system on the set of T-periodic functions is not immediately suited for application of known variational methods to obtain periodic solutions. Assuming V to be an even or a convex function, it is shown in this paper that it is possible to apply these methods after introducing certain subsets (called natural constraints) which have the property that critical points of the action functional restricted to these subsets also provide T-periodic solutions. Using specific natural constraints, the existence of superharmonic solutions, i.e. solutions which have period $T/2, T/3, \ldots$, is also investigated. In the paper the case of superquadratic growth at infinity, as well as the case that V is subquadratic at infinity and satisfies conditions at the origin, are investigated.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

APPLICATIONS OF NATURAL CONSTRAINTS IN CRITICAL POINT THEORY TO PERIODIC SOLUTIONS OF NATURAL HAMILTONIAN SYSTEMS

E. W. C. van Groesen*

1. Introduction and Results.

In this paper we shall consider the problem of finding periodic solutions with a prescribed period $T > 0$ of the autonomous system of second order equations

$$(1) \qquad -\ddot{x} = V'(x) \quad , \quad x(t) \in \mathbb{R}^n \quad ,$$

where $V \in C^2(\mathbb{R}^n,\mathbb{R})$ is a potential energy function, normalized such that $V(0) = 0$. Equations (1) correspond to a Hamiltonian system with a "natural" Hamiltonian (of the form kinetic plus potential energy) given by

$$H(x,p) = \frac{1}{2}|p|^2 + V(x) \quad , \quad (x,p) \in \mathbb{R}^n \times \mathbb{R}^n \quad ,$$

where, here and in the following, $|\ |$ denotes the Eucledian norm in $\mathbb{R}^n$ (occasionally we shall also write p^2 for $|p|^2$). In the results to be presented, some of the next conditions will be required. In the formulation we let $\lambda_1 := (2\pi/T)^2$ and $j, \ell \in \mathbb{N}$.

(V_1) $\quad \limsup\limits_{|x|\to\infty} \dfrac{V(x)}{x^2} < \dfrac{1}{2}\lambda_1$.

$(V_2)_j$ $\quad \liminf\limits_{|x|\to 0} \dfrac{V(x)}{x^2} > \dfrac{1}{2}\lambda_1 j^2$.

$(V_3)_\ell$ $\quad \dfrac{V(x)}{x^2} < \dfrac{1}{2}\lambda_1(\ell+1)^2$ for all $x \in \mathbb{R}^n$.

(V_4) There exist numbers $\mu > 2$ and $R > 0$ such that

---

* Mathematical Institute, Catholic University of Nÿmegen, The Netherlands.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041. This research has been partially supported by the Niels Stensen Stichting, Amsterdam.

$$V'(x)\cdot x \geq \mu V(x) \quad \text{for all} \quad x \in \mathbb{R}^n \quad \text{for which} \quad |x| \geq R \ .$$

The first result deals with subquadratic potentials, i.e. condition (V_1) holds, which satisfy for some $j \in \mathbb{N}$ condition $(V_2)_j$. Clark [1] for the case that V is even, and Costa and Willem [2] for the case that V is convex, obtained the existence of at least $j\cdot n$ distinct T periodic solutions ("distinct solutions" will mean, here and in the following, solutions that have distinct trajectories). See also Amann and Zehnder [3] for a multiplicity result in case V is even and asymptotically linear at infinity. These results do not give any information about the minimal period of these solutions. In that respect, Clarke and Ekeland [4] established the existence of at least one solution with minimal period T for the case of general Hamiltonian systems with convex Hamiltonians which satisfy conditions like (V_1) and $(V_2)_j$. (Their method can be modified to be applicable to the natural Hamiltonians we are considering here, but their results do not seem to cover this case directly.) See also Ambrosetti and Mancini [5] for related results.

In the formulation of the next theorem, and in the following, we use the notation $[a]$ for $a > 0$ to denote the integer part of a:
$[a] = \max\{k : k \in \mathbb{N} \cup \{0\}, k \leq a\}$.

THEOREM 1.

Suppose that V is even or strictly convex, and that V satisfies for some $j \in \mathbb{N}$ conditions (V_1) and $(V_2)_j$. Then we have:

(i) For each $k \in \mathbb{N}$, $1 \leq k \leq j$, there exists at least one solution of equation (1) with minimal period T/k.

(ii) Equation (1) has at least $j\cdot n$ distinct (non-constant) solutions of period T.

If, in addition, V satisfies $(V_3)_j$, then we have furthermore:

(iii) Equation (1) has at least $j \cdot n$ distinct (non-constant) solutions of period T with minimal period not less than T/j.

(iv) For each $k \in \mathbb{N}$, with $[j/2] + 1 \leq k \leq j$, there exist at least n distinct solutions of equation (1) that have minimal period precisely T/k.

The T-periodic solutions which have minimal period less than T, thus having period $T/2, T/3, \ldots$, shall be called <u>superharmonic</u> solutions (compare these with the <u>sub</u>harmonic solutions considered e.g. by Rabinowitz [6]). The other result deals with the case that V is superquadratic at infinity, i.e. V satisfies (V_4). In that case it is well-known that for arbitrary (large) $A > 0$, equation (2) has a periodic solution with period T and with L_∞-norm larger than A (cf. Rabinowitz [7,8]).

We shall show, at least when V is even or convex, that there exists such a solution that has the additional property that its minimal period is arbitrary small.

<u>THEOREM 2.</u>

Suppose that V is even or strictly convex, and that V satisfies condition (V_4).

Then for any $T > 0$ there exist a number $k_0 \in \mathbb{N}$ and a sequence $\{x_k\}$, $k \in \mathbb{N}$, $k \geq k_0$, of T-periodic non-constant solutions of equation (1) for which $|x_k|_{L_\infty} \to \infty$ as $k \to \infty$ and for which the minimal period, to be denoted by τ_k, satisfies $\tau_k \leq T/k$.

All the results stated above will be obtained by applying variational methods directly to the action functional of which (1) is the Euler-Lagrange equation, i.e.

(2) $$\phi(x) = \frac{1}{2}\int \dot{x}^2 - \int V(x) \quad ,$$

where, here and in the following, $\int$ denotes integration with respect to t over an interval of one period, to be taken to be the interval $[-T/2, T/2]$.

If E denotes the set of T-periodic functions, a direct treatment of ϕ on E is known to cause difficulties, mainly because E contains the set of constant vectorfunctions $(\simeq \mathbb{R}^n)$. However, it will be shown that, by restricting ϕ to suitable subsets of E, namely natural constraints, these difficulties can be overcome and, in fact, more specific information about the set of critical points is obtained.

To clarify the underlying idea of natural constraints, let, more generally, E be any set and $\phi \in C^1(E,\mathbb{R})$ a given functional on E. We denote by $S(\phi,E)$ the set of critical points of ϕ on E.

Definition 1.

A subset $\tilde{E}$ of E will be called a natural constraint for the couple (ϕ,E) if:

$$S(\phi,\tilde{E}) \subset S(\phi,E) \quad ,$$

i.e. any critical point of the functional ϕ restricted to the set $\tilde{E}$ is also a critical point of ϕ on E.

Note that, since any critical point of ϕ on E that belongs to $\tilde{E}$ is also a critical point of ϕ on $\tilde{E}$, an equivalent definition is

$$S(\phi,\tilde{E}) = S(\phi,E) \cap \tilde{E} \quad .$$

The notion of natural constraint is known in the literature. In a more restricted sense, requiring at least $S(\phi,E) = S(\phi,\tilde{E})$ it has been studied and propagated most strongly by Berger [9,10,11,12,13,14], but incidentally this method turns up at several places (c.f. for instance Nehari [15], Coffmann [16], Hempel [17], Ambrosetti and Mancini [5,18]).

To obtain more specific information about the solution set $S(\psi,E)$ we shall use natural constraints also in a more restricted sense. For the applications we have in mind, this more restricted use can be formulated as follows.

Definition 2.

A mapping $\Phi : E \to \tilde{E} \subset E$ will be called a natural embedding for the couple (ψ,E), and $\tilde{E}$ defined by $\tilde{E} := \Phi E$ will be called a naturally embedded set for (ψ,E), if

(i) the functional $\tilde{\psi}$, defined by $\tilde{\psi} := \psi\Phi : E \to \mathbb{R}$ belongs to $C^1(E,\mathbb{R})$, and

(ii) for every $x \in S(\tilde{\psi},E)$ it holds that $\Phi x \in S(\psi,E)$.

It is clear that a naturally embedded set $\tilde{E}$ is a natural constraint in the sense of definition 1.

One of the reasons to consider natural constraints is that more specific information can be obtained for the set $S(\psi,E)$. In fact, merely the fact that there are critical points that belong to a certain subset $\tilde{E}$ may provide already specific characterizations or properties of these critical points. Furthermore, and this is especially important from a "constructive" point of view, critical points which are "saddle points" for ψ on E may be minimal points for ψ when ψ is restricted to some natural constraint. It must be remarked, however, that the determination of useful natural constraints for a given problem is not constructive, but requires some a-priori reflection on potentially useful subsets of $S(\psi,E)$.

To return to the specific Hamiltonian problem under consideration, if V is an even function, we shall consider the action functional ψ on the natural constraint E^o which consists of the odd T-periodic functions. Then

$E^\circ \cap R^n = \{0\}$. Note that $S(\phi,E^\circ) \subset S(\phi,E)$, but simple examples show that these sets need not coincide. [For even potentials one can consider instead of E° an even more restricted set $\tilde{E} \subset E^\circ$ as a natural constraint. The corresponding period solutions in $\tilde{E}$ are called <u>normal modes</u> (c.f. [19,20]).]

In case the potential V is a convex function, one usually considers instead of the functional ϕ, the <u>dual</u> action functional (or variants), see e.g. Ekeland [21], Clarke and Ekeland [4], Ambrosetti and Mancini [5,18], Costa and Willem [2]. We shall show that it is possible to deal directly with the action functional (2), if ϕ is considered on the set $\hat{E}$ defined by

(3) $$\hat{E} := \{x \in E : \int V'(x) = 0\}$$

(c.f. Berger [12]). Then $\hat{E} \cap R^n = \{0\}$. We shall show that $\hat{E}$ is a natural constraint for (ϕ,E), and in this case $S(\phi,\hat{E}) \equiv S(\phi,E)$. (For non-convex potentials V it could happen that $S(\phi,E)$ is a real subset of $S(\phi,\hat{E})$). In fact, we shall show that the constrained variational problem

(4) $$\operatorname{stat}\{\phi(x) : x \in E, \int V'(x) = 0\}$$

is, for strictly convex functions V, an explicit characterization of the unconstrained variational problem

(5) $$\operatorname{stat}\{\max\{\phi(\bar{x}+y) : \bar{x} \in R^n\} : y \in \mathring{E}\}$$

where the set E is written as the direct sum of constant vectorfunctions, R^n, and functions with mean value zero, $\mathring{E}$. In particular, when "stat" is replaced by "inf" in (4), and consequently in (5), which is meaningful if V satisfies (V_1), $(V_2)_1$, the constrained minimization problem (4) is an <u>explicit</u> formulation for the mini-max problem (5).

<u>Remark 1</u>. It is likely that (5), contrary to (4), gives for a larger class of potentials than the convex ones the desired critical points, but this has not been investigated in detail yet.

Remark 2. For the problem of finding T-periodic solutions of more general Hamiltonian systems, it is also possible to introduce a natural constraint if the Hamiltonian is assumed to be (strictly) convex. In fact, defining spaces E and $\hat{E}$ as above for 2n-vector functions x, (4) is the constrained canonical action principle for a system with Hamiltonian V if the functional ϕ is taken to be

$$\phi(x) = \int \frac{1}{2} x \cdot J\dot{x} - \int V(x) \quad ,$$

where J is the usual symplectic matrix $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ with I the identity matrix in $\mathbb{R}^n$.

Writing $\tilde{E}$ for E^* if V is even and for $\hat{E}$ if V is convex, we shall take as naturally embedded sets the sets $\tilde{E}_k \subset E$, $k \in \mathbb{N}$, consisting of k-th superharmonic functions of $\tilde{E}$, i.e. functions in $\tilde{E}$ which have period T/k.

The advantage of using these natural constraints becomes clear from the proof of the theorems. In fact, for Theorem 1, the j solutions referred to in part (i) will be obtained as the minimizing elements of the functional ϕ on $\tilde{E}_k$, $1 \leq k \leq j$. Exploiting invariance properties of the couple $(\phi, \tilde{E})$, the other solutions are obtained using Ljusternik-Schnirelmann theory. For Theorem 2, for each $k \in \mathbb{N}$, $k > k_0$, a non-trivial critical point of ϕ on $\tilde{E}_k$ will be obtained by using the Mountain Pass Lemma (Ambrosetti and Rabinowitz [22]).

Remark 3. In the situation of theorem 2, with an additional (monotonicity) condition on V, it is possible to obtain solutions which have minimal period precisely T/k. Instead of applying the Mountain Pass Lemma, these solutions are obtained as solutions of a specific minimization problem which involves an additional natural constraint. See [20] for the case of normal modes.

Remark 4. The (regularity) assumptions on V may be weakened. In fact, for the case that V is even, the proof of the results only uses $V \in C^1$. For convex, not necessarily strictly convex, C^1 functions V, the results can be established by approximating V by strictly convex C^2 functions (see Berger [9, p. 338-339] for an example of such an approximating procedure).

Other applications of the use of natural constraints in critical point theory, in particular to the problem of multiple solutions in semi-linear boundary value problems on domains with rotation symmetry, shall be dealt with in a forthcoming paper [23].

In Section 2 we shall present the proof of Theorems 1 and 2 for the case that the potential V is an even function, whereas Section 3 deals with the case of convex potentials.

I thank Paul H. Rabinowitz for suggestions for improvement of the presentation of the results.

2. Proof of Theorem 1 and 2 for even potentials.

In this section we consider the case that V is an even function. Let E be the set of T-periodic functions:

$$E = \{x \in H_1(\mathbb{R},\mathbb{R}^n) : x \text{ is T-periodic}\} \quad ,$$

where $H_1(\mathbb{R},\mathbb{R}^n)$ is the usual Sobolev space of n-vector functions. E^o will denote the subset of odd T-periodic functions:

$$E^o = \{x \in E : x \text{ is odd}\} \quad .$$

Note that any $x \in E^o$ satisfies $x(0) = x(-T/2) = x(T/2) = 0$. E^o is a Hilbert space with norm denoted by $\| \ \|$:

$$\|x\| := \{\int \dot{x}^2\}^{1/2} \quad \text{for} \quad x \in E^o \quad .$$

For $k \in \mathbb{N}$, we define on E the mapping Φ_k

(2.1) $$\Phi_k : E \to E \quad , \quad \Phi_k x(t) := x(kt) \quad ,$$

and introduce sets E_k^o as the image of E^o under Φ_k:

$$E_k^o := \Phi_k E^o \quad .$$

Then $E_1^o \equiv E^o$, and E_k^o consists of all odd, periodic functions with period T/k (superharmonic functions of E^o).

The action functional ψ, given by (2), is well defined on E and satisfies $\psi \in C^1(E,\mathbb{R})$ and $\psi \in C^1(E^o,\mathbb{R})$. The restriction of ψ to E_k^o defines via $\psi_k := \psi\Phi_k$ a C^1-functional on E^o:

(2.2) $$\psi_k(x) = \frac{1}{2} k^2 \int \dot{x}^2 - \int V(x) \quad .$$

To show that E^o is a natural constraint for the couple (ψ,E), observe that any critical point x of ψ on E^o satisfies for some even function $\xi(t)$ the equation

$$-\ddot{x} = V'(x) + \xi \quad .$$

Taking the inner product of this equation with ξ and integrating over $(-T/2,T/2)$ gives $\int \xi^2 = 0$ because x, and hence $V'(x)$ and $\ddot{x}$ are odd. Thus $\xi \equiv 0$, and x satisfies equation (2), i.e. x is a solution of the *original problem.* Thus $S(\psi,E^\circ) \subset S(\psi,E)$ as required.

That Φ_k is a natural embedding follows from the observation that if $y \in E^\circ$ is a critical point of ψ_k on E°, y satisfies $-k^2\ddot{y} = V'(y)$. Hence, defining $x(t) := \Phi_k y(t)$, x satisfies equation (2) and $x \in E_k^\circ$, which shows that Φ_k is a natural embedding. Note that if $y \in S(\psi_k,E^\circ)$ is known to have <u>minimal</u> period T, then $\Phi_k y \in S(\psi,E^\circ)$ has <u>minimal</u> period T/k. Resuming these results we have

<u>LEMMA 1</u>.

If for some $k \in \mathbb{N}$, $x \in S(\psi_k,E^\circ)$, then $\Phi_k x \in S(\psi,E^\circ)$ and $\psi_k(x) = \psi(\Phi_k x)$. Moreover, if $x \in E^\circ$ has minimal period T, then $\Phi_k x \in E_k^\circ$ has minimal period T/k.

The proof of Theorem 1, part (i), is an immediate consequence of lemma 1 and the next lemma.

<u>LEMMA 2</u>.

Suppose that V satisfies (V_1) and, for some $j \in \mathbb{N}$, $(V_2)_j$. Then, for any $k \in \mathbb{N}$, $1 \le k \le j$, the minimization problem

$$\inf\{\psi_k(x) : x \in E^\circ\} \tag{2.3}$$

has at least one solution x_k, and x_k is nontrivial and has minimal period T.

<u>Proof</u>.

Let us start to recall, for future reference, the Poincaré-Friedrichs inequalities: with $\lambda_1 = (2\pi/T)^2$

$$\|x\|^2 \ge \lambda_1 \int x^2 \quad \text{and} \quad |x|_{C^0} \le \sqrt{T}\,\|x\| \quad , \tag{2.4}$$

valid for all functions x for which $\int x = 0$, so certainly for all $x \in E^\circ$.

As a consequence of condition (V_1) there exist constants α and M, satisfying $0 < \alpha < \lambda_1$ such that

$$V(x) \le \tfrac{1}{2}\alpha|x|^2 + M \quad \text{for all} \quad x \in \mathbb{R}^n \quad . \tag{2.5}$$

Using this, together with (2.4), we have for the functional ψ_k: $(k \in \mathbb{N})$

$$\psi_k(x) \geq \frac{1}{2} k^2 \|x\|^2 - \frac{1}{2} \alpha \int |x|^2 - MT \geq \frac{1}{2} \left(\frac{\lambda_1 - \alpha}{\lambda_1}\right) \|x\|^2 - MT \ .$$

Since $\alpha < \lambda_1$, it follows that ψ_k is coercive on E^*: $\psi_k(x) \to \infty$ as $\|x\| \to \infty$.

It is a standard result that ψ_k is also continuous with respect to the weak convergence in E^*. As a consequence, Weierstrass' theorem provides the existence of a solution of the minimization problem (2.4), for any $k \in \mathbb{N}$.

Remains to show that x_k is non trivial and has minimal period T for $k \in \mathbb{N}$, $1 \leq k \leq j$ if condition $(V_2)_j$ is satisfied. To prove that $x_k \neq 0$ for $k \leq j$, note that $\psi_k(0) = 0$, whereas we shall show that the minimum value of (2.3) is negative. In fact, as a consequence of $(V_2)_j$, there exists numbers $\beta > \lambda_1$ and $\rho_0 > 0$ such that

(2.6) $$V|x| \geq {}^1\!/_2\, \beta \cdot j^2 |x|^2 \quad \text{for} \quad x \in \mathbb{R}^n,\ |x| \leq \rho_0 \ .$$

Hence, with (2.6) we have for $x(t) = \rho \sin\sqrt{\lambda_1}\, t$, $\rho < \rho_0$ and $k \leq j$:

$$\psi_k(x) \leq \frac{1}{2} k^2 \|x\|^2 - \frac{1}{2} \beta j^2 \int x^2 \leq \frac{1}{2} j^2 (\lambda_1 - \beta) \int x^2 < 0 \ .$$

To show that x_k has minimal period T, suppose on the contrary that x_k has minimal period T/m, for some $m \in \mathbb{N}$, $m \geq 2$. Define a function z by $z(t) := x_k(t/m)$. Then $z \in E^*$ and

$$\psi_k(z) = \frac{1}{2} \cdot \frac{k^2}{m^2} \cdot \|x_k\|^2 - \int V(x_k) \ .$$

As x is non-trivial, $\|x_k\| \neq 0$, and thus $\psi_k(z) < \psi_k(x_k)$, contradicting the fact that x_k is a solution of the minimization problem (2.3).

This completes the proof. ∎

For the proof of the other parts of theorem 1 we shall use Ljusternik-Schnirelmann theory. Therefore it is necessary that the Palais-Smale condition is satisfied.

LEMMA 3.

Suppose that V satisfies the condition (V_1). Then, for each $k \in \mathbb{N}$, the functional ψ_k on E^* satisfies the P.S. condition, i.e. if $\{x_n\}$ is any sequence in E^* such that (i) $|\psi_k(x_n)|$ is bounded and (ii) $\psi_k'(x_n) \to 0$ as $n \to \infty$, then $\{x_n\}$ contains a convergent subsequence.

The proof of this result is standard; it is the same as the proof of lemma $\hat{3}$ in the next section if one puts $\phi(y) \equiv 0$ and replaces $\mathring{E}$ by E^* in that proof.

Next observe that E^*, being a linear space, and the functionals ψ_k, which are even since V is an even function, are invariant for the action of the group $\mathbb{Z}_2 = \{id,-id\}$, where id is the identity map in E^*. In the application of the mini-max theory we shall use the genus as index theory (cf. Krasnoselskii [24], Coffmann [16]). The genus of a symmetric, compact subset $A \subset E^*\setminus\{0\}$ will be denoted by $\mathrm{ind}(A)$, and is defined as $\mathrm{ind}(A) = m \in \mathbb{N} \cup \{0\}$ if m is the least number for which there exists an odd, continuous mapping $A \to \mathbb{R}^m\setminus\{0\}$, and $\mathrm{ind}(A) = \infty$ if no such mapping exists.

If ϕ is any even functional on E^*, satisfying the P.S. condition, $\phi(0) = 0$, and $a := \inf\{\phi(x) : x \in E^*\}$ is finite and negative, the results of the Ljusternik-Schnirelmann theory can be summarized as follows (cf. Clark [25]):

The number of $\mathbb{Z}_2$-distinct critical points of ϕ on E^* with values less than or equal to $b < 0$, is not less than $\mathrm{ind}(\phi^{-1}([a,b]))$, where $\phi^{-1}([a,b]) = \{x \in E^* : a \leq \phi(x) \leq b\}$ is the preimage of $[a,b]$ under ϕ.

As a consequence, if one can find some symmetric, compact set $\Sigma \subset E^*\setminus\{0\}$ with $\mathrm{ind}(\Sigma) = \ell \in \mathbb{N}$, the number of distinct critical points of ϕ (with negative critical values) is not less than ℓ if $\phi(\Sigma) < 0$.

For the functionals under consideration we shall show that we can use one of the following sets Σ_ℓ^ρ with appropriate, on k dependent, subscript ℓ and superscript ρ. For $\ell \in \mathbb{N}$ and $\rho > 0$ the set Σ_ℓ^ρ is defined by

$$(2.7)\ \Sigma_\ell^\rho := \{x = \sum_{m=1}^{\ell} e_m \sin(\sqrt{\lambda_1}\, mt) : e_m \in \mathbb{R}^n \text{ for } 1 \le m \le \ell;\ \|x\| = \rho\} .$$

Then $\operatorname{ind}(\Sigma_\ell^\rho) = \ell \cdot n$ for any $\rho > 0$.

LEMMA 4.

Suppose that V satisfies (V_1) and $(V_2)_j$, and let $k \in \mathbb{N}$, $1 \le k \le j$. Then the functional ψ_k has at least $[j/k]\cdot n$ $\mathbb{Z}_2$-distinct critical points on E^* whose critical values are negative.

Proof:

We shall show that for ρ sufficiently small, $\psi_k(\Sigma_\ell^\rho) < 0$ provided $\ell \le [j/k]$. The Ljusternik-Schnirelmann theory summarized above, then gives the result stated in the lemma.

It is readily seen that for $x \in \Sigma_\ell^\rho$ we have

$$\frac{1}{2} k^2 \int \dot{x}^2 \le \frac{1}{2} k^2 \ell^2 \lambda_1 \int x^2 .$$

Together with (2.6) this gives

$$(2.8)\qquad \psi_k(x) \le \frac{1}{2} (k^2\ell^2\lambda_1 - \beta j^2) \cdot \int x^2 \text{ for } x \in \Sigma_\ell^\rho,\ \rho < \rho_0 ,$$

which implies (because Σ_ℓ^ρ is compact) that $\psi_k(\Sigma_\ell^\rho) < 0$ for $\rho < \rho_0$ provided $\ell \le [j/k]$. ∎

Theorem 1, part (ii), follows immediately by taking $k = 1$ in lemma 4.

According to lemma 1, to each set of critical points of the functional ψ_k, which have (not necessarily minimal) period T, there corresponds a set of critical points of ψ with (not necessarily minimal) period T/k.

For the set of critical points obtained in this way, it is always possible to find a lower bound for the minimal period. Indeed, any potential that satisfies condition (V_1), satisfies condition $(V_3)_\ell$ for some $\ell \in \mathbb{N}$ (large enough). The following lemma gives this lower bound and one of the consequences for the problem under consideration.

__LEMMA 5.__

Suppose that V satisfies condition $(V_3)_\ell$ for some $\ell \in \mathbb{N}$. Then we have:

(i) Any critical point of the functional ψ on E^* which has negative critical value, has minimal period not less than T/ℓ.

(ii) For $k \in \mathbb{N}$, $[\ell/2] + 1 \leq k \leq \ell$, any critical point of the functional ψ_k on E^* which has negative critical value, has minimal period T.

__Proof:__

Because of condition $(V_3)_\ell$ we have for $k \in \mathbb{N}$:

(2.9) $\psi_k(x) \geq \frac{1}{2} k^2 \|x\|^2 - \frac{1}{2} \lambda_1 (\ell+1)^2 \int x^2 \geq \frac{1}{2} \lambda_1 (k^2-(\ell+1)^2) \int x^2, \ \forall\, x \in E^*$,

for which it follows that ψ_k is non-negative on E^* for every $k \geq \ell+1$.

For $k \in \mathbb{N}$, let $x \in E^*$ be a critical point of ψ_k for which $\psi_k(x) < 0$, and suppose that x has T/m as minimal period, for some $m \in \mathbb{N}$. Define a function z by $z(t) := x(t/m)$. Then $z \in E^*$, z is a critical point of the functional ψ_{mk}, and $\psi_{mk}(z) = \psi_k(x)$. Since $\psi_k(x) < 0$ it follows that $m \cdot k$ must satisfy $m \cdot k \leq \ell$.

Taking $k = 1$ in this argument (thus $\psi_1 \equiv \psi$), gives the result (i), whereas if $k \geq [\ell/2] + 1$, this inequality can only be satisfied for $m = 1$, which proves part (ii) of the lemma. ∎

Now we are able to complete the proof of theorem 1. Part (iii) of theorem 1 is an immediate consequence of Lemma 5 (i) with $\ell = j$, and the observation that Lemma 4 gives the existence of at least $j \cdot n$ distinct critical points with negative critical values. In the same way combining Lemma 4 and Lemma 5 (ii) gives the proof of part (iv) of Theorem 1. The proof of Theorem 1 is therefore completed.

To prove theorem 2 we shall show that there exists a number $k_0 \in \mathbb{N}$ such that for each $k \in \mathbb{N}$ with $k > k_0$, the functional ϕ_k has a non-trivial critical point $y_k \in E^*$ and that $|y_k|_{L_\infty} \to \infty$ as $k \to \infty$.

Once the existence of a sequence $\{y_k\}$ with these properties has been shown, the proof of theorem 2 follows immediately from the fact that E_k^* is naturally embedded in E^*: defining $x_k := \phi_k y_k$ for $k > k_0$, each x_k is a critical point of ϕ on E^*, $x_k \in E_k^*$ has minimal period not greater than T/k and $|x_k|_{L_\infty} = |y_k|_{L_\infty}$.

In fact we shall prove somewhat more than required. We shall prove that for all $\varepsilon > 0$ sufficiently small, the functional $\psi^\varepsilon \in C^1(E^*, \mathbb{R})$ defined by

$$\psi^\varepsilon(x) := \frac{1}{2} \int \dot{x}^2 - \varepsilon^2 \int V(x) \tag{2.10}$$

has a non-trivial critical point x^ε, and that $|x^\varepsilon|_{L_\infty} \to \infty$ as $\varepsilon \to 0$. Taking a sequence $\varepsilon_k = 1/k \to 0$ as $k \to \infty$ gives the required result as a special case.

PROPOSITION 1.

Suppose that the function V satisfies condition (V_4) and let ψ^ε be defined by (2.10). Then there exists a number $\varepsilon_0 > 0$ such that for each ε, with $0 < \varepsilon < \varepsilon_0$, the functional ψ^ε has on E^* a non-trivial critical point x^ε with the property that $|x^\varepsilon|_{L_\infty} \to \infty$ as $\varepsilon \to 0$.

Proof:

For the proof of the existence of a critical point we use the celebrated Mountain Pass Lemma (Ambrosetti and Rabinowitz [22]). (Note that ψ^ε is neither bounded from below nor from above on E^*.) For the applicability of this lemma the Palais-Smale condition has to be verified.

Lemma 6

Let the function V satisfy (V_4). Then the functional ψ^ε on E^* satisfies the P.S.-condition.

The proof of this result is standard; it is the same as the proof of lemma $\hat{6}$ in the next section if one puts $\phi(y) \equiv 0$ and replaces $\dot{E}$ by E^* in that proof.

Next, define a number c^ε as:

$$c^\varepsilon := \inf \max \psi^\varepsilon(x) \quad ,$$

where "max" is taken over the points of a continuous path in E^* connecting two points $x_0, x_1 \in E^*$, and "inf" is taken over all the paths with this property. Then the mountain pass lemma states that c^ε is a critical value of ψ^ε if

$$c^\varepsilon > \max(\psi^\varepsilon(x_0), \psi^\varepsilon(x_1)) \quad .$$

We shall take $x_0 \equiv 0$, and for x_1 any point in E^* with sufficiently large norm, say $\|x_1\| > 1$, for which $\psi^\varepsilon(x_1) < \psi^\varepsilon(0) = 0$. The existence of points x_1 with this property is a consequence of condition (V_4). Indeed, this condition implies that there exist constants $\alpha > 0$ and $A \in \mathbb{R}$ such that

$$V|x| > \alpha|x|^\mu - A \quad \text{for all} \quad x \in \mathbb{R}^n \quad . \tag{2.11}$$

Hence, for any $y \in E^*\backslash\{0\}$, and $\sigma > 0$ we have the estimate

$$\psi^\varepsilon(\sigma y) < -\sigma^\mu(\varepsilon^2\alpha \int |y|^\mu - \frac{1}{2}\sigma^{2-\mu} \int \dot{y}^2) + \varepsilon^2 AT \quad .$$

Since $\mu > 2$, it follows that $\psi(\sigma y) \to -\infty$ as $\sigma \to \infty$. Having chosen x_1

such that $\|x_1\| > 1$, $\psi^\varepsilon(x_1) < 0$, it remains to show that $c^\varepsilon > 0$. But this is easy: for $x \in S_\rho := \{x \in E^\circ : \|x\| = \rho\}$, it follows with (2.4) that

$$\psi^\varepsilon(x) > \frac{1}{2}\rho^2 - \varepsilon^2 T M(\rho), \quad \text{for} \quad x \in S_\rho \quad ,$$

where $M(\rho) := \max\{V(x) : x \in \mathbb{R}^n, |x| < \sqrt{T}\,\rho\}$. For given ρ, say $\rho = 1$, it follows that

$$\psi_\varepsilon(x) > \frac{1}{2} - \varepsilon^2 T M(1) > \frac{1}{4} \quad \text{for} \quad \|x\| = 1 \quad ,$$

for all $\varepsilon > 0$ satisfying $\varepsilon < \varepsilon_0 := \frac{1}{2}(TM(1))^{-1/2}$. Hence, for all ε, $0 < \varepsilon < \varepsilon_0$, $c^\varepsilon > \frac{1}{4}$, and c^ε is a critical value by the mountain pass lemma.

For $\varepsilon < \varepsilon_0$, let x^ε be any "mountain pass point", i.e. a critical point of ψ^ε with $\psi^\varepsilon(x^\varepsilon) = c^\varepsilon$. As $c^\varepsilon > 0$, x^ε is nontrivial. To show that x^ε satisfies $|x^\varepsilon|_{L_\infty} \to \infty$ as $\varepsilon \to 0$, take the L_2-innerproduct of the function x^ε with the equation satisfied by x^ε:

$$-\ddot{x}^\varepsilon = \varepsilon^2 V'(x^\varepsilon) \quad .$$

A partial integration yields $\|x^\varepsilon\|^2 = \varepsilon^2 \int V'(x^\varepsilon)\cdot x^\varepsilon$, and consequently

$$\psi^\varepsilon(x^\varepsilon) = \varepsilon^2 \int \left(\frac{1}{2} V'(x^\varepsilon)\cdot x^\varepsilon - V(x^\varepsilon)\right) \quad .$$

Since $\psi^\varepsilon(x^\varepsilon) > 1/4$ for all $\varepsilon > 0$, it follows that

$$\int \left(\frac{1}{2} V'(x^\varepsilon)\cdot x^\varepsilon - V(x^\varepsilon)\right) > (2\varepsilon)^{-2} \to \infty \quad \text{as} \quad \varepsilon \to 0 \quad .$$

From this result one concludes that $|x^\varepsilon|_{L_\infty}$ is unbounded as $\varepsilon \to 0$. This completes the proof of the proposition. ∎

3. Proof of theorem 1 and 2 for convex potentials.

In this section we consider the case that V is a strictly convex function and $V \in C^2(\mathbb{R}^n,\mathbb{R})$. Without loss of generality we may, and shall, assume that V attains its minimum value at the origin and that this value is zero (as before):

$$V(0) = 0 < V(x) \quad \text{for all} \quad x \in \mathbb{R}^n \quad .$$

For the space E defined in section 2, we shall use the decomposition $E = \mathbb{R}^n \oplus \mathring{E}$, where $\mathring{E}$ is given by

$$\mathring{E} = \{y \in E : \int y = 0\} \quad .$$

$\mathring{E}$ is a Hilbert space with the norm $\| \ \|$ as in section 2: $\|y\| = (\int \dot{y}^2)^{1/2}$.

For the norm in E we shall take

$$|||x||| := \{|\bar{x}|^2 + \|y\|^2\}^{1/2} \quad \text{for} \quad x = \bar{x}+y, \ \bar{x} \in \mathbb{R}^n, \ y \in \mathring{E} \quad .$$

In the space E we consider the subset $\hat{E}$:

(3.1) $$\hat{E} := \{x \in E : \int V'(x) = 0\} \quad .$$

We shall investigate this set in detail below, but note already that $\hat{E} \cap \mathbb{R}^n = \{0\}$ and that $\hat{E}$ is a regular manifold in E, i.e. for every $x \in \hat{E}$, the linear mapping

$$\xi \mapsto \int V''(x)\xi \quad \text{maps} \quad E \quad \text{onto} \quad \mathbb{R}^n \quad .$$

The mapping Φ_k, $k \in \mathbb{N}$, given by (2.1), defines a set $\hat{E}_k := \Phi_k\hat{E}$. As is easily verified, $\hat{E}_k \subset \hat{E}$.

The action functional ψ belongs to $C^1(\hat{E},\mathbb{R})$ and defines via the definition $\psi_k := \psi\Phi_k$ functionals $\psi_k \in C^1(\hat{E},\mathbb{R})$, explicitly given by (2.2).

To show that $\hat{E}$ is a natural constraint for the couple (ψ,E), observe that any critical point x of ψ on $\hat{E}$ satisfies for some multiplier $\sigma \in \mathbb{R}^n$ the equation

$$- \ddot{x} = V'(x) + V''(x)\sigma \quad ,$$

together with periodicity conditions $x(-T/2) = x(T/2)$, $\dot{x}(-T/2) = \dot{x}(T/2)$.

Taking the $\mathbb{R}^n$-innerproduct of this equation with the vector σ, and integrating over $(-T/2,T/2)$ gives $\int V''(x)\sigma\cdot\sigma = 0$. Since $V''(x(t))$ is for every $t \in (-T/2,T/2)$ a positive definite matrix, it follows that $\sigma \equiv 0$. Thus x satisfies equation (2), i.e. $S(\psi,\hat{E}) \subset S(\psi,E)$ as required.

In the same way as in section 2, it is readily verified that ϕ_k is a natural embedding. Consequently we have:

Lemma $\hat{1}$ ≡ Lemma 1, with E^* replaced by $\hat{E}$.

In order to be able to prove theorems 1 and 2 for the convex case along the same lines as in section 2, we need more information about the set $\hat{E}$. The following proposition gives a characterisation of this set $\hat{E}$, and may be of interest in itself.

Proposition 2.

Let $V \in C^2(\mathbb{R}^n,\mathbb{R})$ be strictly convex, $V(0) = 0 < V(x)$ on $\mathbb{R}^n$, and let $\hat{E}$ be defined by (3.1). Then we have:

(i) $\hat{E}$ is an unbounded subset of E and $\hat{E}$ is closed with respect to weak convergence in E.

(ii) There exists a single-valued mapping $\phi : \mathring{E} \to \mathbb{R}^n$ such that

(3.2) $$\hat{E} = (id+\phi)\,\mathring{E} \quad ,$$

where id is the identity mapping. In fact, for any $y \in \mathring{E}$, $\phi(y)$ is uniquely determined as the solution of the minimisation problem

(3.3) $$\min\{\int V(\bar{x}+y) \;:\; \bar{x} \in \mathbb{R}^n\} \quad .$$

Furthermore, ϕ is differentiable and continuous with respect to weak convergence in $\mathring{E}$.

(iii) Let S_ρ denote the sphere of radius ρ in E, and $\mathring{B}_\rho$ the ball of radius ρ in $\mathring{E}$. Then there exists a monotonically increasing function $a : \mathbb{R}_+ \to \mathbb{R}_+$ with $a(0) = 0$, $a(\rho) \to \infty$ as $\rho \to \infty$, such that

$$(\hat{E} \cap S_\rho) \cap \mathring{B}_{a(\rho)} = \emptyset \quad \text{for all} \quad \rho > 0 \quad ;$$

equivalently: for any $x = \bar{x} + y \in \hat{E}$, $\bar{x} \in \mathbb{R}^n$, $y \in \mathring{E}$ it holds:

(3.4) $$|||x||| \geq \|y\| \geq a(|||x|||) \quad .$$

(iv) The following inequalities hold:

(3.5) $$TV(\phi(y)) \leq \int V(y+\phi(y)) \leq \int V(y) \quad \text{for all} \quad y \in \mathring{E} \quad .$$

Remarks.

1. If the functional V is quadratic, then $\phi(y) = 0$ for all $y \in \mathring{E}$, i.e. $\hat{E} = \mathring{E}$. More generally, if the functional V is even, then $E^o \subset \hat{E}$, where E^o is the space of odd periodic functions introduced in section 2; in that case, $\mathring{E} \neq \hat{E}$ in general.

2. The remarks made in the introduction concerning the formulations (4) and (5) are an immediate consequence of this proposition, part (ii).

3. Having defined the mapping ϕ as in the proposition, the problem of finding critical points of Φ on $\hat{E}$ can also be formulated as finding critical points of the functional $\hat{\Phi}$, defined by

(3.6) $$\hat{\Phi}(y) := \int \frac{1}{2} \dot{y}^2 - \int V(y+\phi(y)) \quad , \quad y \in \mathring{E} \quad ,$$

on the set $\mathring{E}$, i.e. $S(\Phi,\hat{E}) \cong S(\hat{\Phi}, \mathring{E})$. (Note that $\hat{\Phi} \in C^1(\mathring{E},\mathbb{R})$). We shall use this observation in the proof of the Palais-Smale condition in the following.

Proof of proposition 2:

Part (i) of the proposition is an immediate consequence of part (ii). To prove (ii), let $y \in \mathring{E}$ be fixed, and consider the function

$$v_y : \mathbb{R}^n \to \mathbb{R} \quad , \quad v_y(\bar{x}) := \int V(\bar{x}+y) \quad .$$

It is a simple matter to verify that $v_y \in C^2(\mathbb{R}^n,\mathbb{R})$ and that it is strictly convex. This function attains its minimum value at a unique point, to be denoted by $\phi(y)$. The stationarity condition $v_y'(\phi(y)) = 0$ is precisely $\int V'(y+\phi(y)) = 0$, i.e. $y + \phi(y) \in \hat{E}$. This shows $(id+\phi)\,\mathring{E} \subset \hat{E}$.

If x is any point in $\hat{E}$, then $x = \bar{x}+y$, with $\bar{x} \in \mathbb{R}^n$ and $y \in \mathring{E}$ uniquely determined. Since $\int V'(\bar{x}+y) = 0$, $\bar{x}$ is a (and consequently the unique) critical point of the function v_y, i.e. $\bar{x} = \phi(y)$. This shows $\hat{E} \subset (id+\phi)\,\mathring{E}$, and together with the other inclusion this gives (3.2).

Now consider the function $F \in C^1(\mathbb{R}^n \times \mathring{E}, \mathbb{R}^n)$ defined by

$$F(\bar{x},y) := \int V'(\bar{x}+y) \quad .$$

Since V is strictly convex, the derivative of F with respect to $\bar{x}$ is non-singular: $D_{\bar{x}} F(\bar{x},y) : \xi \mapsto \int V''(\bar{x}+y)\xi$ is a non-singular mapping from $\mathbb{R}^n$ onto $\mathbb{R}^n$. The implicit function theorem can thus be applied, and gives that, for any $\bar{x}_0 + y_0 \in \hat{E}$, there exists a differentiable mapping, to be denoted by $\tilde{\phi}$, from a neighbourhood U of y_0 into $\mathbb{R}^n$ such that $\tilde{\phi}(y) + y \in \hat{E}$ for all $y \in U$. As $\phi(y)$ is the unique point in $\mathbb{R}^n$ such that $y + \phi(y) \in \hat{E}$, it follows that $\phi \equiv \tilde{\phi}$, which shows that ϕ is differentiable. To show that ϕ is continuous with respect to weak convergence in $\mathring{E}$, let $y_n \to y$ weakly in $\mathring{E}$. Since $\mathring{E}$ is compactly embedded in $\mathring{C} := \{y \in C^{\circ}(\mathbb{R},\mathbb{R}^n) : y \text{ is } T\text{-periodic}, \int y = 0\}$, it follows that $|y_n - y|_{C^{\circ}} \to 0$ as $n \to \infty$.

Another application of the implicit function theorem to the function F, now considered as a mapping $F \in C^1(\mathbb{R}^n \times \mathring{C}, \mathbb{R}^n)$ gives the required result: $\phi(y_n) \to \phi(y)$ as $n \to \infty$.

The essential contents of part (iii) is that for given $\bar{x} \neq 0$, the set of functions $y \in \mathring{E}$ such that $\bar{x} + y \in \hat{E}$ is bounded away from zero. To state this precisely, consider the function $b : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$b(r) := \inf\{\|y\|^2 : y \in \mathring{E}, \phi(y) = r\}, \; r > 0 \quad .$$

As ϕ is continuous with respect to weak convergence in $\mathring{E}$, this minimization problem makes sense, and in fact has a solution (by Weierstrass' theorem for coercive, weakly lower semicontinuous functionals on weakly closed sets). It follows that $b(r) > 0$ for $r > 0$: if $b(r)$ were zero for $r > 0$, then $\phi(0) = r > 0$, contrary to the fact that $V'(\bar{x}) = 0$ on $\mathbb{R}^n$ iff $\bar{x} = 0$. Furthermore, as $\|y\|^2$ has no critical points in $\mathring{E}\setminus\{0\}$, it follows that b is also given by $b(r) = \inf\{\|y\|^2 : y \in \mathring{E},\ \phi(y) \geq r\}$, and that b is a continuous, monotonically increasing function with $b(r) \to \infty$ as $r \to \infty$. Hence the function b has a monotonically increasing inverse $c := b^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$, satisfying $c(0) = 0$ and $c(R) \to \infty$ as $R \to \infty$. (In fact, c is explicitly given by the inverse extremum formulation

$$c(R) = \sup\{\phi(y) : y \in \mathring{E},\ \|y\|^2 \leq R\} ,$$

cf. [26].) Thus $c(\|y\|^2) \geq \phi(y)$ for all $y \in \mathring{E}$, and hence

$$|||y+\phi(y)|||^2 = \|y\|^2 + \phi^2(y) \leq f^2(\|y\|^2) ,$$

where f is the function given by $f(R) := (R + c^2(R))^{1/2}$. With c, the function f is monotonically increasing, and (3.4) follows with the function a defined as the square root of the inverse function of f.

Finally, to prove (iv), recall the well-known inequality for convex functions:

$$V(u) - V(v) \geq V'(v) \cdot (u-v) \quad \text{for all} \quad u, v \in \mathbb{R}^n .$$

Taking $u = y+\phi(y)$, $v = \phi(y)$, and $u = y$, $v = y+\phi(y)$ in this inequality, an integration over $(-T/2,T/2)$ gives the result (3.5). This completes the proof of proposition 2. ■

Having established the foregoing proposition, from now on the proof of theorems 1 and 2 for the case under consideration resembles the proof given in section 2 in many ways. Therefore we shall restrict ourselves in the following to the essentials.

<u>Lemma $\hat{2}$</u> ≡ Lemma 2, with E^* replaced by $\hat{E}$.

Proof:

The most difficult part of the proof is to show that the functionals ψ_k are coercive on $\hat{E}$. (Note that ψ_k is certainly not coercive on E.) Writing $\bar{x} = \phi(y)$, we can use one of the inequalities of (3.5) to obtain

$$\psi_k(\bar{x}+y) = \frac{1}{2} k^2 \int \dot{y}^2 - \int V(\bar{x}+y) \geq \frac{1}{2} k^2 \int \dot{y}^2 - \int V(y) \quad .$$

Now using the inequality (2.5) for $\int V(y)$ (instead of for $\int V(\bar{x}+y)$) gives the existence of constants $\beta > 0$ and $\mu \in \mathbb{R}$, such that $\psi_k(y+\phi(y)) \geq \beta\|y\|^2 - \mu$. The coercivity of ψ_k on $\hat{E}$ then follows because of (3.4): $\psi_k(x) \to \infty$ as $x \in \hat{E}$, $|||x||| \to \infty$.

As a consequence of proposition 2, (i), Weierstrass' theorem for functionals on weakly closed sets, can be applied to provide the existence of a solution of the minimization problem

$$\inf\{\psi_k(x) : x \in \hat{E}\} \quad .$$

That the solutions of this problem are non-trivial and have minimal period T for $1 \leq k \leq j$, follows as in the proof of lemma 2. ■

Lemma $\hat{3}$ ≡ Lemma 3, with E^* replaced by $\hat{E}$.

Proof:

In view of proposition 2 and remark 3 following it, the proof amounts to showing that the functional $\hat{\psi}_k$ satisfies the Palais-Smale condition on $\dot{E}$, where $\hat{\psi}_k$ is defined by

$$\hat{\psi}_k(y) := \int \frac{1}{2} \dot{y}^2 - \int V(y+\phi(y)), \; y \in \dot{E} \quad .$$

Let $\{y_n\} \subset \dot{E}$ be any sequence for which $|\psi_k(y_n)|$ is uniformly bounded and $\hat{\psi}'_k(y_n) \to 0$ as $n \to \infty$. Since ψ_k is coercive on $\hat{E}$, $\hat{\psi}_k$ is coercive on $\dot{E}$ and hence, $\{y_n\}$ is uniformly bounded in $\dot{E}$ and thus has a subsequence, again to be denoted by y_n, which converges weakly in $\dot{E}$ and strongly in C° to some $y \in \dot{E}$. From proposition 2 it follows that $\phi(y_n) \to \phi(y)$, and thus $\int V(y_n+\phi(y_n)) \to \int V(y+\phi(y))$. Since $\hat{\psi}'_k(y_n) \to 0$ implies

$< \psi_k'(y_n) - \hat{\psi}_k'(y),\ y_n - y > \to 0$ as $n \to \infty$, one readily obtains $\|y_n - y\| \to 0$, which has to be proved. ∎

In order to apply mini-max theory, observe that the set $\hat{E}$ and the functionals ψ_k are invariant for the action of the group $G = S^1 \times Z_2$, where S^1 is the group provided by time-translations:

$$S_\theta x(t) := x(t+\theta), \quad \text{for} \quad \theta \in (-T/2, T/2] \quad ,$$

and $Z_2 = \{id, inv\}$, with id the identity and inv time-inversion:

$$inv\ x(t) := x(-t) \quad .$$

The index of a compact G-invariant set $A \subset \hat{E}\backslash\{0\}$ will again be denoted by ind(A), and is defined as (cf. [27, [28], [29]) $ind(A) = m \in \mathbb{N} \cup \{0\}$ if m is the least number for which there exists an equivariant, continuous mapping $h : A \to \mathbb{C}^m\backslash\{0\}$, and $ind(A) = \infty$ if no such mapping exists.

(The mapping $h : A \to \mathbb{C}^m | \{0\}$ is equivariant if

$$h(S_\theta x) = R_\theta h(x),\ h(inv\ x) = -h(x) \quad ,$$

where $R_\theta : \mathbb{C}^m \to \mathbb{C}^m$ is defined for $z = (z_1, z_2, \ldots, z_m) \in \mathbb{C}^m$ as

$$R_\theta z = (e^{i\bar{\theta}} z_1,\ e^{i\bar{\theta}} z_2, \ldots, e^{i\bar{\theta}} z_m) \quad \text{with} \quad \bar{\theta} = \frac{2\pi}{T}\theta \quad .)$$

As a consequence of the Ljusternik-Schnirelmann theory for invariant functionals which are bounded from below on invariant sets, the number of (G -) distinct critical points of ψ_k on $\hat{E}$ is not less than ℓ if some compact, invariant set $\Sigma \subset \hat{E}\backslash\{0\}$ can be found for which $ind(\Sigma) = \ell$ and $\psi_k(\Sigma) < 0$.

Appropriate sets Σ for this case will be (compare with (2.7))

$$\hat{\Sigma}_\ell^\rho := \{y + \phi(y) : y \in \bar{\Sigma}_\ell^\rho\} \quad ,$$

where ϕ is the mapping introduced in proposition 2, and $\bar{\Sigma}_\ell^\rho$ is a subset of $\hat{E}$ defined as $\bar{\Sigma}_\ell^\rho = \bar{\Sigma}_\ell^\rho(+) \cup \bar{\Sigma}_\ell^\rho(-)$ with

$$\overline{\Sigma}^{\rho}_{\ell}(\pm) := \{y = \sum_{m=1}^{\ell} e_m[\sin(\sqrt{\lambda_1}\, mt + \sigma) \pm \cos(\sqrt{\lambda_1}\, mt + \sigma)] :$$

$$\sigma \in (-\pi,\pi],\ e_m \in \mathbb{R}^n \ \text{ for } \ 1 \leq m \leq \ell;\ \|y\| = \rho\} \ .$$

Because of the definition of ϕ, $\hat{\Sigma}^{\rho}_{\ell} \subset \hat{E}$, $\hat{\Sigma}^{\rho}_{\ell}$ is compact (proposition 2, (ii)) and does not contain the zero-element for $\rho > 0$ (proposition 2, (iii)). For any $\rho > 0$, $\mathrm{ind}(\hat{\Sigma}^{\rho}_{\ell}) = \ell \cdot n$.

<u>Lemma $\hat{4}$</u> $\equiv$ Lemma 4, with E^{*} replaced by $\hat{E}$, and Z_2 by G.

<u>Proof</u>: Almost the same as the proof of lemma 4; instead of (2.8) we have for $y \in \overline{\Sigma}^{\rho}_{\ell}$ (writing $\overline{x} = \phi(y)$):

$$\psi_k(\overline{x}+y) \leq \frac{1}{2}(k^2\ell^2\lambda_1 - \beta j^2) \int y^2 - \frac{1}{2}\beta j^2 T|\overline{x}|^2 \quad ,$$

with the same conclusions. ■

<u>Lemma $\hat{5}$</u> $\equiv$ Lemma 5, with E^{*} replaced by $\hat{E}$.

<u>Proof</u>: As in the proof of lemma $\hat{2}$, first use one of the inequalities of (3.5) before using the inequality implied by $(V_3)_j$. Inequality (2.9) becomes, writing $\overline{x} = \phi(y)$:

$$\psi_k(\overline{x}+y) \geq \frac{1}{2}k^2 \int \dot{y}^2 - \int V(y) \geq \frac{1}{2}\lambda_1(k^2 - (\ell+1)^2) \int y^2 \quad ,$$

with the same conclusions. ■

As in section 2, the proof of theorem 1 can easily be completed.

For the result of theorem 2 we shall consider, as in section 2, the more general problem for the functional ψ^{ε}, defined by (2.10), on the set $\hat{E}$. Then $\psi^{\varepsilon} \in C^1(\hat{E},\mathbb{R})$, and Theorem 2 follows as a special case of the following proposition.

<u>PROPOSITION $\hat{1}$</u> $\equiv$ Proposition 1, with E^{*} replaced by $\hat{E}$.

For the proof of this proposition we have to verify

<u>Lemma $\hat{6}$</u> $\equiv$ Lemma 6, with E^{*} replaced by $\hat{E}$.

Proof: As in the proof of Lemma $\hat{3}$ we shall show that the functional $\hat{\psi}^\varepsilon \in C^1(\mathring{E},\mathbb{R})$ defined by

$$\hat{\psi}^\varepsilon(y) := \int \frac{1}{2} \dot{y}^2 - V(y+\phi(y)) \quad , \quad y \in \mathring{E} \quad ,$$

satisfies the P.S. condition. We shall show that any sequence $\{y_n\} \subset \mathring{E}$ for which there exists $M > 0$ such that $|\hat{\psi}^\varepsilon(y_n)| < M$ and $\hat{\psi}^{\varepsilon\prime}(y_n) \to 0$ as $n \to \infty$, is uniformly bounded in $\mathring{E}$. The rest of the proof is then the same as in the proof of lemma $\hat{3}$. Writing $x_n = y_n + \bar{x}_n$ with $\bar{x}_n = \phi(y_n)$, from $\hat{\psi}^{\varepsilon\prime}(y_n) \to 0$ it follows that for arbitrary small $\delta > 0$, for n sufficiently large $|< \hat{\psi}^{\varepsilon\prime}(y_n), y_n >| < \delta \|y_n\|$. Since $\int V'(x_n) = 0$, this implies

$$\left|\int \dot{y}_n^2 - \int V'(x_n) \cdot x_n\right| < \delta\|y_n\| \quad .$$

Because of condition (V_4) there exists a constant $A(<0)$ such that $\frac{1}{\mu} V'(x) \cdot x > V(x) + A$ for all $x \in \mathbb{R}^n$, and hence

$$\frac{1}{\mu} \int \dot{y}_n^2 > \int V(x_n) + A - \delta\|y_n\| \quad .$$

Subtracting this result from $\frac{1}{2} \int \dot{y}_n^2 < \int V(x_n) + M$, there results

$$\left(\frac{1}{2} - \frac{1}{\mu}\right) \|y_n\|^2 < M - A + \delta\|y_n\| \quad .$$

Since $\mu > 2$ it follows that $\|y_n\|$ is uniformly bounded and the proof can be completed. ∎

Proof of proposition $\hat{1}$.

For the application of the Mountain Pass Lemma we first show the existence of an element x_1 with the properties:

(3.7) $\quad x_1 = \bar{x}_1 + y_1 \in \hat{E},\ \bar{x}_1 \in \mathbb{R}^n,\ y_1 \in \mathring{E},\ \|y_1\| > 1$ and $\psi^\varepsilon(x_1) < 0$.

To that end, take any $y \in \mathring{E}$, $y \neq 0$, and consider for $\sigma > 0$, $x_\sigma := \bar{x}_\sigma + \sigma y$, where $\bar{x}_\sigma := \phi(\sigma y)$ (ϕ defined in proposition 2). Then $x_\sigma \in \hat{E}$ and with (2.11) it follows

$$\psi^\varepsilon(\bar{x}_\sigma + \sigma y) < \frac{1}{2} \sigma^2 \int \dot{y}^2 - \alpha\varepsilon^2 \int |\bar{x}_\sigma + \sigma y|^\mu + \varepsilon^2 AT \quad .$$

Since μ satisfies $\mu > 2$, a special case of Jensen's inequality, viz

$$\left(\int |g|\right)^{\mu/2} < c \cdot \int |g|^{\mu/2} \quad , \quad \text{with} \quad c = T^{\mu-2/2} \quad ,$$

can be used to obtain the estimate

$$\int |\bar{x}_\sigma + \sigma y|^\mu \geq \frac{1}{c} \{\int |\bar{x}_\sigma + \sigma y|^2\}^{\mu/2} \geq \frac{1}{c} \cdot \sigma^\mu \{\int y^2\}^{\mu/2} \quad ,$$

and one easily concludes $\psi^\varepsilon(\bar{x}_\sigma + \sigma y) \to -\infty$ as $\sigma \to \infty$, from which the existence of elements x_1 with the desired properties (3.7) follows.

Next, to show $c^\varepsilon > 0$, let $y \in S_\rho := \{y \in \hat{X} : \|y\| = \rho\}$, and consider $x = y + \phi(y)$, ϕ defined in proposition 2. With (3.4) it follows that $|||x||| \leq a^{-1}(\rho)$, where a^{-1} is the inverse of the function a. As $\min(1,T) \cdot |x|^2_{C^0} \leq |||x|||^2$, we find, taking $\rho = 1$ and writing

$b = \{\frac{1}{T} \cdot \frac{a^{-1}(1)}{\min(1,T)}\}^{1/2}$:

$$\psi^\varepsilon(y + \phi(y)) \geq \frac{1}{2} - \varepsilon^2 TM(b) \geq \frac{1}{4} \text{ for all } y \in S_\rho \quad ,$$

for all $\varepsilon > 0$ satisfying $\varepsilon < \varepsilon_0 = \{TM(b)\}^{1/2}$.

As a consequence, on any continuous path connecting 0 and a point x_1 with the properties (3.7), the functional ψ^ε attains values not less than $\frac{1}{4}$ if $\varepsilon < \varepsilon_0$. Hence $c^\varepsilon \geq \frac{1}{4}$ for all ε, $0 < \varepsilon < \varepsilon_0$.

Along the same lines as in the proof of proposition 1, one shows that the mountain pass points x^ε satisfy $|x^\varepsilon|_{L_\infty} \to \infty$ as $\varepsilon \to 0$. This completes the proof of proposition $\hat{1}$. ∎

REFERENCES

1. CLARK, D. C., On periodic solutions of autonomous Hamiltonian systems of ordinary differential equations; Proc. A.M.S. 39 (1973) 579-584.
2. COSTA, D. G. and M. WILLEM, Multiple critical points of invariant functionals and applications; MRC Report #2532, June 1983.
3. AMANN, H. and E. ZEHNDER, Periodic solutions of asymptotically linear Hamiltonian systems; Manuscripta Math. 32 (1980) 149-189.
4. CLARKE, F. and I. EKELAND, Hamiltonian trajectories having prescribed minimal period; Comm. Pure Appl. Math. 33 (1980) 103-116.
5. AMBROSETTI, A. and G. MANCINI, Solutions of minimal period for a class of convex Hamiltonian Systems; Math. Annalen 255 (1981) 405-421.
6. RABINOWITZ, P. H., On subharmonic solutions of Hamiltonian systems; Comm. Pure Appl. Math. 33 (1980) 609-633.
7. RABINOWITZ, P. H., A variational method for finding periodic solutions of differential equations; in: Nonlinear Evolution Equations, M. G. Crandall, ed.; Academic Press, New York, 1978, 225-251.
8. RABINOWITZ, P. H., Periodic solutions of large norm of Hamiltonian systems; to appear in Jrnl. Diff. Eqns.
9. BERGER, M. S., Nonlinearity and Functional Analysis; Academic Press, New York 1977.
10. BERGER, M. S., Creation and breaking of self duality symmetry - A modern aspect of Calculus of Variations; Contemporary Mathem. 17 (1983) 379-394.
11. BERGER, M. S., On periodic solutions of second order Hamiltonian systems (1); Jrnl. Math. Anal. Appl. 29 (1970) 512-522.
12. BERGER, M. S., Periodic solutions of second order dynamical systems and isoperimetric variational problems; Amer. J. Math. 93 (1971) 1-10.

13. BERGER, M. S. and E. BOMBIERI, On Poincaré's Isoperimetric Problem; Jrnl. Funct. Anal. 42 (1981) 274-298.

14. BERGER, M. S. and M. SCHECHTER, On the solvability of semilinear gradient operator equations; Advances in Math. 25 (1977) 97-132.

15. NEHARI, Z., On a class of nonlinear second order differential equations; Trans. A.M.S. 95 (1960) 101-123.

16. COFFMANN, C. V, A minimum-maximum principle for a class of nonlinear integral equations; J. Analyse Math. 22 (1969) 391-419.

17. HEMPEL, J. A., Multiple solutions for a class of nonlinear boundary value problems; Ind. Univ. Math. Jrnl. 20 (1971) 983-996.

18. AMBROSETTI, A. and G. MANCINI, On a theorem by Ekeland and Lasry concerning the number of periodic Hamiltonian trajectories; Jrnl. Diff. Eqns. 43 (1982) 249-256.

19. VAN GROESEN, E. W. C., Existence of multiple normal mode trajectories on convex energy surfaces of even, classical Hamiltonian systems; Jrnl. Diff. Eqs., to appear.

20. VAN GROESEN, E. W. C., Multiple normal modes in natural Hamiltonian systems; in: Topological Methods in Nonlinear Analysis, proceedings of 22-nd session of Séminaire de Mathématiques Supericures, A. GRANAS, ed., Université de Montréal, summer 1983, to appear.

21. EKELAND, I., Periodic solutions of Hamiltonian equations and a theorem of P. Rabinowitz; Jrnl. Diff. Eqns. 34 (1979), 523-534.

22. AMBROSETTI, A. and P. H. RABINOWITZ, Dual variational methods in critical point theory and applications; Jrnl. Funct. Anal. 14 (1973) 349-381.

23. VAN GROESEN, E. W. C., Applications of natural constraints in critical point theory to boundary value problems on domains with rotation symmetry; MRC Report #2594, November 1983.

24. KRASNOSELSKII, M. A., Topological methods in the theory of nonlinear integral equations; Macmillan, New York, 1964.

25. CLARK, D. C., A variant of Lusternik-Schnirelman theory; Indiana Univ. Math. J. 22 (1972) 65-74.

26. VAN GROESEN, E. W. C., Dual and inverse formulations of constrained extremum problems; Jrnl. Math. Modelling, 1 (1980) 237-254.

27. FADELL, E. and P. H. RABINOWITZ, Generalized cohomological index theories for Lie group action with an application to bifurcation questions for Hamiltonian systems; Invent. Math. 45, (1978) 134-174.

28. BENCI, V., A geometrical index for the group S^1 and some applications to the study of periodic solutions of ordinary differential equations; Comm. Pure Appl. Math., 34 (1981) 393-432.

29. BENCI, V., On critical point theory for indefinite functionals in the presence of symmetries; Trans. A.M.S. 274 (1982) 533-572.

EWCV/jvs

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #2593	2. GOVT ACCESSION NO. AD-A136348	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Applications of Constraints in Critical Point Theory to Periodic Solutions of Natural Hamiltonian Systems		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) E. W. C. van Groesen		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE November 1983
		13. NUMBER OF PAGES 30
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

periodic solution, Hamiltonian system, variational methods, natural constraints

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper deals with periodic solutions of Hamiltonian systems of the form $-\ddot{x} = V'(x)$ with V a given function. Assuming V to be either a convex or an even function, and prescribing the period, existence results are obtained for the number of solutions in relation to the minimal period of these solutions, assuming superquadratic growth at infinity only, or subquadratic growth at infinity together with specific behaviour at the origin for V. By introducing natural constraints, these results are obtained by applying variational methods directly to the action functional.

DD FORM 1473 1 JAN 73 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

LM