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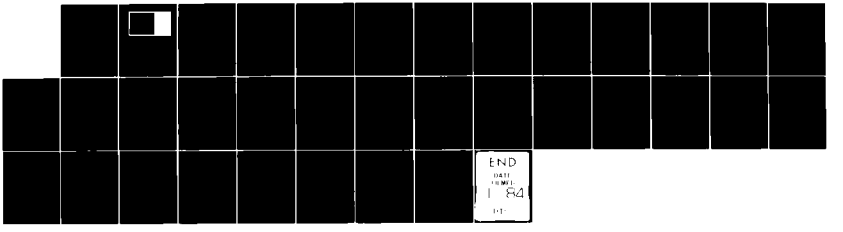

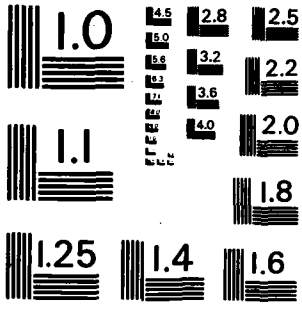

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MRC Technical Summary Report #2583

INTERVAL METHODS FOR FIXED-POINT PROBLEMS

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October 1983

(Received September 21, 1983)

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U. S. Army Research Office
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UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

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ABSTRACT

Interval analysis is applied to the fixed-point problem $x = \phi(x)$ for continuous ϕ: S → S, where the space S is constructed from Cartesian products of the set R of real numbers, with componentwise definitions of arithmetic operations, ordering, and the product topology. Spaces of this kind include many of the usual Hilbert and Banach spaces important in analysis. With the aid of an interval inclusion Φ: IS → IS in the interval space IS corresponding to S, the interval iteration process $X_{N+1} = X_N \cap \Phi(X_N)$ is shown to converge if the initial interval X_0 contains a fixed point x^* of ϕ; on the other hand, divergence of the iteration ($X_{N+1} = \emptyset$ for *some* N) *proves* that X_0 contains no fixed points of ϕ, while $\Phi(X_N) \subset X_N$ for some N establishes the existence of a fixed point $x^* \in X_0$ and guarantees the convergence of the interval iteration. Each step of interval iteration provides lower and upper bounds for fixed points of ϕ in the initial interval, from which approximate values and guaranteed error bounds for them can be obtained directly. In addition to interval iteration, operator equation and dissection methods are also considered briefly.

Since the theory of interval iteration applies directly when only finite subsets of S, IS are used, it is adaptable immediately to actual computation. A numerical example is given of the use of interval iteration for the approximate solution of a nonlinear integral equation of radiative transfer. It is shown that numerical results with acceptable, guaranteed accuracy can be obtained with a modest amount of computation for an extended range of the parameter involved.

AMS (MOS) Subject Classifications: 47H10, 58C30, 65G10, 65J10, 65J20, 65R20

Key words: Fixed-point problems, Interval iteration, Error analysis, Nonlinear integral equations

Work Unit Number 3 - Numerical Analysis and Scientific Computing

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

Scientific computation is devoted largely to processes for numerical solution of ordinary and partial differential equations, integral equations, and finite systems of linear and nonlinear equations, all problems which can be formulated abstractly as a fixed-point problem in some suitable space. Since the computation is carried out on a computer, and not in the space in which the problem is posed, methods of discrete approximation are used, and ordinarily detailed analysis is required to estimate the reliability and accuracy of the flood of floating-point numbers produced, as well as their interpretation in terms of the original problem. In this report, which is the text of an invited paper for the Special Session on Fixed Points and Operator Equations at the 1983 Summer Meeting of the American Mathematical Society in Albany, New York, it is shown that interval methods can be used to overcome many of the difficulties of this kind which arise in scientific computation. The goal is to have the computation itself provide the required information about the reliability and accuracy of the computed results.

It is shown that interval iteration can be used to prove existence or nonexistence of solutions in given regions, and to obtain lower and upper bounds for solutions in those regions. The transition to actual computation consists simply of identification of the floating-point numbers available on a computer with a finite set of elements of the space in question, for example, as step functions with floating-point values, and directed rounding from the space to this finite subspace. Naturally, the computer used has to support interval arithmetic by having both upward and downward rounding of arithmetic operations available, as required by the IEEE Standard for floating-point arithmetic. This capability is essential for accurate, reliable computation.

A numerical example is given in which a nonlinear integral equation of radiative transfer is solved for an extended range of parameters on a personal-size microcomputer in only a few seconds. For suitable initial intervals, this modest computation gives a guarantee of the existence of the solution of the integral equation, and proves that the numerical results obtained are accurate to better than 10%. In other cases, the computation shows that the initial interval is not large enough to contain a solution of the integral equation, which is also valuable information when exploring for regions in which solutions will lie.

INTERVAL METHODS FOR FIXED-POINT PROBLEMS

L. B. Rall

1. Real spaces. Here, the fixed-point problem

$$x = \phi(x) \tag{1.1}$$

will be considered for continuous operators ϕ: S → S acting in spaces S which can be constructed in a simple way from the set R of real numbers. This construction is based on the Cartesian product

$$P = \prod_A R\ , \tag{1.2}$$

of the real numbers over an index set A. Elements f of P are vectors with real components denoted by f_α or $f(\alpha)$, $\alpha \in A$.

For example, if $A = \{1,2,\ldots,n\}$ is the set of the first n positive integers, then P is the set R^n of n-dimensional real vectors $f = (f_1,f_2,\ldots f_n)$; if $A = [a,b]$, an interval, then P is the set $R[a,b]$ of all real-valued functions f on [a,b], with $f_\alpha = f(\alpha)$.

In general, the product set P can thus be considered to be the set of all real-valued functions on A, $P = \{f \mid f: A \to R\}$ [1].

The set P will now be equipped with arithmetic operations, a partial ordering, and a topology, all taken from the underlying set R of real numbers.

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Definition 1.1. The product set P given by (1.1) is called a **basic real space** over A if:

(i) The arithmetic operations +, - , • , / are defined in P by

$$(f * g)_\alpha = f_\alpha * g_\alpha, \quad \alpha \in A, \tag{1.3}$$

for $* \in \{+, -, \cdot, /\}$, with divisors restricted to the set

$$D = \{g \mid g_\alpha \neq 0, \quad \alpha \in A\}; \tag{1.4}$$

(ii) P is partially ordered by $\leq$, where

$$f \leq g \text{ iff } f_\alpha \leq g_\alpha, \quad \alpha \in A; \tag{1.5}$$

(iii) The topology of P is the product topology of the ordinary (metric) topology of the reals [12].

Properties of arithmetic operations in a basic real space P, such as commutivity of addition and multiplication, thus follow from the corresponding properties of real arithmetic, and the partial ordering induced in P by the order relationship for real numbers results in P being a complete lattice [5].

A basic real space will be a linear space if multiplication of a vector $f \in P$ by a **scalar** $a \in R$ is defined by multiplication of f by the vector in P with all components equal to a. Vectors which have all components equal (that is, which are constant on A), will be denoted on occasion by this common value, in particular, the notation 0 and 1 will be used for the vectors such that $0_\alpha = 0$ and $1_\alpha = 1$ for all $\alpha \in A$, respectively. Thus, for example, for A = [0,1], the basic real space P = R[0,1] includes the elements of the Banach space C[0,1] of continuous real functions on [0,1], and the Hilbert space $L_2[0,1]$ consisting of

all real functions which are square-integrable in the sense of Lebesgue on [0,1].

Instead of the entire space P, attention can be restricted to the subspace of P consisting of **bounded** elements f, for which

$$\|f\| = \sup_{\alpha \in A} |f_\alpha| < +\infty . \tag{1.6}$$

On this subspace, divisors have to be restricted to the set of elements g which have bounded reciprocals 1/g.

The absolute value | | and other functions of elements of basic real spaces are again defined componentwise, provided this makes sense for the given function.

In order to consider higher-dimensional problems, a more general class of spaces is introduced by the following recursive definition.

Definition 1.2. A **real space** S is either a basic real space or the Cartesian product of real spaces. In the latter case, arithmetic operations and partial ordering of the product space are defined componentwise on the basis of the corresponding operations and relations in the factor spaces, and the topology for the product space is the product of the topologies of its factors.

For example, the real space $S = R^n \times R^m$ consists of all functions which map the n-dimensional space R^n into the m-dimensional space R^m.

Real spaces will sometimes be called **R-spaces** for brevity.

As is generally known, an overwhelming number of theoretical and applied mathematical problems can be expressed in the deceptively simple form (1.1). In this paper, this general problem will be discussed from the standpoint of interval analysis [17], [18], [3], [4]. The next sections will introduce the necessary ideas of interval spaces and operators.

2. Interval spaces. Given a real space S, the corresponding **interval space** IS is constructed in the same way as the set IR of nonempty, closed, bounded intervals

$$[a,b] = \{x \mid -\infty < a \leq x \leq b < +\infty\} \tag{2.1}$$

is obtained from the set of real numbers R. In IR, the operations of **interval arithmetic** [17], [18] are defined by

$$[a,b] * [c,d] = \{z \mid z = x * y,\ x \in [a,b],\ y \in [c,d]\}, \tag{2.2}$$

$* \in \{+, -, \cdot, /\}$, with divisors restricted to the set

$$D = \{[c,d] \mid cd > 0\}, \tag{2.3}$$

that is, to the set of intervals which do not contain zero. Intervals [c,d] belonging to D will have **bounded reciprocals**

$$[c,d]^{-1} = [1/d\ ,\ 1/c]. \tag{2.4}$$

It is important to note that intervals do not form a group with respect to addition, since additive inverses do not exist in general; for example, one has $[0,1] - [0,1] = [-1,1]$ instead of $[0,0]$.

A partial ordering can be defined in IR by, for example

$$[a,b] \leq [c,d] \text{ iff } b \leq c, \tag{2.5}$$

([17], p. 7). There is also a metric topology for intervals [17]. Use of these ideas leads to the following definition.

Definition 2.1. The **interval space** IS corresponding to a real space S is the set of intervals [a,b] analogous to (2.1), where a,b are bounded elements of S and $a \leq b$ in the partial ordering on S, with componentwise definition of interval arithmetic and partial ordering based on (2.2) and (2.5) respectively, and topologized by the product of the metric topology for intervals ([17], Chap. 4).

In IS, division is restricted to intervals with bounded reciprocals. Elements x of S can be identified with the **degenerate** intervals $[x,x] \in IS$ having equal lower and upper endpoints. The notation

$$x = [x,x] \tag{2.6}$$

will be used throughout to indicate this identification.

The interval space IS corresponding to a real space S can also be constructed from the intervals IR of real numbers R according to the recipe for S, keeping only intervals with bounded endpoints. Although the real space S is a linear space, the corresponding interval space IS will not be a linear space, because interval addition does not have the group property, as noted above.

Since intervals are sets, the inclusion relation $\subset$ is defined in IR, and provides another partial ordering of this set [5]. Furthermore, the intersection $[a,b] \cap [c,d]$ of two intervals in IR is either an interval or the empty set $\emptyset$. These ideas extend componentwise to interval spaces.

Definition 2.2. If $F,G \in IS$, where IS is an interval space, then $F \subset G$ means

$$F_\alpha \subset G_\alpha, \quad \alpha \in A. \tag{2.7}$$

Definition 2.3. If $F,G \in IS$, an interval space, and $F_\alpha \cap G_\alpha = \emptyset$ for some $\alpha \in A$, then the intersection of F and G is said to be **empty**, and one writes

$$F \cap G = \emptyset. \tag{2.8}$$

Thus, the intersection of elements of an interval space will either be an element of that space or empty. On occasion, elements of an interval space IS will be referred to simply as **intervals**.

Since the intersection of a nested sequence of closed, nonempty subsets of R is nonempty [33], it follows that for intervals $X_i \in IR$ such that

$$X_1 \supset X_2 \supset X_3 \supset \ldots \supset X_n \supset \ldots, \tag{2.9}$$

one has

$$\bigcap_{n=0}^{\infty} X_n \neq \emptyset, \tag{2.10}$$

since elements of IR are closed and nonempty by definition. It follows from Definition 2.4 by construction and the Cantor theorem [33] that the same result holds in interval spaces IS, that is, if (2.9) holds for $X_i \in IS$, $i = 1,2,3,\ldots$, then (2.10) is true [28].

3. Interval inclusions of real operators. Suppose that S,T are real spaces, and IS, IT are the corresponding interval spaces. An operator $f: S \to T$ will be called a **real operator**, and an operator $F: IS \to IT$ will be called an **interval operator**. Although real operators can be classified as linear or nonlinear, this cannot be done for interval operators, since the interval spaces IS, IT are not linear spaces, and the definition of linearity for an operator depends on a linear substructure for the spaces involved.

Definition 3.1. The interval operator F: IS → IT is said to be **(inclusion) monotone** if for X,Z ∈ IS,

(3.1) $$X \subset Z \text{ implies } F(X) \subset F(Z).$$

Definition 3.2. The interval operator F: IS → IT is said to **contain** the real operator f: S → T if

(3.2) $$f(X) = \{y \mid y = f(x),\ x \in X\} \subset F(X), \quad X \in IS.$$

This is symbolized by f ∈ F, and F is said to be an **interval inclusion** of f. If F is monotone, then it is called a **monotone inclusion** of f.

Definition 3.3. The real operator f: S → T is said to be a **restriction** of the interval operator F: IS → IT if

(3.3) $$f(x) = F(x) = F([x,x]), \quad x \in X,$$

in the sense of the identification (2.6) of elements of S with degenerate intervals in IS. If (3.3) holds for a (monotone) inclusion F of f, then F is called a (monotone) **interval extension** of f [17].

Interval extensions of rational functions can be constructed by the use of interval arithmetic [17], [18], [3], [4]. The class of operators for which interval extensions are available can be broadened by the use of interval extensions of elementary functions, and so on. Integral operators can be extended by the use of interval integration [9], [26]. In actual numerical computation, it is usually impossible to realize the **restriction property** (3.3); however, interval arithmetic with directed rounding can be used to construct interval inclusions of real operators [15], [17], [18]. The results given below

require only the availability of interval inclusions F of real operators f, and do not depend on the restriction property.

4. Interval iteration. Suppose that ϕ: S → S is a continuous operator in a real space S, and Φ: IS → IS is an interval inclusion of ϕ.

Definition 4.1. The sequence $\{X_n\}$ of elements of IS defined by

$$X_{n+1} = X_n \cap \Phi(X_n), \quad n = 0,1,2,\ldots, \tag{4.1}$$

is said to be generated by **interval iteration** with **initial interval** X_0 and **iteration operator** Φ.

Definition 4.2. If a positive integer N exists such that

$$X_N \cap \Phi(X_N) = \emptyset, \tag{4.2}$$

then the interval iteration (4.1) is said to **diverge**; otherwise, it **converges** to the **limit**

$$X^* = \bigcap_{n=0}^{\infty} X_n \in IP. \tag{4.3}$$

The existence of X^* in the case of convergence follows from the theorem of Cantor [33] cited in §2, since the **iteration sequence** $\{X_n\}$ is a nested sequence (2.9) of nonempty intervals in S. If the iteration diverges, then the iteration sequence will consist only of the N intervals $X_0, X_1, \ldots, X_N$. The following theorems relate the convergence or divergence of the interval iteration (4.1) to the existence or nonexistence of a fixed point of ϕ in the initial interval X_0 [28].

Theorem 4.1. If the initial interval X_0 contains a fixed point of ϕ, then the interval iteration (4.1) will converge to a limit X^* which contains all fixed points x^* of ϕ which belong X_0.

The proof of this theorem [28] follows from the simple observation that if $x^* \in X_n$ is a fixed point of ϕ, then $x^* \in \Phi(X_n)$, so $x^* \in X_{n+1}$, and thus, by mathematical induction, the elements of the iteration sequence $\{X_n\}$ will be nonempty and contain all fixed points of ϕ which belong to X_0 .

Theorem 4.2. If the interval iteration (4.1) diverges, then the initial interval X_0 does not contain a fixed point x^* of ϕ.

This is the contrapositive of Theorem 4.1.

Divergence of interval iteration will be observed in a finite number of steps if it occurs. On the other hand, interval iteration will converge if X_0 is large enough to include at least one fixed point of ϕ. Interval and ordinary iteration will be compared in more detail in the next section. The next theorems give sufficient conditions for the convergence of interval iteration.

Theorem 4.3. If

$$\Phi(X_N) \subset X_N \tag{4.4}$$

for some positive integer N, then a fixed point $x^* \in X_N \subset X_0$ of ϕ exists, and the interval iteration (5.1) converges to the limit X^* given by (4.3), and $x^* \in X^*$.

The existence of $x^* \in X_N$ follows from (4.4) by the Schauder fixed point theorem [32], since ϕ is assumed to be continuous, and, as a subset of S, X_N is closed, convex [29], and compact. The convergence of the the interval iteration to the limit X^* with $x^* \in X^*$ then follows from Theorem 4.2.

Theorem 4.4. If

$$X_N \subset \phi(X_N) \tag{4.5}$$

for some positive integer N, then the interval iteration (4.1) converges to the limit $X^* = X_N$.

It follows from (4.1) and (4.5) that $X_{N+1} = X_N$, and thus $X_n = X_N$ for $n \geq N$, which gives $X^* = X_N$ by (4.3).

Theorems 4.3 and 4.4 show that it is sufficient for the convergence of interval iteration for X_N and $\phi(X_N)$ to be comparable for the inclusion relation $\subset$ for some positive integer N.

Note that convergence of the interval iteration under condition (4.5) does not imply the existence of a fixed point $x^* \in X_N$ of ϕ. It is easy to construct examples in which (4.5) holds, but the initial interval X_0 does not contain a fixed point of ϕ. In this case, it seems more appropriate to say that the interval iteration has "stalled" instead of "converged".

An example of the application of interval iteration to a nonlinear integral equation will be given below.

5. Comparison with ordinary iteration. A customary approach to the fixed point problem (1.1) is by ordinary iteration, that is, generation of the iteration sequence $\{x_1\}$ by

$$x_{n+1} = \phi(x_n), \quad n = 0,1,2,\ldots, \tag{5.1}$$

starting from the initial point x_0.

It is well-known that if ϕ is continuous and the iteration sequence converges to x^*, then x^* is a fixed point of ϕ; conversely, if ϕ does not have a fixed point, then the iteration sequence is divergent in the sense that it

cannot converge to an element of the space. Comparison of these results with those of §4 for interval iteration indicates that the conditions relating existence or nonexistence of fixed points to convergence or divergence of the corresponding iteration sequences are converse in a certain sense [28]. For example, existence of a fixed point x^* of ϕ in the initial interval X_0 is a **sufficient** condition for the convergence of interval iteration (Theorem 4.1), while this is a **necessary** condition for the convergence of ordinary iteration. Similarly, divergence of ordinary iteration is a **necessary** condition for the nonexistence of a fixed point, while divergence of interval iteration is a **sufficient** condition for the nonexistence of x^* in the initial interval X_0.

Thus, interval and ordinary iteration complement each other in a certain sense, and one or the other can be used as appropriate to a given fixed point problem. One advantage of interval iteration for numerical purposes is that it provides lower and upper bounds for fixed points at each step. One can write

$$X_n = [a_n, b_n], \quad n = 0, 1, 2, \ldots, \tag{5.1}$$

where $a_n, b_n \in S$. For each fixed point $x^* \in X_0$ of ϕ, one has

$$a_0 \leq a_1 \leq \cdots \leq a_n \leq \cdots \leq x^* \leq \cdots \leq b_n \leq \cdots \leq b_1 \leq b_0, \tag{5.2}$$

and in the limit, for $X^* = [a^*, b^*]$,

$$a^* \leq x^* \leq b^*. \tag{5.3}$$

Thus, if one knows initially that X_0 contains at least one fixed point x^* of ϕ, then interval iteration provides a way to obtain improved approximations to x^*, with guaranteed componentwise error bounds. For

(5.4) $$x_n = \frac{1}{2}(a_n + b_n) ,$$

one has

(5.5) $$|x^* - x_n|_\alpha \leq \frac{1}{2}(b_n - a_n)_\alpha , \quad \alpha \in A, \quad n = 0,1,2,\ldots,$$

componentwise, or the global error bound

(5.6) $$\|x^* - x_n\| \leq \frac{1}{2}\|b_n - a_n\| , \quad n = 0,1,2,\ldots,$$

where the norm is defined by (1.6).

Error bounds for ordinary iteration (5.1), on the other hand, are not usually obtained from the iteration itself, but require the construction of some majorant function E(n) such that

(5.7) $$\|x^* - x_n\| \leq E(n), \quad n = 0,1,2,\ldots .$$

Another distinction between ordinary and interval iteration arises in actual computation, due to the fact that the sequence $\{x_n\}$ cannot be obtained exactly, but has to be approximated by a sequence $\{z_n\}$. For ordinary iteration, this means that $\|x^* - z_n\|$ has to be estimated as a separate issue, for example, by use of (5.7) and a bound for the approximation error $\|x_n - z_n\|$. Thus, the additional analysis which has to be brought to bear on the error estimation problem can be extensive. In the case of interval iteration, however, the lower and upper bounds $\{a_n\}$, $\{b_n\}$ are simply selected from a finite subset of $S_\#$ of S, and the results (5.2)-(5.6) hold without modification. Furthermore, this means that a positive integer N will exist for which one observes either (4.2)

(divergence) or (4.5) (convergence). Hence, convergence or divergence of interval iteration is **observable** in practice, at least in principle, while the approximate iteration sequence $\{z_n\}$ obtained by ordinary iteration can wander indefinitely without revealing whether it is convergent or not.

It is also usual in application of ordinary iteration that the initial point x_0 has to be chosen "close enough" to x^* to obtain convergence, and, indeed, some fixed points repel the iteration sequence $\{x_n\}$ for x_0 arbitrarily close to x^*. As mentioned above, the initial interval X_0 simply has to be "big enough" to include x^* for interval iteration to converge. Of course, the interval iteration may not converge to anything useful if its limit X^* is too large.

The challenge in interval iteration is to construct the interval inclusion Φ of ϕ in such a way that $\Phi(X)$ is not grossly larger than $\phi(X)$, so that approximations and error bounds which are obtained from the computation itself will be as accurate as possible. In particular, if ϕ is a contraction operator in the ordinary sense, one would want Φ to be an **interval contraction** [6], or as close to one as can be obtained with outward rounding. Mean value and Taylor forms [7], [29] have been found to be very useful for the construction of accurate interval inclusions of sufficiently differentiable operators in real spaces.

Before leaving the subject of interval iteration, it should be noted that a substantial number of the 1200 references in a recent bibliography on interval mathematics [12] deal with some aspects of the theory or examples of application of this technique. Most of these studies focus on conditions under which the inclusion (4.4) will hold, so that the interval iteration will yield a guarantee of existence of the fixed point, as well as lower and upper bounds for it. However, as pointed out above, interval iteration can also be useful to prove nonexistence of fixed points, as well as to obtain bounds for fixed points known

to belong to X_0 on the basis of some other argument. In addition to the pioneering work of Moore [17], [18] substantial contributions by Alefeld [2], Krawczyk [13], [14], Nickel [20], [21], and Wisskirchen [34] should be noted, among others. The books by Alefeld and Herzberger [3], [4] also contain much valuable information on this topic. The interval bibliography [11] also contains references to proceedings of conferences on recent developments in this rapidly growing field.

6. Operator equation methods. Interval methods can also be based on the equivalence of the fixed point problem (1.1) to an **operator equation**

$$f(x) = 0, \tag{6.1}$$

where $f: S \to S$ is such that $f(x^*) = 0$ if and only if $x^* = \phi(x^*)$. Such equivalent problems can be constructed in many ways, for example, simply by taking f to be defined by

$$f(x) = x - \phi(x). \tag{6.2}$$

In many cases, an equivalence between (1.1) and (6.1) is exploited in the other direction; one attempts to solve (6.1) for $x = x^*$ by constructing an **iteration operator** ϕ for which x^* is a fixed point, and then using ordinary iteration (5.1). Typical examples of this are furnished by the use of Newton's methods and its variants for the solution of operator equations [24].

Suppose that $F: IS \to IS$ is an interval inclusion of the operator $f: S \to S$ appearing in (6.1). In searching for an initial interval X_0 which contains a fixed point x^* of an equivalent operator ϕ, the following **exclusion theorem** is often useful.

Theorem 6.1. If

$$0 \notin F(X), \tag{6.3}$$

then the interval X does not contain a solution $x = x^*$ of equation (6.1).

Since F is an inclusion of f, (6.3) contradicts the existence of an element $x^* \in X$ such that $f(x^*) = 0$.

7. Dissection methods. The search for fixed points of the operator $\phi: S \to S$ in intervals $X \in IS$ can be carried out with the aid of a **three-valued logic** $\Lambda = \{\text{TRUE}, ?, \text{FALSE}\}$ and a **test** T_ϕ such that

(7.1)
(a) $T_\phi(X) = \text{TRUE}$ means X contains a fixed point x^* of ϕ;
(b) $T_\phi(X) = ?$ means the test is inconclusive.
(c) $T_\phi(X) = \text{FALSE}$ means X does not contain a fixed point x^* of ϕ;

As shown above in §4, interval iteration starting from $X_0 = X$ provides an example of such a test, with (7.1)(a) holding if (4.4) is observed, (7.1)(c) if (4.2) is observed, and no conclusion about existence or nonexistence of x^* can be drawn if the interval iteration simply converges without (4.4) being true at some step, for example, if (4.5) occurs with proper inclusion. In actual computation on finite subsets $S_\#$ and $IS_\#$ of S and IS, one of these alternatives will be observed in a finite number of steps, so that interval iteration can be used as the test T_ϕ. Of course, other criteria for existence or nonexistence of fixed points x^* of ϕ in X can be applied instead of interval iteration, or to supplement it.

Suppose now that X is an interval such that $T_\phi(X) = ?$, and it is desired to determine whether or not X actually contains a fixed point x^* of ϕ. Then, X can be **dissected** into intervals $X^{(1)}, X^{(2)}, \ldots, X^{(k)}$, such that

(7.2) $$X = X^{(1)} \cup X^{(2)} \cup \ldots \cup X^{(k)}.$$

(The usual case k = 2 is called **bisection** of X.) The test T_ϕ can now be applied to the $X^{(i)}$, i = 1,2,...,k, in turn. If no positive conclusion is reached, then the subintervals $X^{(j)}$ for which $T_\phi(x^{(j)})$ = FALSE can be discarded, and dissection applied again to the remaining intervals until one is found which contains a fixed point of ϕ, or all are rejected. In the latter case, of course, the original interval X is fixed-point free.

Once a subinterval has been found which contains fixed points, then interval iteration or some other method can be used with confidence to obtain their values, at least approximately. Dissection is most appropriate in finite-dimensional spaces, however, even in spaces of low dimension, considerable bookkeeping can be required to keep track of the subintervals still under consideration, and is thus a task suitable for a computer. Furthermore, it has been shown by Moore and Jones [19] that bisection will yield a definite conclusion (7.1)(a) or (7.1)(c) for certain fixed point problems in a finite number of steps. In particular, there are classes of fixed points for which well-known sufficient conditions for their existence hold in regions surrounding the fixed point, and these regions contain intervals in which the fixed point lies. In this case, a positive conclusion will be reached after the dissection has been carried out to the point that the subintervals under consideration are sufficiently small. An example of this situation is furnished by the famous theorem of Kantorovich on the convergence of Newton's method [24], [23], [25].

8. A nonlinear integral equation. The above results will be illustrated by application to the numerical solution of the integral equation

(8.1) $$H(\mu) = \frac{1}{1 - \frac{\lambda}{2}\int_0^1 \frac{\mu H(\mu')}{\mu + \mu'}\, d\mu'}, \quad 0 \le \mu \le 1,$$

which is a form of the H-equation of radiative transfer [10]. Equation (8.1) will be considered to be the fixed point problem

$$H = \phi(H) \tag{8.2}$$

in the space $R[0,1]$ of real-valued functions on the interval $[0,1]$. For numerical purposes, the subset $R_\#[0,1]$ will be taken to the set of step functions with constant floating-point values on the subintervals

$$\mu_{i-1} = (i-1)/10 \leq \mu \leq i/10 = \mu_i, \quad i = 1,2,\ldots,10. \tag{8.3}$$

This space and the corresponding interval space $IR_\#[0,1]$ give a "low resolution" view of $R[0,1]$ and $IR[0,1]$, respectively. For computational purposes, these spaces of piecewise constant functions and intervals can be identified with the vector space $R_\#^{10}$ of vectors $x = (x_1,x_2,\ldots,x_{10})$ with floating-point numbers as components, and the corresponding interval space $IR_\#^{10}$ of **interval vectors** $X = (X_1,X_2,\ldots,X_{10})$, respectively.

An interval inclusion Φ of the operator ϕ defined by the right side of (8.1) will be constructed with the aid of interval integration [9], [26], as in [31]. Identifying an element x of $R_\#[0,1]$ by the vector x in $R_\#^{10}$ of its values, the integral transform of x is given by

$$\int_0^1 \frac{\mu x(\mu')}{\mu + \mu'}\, d\mu' = \frac{\lambda}{2}\, \mu x_i \cdot \sum_{j=1}^{10} x_j \ln\{(\mu + \mu_j)/(\mu + \mu_{j-1})\}, \tag{8.4}$$

$\mu_{i-1} \leq \mu \leq \mu_i$, $i = 1,2,\ldots,10$. In each subinterval $[\mu_{i-1},\mu_i]$, the function (8.4) is rounded downward or upward to a floating-point constant to obtain an element $I(x)$ of $R_\#[0,1]$. Applied to the lower and upper endpoint functions of

an element X of $IR_{\#}[0,1]$, this technique yields an inclusion $I(X) \in IR_{\#}[0,1]$ of the interval integral transform of an element $X \in IR_{\#}[0,1]$. The rest of the construction of Φ: $IR_{\#}[0,1] \to IR_{\#}[0,1]$ defined by

$$\Phi(X) = \frac{1}{1 - \frac{\lambda}{2} I(X)}, \quad X \in IR_{\#}[0,1], \tag{8.5}$$

is now done by rounded interval arithmetic in $R_{\#}^{10}$ [4], [15], [17], [18], with the division in (8.5) carried out componentwise.

The range of values of λ of physical interest is $0 \leq \lambda \leq 1$ [10]. By ordinary methods of functional analysis, it is easy to establish that equation (8.1) has a solution for $0 \leq \lambda \leq 1/(2 \ln 2) \approx 0.721...$ [24]. The same bound has been obtained for interval iteration applied to the standard form of (8.1), using a cruder form of interval integration than given here [22], the basic result being that Theorem 4.3 holds for the initial interval

$$X = \left[1 , \frac{1}{\lambda \ln 2} \{1 - \sqrt{1 - 2 \lambda \ln 2}\} \right]. \tag{8.6}$$

Here, interval iteration will be applied to (8.1) for $0.74 \leq \lambda \leq 1.00$ in order to extend the range of λ beyond that established previously. The iteration is performed in the "Gauss-Seidel" mode recommended by Caprani and Madsen [8] for integral equations, in which as much updated information is used at each step as possible, that is, the ith component $\Phi(X)_i$ of $\Phi(X)$ is computed by

$$\Phi(X)_i = \Phi(\Phi(X)_1,\ldots,\Phi(X)_{i-1},X_i,X_{i+1},\ldots,X_{10}), \tag{8.7}$$

$i = 1,2,\ldots,10$. Wisskirchen [34] has established the significant result that interval iteration performed in this way converges to the same limit as when

straightforward evaluation of $\Phi(X_1,X_2,\ldots,X_{10})$ is used. However, since (8.7) uses smaller component intervals at each substage of the evaluation of $\Phi(X)$, one can expect more rapid convergence. (This is another distinction between interval and ordinary iteration, since a Gauss-Seidel version of a convergent ordinary iteration method may diverge or converge to a different limit.)

The results are given in Table 8.1. For each initial interval and values of λ listed, the three following columns report the number $N + 1$ of the iteration at which (4.4) is observed (Existence), successive intervals agree to five decimal places (Convergence), or (4.2) happens (Nonexistence). Convergence in the sense of Theorem 4.4 will, of course, follow (4.4) after a certain number of iterations, but can also occur without a guarantee of existence of a solution if the inclusion in (4.5) is proper.

Examination of the given results reveals that relatively few iterations are required to establish existence or nonexistence of a fixed point in the initial interval when one of these outcomes obtains. The number of iterations needed for convergence, on the other hand, depends on the number of digits to which the endpoints of successive intervals are required to agree, that is, on the precision of the floating-point number system used. Experiments with this example indicate that this dependence is linear, with twice as many iterations needed to obtain convergence to twelve significant decimal digits as were observed for six. Since step functions are relatively crude approximations to continuous functions, particularly on the coarse subdivision of [0,1] into only 10 subintervals, it is a waste of time to insist that this interval iteration converge to more than a few decimal places.

As observed in [22], [31], better approximations to the solution of the integral equation are obtained at the break points $\mu_i = (0.1)i$, $i = 1,2,\ldots,9$, than in the corresponding subintervals, since the solution evaluated at these points has to satisfy

(8.8) $$x^*(\mu_i) \in X_i \cap X_{i+1} = Z_i, \quad i = 1,2,\ldots,9.$$

An approximation to $x^*(\mu_i)$ can be taken to be the harmonic mean $2ab/(a + b)$ of the endpoints of $Z_i = [a,b]$, and a bound for its percentage error is then 100 times the relative width $(b - a)/(a + b)$ of Z_i [30]. The maximum percentage error of these approximations in the cases in which existence was proved by the interval iteration is shown in Table 8.2. In all of these cases, accuracy to better than 10% was achieved, and thus the results are guaranteed to be of accuracy which is sufficient for many practical purposes, or what is sometimes called "engineering accuracy". This accuracy was not increased by requiring the interval iteration to converge to more decimal places.

The calculations were carried out on a Z80-based microcomputer, using the Pascal-SC compiler of Kulisch and Wippermann [16], which provides accurately rounded floating-point and interval arithmetic. The results clearly indicate that by going to "higher resolution" (more subintervals of [0,1]), the range of λ for which existence of the solution of (8.1) can be proved can be extended, and the accuracy of the numerical approximations obtained can be increased.

Computed results are given in Appendix I for $\lambda = 0.92$, where it is shown that the initial interval $X_0 = [1,1.75]$ does not contain a solution of the integral equation, the interval iteration converges for the initial interval $X_0 = [1,2.00]$, but the existence test is inconclusive, while $X_0 = [1,2.25]$ is proved to contain a solution of the integral equation which is bounded by the converged values of the interval iteration. Appendix II gives the source code of the Pascal-SC program CHANSEK5 with which the calculations were performed. The microcomputers used were a Zilog MCZ-1 with the RIO™ 2.06 operating system and an IMS 8000 SX with the CP/M™ 2.24 operating system. Both systems gave identical results.

Table 8.1

Initial Interval	Lambda	Existence	Convergence	Nonexistence
[1,1.25]	0.74			1
[1,1.50]	0.74		7	
	0.76		7	
	0.78			2
[1,1.75]	0.74	1	8	
	0.76	1	8	
	0.78	1	8	
	0.80	1	9	
	0.82	1	9	
	0.84	2	9	
	0.86		8	
	0.88		9	
	0.90		9	
	0.92			3
[1,2.00]	0.86	1	11	
	0.88	2	12	
	0.90	3	13	
	0.92		10	
	0.94		10	
	0.96		11	
	0.98		12	
	1.00			3
[1,2.25]	0.92	2	18	
	0.94		16	
	0.96		14	
	0.98		12	
	1.00		14	
[1,2.50]	0.94	4	28	
	0.96		18	
	0.98		16	
	1.00		14	
[1,2.75]	0.96	*	27	
	0.98		14	
	1.00		16	
[1,3.00]	0.96		26	
	0.98		20	
	1.00		15	

Table 8.1 (Continued)

Initial Interval	Lambda	Existence	Convergence	Nonexistence
[1,3.25]	0.96 0.98 1.00		26 20 15	
[1,3.50]	0.96 0.98 1.00		20 ** **	
[1,3.75]	0.96		**	

* Existence established in 4 iterations using 20 subdivisions of [0.1].

** Φ undefined for the initial interval.

Table 8.2

Initial Interval	Lambda	Iterations	Maximum Percentage Error
[1,1.75]	0.74 0.76 0.78 0.80 0.82 0.84	8 8 8 9 9 9	1.52 % 1.72 % 1.97 % 2.26 % 2.62 % 3.05 %
[1,2.00]	0.86 0.88 0.90	11 12 13	3.61 % 4.34 % 5.36 %
[1,2.25]	0.92	18	6.91 %
[1,2.50]	0.94	28	9.87 %

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APPENDIX I

Computed Results for $\lambda = 0.92$

```
C>XQPC CHANSEK5
ENTER INITIAL INTERVAL
*[1,1.75]
ENTER LAMBDA
*0.92
EMPTY INTERSECTION AT ITERATION NUMBER   3

FOR LAMBDA =  0.92,
INTEGRAL EQUATION HAS NO SOLUTION IN THE INTERVAL H0 = [ 1.000, 1.750]

H[ 1] = [ 1.00000000000, 1.19034842076]
H[ 2] = [ 1.16381274032, 1.32334334049]
H[ 3] = [ 1.27655952657, 1.43596418062]
H[ 4] = [ 1.37089212268, 1.53512110298]
H[ 5] = [ 1.45333391818, 1.62409194952]
H[ 6] = [ 1.52700493008, 1.70485377468]
H[ 7] = [ 1.59374951728, 1.75000000000]
H[ 8] = [ 1.65476948954, 1.75000000000]
H[ 9] = [ 1.71089034579, 1.75000000000]
H[10] = [ 1.76270273459, 1.75000000000]  **IMPROPER INTERVAL**

ENTER "I" FOR NEW INTERVAL, "L" FOR NEW LAMBDA, "Q" TO QUIT
*I
ENTER INITIAL INTERVAL
*[1,2.00]
ENTER LAMBDA
*0.92

FOR LAMBDA =  0.92,
INTERVAL ITERATION CONVERGED AT ITERATION NUMBER  10 TO:

H[ 1] = [ 1.00000, 1.19869]
H[ 2] = [ 1.16670, 1.34164]
H[ 3] = [ 1.28205, 1.46512]
H[ 4] = [ 1.37845, 1.57570]
H[ 5] = [ 1.46232, 1.67643]
H[ 6] = [ 1.53677, 1.76911]
H[ 7] = [ 1.60369, 1.85498]
H[ 8] = [ 1.66437, 1.93493]
H[ 9] = [ 1.71977, 2.00000]
H[10] = [ 1.77063, 2.00000]

**EXISTENCE OF SOLUTION NOT GUARANTEED**

ENTER "I" FOR NEW INTERVAL, "L" FOR NEW LAMBDA, "Q" TO QUIT
```

```
*I
ENTER INITIAL INTERVAL
*[1,2.25]
ENTER LAMBDA
*0.92

FOR LAMBDA =  0.92,
HYPOTHESES OF SCHAUDER THEOREM VERIFIED AT ITERATION NUMBER   2
SOLUTION OF INTEGRAL EQUATION IS BOUNDED BY:

H[ 1] = [ 1.00000, 1.22918]
H[ 2] = [ 1.15545, 1.39469]
H[ 3] = [ 1.26068, 1.53896]
H[ 4] = [ 1.34893, 1.66824]
H[ 5] = [ 1.42696, 1.78559]
H[ 6] = [ 1.49798, 1.89285]
H[ 7] = [ 1.56386, 1.99104]
H[ 8] = [ 1.62568, 2.08055]
H[ 9] = [ 1.68405, 2.16095]
H[10] = [ 1.73910, 2.23799]

DO YOU WANT TO ITERATE (Y/N)?
*Y

FOR LAMBDA =  0.92,
INTERVAL ITERATION CONVERGED AT ITERATION NUMBER  18 TO:

H[ 1] = [ 1.00000, 1.20069]
H[ 2] = [ 1.16670, 1.34595]
H[ 3] = [ 1.28206, 1.47194]
H[ 4] = [ 1.37845, 1.58519]
H[ 5] = [ 1.46232, 1.68868]
H[ 6] = [ 1.53677, 1.78419]
H[ 7] = [ 1.60369, 1.87291]
H[ 8] = [ 1.66437, 1.95573]
H[ 9] = [ 1.71977, 2.03334]
H[10] = [ 1.77063, 2.10628]

**EXISTENCE OF SOLUTION GUARANTEED**

ENTER "I" FOR NEW INTERVAL, "L" FOR NEW LAMBDA, "Q" TO QUIT
*Q
KL/P-STOP
```

APPENDIX II

Source Code for the Pascal-SC Program CHANSEK5

```
C>
PROGRAM CHANSEK5(INPUT,OUTPUT);
    CONST DIM = 10;
     TYPE DIMTYPE = 1..DIM;
          INTERVAL = RECORD INF,SUP: REAL END;
          IVECTOR = ARRAY[DIMTYPE]OF INTERVAL;
          RVECTOR = ARRAY[DIMTYPE]OF REAL;
          RMATRIX = ARRAY[DIMTYPE]OF RVECTOR;
      VAR C: CHAR;ALTER,ITER: INTEGER;HTEST,SUM,LAMBDA: REAL; I,J: DIMTYPE;
          HINF,HSUP: RVECTOR;LO,HI: RMATRIX;HO,COEF: INTERVAL;
          GO,EXIST,EMPTY: BOOLEAN;

FUNCTION INTPT ( RA: REAL ) : INTERVAL;
  EXTERNAL 41;
OPERATOR / ( A,B: INTERVAL ) RES: INTERVAL;
  EXTERNAL 85;
FUNCTION ILN ( Y: INTERVAL ) : INTERVAL;
  EXTERNAL 107;
PROCEDURE IREAD ( VAR F: TEXT; VAR A: INTERVAL );
  EXTERNAL 92;


BEGIN  (*MAIN PROGRAM*)

    FOR J:=1 TO DIM DO   (*Generate coefficient matrices*)
      BEGIN
        COEF:=ILN(INTPT(J+1)/INTPT(J));  (*First rows*)
        HI[1,J]:=COEF.SUP;LO[1,J]:=COEF.INF;
      END;                                (*First rows*)
    FOR I:=2 TO DIM DO
    BEGIN      (*Rows 2,...,DIM*)
       FOR J:=1 TO (DIM-1) DO
        BEGIN
          HI[I,J]:=HI[I-1,J+1];LO[I,J]:=LO[I-1,J+1];
        END;
       COEF:=ILN(INTPT(DIM+I)/INTPT(DIM+I-1));
       HI[I,DIM]:=COEF.SUP;LO[I,DIM]:=COEF.INF;
    END;                  (*Matrix generation*)

C:='I';WHILE C = 'I' DO  (*Restart with new initial interval*)
BEGIN
WRITELN('ENTER INITIAL INTERVAL');
IREAD(INPUT,HO);

C:='L';WHILE C = 'L' DO  (*Restart with new LAMBDA*)

    BEGIN
       WRITELN('ENTER LAMBDA');READ(LAMBDA);

  FOR I:=1 TO DIM DO
    BEGIN
      HINF[I]:=HO.INF;HSUP[I]:=HO.SUP
    END;
```

```
        HTEST:=HO.SUP;
        GO:=TRUE;EMPTY:=FALSE;EXIST:=FALSE;ITER:=1;WHILE GO DO
        BEGIN    (*GO Loop*)
        (*Update HSUP[1]*)
        ALTER:=1;
        SUM:=1->(LAMBDA/>(2*DIM)*>SCALP(HI[1],HSUP,1));
        SUM:=1/>SUM;
        IF SUM > (HSUP[1]-5.0E-06) THEN ALTER:=ALTER+1;
        IF SUM < HSUP[1] THEN HSUP[1]:=SUM;
        (*Update HINF[I],HSUP[I], I = 2..DIM*)
        FOR I:=2 TO DIM DO
          BEGIN
            SUM:=1-<(LAMBDA*<(I-1)/<(2*DIM))*<SCALP(LO[I-1],HINF,-1);
            SUM:=1/<SUM;
            IF SUM < (HINF[I]+5.0E-06) THEN ALTER:=ALTER+1;
            IF SUM > HINF[I] THEN HINF[I]:=SUM;
            SUM:=1->(LAMBDA*>I/>(2*DIM))*>SCALP(HI[I],HSUP,1);
            SUM:=1/>SUM;
            IF SUM > (HSUP[I]-5.0E-06) THEN ALTER:=ALTER+1;
            IF SUM < HSUP[I] THEN HSUP[I]:=SUM;
            IF HSUP[I] < HINF[I] THEN EMPTY:=TRUE;
          END;  (*Update of HINF[I],HSUP[I] for I = 2..DIM*)
        IF EMPTY THEN
                    BEGIN  (*Empty intersection*)
              WRITELN('EMPTY INTERSECTION AT ITERATION NUMBER ',ITER:3);
              WRITELN;WRITELN('FOR LAMBDA = ',LAMBDA:5:2,',');
              WRITELN('INTEGRAL EQUATION HAS NO SOLUTION
IN THE INTERVAL HO = [',HO.INF:6:3,',',HO.SUP:6:3,']');
    WRITELN;FOR I:=1 TO DIM DO
    BEGIN (*Improper interval output*)
        WRITE('H[',I:2,'] = [',HINF[I]:14:11:-1,',',HSUP[I]:14:11:1,']');
        IF HSUP[I] < HINF[I] THEN WRITE('  **IMPROPER INTERVAL**');
        WRITELN;
    END;  (*Improper interval output*)
            GO:=FALSE;
            END;    (*Empty intersection*)
        IF HSUP[DIM] < HTEST THEN
    BEGIN  (*Existence*)
        EXIST:=TRUE;
        WRITELN;WRITELN('FOR LAMBDA = ',LAMBDA:5:2,',');
        WRITELN('HYPOTHESES OF SCHAUDER THEOREM VERIFIED AT
ITERATION NUMBER ',ITER:3);
        WRITELN('SOLUTION OF INTEGRAL EQUATION IS BOUNDED BY:');WRITELN;
        FOR I:=1 TO DIM DO   (*Output of results*)
          BEGIN
            WRITELN('H[',I:2,'] = [',HINF[I]:8:5:-1,',',HSUP[I]:8:5:1,']');
          END;               (*Output of results*)
```

```
        WRITELN;WRITELN('DO YOU WANT TO ITERATE (Y/N)?');READ(C,C);
        IF C <> 'Y' THEN GO:=FALSE ELSE HTEST:=0;
    END;    (*Existence*)

    IF ALTER = 20 THEN
    BEGIN    (*Convergence*)
        GO:=FALSE;
        WRITELN;WRITELN('FOR LAMBDA = ',LAMBDA:5:2,',');
        WRITELN('INTERVAL ITERATION CONVERGED AT ITERATION
 NUMBER ',ITER:3,' TO:');
        WRITELN;
    FOR I:=1 TO DIM DO     (*Output of results*)
      BEGIN
        WRITELN('H[',I:2,'] = [',HINF[I]:8:5:-1,',',HSUP[I]:8:5:1,']');
      END;                 (*Output of results*)
        WRITELN;
        IF EXIST THEN WRITELN('**EXISTENCE OF SOLUTION GUARANTEED**')
        ELSE WRITELN('**EXISTENCE OF SOLUTION NOT GUARANTEED**');
    END;        (*CONVERGENCE*)

    ITER:=ITER+1;
  END; (*GO Loop*)
        WRITELN;
        WRITELN('ENTER "I" FOR NEW INTERVAL, "L" FOR NEW LAMBDA, "Q" TO QUIT');
        READ(C,C);
    END; (*Restart with new LAMBDA*)
  END;    (*Restart with new initial interval*)
END.  (*MAIN PROGRAM*)
```

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2583	2. GOVT ACCESSION NO. AD-A136323	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) INTERVAL METHODS FOR FIXED-POINT PROBLEMS		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) L. B. Rall		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 3 - Numerical Analysis and Scientific Computing
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P. O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE October 1983 13. NUMBER OF PAGES 29
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Fixed-point problems
Interval iteration
Error analysis
Nonlinear integral equations

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Interval analysis is applied to the fixed-point problem $x = \phi(x)$ for continuous $\phi: S \to S$, where the space S is constructed from Cartesian products of the set R of real numbers, with componentwise definitions of arithmetic operations, ordering, and the product topology. Spaces of this kind include many of the usual Hilbert and Banach spaces important in analysis. With the aid of an interval inclusion $\Phi: IS \to IS$ in the interval space IS corresponding to S, the interval iteration process $X_{N+1} = X_N \cap \Phi(X_N)$ is

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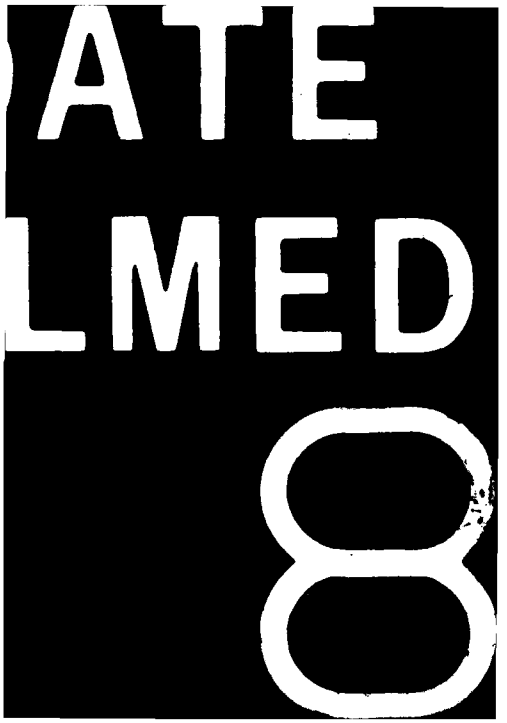
20. ABSTRACT - cont'd.

shown to converge if the initial interval X_0 contains a fixed point x^* of ϕ; on the other hand, divergence of the iteration ($X_{N+1} = \emptyset$ for some N) proves that X_0 contains no fixed points of ϕ, while $\phi(X_N) \subset X_N$ for some N establishes the existence of a fixed point $x^* \in X_0$ and guarantees the convergence of the interval iteration. Each step of interval iteration provides lower and upper bounds for fixed points of ϕ in the initial interval, from which approximate values and guaranteed error bounds for them can be obtained directly. In addition to interval iteration, operator equation and dissection methods are also considered briefly.

Since the theory of interval iteration applies directly when only finite subsets of S, IS are used, it is adaptable immediately to actual computation. A numerical example is given of the use of interval iteration for the approximate solution of a nonlinear integral equation of radiative transfer. It is shown that numerical results with acceptable, guaranteed accuracy can be obtained with a modest amount of computation for an extended range of the parameter involved.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