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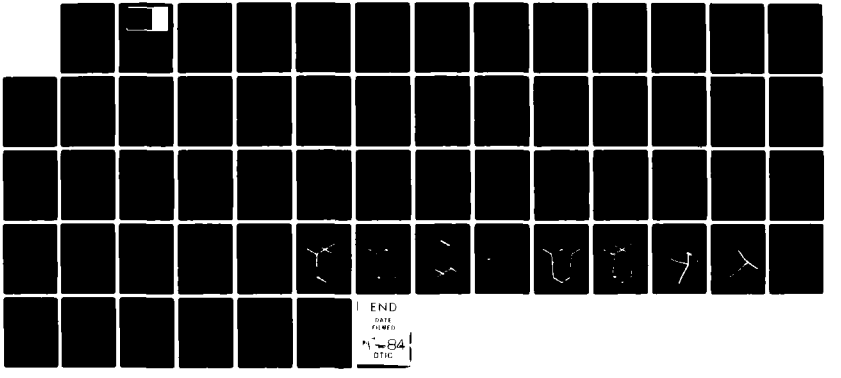

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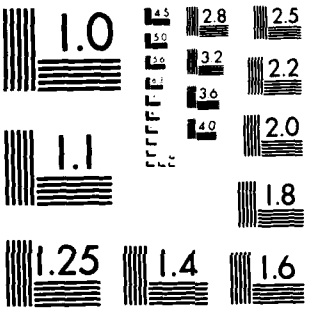


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# A SINGULAR PERTURBATION ANALYSIS OF THE FUNDAMENTAL SEMICONDUCTOR DEVICE EQUATIONS

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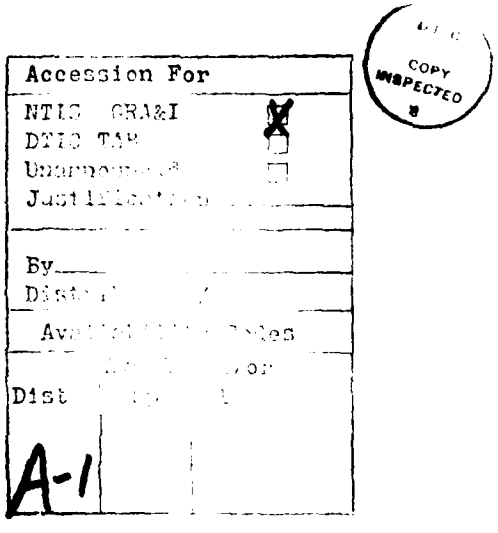

UNIVERSITY OF WISCONSIN-MADISON
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# A SINGULAR PERTURBATION ANALYSIS OF THE FUNDAMENTAL SEMICONDUCTOR DEVICE EQUATIONS

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## ABSTRACT

In this paper we present a singular perturbation analysis of the fundamental semiconductor device equations which form a system of three second order elliptic differential equations subject to mixed Neumann-Dirichlet boundary conditions. The system consists of Poisson's equation and the continuity equations and describes potential and carrier distributions in an arbitrary semiconductor device.

The singular perturbation parameter is the minimal Debye-length of the device under consideration.

Using matched asymptotic expansions we demonstrate the occurrence of internal layers at surfaces across which the impurity distribution which appears as an inhomogeneity of Poisson's equation has a jump discontinuity (these surfaces are called 'junctions') and the occurrence of boundary layers at semiconductor-oxide interfaces. We derive the layer-equations and the reduced problem (charge-neutral-approximation) and give existence proofs for these problems. They layer solutions which characterize the solution of the singularly perturbed problem close to junctions and interfaces resp. are shown to decay exponentially away from the junctions and interfaces resp.

We show that, if the device is in thermal equilibrium, then the solution of the semiconductor problem is close to the sum of the reduced solution and the layer solution assuming that the singular perturbation parameter is small. Numerical results for a two-dimensional diode are presented.

AMS (MOS) Subject Classifications: 78A30, 35J65, 35B25, 35C20

Key Words: Semiconductor Device Equations, Elliptic Boundary Value Problems, Singular Perturbations, Asymptotic Expansions

Work Unit Number 1 (Applied Analysis)

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## SIGNIFICANCE AND EXPLANATION

In this paper qualitative properties of the solutions of the system of partial differential equations which describes potential and carrier distributions in an arbitrary semiconductor device are discussed. The system consists of Poisson's equations for the potential and of the continuity equations for the carrier densities. The problem can be classified as elliptic (in the static case). After appropriate scaling a small parameter $\lambda^2$, which is physically identified as the square of the normed minimal Debyelength of the device under consideration, appears as multiplier of the Laplace-operator in Poisson's equation. Therefore, the system is singularly perturbed.

We investigate the asymptotic behaviour of solutions as the singular perturbation parameter $\lambda^2$ converges to zero and demonstrate the occurrence of internal layers (thin regions of fast variations) in the potential and the carrier densities. These layers occur at curves in the device where n-regions (i.e. regions in which the doping profile is positive) and p-regions (i.e. regions in which the doping profile is negative) meet. In the analysis we also prove existence theorems for the reduced problem ($\lambda^2$ set to zero) and for the layer-problems.

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# A SINGULAR PERTURBATION ANALYSIS OF THE FUNDAMENTAL SEMICONDUCTOR DEVICE EQUATIONS

Peter A. Markowich*

## 1. INTRODUCTION

An analysis of the two-dimensional, static equations which describe potential distribution and current flow in a semiconductor device is presented.

The basis semiconductor device equations are (see Van Roosbroeck (1950)):

$$\text{(1.1)(a)} \quad \operatorname{div}(\varepsilon_s \nabla\psi) = q(n - p - C(x,y)) \qquad \text{Poisson's equation}$$

$$\text{(1.1)(b)} \quad \operatorname{div}(D_n \nabla n - \mu_n n \nabla\psi) = R \qquad (x,y) \in \Omega \quad \text{electron continuity equation}$$

$$\text{(1.1)(c)} \quad \operatorname{div}(D_p \nabla p + \mu_p p \nabla\psi) = R \qquad \text{hole continuity equation}$$

where the dependent variables are

$\psi$ : electrostatic potential

$n$ : electron density

$p$ : hole density

$\Omega$ is a bounded domain in $\mathbb{R}^2$ representing the device geometry; $\varepsilon_s$ is the semiconductor permittivity (which will be assumed to be constant in the sequel); $\mu_n, \mu_p$ are the electron and hole mobilities resp.; $D_n, D_p$ are the electron and hole diffusion coefficients resp. and $q$ is the elementary charge.

$C(x,y)$ is the doping profile, that means $C(x,y)$ is the difference of the electrically active concentration of donors and the electrically active concentration of acceptors. $R$ is the recombination generation rate. In the sequel we will neglect recombination-generation effects, that means we set $R \equiv 0$. Of course only solutions with $n > 0$, $p > 0$ are admitted

We assume the validity of Einstein's relation

$$\text{(1.2)} \quad \frac{D_n}{\mu_n} = \frac{D_p}{\mu_p} = U_T (\equiv \text{const})$$

where $U_T$ is the thermal voltage.

The electron and hole current densities $J_n$ and $J_p$ are given by

$$\text{(1.3)(a)} \quad J_n = q(D_n \nabla n - \mu_n n \nabla\psi)$$

$$\text{(1.3)(b)} \quad J_p = -q(D_p \nabla p + \mu_p p \nabla\psi).$$

Numerical values for the parameters (for Silicon at roomtemperature) are given in Table 1.

The elliptic system of differential equations (1.1) has to be supplemented by appropriate boundary conditions for $\psi, n, p$ determined by the device under consideration. We assume that $\partial\Omega$ splits up into three dispoint parts, namely $\partial\Omega_C$, $\partial\Omega_{iS}$ and $\partial\Omega_{oS}$. $\partial\Omega_{iS}$ and $\partial\Omega_{oS}$ are open, $\partial\Omega_{oS}$ is connected and $\partial\Omega_C = \bigcup_{k=0}^{r} C_k, r \geq 0$ where the $C_k$ are closed and connected arcs with positive (one-dimensional) Lebesguemeasure.

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Dirichlet boundary conditions for $\psi$,n,p are given on $\partial\Omega_C$ (the $C_k$ are Ohmic contacts) and zero Neumann boundary conditions are prescribed on $\partial\Omega_{is}$ (insulating segments). $\partial\Omega_{os}$ represents a semiconductor-oxide interface occuring in MOS-technology (see Sze (1981) for a survey on MOS-devices). The oxide is located in a bounded domain $\phi$ which is such that $\bar{\phi}\cap\bar{\Omega} = \overline{\partial\Omega}_{os}$ and

$$(1.3) \qquad \Delta\psi = 0, \qquad (x,y) \in \phi$$

holds. The carrier densities n,p only exist in $\bar{\Omega}$. Usually $\partial\phi$ splits into three parts, namely $\partial\phi_C$ (oxide-contact) where a Dirichlet condition for $\psi$ is prescribed, $\partial\phi_{is}$ (insulating segment)where a zero Neumann condition for $\psi$ holds,and the interface $\partial\Omega_{os}$. $\psi$ has to be continous accross $\partial\Omega_{os}$ and

$$(1.4) \qquad [\epsilon\,\nabla\psi.\vec{n}]_{\partial\Omega_{os}} = 0, \qquad \epsilon(x,y) = \begin{cases} \epsilon_o, & (x,y)\in\phi \\ \epsilon_s, & (x,y)\in\Omega \end{cases}$$

holds where $\epsilon_o$ is the oxide permittivity ($\frac{\epsilon_o}{\epsilon_s}\approx\frac{1}{3}$) ($[f]_\Gamma$ denotes the jump of the function f accross the curve $\Gamma$). $\vec{n}$ is the exterior unit normal vector of $\partial\Omega$. The condition (1.4) represents the continuity of the electrical displacement accross the semiconductor oxide interface.

The electron and hole current density components $J_n\cdot\vec{n}$ and $J_p\cdot\vec{n}$ (perpendicular to $\partial\Omega_{os}$) vanish on $\partial\Omega_{os}$. This gives boundary conditions for n and p (by using (1.3)) at the interface.

The Dirichlet boundary conditions for n and p at the Ohmic contacts are given by the vanishing-space-charge condition

$$(1.5) \qquad (n - p - C(x,y))|_{\partial\Omega_C} = 0$$

and the thermal equilibrium condition

$$(1.6) \qquad np|_{\partial\Omega_C} = n_i^2$$

where $n_i$ is the intrinsic number of the semiconductor.

For the following we assume that $\Omega$ splits up into N+1 connected subdomains $\Omega_i$, ($\bar{\Omega} = \bigcup_{i=0}^{N} \bar{\Omega}_i$), such that C does not change sign in each of the $\Omega_i$ and C has jump-discontinuities accross the curves $\Gamma_i = \bar{\Omega}_{i-1}\cap\bar{\Omega}_i$ (abrupt doping). $\Gamma_i\cap\Gamma_j = \{\}$ holds for $i\neq j$. $\Omega_i$ is called a *n region* if $C|_{\bar{\Omega}_i} > 0$ and it is called a p region if $C|_{\bar{\Omega}_i} < 0$. $\Gamma_i$ is a pn junction if it is the joint boundary of a p and an n region and it is an nn (pp) junction if it is the joint boundary of two n (p) regions.

We also assume that the Ohmic contacts $C_k$ have positive distance from the junctions $\Gamma_i$.

The performance by the device under consideration is mainly determined by the location of the subdomains, of the oxide (for MOS-devices) and by the location of the

Ohmic contacts. The boundary conditions for the potential (at the Ohmic contacts) are

(1.7) (a) $\psi|_{C_k} = U_T \ln \frac{n}{n_i}|_{C_k} + U_k$, if $C_k \subset \bar{\Omega}_i$ and $C|_{\bar{\Omega}_i} > 0$

in n-regions and

(1.7) (b) $\psi|_{C_k} = U_T \ln \frac{n_i}{p}|_{C_k} + U_k$ if $C_k \subset \bar{\Omega}_i$ and $C|_{\bar{\Omega}_i} < 0$,

in p-regions where $U_k$ represents the potential applied to the Ohmic contact $C_k$. We remark that there are devices which are such that not every n or p region "has" an Ohmic contact (for example thyristors, see Sze (1981)).

An externally applied potential $U_G$ is given at the oxide (gate) contact $\partial\Phi_C$:

(1.7)(c) $\psi|_{\partial\Phi_C} = U_G - U_F$

where (the flat band voltage) $U_F$ is a constant which depends on the semiconductor, on the oxide contact and on the doping. The applied potentials $U_k$, $U_G$ are constants, too.

As illustration for the notation of the device geometry we show a typical MOS-transistor in Figure 1.

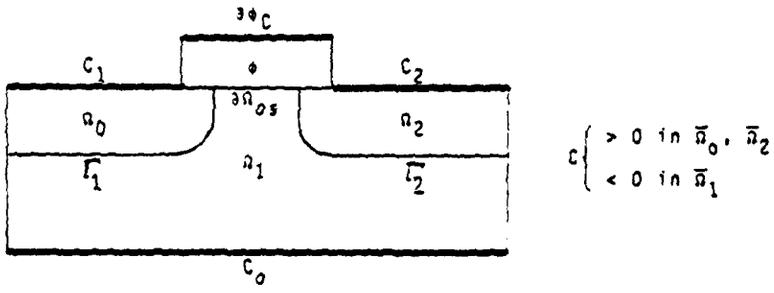



Figure 1. Mos-Transistor

There are two n-regions and one p-region (n-channel-transistor), three Ohmic contacts ($C_1$: source-contact, $C_2$: drain contact, $C_0$ bulk contact) and one oxide contact ($\partial\Phi_C$: gate contact).

The vertical boundaries of $\Omega$ and $\Phi$ are insulating ($\partial\Omega_{is}$ and $\partial\Phi_{is}$ resp.).

We remark that the rectangular shape of $\Omega$ and $\Phi$ as shown in Figure 1 is a simplification commonly used for numerical simulation (see Selberherr(1980)). The following theory however is not restricted to particular shapes of domains.

The problem (1.1) can be put into a simpler form by the transformation (called Boltzmann Statistics)

$$(1.8) \qquad n = n_i e^{\frac{\psi}{U_T}} u , \qquad p = n_i e^{-\frac{\psi}{U_T}} v.$$

Here $u = \exp(-\frac{\phi_n}{U_T})$, $v = \exp(\frac{\phi_p}{U_T})$ where $\phi_n$, $\phi_p$ are the electron and hole quasifermilevels resp. ($u > 0$, $v > 0$ has to hold). Then (1.1) takes the form (by using (1.2), assuming $R \equiv 0$ and $\varepsilon_s$ to be constant)

$$\left.\begin{aligned} &(1.9)(a) \quad \varepsilon_s \Delta \psi = q(n_i e^{\frac{\psi}{U_T}} u - n_i e^{-\frac{\psi}{U_T}} v - C(x,y)) \\ &(1.9)(b) \quad \operatorname{div}(\mu_n e^{\frac{\psi}{U_T}} \nabla u) = 0 \\ &(1.9)(c) \quad \operatorname{div}(\mu_p e^{-\frac{\psi}{U_T}} \nabla v) = 0. \end{aligned}\right\} \quad (x,y) \in \Omega.$$

The continuity equations (1.9)(b),(c) are in self-adjoint form.

There have been many analytical and numerical investigations of (1.1) ((1.9)). Mock (1972) showed the existence of a solution of (1.9) subject to the mixed set of boundary conditions and he proved that this solution is unique if the applied potentials $U_k$ are sufficiently small. He only assumed $C \in L^\infty(\Omega)$. Continuous dependence of the solutions on the boundary data was also shown in this paper.

A very similar existence proof was given recently by Bank, Jerome and Rose (1982).

The parabolic semiconductor problem (with homogenous Neumann boundary conditions on $\partial\Omega$) was investigated by Mock (1974). Finite difference methods are discussed in Mock (1973), (1981).

In this paper we scale the problem (1.9) appropriately and obtain a singular perturbation problem. The singular perturbation parameter $\lambda$ is the minimal normed Debye length of the device under consideration.

Using matched asymptotic expansions (as $\lambda \to 0+$) we demonstrate the occurance of a boundary layer in $\psi$ at oxide-semiconductor interfaces and the occurance of internal layers (in $\psi$) at pn, nn and pp junctions. $u$ and $v$ are the slow variables, that means they do not exhibit zero-order layers.

We derive the reduced problem (vanishing space charge approximation) which is obtained by setting the singular perturbation parameter to zero, and the (boundary and internal) layer equations and give existence proofs for these problems. We discuss the asymptotic behavior of the current densities $J_n$, $J_p$ (as $\lambda \to 0+$) and show the validity of the asymptotic expansions for the equilibrium problem (zero external potential applied at the contacts).

The singular perturbation approach was applied to the one dimensional semiconductor

problem by Vasileva and Stelmakh (1977), Vasileva and Butuzow (1978); Markowich, Ringhofer,Selberherr and Langer (1982 a,b).

The main advantage of the singular perturbation approach is that it gives qualitative information on the behaviour of the solutions.This a-priori information can be used to construct appropriate discretisation methods for the numerical solution of the semiconductor device equations. In particular efficient mesh-strategies employing only a reasonable number of grid points (but still giving accurate numerical approximations even in layer regions) can be obtained (see Markowich, Ringhofer and Selberherr (1982) and Section 6).

All results in this paper also hold for the one and three dimensional static semiconductor problems after obvious modification of assumptions (e.g. junctions are then represented by points and surfaces resp.). We chose the two-dimensional semiconductor problem for the presentation since it is most often used for numerical simulation.

The paper is organized as follows. In Section 2 we perform the scaling which leads to the singular perturbation problem , prove an existence and a regularity theorem and derive a-priori estimates of the solutions. In Section 3 we derive (the zeroth order terms of) the asymptotic expansions, in Section 5 we give the existence proofs for the reduced problem and for the layer problems and the equilibrium problem is discussed in Section 5. Section 6 is concerned with numerical examples and with possible extensions of the theory.

Table 1. Numerical Values of the Parameters for Silicon and Silicon oxide at roomtemperature T ≈ 300 k.

| Parameter | Physical Meaning | Numerical Value |
|---|---|---|
| $q$ | elementary charge | $10^{-19}$As |
| $\varepsilon_s$ | semiconductor permittivity constant | $10^{-12}$As/Vcm |
| $\varepsilon_0$ | oxide permittivity constant | $\frac{1}{3} \cdot 10^{-12}$As/Vcm |
| $\mu_n$ | electron mobility | $10^3 cm^2/Vs$ |
| $\mu_p$ | hole mobility | $10^3 cm^2/Vs$ |
| $D_n$ | electron diffusion constant | $25\, cm^2/s$ |
| $D_p$ | hole diffusion constant | $25\ cm^2/s$ |
| $n_i$ | intrinsic number | $10^{10}/cm^3$ |
| $U_T$ | thermal voltage | 0.025 V |

The numerical values given for $\mu_n, \mu_p, D_n, D_p$ have to be understood as averages, since these quantities are generally modelled by functions of x and y.

## 2. THE SINGULARLY PERTURBED PROBLEM

We assume that C is bounded in $\Omega$ and set

$$(2.1) \qquad \bar{C} = \sup_{\Omega} |C(x,y)|, \quad D = \frac{C}{\bar{C}}$$

and

$$(2.2) \qquad l = \text{diam } (\Omega).$$

The dependent variables are scaled as follows

$$(2.2) \qquad \psi_s = \frac{\psi}{U_T}, \quad n_s = \frac{n}{\bar{C}}, \quad p_s = \frac{p}{\bar{C}}, \quad u_s = u, \quad v_s = v$$

and the independent variables

$$(2.3) \qquad x_s = \frac{x}{l}, \quad y_s = \frac{y}{l}, \quad (x_s, y_s) \in \bar{\Omega}_s \cup \bar{\phi}_s$$

Then (1.9) transforms to (after dropping the subscript s):

$$\left.\begin{aligned} &(2.4)(a) \quad \lambda^2 \Delta\psi = \delta^2 e^{\psi} u - \delta^2 e^{-\psi} v - D \\ &(2.4)(b) \quad \text{div}(e^{\psi} \nabla u) = 0 \\ &(2.4)(c) \quad \text{div}(e^{-\psi} \nabla v) = 0 \end{aligned}\right\} \quad (x,y) \in \Omega$$

and (1.3) remains unchanged:

$$(2.5) \qquad \Delta\psi = 0, \quad (x,y) \in \phi$$

where

$$(2.6) \text{ (a) } \lambda^2 = \left(\frac{\lambda_D}{l}\right)^2 = \frac{\epsilon_s U_T}{l^2 q \bar{C}}, \qquad \text{(b) } \delta^2 = \frac{n_i}{\bar{C}}$$

holds. $\lambda_D$ is the minimal Debye length of the device.

For (2.4) we assumed that the mobilities $\mu_n, \mu_p$ are constant throughout the device (for numerical values see Table 1 ).

The following theory however carries over to the case that $\mu_n, \mu_p$ are smooth and positive functions of x and y.

The (scaled) boundary conditions are

$$(2.7)(a) \qquad \nabla\psi \cdot \vec{n}|_{\partial\Omega_{is}} = \nabla u \cdot \vec{n}|_{\partial\Omega_{is}} = \nabla v \cdot \vec{n}|_{\partial\Omega_{is}} = 0$$

(the (unit) vector $\vec{n}$ is perpendicular to $\partial\Omega$ and is assumed to exist almost everywhere) and

$$(2.7)(b) \quad u|_{C_k} = e^{-\frac{U_k}{U_T}}, \qquad v|_{C_k} = e^{\frac{U_k}{U_T}}$$

$$(2.7)(c) \quad \psi|_{C_k} = \ln\left[\frac{D + \sqrt{D^2+4\delta^4}}{2\delta^2}\right]\Big|_{C_k} + \frac{U_k}{U_T}$$

for $k = 0,..,r$.

(2.7) is derived from (1.5), (1.6), (1.7) (a,b) by using (1.8).

Boundary conditions on $\partial\phi$ are

$$(2.8)(a) \quad \nabla\psi\cdot\vec{\zeta}|_{\partial\phi_{is}} = 0$$

($\vec{\zeta}$ denotes the exterior unit normal vector of $\partial\phi$ )

$$(2.8)(b) \quad \psi|_{\partial\phi_C} = \frac{U_G}{U_T} - \frac{U_F}{U_T} =: \psi_G$$

and

$$(2.8)(c) \quad [\psi]_{\partial\Omega_{os}} = 0$$

$$(2.8)(d) \quad [\varepsilon^*\nabla\psi\cdot\vec{n}]_{\partial\Omega_{os}} = 0, \qquad \varepsilon^* = \begin{cases} 1, & (x,y) \in \Omega \\ \frac{\varepsilon_o}{\varepsilon_s}, & (x,y) \in \phi \end{cases}.$$

$$(2.8)(e) \quad \nabla u\cdot\vec{n}|_{\partial\Omega_{os}} = \nabla v\cdot\vec{n}|_{\partial\Omega_{os}} = 0$$

$u$ and $v$ are only defined in $\bar{\Omega}$.

For modern devices $\bar{C} \geq 10^{17}\text{cm}^{-3}$. With the realistic value $l = 5 \times 10^{-3}$cm and the numerical values for $q$, $\varepsilon_s$, $U_T$ given in Table 1 we get $\lambda^2 \leq 10^{-7} << 1$. Therefore the problem (2.4), (2.5), (2.7), (2.8) constitutes a singularly perturbed quasilinear elliptic system of differential equations (subject to mixed Neumann-Dirichlet boundary and interface conditions).

The parameter $\delta^2 << 1$, too (normally $\delta^2 \leq 10^{-7}$ holds). This however gets compensated by the Dirichlet boundary conditions (2.7)(d),(e) which imply that $\delta^2(e^{\psi}u - e^{-\psi}v) = O(1)$ at Ohmic contacts as $\delta^2 \to 0$. Note that the potential difference between an Ohmic contact in a n-region and an Ohmic contact in a p-region behaves asymptocally (as $\delta^2 \to 0$) like $\ln \frac{1}{\delta^4}$.

We regard $\delta^2$ as (fixed) parameter and investigate the asymptotic properties of the solutions as $\lambda \to 0+$. We will then show that the asymptotics are uniform in $\delta$ as long as $\delta$ does not converge to zero too fast (compared to $\lambda$ ).

The scaling factors for the current densities $J_n$, $J_p$ are $qu_n\tilde{C}U_T$ and $qu_p\tilde{C}U_T$ resp. Then the scaled current densities are given by

(2.9)(a) $J_n = \delta^2 e^{\psi}\nabla u$, (b) $J_p = -\delta^2 e^{-\psi}\nabla v$.

The scaled carrier densities follow from (2.2)

(2.8)(c) $n = \delta^2 e^{\psi}u$, (d) $p = \delta^2 e^{-\psi}v$.

We now give definitions which will be needed in the sequel.

We denote by $L^q(\Omega)$ the space of q-integrable real valued functions defined on $\Omega$ with the norm

$$\|f\|_{q,\Omega} := \left(\int_\Omega |f(x,y)|^q dxdy\right)^{\frac{1}{q}}$$

and by $L^\infty(\Omega)$ the space of bounded functions on $\Omega$ and

$$\|f\|_{\infty,\Omega} = \sup_\Omega |f(x,y)| .$$

$C^m(\Omega)$ for $m \in N_0$ is the space of all functions defined on $\Omega$ which together with their partial derivations of order up to m are continuous in $\Omega$ .

$C^m(\bar{\Omega})$ is the space of all functions which are in $C^m(\Omega)$ and which together with their partial derivatives of order up to m are bounded and uniformly continous in $\Omega$. A norm on $C^m(\bar{\Omega})$ is given by

$$\|f\|^{m,\bar{\Omega}} = \max_{0\le|\alpha|\le m} \sup_\Omega \left| \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\partial y^{\alpha_2}} f(x,y)\right|$$

where $\alpha=(\alpha_1,\alpha_2)$ and $|\alpha| =\alpha_1+\alpha_2$ .

Spaces of Hölder continuous function $C^{m,\beta}(\Omega)$, $C^{m,\beta}(\bar{\Omega})$ for $0 \le \beta \le 1$ and their norms are as in Adams (1975).

We define the functional

$$\|f\|_{m,2,\Omega} = \left(\sum_{0\le|\alpha|\le m} \left\| \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1}\partial y^{\alpha_2}} f\right\|^2_{2,\Omega}\right)^{\frac{1}{2}}$$

and write $H^m(\Omega)$ for the completion of $\{u \in C^m(\Omega): \|u\|_{m,2,\Omega}<\infty\}$ with respect to the norm $\|\cdot\|_{m,2,\Omega}$.

For $\Gamma\subset\partial\Omega$ we denote by $H_0^m(\Omega\cup\Gamma)$ the completion of $\{u \in C^m(\Omega): \|u\|_{m,2,\Omega} < \infty$ and $u$ vanishes in a neighbourhood of $\Gamma\}$ with respect to $\|\cdot\|_{m,2,\Omega}$.

For the following we set $\Lambda = \Omega \cup \partial\Omega_{os} \cup \phi$ and state the existence theorem which is basic for the theory of the semiconductor equations.

Theorem 2.1 Assume that $D$ is defined in $\bar{\Omega}$, $D \in L^\infty(\Omega)$ and that $D$ is Lipschitzcontinuous in a neighbourhood of $\partial\Omega_C$. Also assume that $\partial\Lambda$ and $\partial\Omega$ are Lipschitzcontinuous and piecewise $C^\infty$ and that the (one-dimensional) measure of $\partial\Omega_C$ is positive.
Then the problem (2.4),(2.5),(2.7),(2.8) has a weak solution $(\psi,u,v)$ for which $\psi \in H^1(\Lambda) \cap L^\infty(\Lambda)$, $u,v \in H^1(\Omega) \cap L^\infty(\Omega)$ holds.

Every solution $(\psi,u,v) \in H^1(\Lambda) \times (H^1(\Omega)^2)$, $\psi \in L^\infty(\Lambda)$ fulfills the a priori estimates

$$(2.10)(a) \quad u_- := \min_k e^{-\frac{U_k}{U_T}} \leq u(x,y) \leq \max_k e^{-\frac{U_k}{U_T}}$$

$$(2.10)(b) \quad v_- := \min_k e^{\frac{U_k}{U_T}} \leq v(x,y) \leq \max_k e^{\frac{U_k}{U_T}}$$

for $(x,y) \in \bar{\Omega}$ and

$$(2.10)(c) \quad \psi_- := \min\left(\tilde{\psi}_-, \inf_{\partial\Phi_C}\psi\right) \leq \psi(x,y) \leq \max\left(\tilde{\psi}_+, \sup_{\partial\Phi_C}\psi\right) := \psi_+$$

for $(x,y) \in \bar{\Lambda}$ where

$$(2.11)(a) \quad \tilde{\psi}_+ := \ln\left[\frac{D_+ + \sqrt{D_+^2 + 4\delta^4 u_+ v_+}}{2\delta^2 u_-}\right], \quad D_+ = \sup_\Omega D(x,y)$$

$$(2.11)(b) \quad \tilde{\psi}_- := \ln\left[\frac{D_- + \sqrt{D_-^2 + 4\delta^4 u_- v_-}}{2\delta^2 u_+}\right], \quad D_- = \inf_\Omega D(x,y)$$

Proofs of slighty weaker existence theorems were given by Mock (1974) and Bank, Jerome and Rose (1982). We therefore only sketch the

Proof: The a priori estimates (2.10)(a)(b) follow immediately by on application of the maximum principle for $H^1$-solutions(see Gilbarg and Trudinger (1977), Chapter 8) to (2.4)(b),(c), (2.7)(a),(b) and (2.8).

We set

$$Q(\psi) = \lambda^2 \operatorname{div}(\varepsilon^* \nabla\psi) - (\delta^2 e^{\psi} u^* - \delta^2 e^{-\psi} v^* - D^*)$$

$$\text{with} \quad u^* = \begin{cases} u & \text{in } \Omega \\ 0 & \text{in } \phi \end{cases}, \quad v^* = \begin{cases} v & \text{in } \Omega \\ 0 & \text{in } \phi \end{cases}, \quad D^* = \begin{cases} D & \text{in } \Omega \\ 0 & \text{in } \phi \end{cases}.$$

The weak formulation of $Q(\psi) = 0$ subject to (2.7)(a),(2.8)(a),(2.8)(c),(2.8)(c),(d) is

$$Q(\psi,\varphi) = \int_{\Lambda}(\lambda^2\varepsilon^*\nabla\psi\cdot\nabla\varphi + (\delta^2e^{\psi}u^*-\delta^2e^{-\psi}v^*-D^*)\varphi)dxdy$$

$$= \int_{\Omega}(\lambda^2\nabla\psi\cdot\nabla\varphi+(\delta^2e^{\psi}u-\delta^2e^{-\psi}v-D)\varphi dxdy + \lambda^2\frac{\varepsilon_0}{\varepsilon_s}\int_{\phi}\nabla\psi\cdot\nabla\varphi dxdy$$

$$= 0$$

for all $\varphi \in H_0^1(\Lambda \cup \partial\Omega_{is} \cup \partial\phi_{is})$ since if $\psi,\varphi$ are sufficiently smooth on $\bar{\Lambda}$ then the arbitrariness of $\varphi$ implies (after integration by parts) that $Q(\psi) = 0$ in $\Omega \cup \phi$ and that the zero-Neumann boundary condition on $\partial\Omega_{is} \cup \partial\phi_{is}$ and the interface conditions (2.8)(c),(d) are fulfilled ((2.8)(c),(d) are the"natural" interface conditions for the problem $Q(\psi)=0$).

A simple calculation shows that $Q(\psi_+) \leq 0, \psi|_{\partial\Omega_C \cup \partial\phi_C} \leq \psi_+$ and $Q(\psi) \geq 0, \psi|_{\partial\Omega_C \cup \partial\phi_C} \geq \psi_-$. Therefore we get for every weak solution of $Q(\psi) = 0$

$$Q(\psi) - Q(\psi_+) = \lambda^2 div(\varepsilon^*\nabla(\psi-\psi_+)) - f(x,y)(\psi-\psi_+) \geq 0$$

and

$$Q(\psi) - Q(\psi_-) = \lambda^2 div(\varepsilon^*\nabla(\psi-\psi_-)) - g(x,y)(\psi-\psi_-) \leq 0$$

where $f(x,y) = \delta^2(e^{\psi_1}u^*+e^{-\psi_1}v^*) \geq 0$, $g(x,y) = \delta^2(e^{\psi_2}u^*+e^{-\psi_2}v^*) \leq 0$ and $\psi_1(x,y)$ $(\psi_2(x,y))$ is between $\psi(x,y)$ and $\psi_+(\psi_-)$. As for the continuity equations the maximum principle yields (2.10)(c).

The existence statement of Theorem 2.1 follows by using Schauder's fixed point theorem as in Bank, Jerome and Rose (1982). □

We remark that $\psi_-=\tilde{\psi}_-$ and $\psi_+=\tilde{\psi}_+$ holds if $\phi$ is empty and that the estimates (2.10) imply that the solution is unique if $U_k=0$ for $k=0,\ldots,r$. Then $u=v=1$ in $\Omega$ which implies $J_n=J_p=0$. The whole device is in thermal equilibrium.

Mock (1974) showed (under slightly more stringent smoothness assumptions on the boundary conditions) that the solution is unique if $|U_k|$ are sufficiently small. Uniqueness for arbitrary $U_k$ cannot be expected since there are wellknown devices (like thyristors,see Sze (1981)) which exhibit multiple solutions.

We now show (for devices without oxide-regions) that any solution of the semiconductor problem is classical if the doping profile D is piecewise Hödercontinuous.

We denote by $CR(\partial\Omega)$ the set ofcritical points of $\partial\Omega$, that is the set of all points $P \in \partial\Omega$ for which either $P \in \partial\Omega_C \cap \overline{\partial\Omega}_{is}$ (Ohmic contacts and insulating segments meet),

$P \in \bar{\Phi} \cap \partial\Omega_C$ (Ohmic contacts and the oxide boundary meet) or $\partial\Omega \cap B_\rho(P)$ (where $B_\rho(P)$ is the open ball with radius $\rho$ centred at P) does not have a $C^{2,\alpha}$ parametrisation for any $\rho > 0$.

Theorem 2.2: Let $\phi=(\ )$ and assume that $\partial\Omega$ is Lipschitzcontinuous and piecewise $C^\infty$ and that $\Gamma_i$ are $C^\infty$ curves for i=1,...,N. Also assume that D is defined on $\bar{\Omega}$, that $D \in L^\infty(\Omega)$ and that $D|_{\Omega_i} \in C^{0,\alpha}(\Omega_i)$ for i=0,...,N and some $\alpha>0$.
Moreover D is Lipschitzcontinuous in a neighbourhood of $\partial\Omega_C$. Then every weak solution $(\psi,u,v)$ of the semiconductor problem (2.4), (2.7) for which $\psi \in L^\infty(\Omega)$ holds, fulfills

(2.12)(a) $(\psi,u,v) \in (C^1(\bar{\Omega} - CR(\partial\Omega)))^3$

(2.12)(b) $\psi|_{\Omega_i} \in C^2(\Omega_i), \quad i=0,\ldots,N$

and

(2.12)(c) $(u,v) \in (C^2(\Omega))^2$.

Also all second derivatives of $\psi,u,v$ are square integrable over any subregion of $\Omega$ which has positive distance from the critical points of $\partial\Omega$ .

Proof: For a given weak solution $(\psi^*,u^*,v^*)$ we set
$F(x,y) = \lambda^2(e^{\psi^*(x,y)}u^*(x,y) - e^{-\psi^*(x,y)}v^*(x,y)) - D(x,y)$.
Since $\psi^*,u^*,v^* \in L^\infty(\Omega)$ we have $F \in L^\infty(\Omega)$. $\psi^*$ is the unique solution of

(2.13) $\lambda^2\Delta\psi = F(x,y), \quad (x,y) \in \Omega$

subject to the mixed Dirichlet-Neumann conditions (2.7)(a), (2.7)(d),(e). From Kawohl (1980) we obtain that $\psi^* \in H^2(\Omega')$ where $\Omega'$ is any subdomain of $\Omega$ with positive distance from the critical points of $\Omega$ and that $\psi$ fulfills (2.13) and the boundary conditions almost everywhere (with respect to the two and one dimensional Lebesgue measures resp.). Therefore $\psi^* \in C(\bar{\Omega}\text{-}CR(\partial\Omega))$. Theorem 15.1 in Ladyzenskaja and Ural'tseva(1968, pp.203) implies $\psi^* \in C^{1,\alpha}(\bar{\Omega}')$ and $\psi^* \in C^1(\bar{\Omega}\text{-}CR(\partial\Omega))$.

The same theorem implies that the solutions $u^*,v^*$ of

$$\operatorname{div}(e^{\psi^*(x,y)}\nabla u) = 0$$

and

$$\operatorname{div}(e^{-\psi^*(x,y)}\nabla v) = 0$$

fulfill $u^*,v^* \in C^{1,\alpha}(\bar{\Omega}')$ and therefore $u^*,v^* \in C^1(\bar{\Omega}\text{-}CR(\partial\Omega))$. Theorem 6.24 in Gilbarg and Trudinger (1977, page 6.24) yields $u^*,v^* \in C^2(\Omega)$.

Let $\Omega''$ be a subdomain of $\Omega_i$ with Lipschitzcontinuous boundary and with a positive distance from the critical points of $\partial\Omega_i$. Then $\psi^*,u^*,v^*,D^*\in C^{0,\alpha}(\Omega'')$ and therefore $F\in C^{0,\alpha}(\Omega'')$. Since $\psi^*$ solves

$$\lambda^2\Delta\psi = F(x,y), \qquad (x,y)\in\Omega''$$

$$\psi|_{\partial\Omega''} = \psi^*|_{\partial\Omega''}$$

we get $\psi^*\in C^{2,\alpha}(\Omega'')$ which implies $\psi^*\in C^2(\Omega_i)$. □

$\psi$,u and v may have singularities at the critical points of $\partial\Omega$ and second derivatives of $\psi$ are discontinuous accross the junction $\Gamma_i$ if $[D]_{\Gamma_i}\neq 0$. The extension of Theorem 2.2 to MOS devices is straight-forward using the methods of Ladyzenskaja and Ural'tseva(1968, Chapter 3, Section 16).

We remark that (2.10)(a),(b) yield an a-priori estimate on the number of active carrier-pairs. (2.9)(c),(d) gives

$$np = \delta^4 uv \quad \text{in } \bar{\Omega} \tag{2.14}$$

and (2.10)(a),(b) imply:

$$\delta^4\exp\left(-\frac{|V_{max}|}{T}\right)\leq np\leq\delta^4\exp\left(\frac{|V_{max}|}{T}\right) \quad \text{in } \bar{\Omega} \tag{2.15}$$

where

$$|V_{max}| = \max_{l,k}|U_k - U_l|$$

$|V_{max}|$ is the largest(in absolute value) voltage applied to two Ohmic contacts. This estimate was anticipated by De Mari (1968) and it was proven by Markowich, Ringhofer, Selberherr and Langer (1982 a,b) for the one-dimensional case.

Also estimates for n and p follow from (2.9)(c).(d) and (2.1). We get for non-MOS devices

$$\frac{D_- + \sqrt{D_-^2+4\delta^4\exp(-|V_{max}|/U_T)}}{2}\exp\left(-\frac{|V_{max}|}{U_T}\right)\leq n\leq$$
$$\leq\frac{D_+ + \sqrt{D_+^2+4\delta^4\exp(|V_{max}|/U_T)}}{2}\exp\left(\frac{|V_{max}|}{U_T}\right) \text{ in } \bar{\Omega} \tag{2.16)(a}$$

and

$$\frac{2\delta^4\exp\left(-\frac{|V_{max}|}{U_T}\right)}{D_+ + \sqrt{D_+^2+4\delta^4\exp(|V_{max}|/U_T)}}\leq p\leq\frac{2\delta^4\exp\left(\frac{|V_{max}|}{U_T}\right)}{D_- + \sqrt{D_-^2+4\delta^4\exp(-|V_{max}|/U_T)}} \text{ in } \bar{\Omega}. \tag{2.16)(b}$$

If $D_- < 0$, $D_+ > 0$ then the lower bounds in (2.16) are $O(\delta^4)$ as $\delta \to 0$, the upper bounds are $O(1)$ (for fixed $|V_{max}|$).

There is numerical evidence (see Markowich, Ringhofer, Selberherr and Langer (1982 a,b)) that the estimates (2.16) are not sharp. The factors $\exp\left(-\frac{|V_{max}|}{U_T}\right)$, $\exp\left(\frac{|V_{max}|}{U_T}\right)$ can probably be omitted.

## 3. ASYMPTOTIC EXPANSIONS

We are now concerned with the asymptotic behavior of the solutions of (2.4),(2.5), (2.7), (2.8) as $\lambda \to 0+$.

When we set $\lambda=0$ in (2.4)(a) we see that $\psi(\lambda=0)$ has to be discontinuous at pn, nn and pp junctions (since D is discontinuous at these junctions) assuming that $u(\lambda=0)$ and $v(\lambda=0)$ are continuous in $\Omega$ (note that (2.4)(b),(c) only admit weak solution in $C(\Omega)$ if $\psi \in L^\infty(\Omega)$). Therefore, standard singular perturbation theory implies that we have to expect an internal layer in $\psi$ (that is a region of fast variation of $\psi$ ) at these junctions. Also boundary layers occur at $\partial\Omega$ if the reduced solutions (i.e. the solutions of (2.4) with $\lambda=0$) do not fulfill the boundary conditions.

For the following analysis we assume that the profile D is discontinuous accross only one (open) $C^\infty$-curve $\Gamma$ which splits $\Omega$ into two connected subdomains $\Omega_+$ and $\Omega_-$. Also we assume that $D|_{\bar\Omega_+} \in C^{0,\alpha}(\bar\Omega_+)$, $D|_{\bar\Omega_-} \in C^{0,\alpha}(\bar\Omega_-)$ for some $\alpha > 0$ and $[D]_\Gamma \neq 0$. Also $\Gamma \cap \partial\Omega_C = \{\}$. D does not change sign in $\bar\Omega_+$ and in $\bar\Omega_-$.

We denote by $t(x,y)$ the oriented distance of $(x,y)$ from $\Gamma$, that means $t>0$ in $\Omega_+$ and $t<0$ in $\Omega_-$. $s(x,y)=(s_1(x,y),s_2(x,y))$ is the point on $\Gamma$ which is closest to $(x,y)$ ($s$ is unique in a sufficiently small strip about $\Gamma$). Similarly $r(x,y) \geq 0$ denotes the distance of $(x,y)$ to $\partial\Omega$ ($\partial\Omega$ fulfills the assumptions of Theorem 2.1) and $q(x,y)=(q_1(x,y),\ q_2(x,y))$ denotes the point on $\partial\Omega$ closest to $(x,y)$. Note that $\nabla r|_{\partial\Omega}=-\vec{n}$ and $\nabla t|_\Gamma$ is the unit-normal vector of $\Gamma$ pointing into $\Omega_+$.
For a function $f$ defined on $\Omega$ (or $\bar\Omega$) we set

$$f^\Gamma(t,s) = f(x,y)$$

and

$$f^\partial(r,q) = f(x,y)$$

in neighbourhoods of $\Gamma$ and $\partial\Omega$ where $s$ and $q$ resp. are unique.

We define for some $s \in \Gamma$

$$f^\Gamma(0+,s) := \lim_{\substack{(a,b)\to s \\ (a,b)\in\Omega_+}} f(a,b)$$

$$f^\Gamma(0-,s) := \lim_{\substack{(a,b)\to s \\ (a,b)\in\Omega_-}} f(a,b)$$

(assuming that the limits exist).

We also set

$$\frac{\partial s}{\partial(x,y)} = \begin{pmatrix} \frac{\partial s_1}{\partial x} & \frac{\partial s_2}{\partial x} \\ \frac{\partial s_1}{\partial y} & \frac{\partial s_2}{\partial y} \end{pmatrix}, \quad \frac{\partial q}{\partial(x,y)} = \begin{pmatrix} \frac{\partial q_1}{\partial x} & \frac{\partial q_2}{\partial x} \\ \frac{\partial q_1}{\partial y} & \frac{\partial q_2}{\partial y} \end{pmatrix}, \quad \nabla_s f^\Gamma = \begin{pmatrix} \frac{\partial f^\Gamma}{\partial s_1} \\ \frac{\partial f^\Gamma}{\partial s_2} \end{pmatrix}, \quad \nabla_q f^\partial = \begin{pmatrix} \frac{\partial f^\partial}{\partial q_1} \\ \frac{\partial f^\partial}{\partial q_2} \end{pmatrix}$$

and remark that

$$\left(\frac{\partial s}{\partial(x,y)}\right)^T \nabla t|_\Gamma = \left(\frac{\partial q}{\partial(x,y)}\right)^T \nabla r|_{\partial\Omega} = 0$$

holds (the superscript T denotes transposition).

Following standard singular perturbation theory we make the ansatz':

$$(3.1)\quad \begin{pmatrix} \psi(x,y,\lambda) \\ u(x,y,\lambda) \\ v(x,y,\lambda) \end{pmatrix} \sim \sum_{i=0}^{\infty} \lambda^i \begin{pmatrix} \bar{\psi}_i(x,y) \\ \bar{v}_i(x,y) \\ \bar{u}_i(x,y) \end{pmatrix} + \begin{pmatrix} \hat{\psi}_i(\frac{t(x,y)}{\lambda},\ s(x,y)) \\ \hat{u}_i(\frac{t(x,y)}{\lambda},\ s(x,y)) \\ \hat{v}_i(\frac{t(x,y)}{\lambda},\ s(x,y)) \end{pmatrix} + \begin{pmatrix} \tilde{\psi}_i(\frac{r(x,y)}{\lambda},\ q(x,y)) \\ \tilde{u}_i(\frac{r(x,y)}{\lambda},\ q(x,y)) \\ \tilde{v}_i(\frac{r(x,y)}{\lambda},\ q(x,y)) \end{pmatrix}$$

where the functions marked with '-' are independent of $\lambda$, the functions marked with '$\wedge$' are defined on $(-\infty,\infty) \times \Gamma$ and decay to zero as $\tau = \frac{t}{\lambda} \to \pm\infty$ (internal layer terms), the functions marked with '$\sim$' are defined on $[0,\infty) \times \partial\Omega$ and decay to zero as $\rho = \frac{r}{\lambda} \to \infty$ (boundary layer terms).

We insert (3.1) into (2.4), (2.5), (2.7), (2.8) and obtain equations for the i-th term in the series (3.1) by comparing coefficients of $\lambda$. We start with

A) <u>The Reduced Problem</u>

Evaluation of (2.4) away from $\Gamma$ and $\partial\Omega$ and comparing O(1) terms gives (after dropping the index 0) the zeroth order reduced problem

$$\left.\begin{array}{ll} (3.2)(a) & 0 = s^2 e^{\bar{\psi}} \bar{u} - s^2 e^{-\bar{\psi}} \bar{v} - D(x,y) \\ (3.2)(b) & \operatorname{div}(e^{\bar{\psi}} \nabla \bar{u}) = 0 \\ (3.2)(c) & \operatorname{div}(e^{-\bar{\psi}} \nabla \bar{v}) = 0 \end{array}\right\} (x,y) \in \Omega - \Gamma$$

In the context of semiconductor device physics this problem is referred to as 'zero-space charge approximation'.

By investigating the internal and boundary layer problems we will supplement (3.2) by interface and boundary conditions.

B) The Internal Layer Problem

We evaluate (2.4)(b) close to $\Gamma$ but away from $\partial\Omega$ and compare $O(\lambda^{-2})$ terms (after carrying out the differentiations).

This yields:

$$\hat{u}_{o\tau\tau} + \hat{u}_{o\tau}\hat{\psi}_{o\tau} = 0$$

(subscripts $\tau$ denote differentiation with respect to $\tau$) and therefore $\hat{u}_{o\tau} = \alpha(s)e^{-\hat{\psi}_{o\tau}}$. From $\hat{u}_o(\tau,s) \to 0$ as $\tau \to \pm\infty$ we get $\hat{u}_o \equiv 0$. Similarly $\hat{v}_o \equiv 0$ follows. $u$ and $v$ do not have zeroth order internal layers.

For the following we set $\hat{u} \equiv \hat{u}_1$, $\hat{v} \equiv \hat{v}_1$ and $\hat{\psi} \equiv \hat{\psi}_o$.

Comparing $O(\lambda^{-1})$ terms in (2.4)(b) (again after differentiation) and evaluating close to $\Gamma$ but in $\Omega_+$ gives

$$\hat{u}_{\tau\tau} + \hat{\psi}_\tau(\bar{u}_t^\Gamma(0+,s) + \hat{u}_\tau) = 0, \quad \tau > 0$$

(subscripts t denote differentiation with respect to t) and evaluation close to $\Gamma$ but in $\Omega_-$:

$$\hat{u}_{\tau\tau} + \hat{\psi}_\tau(\bar{u}_t^\Gamma(0-,s) + \hat{u}_\tau) = 0, \quad \tau < 0.$$

Integration yields

$$\hat{u}_\tau(\tau,s) = \begin{cases} \bar{u}_t^\Gamma(0+,s)(e^{-\hat{\psi}(\tau,s)}-1), & \tau > 0 \\ \bar{u}_t^\Gamma(0-,s)(e^{-\hat{\psi}(\tau,s)}-1), & \tau < 0 \end{cases} \tag{3.3}$$

Proceeding analogously with (2.4)(c) gives

$$\hat{v}_\tau(\tau,s) = \begin{cases} \bar{v}_t^\Gamma(0+,s)(e^{\hat{\psi}(\tau,s)}-1), & \tau > 0 \\ \bar{v}_t^\Gamma(0-,s)(e^{\hat{\psi}(\tau,s)}-1), & \tau < 0 \end{cases} \tag{3.4}$$

We used that $\hat{\psi},\hat{u},\hat{v}$ and their $\tau$-derivatives vanish at $\tau=\pm\infty$ for all $s \in \Gamma$.

The internal-layer problem for $\hat{\psi}$ is obtained by evaluating (2.4)(a) close to $\Gamma$ (but away from $\partial\Omega$) and by comparing $O(1)$ coefficients:

$$(3.5)(a)\qquad \hat{\psi}_{\tau\tau} = \delta^2 e^{\bar{\psi}^\Gamma(0+,s)}+\hat{\psi}\bar{u}^\Gamma(0,s) - \delta^2 e^{-\bar{\psi}^\Gamma(0+,s)}-\hat{\psi}\bar{v}^\Gamma(0-,s)$$
$$- D^\Gamma(0+,s), \qquad \tau > 0$$

$$(3.5)(b)\qquad \hat{\psi}_{\tau\tau} = \delta^2 e^{\bar{\psi}^\Gamma(0-,s)}+\hat{\psi}\bar{u}^\Gamma(0-,s) - \delta^2 e^{-\bar{\psi}^\Gamma(0-,s)}-\hat{\psi}\bar{v}^\Gamma(0-,s)$$
$$- D^\Gamma(0-,s), \qquad \tau < 0$$

Interface condition for (3.5)(a),(b) are derived by using that $\psi \in C^1(\Omega)$ (see Theorem 2.2), which implies $\psi^\Gamma(0+,s)=\psi^\Gamma(0-,s)$ and $\psi_t^\Gamma(0+,s)=\psi_t^\Gamma(0-,s)$. Inserting (3.1) into these relations and comparing $0(1)$ and $0(\lambda^{-1})$ coefficients yields

$$(3.5)(c)\qquad \hat{\psi}(0+,s) - \hat{\psi}(0-,s) = \bar{\psi}^\Gamma(0-,s) - \bar{\psi}^\Gamma(0+,s)$$

$$(3.5)(d)\qquad \hat{\psi}_\tau(0+,s) = \hat{\psi}_\tau(0-,s).$$

$$(3.5)(e)\qquad \hat{\psi}(+\infty,s) = \hat{\psi}(-\infty,s) = 0$$

(3.3) - (3.5) are supposed to hold for all $s \in \Gamma$. $\hat{\psi}$ is discontinuous at $\tau=0$ for all $s \in \Gamma$ since $[\bar{\psi}]_\Gamma \neq 0$.

Theorem 2.2 also provides interface conditions for the reduced problem (3.2) since it implies that $u^\Gamma(0+,s) = u^\Gamma(0-,s)$, $v^\Gamma(0+,s) = v^\Gamma(0-,s)$ and $u_t^\Gamma(0+,s) = u_t^\Gamma(0-,s)$, $v_t^\Gamma(0+,s) = v_t^\Gamma(0-,s)$. Inserting the expansions

$$(3.6)(a)\qquad u(x,y,\lambda) \sim \bar{u}(x,y) + \lambda\hat{u}(\tfrac{t}{\lambda},s) + \ldots$$

$$(3.6)(b)\qquad v(x,y,\lambda) \sim \bar{v}(x,y) + \lambda\hat{v}(\tfrac{t}{\lambda},s) + \ldots$$

(the dots denote a power series in $\lambda$ stating with the $0(\lambda^2)$ term) into these relations and comparing $0(1)$ terms yields

$$(3.7)(a)\qquad [\bar{u}]_\Gamma = 0, \quad [\bar{v}]_\Gamma = 0$$

and using (3.3), (3.4)

$$\bar{u}_t^\Gamma(0+,s)e^{-\hat{\psi}(0+,s)} = \bar{u}_t^\Gamma(0-,s)e^{-\hat{\psi}(0-,s)}$$
$$\bar{v}_t^\Gamma(0+,s)e^{\hat{\psi}(0+,s)} = \bar{v}_t^\Gamma(0-,s)e^{\hat{\psi}(0-,s)}.$$

(3.5)(c) implies

$$u_t^\Gamma(0+,s)e^{\bar{\psi}^\Gamma(0+,s)} = \bar{u}_t^\Gamma(0-,s)e^{\bar{\psi}^\Gamma(0-,s)}$$
$$\bar{v}_t^\Gamma(0+,s)e^{-\bar{\psi}^\Gamma(0+,s)} = \bar{v}_t^\Gamma(0-,s)e^{-\bar{\psi}^\Gamma(0-,s)}.$$

So we obtain the interface conditions for the reduced problem

(3.7)(b) $\quad [e^{\tilde{\psi}}\nabla\tilde{u}\cdot\nabla t]_\Gamma = 0, \quad [e^{-\tilde{\psi}}\nabla\tilde{v}.\nabla t]_\Gamma = 0 .$

since $f_t^\Gamma(0,s) = \nabla f(x,y)\cdot\nabla t(x,y)|_{(x,y)=s\in\Gamma}$ holds.

C) Ohmic Contacts

A straight-forward calculation shows that the Dirichlet boundary conditions on $\partial\Omega_C$ for $\psi$, u and v fulfill (3.2)(a). Therefore we do not expect boundary layers at $\partial\Omega_C$. That means

(3.8) $\quad \tilde{\psi}_o(\rho,q) = \tilde{u}_o(\rho,q) = \tilde{v}_o(\rho,q) = 0 \quad$ for $\rho > 0$ and $q \in \partial\Omega_C$.

To proove (3.8) one has to proceed as Markowich, Ringhofer, Selberherr and Langer (1982 a,b) did for the one dimensional semiconductor problem.

(3.8) implies that we have to impose the same Dirichlet boundary conditions for the reduced problem (3.2) as for the full singularly perturbed problem:

(3.9) $\quad \tilde{\psi}|_{\partial\Omega_C} = \psi|_{\partial\Omega_C}, \quad \tilde{u}|_{\partial\Omega_C} = u|_{\partial\Omega_C}, \quad \tilde{v}|_{\partial\Omega_C} = v|_{\partial\Omega_C}$

where $\psi$,u,v on $\partial\Omega_C$ are given by (2.7).

D) Insulating Segments

We assume that D is differentiable in a neighbourhood of $\partial\Omega$ and that

(3.10) $\quad \nabla D\cdot\vec{n}|_{\partial\Omega_{is}} = 0$

holds. Differentiating (3.2)(a) gives

(3.11) $\quad 0 = \delta^2(e^{\tilde{\psi}}\tilde{u}+e^{-\tilde{\psi}}\tilde{v})\nabla\tilde{\psi} + \delta^2(e^{\tilde{\psi}}\nabla\tilde{u}-e^{-\tilde{\psi}}\nabla\tilde{v}) - \nabla D$

Since $e^{\tilde{\psi}}\tilde{u}+e^{-\tilde{\psi}}\tilde{v} > 0$ in $\bar{\Omega}$ we get

(3.12)(a) $\quad \nabla\tilde{\psi}\cdot\vec{n}|_{\partial\Omega_{is}} = 0$

if

(3.12)(b) $\quad \nabla\tilde{u}\cdot\vec{n}|_{\partial\Omega_{is}} = \nabla\tilde{v}.\vec{n}|_{\partial\Omega_{is}} = 0$

holds. (3.2)(a) is compatible with the zero Neumann conditions for $\psi$,u,v on $\partial\Omega_{is}$ and we get

(3.13) $\tilde{\psi}_0(\rho,q) = \tilde{u}_0(\rho,q) = \tilde{v}_0(\rho,q) = 0 \quad$ for $\rho > 0$, $q \in \partial\Omega_{is}$

as well as

(3.14) $\tilde{\psi}_1(\rho,q) = \tilde{u}_1(\rho,q) = \tilde{v}_1(\rho,q) = 0 \quad$ for $\rho > 0$, $q \in \partial\Omega_{is}$.

No zeroth and first order layers occur at $\partial\Omega_{is}$.

If we did not assume (3.10) then (3.14) and (3.12)(a) would not hold (of course (3.13) would still be valid).

(3.12)(a),(b) define homogenous Neumann boundary conditions for (3.2) on $\partial\Omega_{is}$.

E) Oxide-Semiconductor Interface

As for junctions we get $\tilde{u}_0 = \tilde{v}_0 = 0$ and setting $\tilde{u} = \tilde{u}_1$, $\tilde{v} = \tilde{v}_1$, $\tilde{\psi} = \tilde{\psi}_0$ we obtain

(3.15)(a) $\tilde{u}_0(\rho,q) = \tilde{u}_r^{\partial}(0,q)(e^{-\tilde{\psi}(\rho,q)}-1), \qquad \rho > 0,\ q \in \partial\Omega_{os}$

(3.15)(b) $\tilde{v}_0(\rho,q) = \tilde{v}_r^{\partial}(0,q)(e^{\tilde{\psi}(\rho,q)}-1), \qquad \rho > 0,\ q \in \partial\Omega_{os}$.

Since we set

(3.16)(a) $u(x,y,\lambda) \sim \bar{u}(x,y) + \lambda\hat{u}(\frac{t}{\lambda},s) + \lambda\tilde{u}(\frac{r}{\lambda},q) + \ldots$

(3.16)(b) $v(x,y,\lambda) \sim \bar{v}(x,y) + \lambda\hat{v}(\frac{t}{\lambda},s) + \lambda\tilde{v}(\frac{r}{\lambda},q) + \ldots$

(where the dots denote a power series in $\lambda$ starting with the $O(\lambda^2)$ term), we get by inserting (3.16) into (2.8)(e) and by comparing $O(1)$ coefficients

$$0 = (\nabla\bar{u}\cdot\vec{n})(q) + \tilde{u}_r^{\partial}(0,q)(e^{-\tilde{\psi}(0,q)}-1), \quad q \in \partial\Omega_{os}$$

$$0 = (\nabla\bar{v}\cdot\vec{n})(q) + \tilde{v}_r^{\partial}(0,q)(e^{\tilde{\psi}(0,q)}-1), \quad q \in \partial\Omega_{os}$$

Therefore

$$0 = \tilde{u}_r^{\partial}(0,q) = \tilde{v}_r^{\partial}(0,q), \quad q \in \partial\Omega_{os}$$

and $\tilde{u}(\rho,q) = \tilde{v}(\rho,q) = 0$ for $\rho > 0$, $q \in \partial\Omega_{os}$. Boundary conditions for $\bar{u},\bar{v}$ on $\partial\Omega_{os}$ are

(3.17) $\nabla\bar{u}\cdot\vec{n}|_{\partial\Omega_{os}} = \nabla\bar{v}\cdot\vec{n}|_{\partial\Omega_{os}} = 0.$

If

(3.18) $\nabla D\cdot\vec{n}|_{\partial\Omega_{os}} = 0$

holds then (3.17) implies (as for insulating segments)

$$(3.19) \qquad \nabla\tilde{\psi}\cdot\vec{n}|_{\partial\Omega_{os}} = 0.$$

By evaluating close to $\partial\Omega_{os}$ but away from $\Gamma$ we obtain the boundary layer equation for $\psi|_{\partial\Omega_{os}}$ which is analogous to the interface problem (3.5):

$$(3.20)(a) \qquad \tilde{\psi}_{\rho\rho} = \delta^2 e^{\bar{\psi}^\partial(0,q)+\tilde{\psi}_U^\partial(0,q)} - \delta^2 e^{-\bar{\psi}^\partial(0,q)-\tilde{\psi}_V^\partial(0,q)} - D^\partial(0,q), \quad \rho>0,\ q\in\partial\Omega_{os}$$

$$(3.20)(b) \qquad \tilde{\psi}(\infty,q) = 0, \quad q\in\partial\Omega_{os}.$$

To obtain the boundary condition for (3.20) at $\rho=0$ we solve Laplace's equation in the oxide.

Let $G(x,y,\xi,n)$ denote the Green's function (see Protter and Weinberger (1967), Chapter 2, Section 7) of the problem

$$\Delta\varphi = f \quad \text{in } \Phi$$
$$\varphi|_{\partial\Phi_C} = g, \quad \nabla\varphi\cdot\vec{\zeta}|_{\partial\Phi_{is}} = h, \quad \nabla\varphi\cdot\vec{\zeta}|_{\partial\Omega_{os}} = k$$

(note that $\vec{\zeta}|_{\partial\Omega_{os}} = -\vec{n}|_{\partial\Omega_{os}}$).

Then, since $\psi=\psi_G+\varphi$ fulfills

$$\Delta\varphi = 0 \text{ in}$$
$$\varphi|_{\partial\Phi_C} = 0, \quad \nabla\varphi\cdot\vec{\zeta}|_{\partial\Phi_{is}} = 0, \quad \nabla\varphi\cdot\vec{\zeta}|_{\partial\Omega_{os}} = -\frac{\varepsilon_s}{\varepsilon_o}\nabla\psi\cdot\vec{n}|_{\partial\Omega_{os}}$$

we get

$$\psi = \psi_G - \frac{\varepsilon_s}{\varepsilon_o}\int_{\partial\Omega_{os}} G(x,y,\xi,n)\nabla\psi(\xi,n)\cdot\vec{n}(\xi,n)d(\xi,n)$$

This gives the boundary condition (for the singularly perturbed problem in $\Omega$):

$$(3.21) \qquad \psi|_{\partial\Omega_{os}} + \frac{\varepsilon_s}{\varepsilon_o}\int_{\partial\Omega_{os}} G(x,y,\xi,n)|_{(x,y)\in\partial\Omega_{os}}\nabla\psi(\xi,n)\cdot\vec{n}(\xi,n)d(\xi,n) = \psi_G.$$

For the following we assume that if $\bar{\Gamma}$ hits $\overline{\partial\Omega}_{os}$ (which in fact happens in MOS-technology) then $\bar{\Gamma}$ is perpendicular to $\overline{\partial\Omega}_{os}$, that means

$$(3.22) \qquad \nabla t(S)\cdot\nabla\tau(S) = 0 \quad \text{for } \{S\} = \bar{\Gamma}\cap\overline{\partial\Omega}_{os}.$$

We insert the expansion

$$(3.23) \qquad \psi(x,y,\lambda) \sim \bar{\psi}(x,y) + \hat{\psi}(\tfrac{t}{\lambda},s) + \tilde{\psi}(\tfrac{r}{\lambda},q) + \ldots$$

(the dots denote a power series starting with the $O(\lambda)$ term whose coefficients have the same form as the $O(1)$ terms) into (3.21). Assuming that

(3.24) $$C_1 \leq |\psi_G|\lambda \leq C_2$$

where $C_1, C_2$ are independent of $\lambda$, we obtain by comparing $O(\lambda^{-1})$ terms and by using (3.22):

(3.25) $$\int_{\partial\Omega_{os}} G(x,y,\xi,\eta)|_{(x,y)\in\partial\Omega_{os}} \cdot \tilde{\psi}_\rho(0,q(\xi,\eta))d(\xi,\eta) = \lambda\frac{\epsilon_0}{\epsilon_s}\psi_G.$$

This integral equation is uniquely soluble (for $\tilde{\psi}_\rho(0,q)$) since $\tilde{\psi}_\rho(0,q) = (\nabla w\cdot\vec{\zeta})(q)$ where w is the (unique) solution of

$$\Delta w = 0 \text{ in } \phi$$

$$w|_{\partial\phi_C} = \nabla w\cdot\vec{\zeta}|_{\partial\phi_{is}} = 0, \quad w|_{\partial\Omega_{os}} = \lambda\frac{\epsilon_0}{\epsilon_s}\psi_G$$

Obviously $w\in C^\infty(\bar{\phi}\text{-}CR(\partial\phi))$ holds.

If $\phi$ is a rectangle as in Figure 1 (which is a common assumption in MOS-modelling) with $d = \text{dist}\,(\partial\Omega_{os},\partial\phi_C)$ then w is a linear function and $\nabla w\cdot\vec{\zeta}|_{\partial\Omega_{os}} = -\lambda\frac{\epsilon_0}{\epsilon_s}\psi_G$. In this case

(3.26) $$\tilde{\psi}_\rho(0,q) = -\lambda\frac{\epsilon_0}{d\epsilon_s}\psi_G .$$

No zeroth order layer occurs at the oxide-semiconductor interface if the right hand side of (3.26) is not $O(1)$ as $\lambda\to 0_+$.

(3.25) (or equivalently (3.26) if $\phi$ is a rectangle) provides the missing boundary condition for the interface layer problem (3.20).

Equations for the higher order terms of the expansion (3.1) can be derived in a analogous way.

When the asymptotics of $\psi,u,v$ are known then expansions of $n,p$ and of $J_n$, $J_p$ as given by (2.9) can easily be derived. We get from (3.1)

(3.27) $$n(x,y,\lambda) = \bar{n}(x,y) + \hat{n}(\tfrac{\xi}{\lambda},s) + \tilde{n}(\tfrac{r}{\lambda},q) + \ldots$$

(3,28)(b) $$p(x,y,\lambda) = \bar{p}(x,y) + \hat{p}(\tfrac{\xi}{\lambda},s) + \tilde{p}(\tfrac{r}{\lambda},q) + \ldots$$

(3.16) and (3.23) imply

(3.29)(a) $$\bar{n} = \delta^2 e^{\bar{\psi}}\bar{u}, \quad \hat{n}(\tau,s) = \begin{cases} \delta^2 e^{\bar{\psi}\ (0+,s)}(e^{\hat{\psi}(\tau,s)}-1)\bar{u}^\Gamma(0,s), & \tau > 0 \\ \delta^2 e^{\bar{\psi}^\Gamma(0-,s)}(e^{\hat{\psi}(\tau,s)}-1)\bar{u}^\Gamma(0,s), & \tau < 0 \end{cases}$$

(3.29)(b) $\tilde{n}(\rho,q) = 0$ for $\rho > 0$, $q \in \partial\Omega_C \cup \partial\Omega_{is}$

(3.29)(c) $\tilde{n}(\rho,q) = \delta^2 e^{\tilde{\psi}^3(0,q)}(e^{\tilde{\psi}(\rho,q)}-1)$, $\rho > 0$, $q \in \partial\Omega_{os}$

(3.29)(d) $\tilde{p} = \delta^2 e^{-\tilde{\psi}}\tilde{v}$

and similar expressions for $\hat{p}$ and $\tilde{p}$.

Differentiating (3.16) and using (3.3), (3.4) gives

(3.30)(a) $J_n(x,y,\lambda) = J_n(x,y) + \hat{J}_n(\frac{\xi}{\lambda},s) + \tilde{J}_n(\frac{r}{\lambda},q) + \ldots$

(3.30)(b) $J_p(x,y,\lambda) = J_p(x,y) + \hat{J}_p(\frac{\xi}{\lambda},s) + \tilde{J}_p(\frac{r}{\lambda},q) + \ldots$

with

(3.31)(a) $J_n = \delta^2 e^{\tilde{\psi}}\nabla\tilde{u}$, (b) $J_p = -\delta^2 e^{-\tilde{\psi}}\nabla\tilde{v}$

(3,32)(a) $$\hat{J}_n(\tau,s) = \begin{cases} \delta^2 e^{\tilde{\psi}^\Gamma(0+,s)}(e^{\hat{\psi}(\tau,s)}-1)\left.\frac{\partial s(x,y)}{\partial(x,y)}\right|_{(x,y)=s}\cdot\nabla_s\tilde{u}^\Gamma(0+,s), & \tau > 0 \\ \delta^2 e^{\tilde{\psi}^\Gamma(0-,s)}(e^{\hat{\psi}(\tau,s)}-1)\left.\frac{\partial s(x,y)}{\partial(x,y)}\right|_{(x,y)=s}\cdot\nabla_s\tilde{u}^\Gamma(0-,s), & \tau < 0 \end{cases}$$

(3.32)(b) $\tilde{J}_n(\rho,q) = 0$ for $\rho > 0$, $q \in \partial\Omega_C \cup \partial\Omega_{is}$

(3,32)(c) $\tilde{J}_n(\rho,q) = \delta^2 e^{\tilde{\psi}^3(0,q)}(e^{\tilde{\psi}(\rho,q)}-1)\left.\frac{\partial q(x,y)}{\partial(x,y)}\right|_{(x,y)=q}\nabla_q\tilde{u}^3(0,q)$, $q \in \partial\Omega_{os}$

Analogous expressions hold for $\hat{J}_p$ and $\tilde{J}_p$.

Since $(\frac{\partial s}{\partial(x,y)})^T\nabla t|_\Gamma = (\frac{\partial q}{\partial(x,y)})^T\nabla r|_{\partial\Omega} = 0$ holds we get

(3.33) $\hat{J}_n\cdot\nabla t|_\Gamma = \tilde{J}_n\cdot\nabla r|_{\partial\Omega_{os}} = \hat{J}_p\cdot\nabla t|_\Gamma = \tilde{J}_p\cdot\nabla r|_{\partial\Omega_{os}} = 0$

Therefore the current density components perpendicularto the junctions(to the semiconductor oxide interface) have no zeroth order layers while the current density components parallel to the junctions(to the semiconductor-oxide interface) may very well have zeroth order layers. This phenomen is illustrated by the MOS-transistor simulation performed by Selberherr (1980).

So far we only considered one curve of discontinuity of D. Generally, an internal layer in $\psi$ occurs at each junction $\Gamma_i$ and each layer-term $\hat{\psi}_i$ fulfills the corresponding layer problem (3.5) (with $\Gamma_i$-local coordinates $(s_i,t_i)$).

## 4. Existence of the reduced solution and of the layer solutions

### A. Reduced Solution

We now assume that we have N $C^\infty$-junctions $\Gamma_i$ which are as in Section 1. $[D]_{\Gamma_i} \neq 0$ holds for i=1,...,N. We denote the normalvector to $\Gamma_i$ by $t_i$ ($=\nabla t_i$).
Then the interface conditions (3.7)(a),(b) read

(4.1)(a) $[\tilde{u}]_{\Gamma_i} = 0, \quad [\tilde{v}]_{\Gamma_i} = 0, \quad i=1,\dots,N$

(4.1)(b) $[e^{\tilde{\psi}}\nabla\tilde{u}\cdot t_i]_{\Gamma_i} = 0, \quad [e^{-\tilde{\psi}}\nabla\tilde{v}\cdot t_i]_{\Gamma_i} = 0, \quad i=1,\dots,N.$

Moreover $\partial\Omega$ fulfills the assumptions of Theorem 2.2. We prove:

Theorem 4.1 Assume that

(4.2) $D, \tilde{a}_i \in C(\bar{\Omega}_i), \quad i=1,\dots,N$

holds and that $D, \Gamma_i, \partial\Omega$ fulfill the assumptions given above. Then the reduced problem (3.2),(3.9), (3.12)(b),(3.17),(4.1) has a weak solution $(\tilde{u},\tilde{v}) \in (H^1(\Omega)\cap L^\infty(\Omega))^2$, $\tilde{\psi} \in L^\infty(\Omega)$. Every weak solution fulfills the estimates (2.10)(a),(b) and

(4.3) $\tilde{\psi}_- \le \tilde{\psi}(x,y) \le \tilde{\psi}_+, \qquad (x,y) \in \bar{\Omega}$

(where $\tilde{\psi}_-$, $\tilde{\psi}_+$ are defined in (2.11)(a),(b)). Also every weak solution satisfies

(4.4)(a) $\tilde{\psi} \in C(\bar{\Omega}_i - CR(\partial\Omega)), \qquad$ (b) $(\tilde{u},\tilde{v}) \in (C(\bar{\Omega} - CR(\partial\Omega)))^2$ for i=0,...,N.

Proof: A weak solution of the reduced problem is given by a triple $(\tilde{\psi},\tilde{u},\tilde{v})$ with $(\tilde{u},\tilde{v})\in(H^1(\Omega))^2$, $\tilde{\psi} \in L^\infty(\Omega)$ which solves (3.2)(a) pointwise (almost everywhere), which assumes the Dirichlet boundary conditions on $\partial\Omega_C$ in the sense of $(H^1(\Omega))^3$ and which fulfills

(4.5)(a) $L_1(\psi,u,\varphi) = \int_\Omega e^{\psi}\nabla u\cdot\nabla\varphi dxdy = 0$

(4.5)(b) $L_2(\psi,v,\varphi) = \int_\Omega e^{-\psi}\nabla v\cdot\nabla\varphi dxdy = 0$ for all $\varphi \in H_0^1(\Omega \cup \partial\Omega_{iS} \cup \partial\Omega_{OS})$.

(If $\psi,u,v$ and $\varphi$ are sufficiently smooth, then (4.5)(a),(b) yields (3.2)(b), (c), (3.12)(b), (3.17) and (4.1) using integration by parts.)

The maximum principle for weak solutions of linear equations in divergence form immediately implies that $\tilde{u},\tilde{v}$ fulfill (2.1) (a),(b) if $\tilde{\psi} \in L^\infty(\Omega)$.
(3.2)(a) gives

$$(4.6) \qquad \bar{\psi}(\bar{u},\bar{v}) = \ln\left[\frac{D + \sqrt{D^2+4\delta^4\bar{u}\bar{v}}}{2\delta^2\bar{u}}\right]$$

and (4.3) is immediate. (4.4)(b) follows from a posteriori regularity theorem (see Ladyzenskaja and Ural'tseva(1968), Chapter 3, Section 14) and (4.4)(a) is implied by (4.6).

To establish the existence statement of the theorem we employ Schauder's fixed point theorem similarly to the existence proof of Bank, Jerome and Rose (1982) for the singularly perturbed problem (see Theorem 2.1). The existence proof for the reduced problem isslightly more difficult because the regularising property of Poisson's equation is lost.

We define the following set M:

$$(4.7) \qquad M = \{(u,v) \in (L^\infty(\Omega))^2 \mid u_- \le u \le u_+,\ v_- \le v \le v_+\}$$

and regard M as closed and convex set in $(L^2(\Omega))^2$. A mapping $T: M \to (L^2\Omega))^2$ is defined as follows:

$$(4.8) \qquad T(u_0,v_0) = (u_1,v_1)$$

where $u_1, v_1$ are the weak solution of

$$(4.9)(a) \qquad \mathrm{div}(e^{\bar{\psi}(u_0,v_0)}\nabla u_1) = 0$$

$$(4.9)(b) \qquad \mathrm{div}(e^{-\bar{\psi}(u_0,v_0)}\nabla v_1) = 0$$

subject to the already specified boundary conditions. T is well defined since (4.9)(a),(b) has a unique weak solution $(u_1,v_1) \in (H^1(\Omega))^2$. Clearly every fixed point of T s a weak solution of the reduced problem.

The a priori estimates (2.10)(a),(b) imply that T:M→M( since they are independent of the multipliersof $\nabla u, \nabla v$ in (4.9)). The wellposedness of (4.9) in $(H^1(\Omega))^2$induces the boundedness of Range(T) in $(H^1(\Omega))^2$, therefore Range(T) is relatively compact in $(L^2(\Omega))^2$.

To show that $T \in C(M)$ we take a sequence $(u_n,v_n) \in M$, such that

$$u_n \overset{L_2(\Omega)}{\to} u^*, \quad n \to \infty$$

$$v_n \overset{L_2(\Omega)}{\to} v^*, \quad n \to \infty$$

holds. We will show that every subsequence of $T(u_n,v_n)$ has a subsequence which

converges to $T(u^*,v^*)$ in $(L^2(\Omega))^2$ (which implies $T(u_n,v_n) \to T(u^*,v^*)$ in $(L^2(\Omega))^2$).

Since the range of T is relatively compact in M and since M is closed we conclude that every subsequence of $T(u_n,v_n)$ has a convergent subsequence (with limit in M). Let $T(u_{n_k},v_{n_k})$ be such a convergent subsequence (of a subsequence of $T(u_n,v_n)$) and let $T(u_{n_k},v_{n_k}) \to (a,b) \in M$ in $(L^2(\Omega))^2$ as $k \to \infty$. Obviously

$$\bar{\psi}(u_{n_k},v_{n_k}) \overset{L^2(\Omega)}{\to} \bar{\psi}(u^*,v^*) \quad \text{as } k \to \infty$$

holds. Therefore there is a subsequence $\bar{\psi}(u_{n_{k_j}},v_{n_{k_j}})$ such that

$$e^{\bar{\psi}(u_{n_{k_j}},v_{n_{k_j}})} \to e^{\bar{\psi}(u^*,v^*)}, \quad j \to \infty$$

$$e^{-\bar{\psi}(u_{n_{k_j}},v_{n_{k_j}})} \to e^{-\bar{\psi}(u^*,v^*)}$$

almost everywhere pointwise in $\Omega$.

Theorem 5.4 in Ladyzenskaja and Ural'tseva(1968, Chapter 3, Section 5) implies that the weak solutions of

$$\operatorname{div}\left(e^{\bar{\psi}(u_{n_{k_j}},v_{n_{k_j}})}\nabla u\right) = 0$$

$$\operatorname{div}\left(e^{-\bar{\psi}(u_{n_{k_j}},v_{n_{k_j}})}\nabla v\right) = 0$$

subject to the boundary and interface conditions converge in $H^1(\Omega)$ as $j \to \infty$ to the solutions of the limiting problems

$$\operatorname{div}\left(e^{\bar{\psi}(u^*,v^*)}\nabla u\right) = \operatorname{div}\left(e^{-\psi(u^*,v^*)}\nabla v\right) = 0$$

(subject to the boundary and interface conditions).

This yields $T(u_{n_{k_j}},v_{n_{k_j}}) \to T(u^*,v^*)$ in $(L^2(\Omega))^2$ as $j \to \infty$ and $(a,b) = T(u^*,v^*)$. Therefore T is a continuous and compact mapping from M into M and Schauder's fixed point Theorem implies the existence of a fixed point of T in M. □

Note that $\bar{\psi}$ has jumpdiscontinuities along $\Gamma_i$, $i=1,\ldots,N$.

For the carrier densities $\bar{n}$ and $\bar{p}$ we get from (3.29)(a),(4.6)

$$\bar{n} = \frac{D + \sqrt{D^2 + 4\delta^4 \bar{u}\bar{v}}}{2} \qquad p = \frac{-D + \sqrt{D^2 + 4\delta^4 \bar{u}\bar{v}}}{2}$$

Therefore the a-priori- estimates on $\bar{u}$, $\bar{v}$ yield

$$\bar{n} = \begin{cases} D + O(\delta^4 \exp(\frac{|V_{max}|}{U_T})) \text{ in } \Omega_i \quad \text{if } D|_{\bar{\Omega}_i} > 0 \\ O(\delta^4 \exp(\frac{|V_{max}|}{U_T})) \quad \text{in } \Omega_i \quad \text{if } D|_{\bar{\Omega}_i} < 0 \end{cases} \qquad \text{in } L^\infty(\Omega)$$

and

$$\bar{p} = \begin{cases} O(\delta^4 \exp(\frac{|V_{max}|}{U_T})) \text{ in } \Omega_i \text{ if } D|_{\bar{\Omega}_i} > 0 \\ -D + O(\delta^4 \exp(\frac{|V_{max}|}{U_T})) \text{ in } \Omega_i \text{ if } D|_{\bar{\Omega}_i} < 0 \end{cases} \qquad \text{in } L^\infty(\Omega) .$$

If $\delta^4 \exp(\frac{|V_{max}|}{U_T})$ is small ('low injection condition') then the electron density in the zero space charge approximation is close to the doping profile in n-regions and close to zero in p regions while the hole density is close to zero in n regions and close to the negative doping profile in p regions. n regions are depleted of holes and p-regions are depleted of electrons.

We now investigate the behaviour of $\bar{v}, \bar{u}, \bar{v}$ as $\delta \to 0$. We assume that there is exactly one Ohmic contact $C_i$ in every $\Omega_i$ for $i=0,\ldots,N$ and that no two n (and no two p) regions have a joint boundary, that means

$$\text{(4.10)} \qquad \operatorname{sgn} D|_{\Omega_{i-1}} = -\operatorname{sgn} D|_{\Omega_i}, \quad i=1,\ldots,N.$$

holds.

We define the functions

$$\text{(4.11)} \qquad \bar{u}_o = \begin{cases} e^{-\frac{U_i}{U_T}} \text{ in } \Omega_i \text{ if } D|_{\bar{\Omega}_i} > 0 \text{ (n-region)} \\ \bar{u}_{oi} \text{ in } \Omega_i \text{ if } D|_{\bar{\Omega}_i} < 0 \text{ (p-region)} \end{cases}$$

where $\bar{u}_{oi}$ solves

$$\text{(4.12)} \qquad \begin{aligned} &\text{(a)} \quad \operatorname{div}(-\tfrac{1}{\bar{v}} \nabla \bar{u}_o) = 0 \quad \text{in } \Omega_i \\ &\text{(b)} \quad \nabla \bar{u}_o \cdot \vec{n}|_{\partial\Omega_i \cap (\partial\Omega_{is} \cup \partial\Omega_{os})} = 0 \\ &\text{(c)} \quad \bar{u}_o|_{C_i} = e^{-\frac{U_i}{U_T}} \\ &\text{(d)} \quad [\bar{u}_o]_{\partial\Omega_i - \partial\Omega} = 0 \end{aligned}$$

and

$$(4.13)\qquad \tilde{v}_o = \begin{cases} e^{\frac{U_i}{U_T}} & \text{in } \Omega_i \text{ if } D|_{\bar{\Omega}_i} < 0 \quad \text{(p-region)} \\ \tilde{v}_{oi} & \text{in } \Omega_i \text{ if } D|_{\bar{\Omega}_i} > 0 \quad \text{(n-region)} \end{cases}$$

where $\tilde{v}_{oi}$ solves

$$(4.14)\qquad \begin{aligned} &\text{(a)}\quad \operatorname{div}\left(\tfrac{1}{\delta}\nabla\tilde{v}_o\right) = 0 \quad \text{in } \Omega_i \\ &\text{(b)}\quad \nabla\tilde{v}_o\cdot\vec{n}_i|_{\partial\Omega_i\cap(\partial\Omega_{is}\cup\partial\Omega_{os})} = 0 \\ &\text{(c)}\quad \tilde{v}_o|_{C_i} = e^{\frac{U_i}{U_T}} \\ &\text{(d)}\quad [\tilde{u}_o]_{\partial\Omega_i-\partial\Omega} = 0\ . \end{aligned}$$

The condition (4.10) implies that (4.12)(d), (4.14)(d) give Dirichlet boundary conditions for $\tilde{u}_{oi}$, $\tilde{v}_{oi}$ resp. by using (4.12). The problems (4.12), (4.13) have unique solutions $\tilde{u}_{oi}$, $\tilde{v}_{oi} \in H^1(\Omega_i)$ (see Kawohl (1978))assuming that

$$(4.15)\qquad D|_{\bar{\Omega}_i} \in C^{1,\alpha}(\bar{\Omega}_i), \qquad i=0,\ldots,N.$$

We prove

Theorem 4.2. Let the above mentioned assumption on $\Omega$,D hold and assume that $\partial\Omega$ fulfills the assumption of Theorem 4.1. Also assume that

$$(4.16)\qquad \nu = \delta^4\exp\left(4\max_i\left|\frac{U_i}{U_T}\right|\right)\left(\exp\left(\frac{|V_{max}|}{U_T}\right)-1\right) < \nu$$

where $\nu$ and $\delta$ are sufficiently small (depending on D, $\Omega_0 \ldots \Omega_N$ only) and that $\nabla\tilde{u}_{oi} \in (L^\infty(\Omega_i))^2$, $\nabla\tilde{v}_{oi} \in (L^\infty(\Omega_i))^2$ holds. Then the reduced problem (3.2), (3.9), (3.12)(b), (3.17), (4.1) has a locally unique solution $(\tilde{u},\tilde{v}) \in (H^1(\Omega))^2$ which fulfills

$$(4.17)(a)\quad \tilde{u} = \tilde{u}_o + O(\nu) \qquad (b)\quad \tilde{v} = \tilde{v}_o + O(\nu)$$

and

$$(4.17)(c)\quad \tilde{w} = \begin{cases} \ln\left(\frac{D}{\delta^2}\right) + \frac{U_i}{U_T} + O(\nu+\delta^4) & \text{in } \Omega_i \text{ if } D|_{\bar{\Omega}_i} > 0 \\ \ln\left(\frac{\delta^2}{-D}\right) + \frac{U_i}{U_T} + O(\nu+\delta^4) & \text{in } \Omega_i \text{ if } D|_{\bar{\Omega}_i} < 0 \end{cases}$$

in $L^\infty(\Omega_i)$. The estimates are uniform in $\left|\frac{U_i}{U_T}\right|$ (as long as (4.16) holds).

Proof: We assume that there are only two differently doped regions $\Omega_0,\Omega_1$ separated by $\Gamma$ and that $D|_{\bar{\Omega}_0} < 0$, $D|_{\bar{\Omega}_1} > 0$ holds. The generalisation (under the given

assumptions on $\Omega$ and D) to more than two differently doped regions is straight forward.

We set

$$\tilde{u} = \tilde{u}_0 + E_u, \qquad \tilde{v} = \tilde{v}_0 + E_v$$

$$\tilde{\psi} = E_\psi + \begin{cases} \ln\left(\frac{D}{\delta^2}\right) + \frac{U_1}{U_T} & \text{in } \Omega_1 \\ \ln\left(\frac{\delta^2}{-D}\right) + \frac{U_0}{U_T} & \text{in } \Omega_0 \end{cases}$$

and compute $E_\psi$ from (3.2)(a)

$$E_\psi = E_\psi(E_u, E_v, \delta) = \begin{cases} \ln\left[\frac{D+\sqrt{D^2+4\delta^4 uv}}{2D\exp(\frac{U_1}{U_T})u}\right] & \text{in } \Omega_1 \\ \ln\left[\frac{2D\exp(-\frac{U_0}{U_T})v}{-D+\sqrt{D^2+4\delta^4 uv}}\right] & \text{in } \Omega_0 \end{cases} \tag{4.18}$$

Then we rewrite (3.2)(b),(c), (4.1)(a), (b) as

$$\begin{aligned} &\operatorname{div}\left(De^{E_\psi}\nabla E_u\right) = 0 \quad \text{in } \Omega_1 \\ &\operatorname{div}\left(\delta^4 e^{\frac{U_0-U_1}{U_T}}\left(\frac{1}{-D}\right)e^{E_\psi}(\nabla E_u + \nabla\tilde{u}_0)\right) = 0 \text{ in } \Omega_0 \\ &[E_u]_\Gamma = 0, \quad De^{E_\psi}\nabla E_u\cdot\vec{n}|_{\Gamma(1)} = \delta^4 e^{\frac{U_0-U_1}{U_T}}\left(-\frac{1}{D}\right)(\nabla E_u + \nabla\tilde{u}_0)\cdot\vec{n}|_{\Gamma(0)}, \nabla E_u\cdot\vec{n}|_{\partial\Omega_{is}\cup\partial\Omega_o} = 0 \end{aligned} \tag{4.19}$$

where $f|_{\Gamma(i)}(a,b) = \lim_{\substack{(x,y)\to(a,b)\\(x,y)\in\Omega_i}} f(x,y)$ for $(a,b)\in\Gamma$ and

$$\begin{aligned} &\operatorname{div}\left(\delta^4 e^{\frac{U_0-U_1}{U_T}}\frac{1}{D}e^{-E_\psi}(\nabla E_v+\nabla\tilde{v}_0)\right) = 0 \text{ in } \Omega_1 \\ &\operatorname{div}\left(-De^{-E_\psi}\nabla E_v\right) = 0 \quad \text{in } \Omega_0 \\ &[E_v]_\Gamma = 0, \quad -De^{-E_\psi}\nabla E_v\cdot\vec{n}|_{\Gamma(0)} = \delta^4 e^{\frac{U_0-U_1}{U_T}}\left(\frac{1}{D}\right)e^{-E_\psi}(\nabla E_v+\nabla\tilde{v}_0)\cdot\vec{n}|_{\Gamma(1)}, \nabla E_v\cdot\vec{n}|_{\partial\Omega_{is}\cup\partial\Omega_{os}} = 0 \end{aligned} \tag{4.20}$$

We define the operator $F_\delta: (H_0^1(\Omega\cup\partial\Omega_{is}\cup\partial\Omega_{os}))^2 \to ((H_0^1(\Omega\cup\partial\Omega_{is}\cup\partial\Omega_{os}))^2)^*$

(the superscript '*' denotes the dual space) by

$$(4.21)\quad F_\delta(E_1,E_2)(\varphi_1,\varphi_2) = \int_{\Omega_1} De^{E_\psi(E_1,E_2,\delta)}\nabla E_1\cdot\nabla\varphi_1 dxdy$$
$$+ \delta^4 e^{\frac{U_o-U_1}{U_T}} \int_{\Omega_o} (\tfrac{1}{D}) e^{E_\psi(E_1,E_2,\delta)}(\nabla E_1+\nabla\bar{u}_o)\cdot\nabla\varphi_1 dxdy$$
$$+ \int_{\Omega_o} (-D) e^{-E_\psi(E_1,E_2,\delta)}\nabla E_2\cdot\nabla\varphi_2 dxdy$$
$$+ \delta^4 e^{\frac{U_o-U_1}{U_T}} \int_{\Omega_1} \tfrac{1}{D} e^{-E_\psi(E_1,E_2,\delta)}(\nabla E_2+\nabla\bar{v}_o)\nabla\varphi_2 dxdy$$

for all $(\varphi_1,\varphi_2) \in (H_o^1(\Omega \cup \partial\Omega_{is} \cup \partial\Omega_{os}))^2$.

$$E_u|_{\partial\Omega_C} = E_v|_{\partial\Omega_C} = 0 \quad \text{yields} \quad F_\delta(E_u,E_v) = 0,$$

that means $F_\delta(E_u,E_v) = 0$ is the weak formulation of (4.19), (4.20). In the sequel we equip $(H_o^1(\Omega \cup \partial\Omega_{is} \cup \partial\Omega_{os}))^2$ with the weighted norm

$$\|(F_1,F_2)\|^2_{\delta,1} = \int_{\Omega_1} |\nabla F_1|^2 dxdy + \delta^4 e^{\frac{U_o-U_1}{U_T}} \int_{\Omega_o} |\nabla F_1|^2 dxdy$$
$$+ \int_{\Omega_o} |\nabla F_2|^2 dxdy + \delta^4 e^{\frac{U_o-U_1}{U_T}} \int_{\Omega_1} |\nabla F_2|^2 dxdy$$

and corresponding scalar product. The dual space is equipped with the induced functional norm $\|\cdot\|^*_{\delta,1}$.) Because of (4.11) - (4.14) and (4.18) $F_o(0,0) = 0$ holds and

$$(4.21)\qquad \|F_\delta(0,0)\|^*_{\delta,1} = O(\kappa)$$

where $\kappa = \gamma/\exp(3\max\frac{|U_i|}{U_T})$. The bilinear form obtained by linearizing $F_o$ at (0,0) is given by

$$L(\sigma_1,\sigma_2,\varphi_1,\varphi_2) = \int_{\Omega_1} D\nabla\sigma_1\cdot\nabla\varphi_1 dxdy + \delta^4 e^{\frac{U_o-U_1}{U_T}} \int_{\Omega_o} (\frac{1}{-D})(\nabla\sigma_1\cdot\nabla\varphi_1$$

$$+ e^{-\frac{U_o}{U_T}}\sigma_2\nabla\bar{u}_o\cdot\nabla\varphi_1)dxdy$$

$$+ \int_{\Omega_o}(-D)\nabla\sigma_2\cdot\nabla\varphi_2 dxdy + \delta^4 e^{\frac{U_o-U_1}{U_T}} \int_{\Omega_1}\frac{1}{D}(\nabla\sigma_2\cdot\nabla\varphi_2$$

$$+ e^{\frac{U_1}{U_T}}\sigma_1\nabla\bar{v}_o\cdot\nabla\varphi_2)dxdy.$$

To show that L is coercive we estimate

$|\sigma_2\nabla\bar{u}_o\cdot\nabla\sigma_1| \le \frac{\sqrt{2}}{2}\|\nabla\bar{u}_o\|_{\infty,\Omega}(\omega|\nabla\sigma_1|^2 + \frac{1}{\omega}|\sigma_2|^2)$ and choose $\omega$ such that $1 - \frac{\sqrt{2}}{2} e^{-\frac{U_o}{U_T}} \|\nabla\bar{u}_o\|_{\infty,\Omega}\omega \ge \frac{1}{2}$. Moreover $\|\sigma_2\|^2_{2,\Omega_o} \le C\int_{\Omega_o}|\nabla\sigma_2|^2 dxdy$ holds and we also require $1 - \frac{\sqrt{2}}{2}C\delta^4 e^{-\frac{U_1}{U_T}}\|\nabla\bar{u}_o\|_{\infty,\Omega}/(\min|D|)^2\omega \ge \frac{1}{2}$. The smallness of $\gamma$ (see (4.16)) and the smoothness assumption on $\nabla\bar{u}_o$ imply that there is a $\gamma > 0$ such that both inequalities hold. By proceeding analogously with $\sigma_1\nabla\bar{v}_o\cdot\nabla\sigma_2$ (in the last integral of $L(\sigma_1,\sigma_2,\varphi_1,\varphi_2)$) the mixed terms are estimated below such that L is coercive with a coercivity constant independent of $\delta$ and $|U_1/U_T|$. Therefore

$$(4.22) \qquad \|(D_{E_u,E_v}F_o(0,0))^{-1}\|_{1,\delta} \le \text{const}$$

($D_{E_u,E_v}$ denotes the Frechet-derivative with respect to $(E_u,E_v)$). Since $D_{E_u,E_v}F_\delta$ is uniformly Lipschitzcontinuous the implicit function theorem implies the existence of a locally unique solution $E_u,E_v$ of $F_\delta(E_u,E_v) = 0$ for $\delta$ sufficiently small and

$$\|(E_u,E_v)\|_{1,\delta} = O(\varepsilon)$$

holds. Then

$$(4.23) \qquad \|E_u\|_{1,2,\Omega} + \|E_v\|_{1,2,\Omega} = O(\varepsilon)$$

is easily established by considering the problems(4.19), (4.20) in $\Omega_o$ and $\Omega_1$ separately.

To obtain an $L^\infty$-estimate we write (4.19) as

$$\text{div}\,(a\nabla E_u) = \text{div}\, f$$

with

$$a = \begin{cases} De^{E_u} & \text{in } \Omega_1 \\ \delta^4 \exp(\frac{U_o-U_1}{U_T})\ \frac{1}{(-D)}\ e^{E_u} & \text{in } \Omega_o \end{cases}$$

$$f = \begin{cases} 0 & \text{in } \Omega_1 \\ \delta^4 \exp\left(\frac{U_0 - U_1}{U_T}\right) \frac{1}{D} \left(e^{E_\psi} - 1\right) \nabla \tilde{u}_0 & \text{in } \Omega_0 \end{cases}$$

(note that we used (4.12)(a) to obtain f in this form). The maximum principle in the form given by Gilbarg and Trudinger (1977, Chapter 8, paragraph 5) yields (since $f \in L^\infty(\Omega)$ and $E_u|_{\partial\Omega_C} = 0$)

$$\|E_u\|_{\infty,\Omega} \leq C(\Omega,q) \|\nabla \tilde{u}_0\|_{\infty,\Omega_0} \|e^{E_\psi} - 1\|_{q,\Omega_0}$$

for any $q > 2$. The embedding theorem gives

$$\|E_u\|_{\infty,\Omega} \leq \text{const} \cdot \|\nabla \tilde{u}_0\|_{\infty,\Omega_0} \|E_\psi\|_{1,2,\Omega_0}$$

and (4.23) implies (4.17)(a). (4.17)(b) follows analogously and (4.17)(c) is obtained from (4.18).□

The implicit function theorem could not be applied if there is an n or p region without a contact since $u_0$ or $v_0$ resp. is not uniquely defined in this region. The linearized problem has a zero eigenvalue then.

(3.31) and (4.17) imply

$$J_n = O(\sigma), \qquad J_p = O(\sigma)$$

The reduced current densities are small close to thermal equilibrium.

Theorem 4.2 can easily be extended to the case that two n(or p) regions have a joint boundary. The limiting function for $\psi$ (see (4.17)(c)) remains unchanged.

## B) The Internal Layer Problem

We now investigate the layer problem (3.5) at a fixed curve $\Gamma$ of discontinuity of D and prove:

Theorem 4.3. Let the assumptions on D on $\Omega$ given in Theorem 4.1 hold. Then (for given $\tilde{\psi}, \tilde{u}, \tilde{v}$) the internal layer problem (3.5) has a unique piecewise monotone solution $\hat{\psi}$ for every $s \in \Gamma$ which fulfills for every $0 < \omega < 1$

(4.26)(a) $$|\hat{\psi}(\tau,s)| \leq C_\omega \exp\left((1-\omega) \cdot \sqrt{\tilde{n}^\Gamma(0-,s)\tilde{p}^\Gamma(0-,s)}\,\tau + D_\omega \sqrt{|\tilde{\psi}^\Gamma(0+,s) - \tilde{\psi}^\Gamma(0-,s)|}\right)$$

$$\text{for } \tau < -E_\omega \sqrt{|\tilde{\psi}^\Gamma(0+,s) - \tilde{\psi}^\Gamma(0-,s)|} \quad \text{and}$$

(4.26)(b) $|\hat{\psi}(\tau,s)| \le C_\omega \exp\Big( (-1+\omega)\sqrt{\bar{n}^\Gamma(0+,s)+\bar{p}^\Gamma(0+,s)}\,\tau + D_\omega\sqrt{|\bar{\psi}^\Gamma(0+,s)-\bar{\psi}^\Gamma(0-,s)|}\Big)$

for $\tau < E_\omega\sqrt{|\bar{\psi}^\Gamma(0+,s)-\bar{\psi}^\Gamma(0-,s)|}$ where

$C_\omega, D_\omega, E_\omega > 0$ depend on $\omega$ but not on $[\bar{\psi}]_\Gamma$. Also $\psi \in C((-\infty,0)x\Gamma) \cap C((0,\infty)x\Gamma)$ holds.

Piecewise monotone means monotone on $(-\infty,0)$ and on $(0,\infty)$.

We do not give a proof for Theorem 4.3 since it follows completely the lines of the proof of Theorem 3.3 in Markowich, Ringhofer, Selberherr and Langer (1982 b). We only remark that

$$(4.27)\qquad \hat{\psi}(0+,s) = \frac{D^\Gamma(0-,s)(\bar{\psi}^\Gamma(0+,s)-\bar{\psi}^\Gamma(0-,s)) + \bar{n}^\Gamma(0-,s)-\bar{n}(0+,s) + \bar{p}^\Gamma(0-,s)-\bar{p}^\Gamma(0+,s)}{D^\Gamma(0+,s) - D^\Gamma(0-,s)}$$

$$(4.28)\qquad \hat{\psi}(0-,s) = \frac{D^\Gamma(0+,s)(\bar{\psi}^\Gamma(0+,s)-\bar{\psi}^\Gamma(0-,s)) + \bar{n}^\Gamma(0-,s)-\bar{n}^\Gamma(0+,s) + \bar{p}^\Gamma(0-,s)-\bar{p}^\Gamma(0+,s)}{D^\Gamma(0+,s) - D^\Gamma(0-,s)}$$

holds.

The estimates (4.26) can be used to compute the width $d^\Gamma(s) = d_+^\Gamma(s)+d_-^\Gamma(s)$ of the internal layer of $\psi$ at $s \in \Gamma$

$$(4.29)(a)\qquad d_+^\Gamma(s) = O\left(\frac{\lambda}{\sqrt{\bar{n}^\Gamma(0+,s)+\bar{p}^\Gamma(0+,s)}}\left(\left|\ln\left[\frac{\lambda}{\sqrt{\bar{n}^\Gamma(0+,s)+\bar{p}^\Gamma(0+,s)}}\right]\right| + \sqrt{|\bar{\psi}^\Gamma(0+,s)-\bar{\psi}^\Gamma(0-,s)|}\right)\right)$$

$$(4.29)(b)\qquad d_-^\Gamma(\lambda) = O\left(\frac{\lambda}{\sqrt{\bar{n}^\Gamma(0-,s)+\bar{p}^\Gamma(0-,s)}}\left(\left|\ln\left[\frac{\lambda}{\sqrt{\bar{n}^\Gamma(0-,s)+\bar{p}^\Gamma(0-,s)}}\right]\right| + \sqrt{|\bar{\psi}^\Gamma(0+,s)-\bar{\psi}^\Gamma(0-,s)|}\right)\right)$$

where $d_+^\Gamma(s)$ ($d_-^\Gamma(s)$) is the portion of the layer width which is on that side of $\Gamma$ for which $t > 0$ ($t < 0$) holds. (4.29) is uniform in $[\bar{\psi}]_\Gamma$. The width of the depletion layer depends linearly on the square root of the potential drop accross the junction.

$d_+, d_-$ can be estimated by using the a-priori estimates on $\bar{n}$, $\bar{p}$, $\bar{u}$ and $\bar{v}$. We obtain

$$(4.30)\qquad |D| \le \bar{n} + \bar{p} \le |D| + O(s^4 e^{\frac{|V_{max}|}{U_T}}).$$

If $\operatorname{sgn} D^\Gamma(0+,s) = -\operatorname{sgn} D^\Gamma(0-,s)$ (that means $\Gamma$ is a pn-junction) then (4.6) yields

$$(4.31)\qquad \bar{\psi}^\Gamma(0+,s) - \bar{\psi}^\Gamma(0-,s) = \ln\left[\frac{|D^\Gamma(0+,s)\cdot D^\Gamma(0-,s)|}{s^4}\right] - \ln \bar{u}^\Gamma(0,s) - \ln \bar{v}^\Gamma(0,s) + O(s^4 e^{\frac{|V_{max}|}{U_T}}).$$

If $\operatorname{sgn} D^\Gamma(0+,s) = \operatorname{sgn} D^\Gamma(0-,s)$ ($\Gamma$ is an nn or pp junction) then $[\bar{\psi}]_\Gamma$ is uniformly

bounded as $\delta \to 0$. We get from (4.29)

$$(4.32)\qquad d_+^\Gamma(s) = O\left(\frac{\lambda}{\sqrt{D^\Gamma(0+,s)}}\left(\left|\ln\frac{\lambda}{\sqrt{D^\Gamma(0+,s)}}\right| + \delta^4 e^{\frac{|V_{max}|}{U_T}}\right.\right.$$

$$\left.\left. + \sqrt{\ln\left[\frac{|D^\Gamma(0+,s)\cdot D^\Gamma(0-,s)|}{\delta^4}\right]} + \frac{|V_{max}|}{U_T}\right)\right)$$

(if $\Gamma$ is a pn-junction). An analogous formula holds for $d_-^\Gamma(s)$. The square root drops out if $\Gamma$ is an nn (or pp) junction.

C) The Semiconductor-Oxide Interface Problem

Theorem 4.4 The problem (3.20), (3.25) has a unique monotone solution $\tilde\psi$ (for every $q \in \partial\Omega_{os}$) which fulfills for every $0 < \omega < 1$

$$(4.33)\qquad |\tilde\psi(\rho,q)| \le C_\omega \exp((-1+\omega)\sqrt{\tilde n^2(0,q)+\tilde p^2(0,q)}\,\rho + D_\omega\sqrt{|\tilde\psi(0,q)|}$$

for $\rho > E_\omega\sqrt{|\tilde\psi(0,q)|}$ where $C_\omega, D_\omega, E_\omega > 0$ are independent of $\tilde\psi(0,q)$. Also

$$(4.34)\qquad |\tilde\psi(0,q)| \le \eta(|\tilde\psi_\rho(0,q)|^2,q), \qquad (b)\quad \operatorname{sgn}\tilde\psi(0,q) = -\operatorname{sgn}\tilde\psi_\rho(0,q)$$

holds, where $\eta(\cdot,q):[0,\infty) \to [0,\infty)$ is a monotonely increasing function which fulfills

$$\eta(t,q) \sim \frac{\sqrt{t}}{\sqrt{\tilde n^2(0,q)+\tilde p^2(0,q)}} \quad \text{as } t \to 0 \text{ and } \eta(\infty,q) = \infty,\ \eta_t(\infty,q) = 0.$$

Proof: From Fife (1974) we derive that there is a unique monotone solution of (3.20) with given boundary value $\tilde\psi(0,q)$. At first we assume that $\tilde\psi(0,q) > 0$. Then $\tilde\psi(\rho,q)$ fulfills

$$(4.35)(a)\qquad \rho = \int_{\tilde\psi(\rho,q)}^{\psi(0,q)} \frac{d\alpha}{\sqrt{2G(\alpha,q)}}$$

where

$$(4.35)(b)\qquad G(\alpha,q) = \int_0^\alpha (\tilde n^2(0,q)e^t - \tilde p^2(0,q)e^{-t} - D^2(0,q))\,dt$$

$$= \tilde n^2(0,q)(e^\alpha - 1) + \tilde p^2(0,q)(e^{-\alpha}-1) - D^2(0,q)\alpha$$

holds (see Fife (1974)).

Differentiating (4.35)(a) with respect to $\rho$ and evaluating at $\rho=0$ gives the equation

$$(4.36) \qquad -\tilde{\psi}_\rho(0,q) = \sqrt{2G(\tilde{\psi}(0,q),q)}.$$

Therefore the interface problem (3.20), (3.25) has a unique monotone solution for $\tilde{\psi}_\rho(0,q) < 0$ if (4.36) has a unique positive solution $\tilde{\psi}(0,q)$.

For $\tilde{\psi}(0,q) < 0$ we apply the same argument to $-\tilde{\psi}$ and find that (3.20), (3.25) has a unique monotone solution for $\tilde{\psi}_\rho(0,q) > 0$ if the equation

$$(4.37) \qquad \tilde{\psi}_\rho(0,q) = \sqrt{2G(\tilde{\psi}(0,q),q)}$$

has a unique negative solution $\tilde{\psi}(0,q)$. From $\tilde{\psi}_\rho(0,q) = 0$ we immediately get $\tilde{\psi}(\rho,q) \equiv 0$. A simple calculation shows that the function

$$H(s,q) = \sqrt{2G(s,q)}$$

*is decreasing for* $s \in (-\infty,0]$, *increasing for* $s \in [0,\infty)$ and $H(-\infty,q) = H(\infty,q) = \infty$ for every $q \in \partial\Omega_{os}$.

This settles the existence statement of the theorem and (4.34)(b).

The estimate (4.33) follows by proceeding as in Markowich, Ringhofer, Selberherr and Langer (1982 b, Theorem 3.3). To prove (4.34)(a) we set

$\beta = \frac{\tilde{\psi}_\rho^2(0,q)}{2}$ and rewrite (4.36), (4.37) as

$$G(Z,q) = \beta, \qquad Z := \tilde{\psi}(0,q)$$

for fixed $q$. At first we investigate the root $Z_1 > 0$. Differentiation with respect to $\beta$ yields

$$\frac{\partial G}{\partial s}(Z_1(\beta),q) Z_1'(\beta) = 1 .$$

$Z_1(\infty) \in (0,\infty)$ implies $Z_1'(\infty) \in (0,\infty)$, which is a contradiction. Therefore $Z_1(\infty) = \infty$ and $\frac{\partial G}{\partial s}(\infty,q) = \infty$ implies $Z_1'(\infty) = 0$. The root $Z_2 < 0$ is investigated analogously ($Z_2(\infty) = -\infty$, $Z_2'(\infty) = 0$ hold).

Taylor expansion of $G$ for $\beta \to 0$ gives the behaviour of $Z$ close to zero. □

The width $d_0^3(q)$ of the interface layer is

$$(4.38) \qquad d_{os}^3(q) = 0\left(\frac{1}{\sqrt{|D^3(0,q)|}}\left|\ln\frac{1}{\sqrt{|D^3(0,q)|}}\right| + \sqrt{|\tilde{\psi}(0,q)|} + \varepsilon^4 e^{-\frac{|V_{max}|}{U_T}}\right).$$

The first order term of the expansion $\bar{\psi} + \hat{\psi} + \tilde{\psi}$ cannot be expected to approximate the solution of the singularly perturbed problem close to the point $S$ with $\{S\} = \bar{\Gamma} \cap \overline{\partial\Omega}_{OS}$ since $\tilde{\psi}(0,\cdot)$ is discontinuous at $q = S$ unless $\tilde{\psi}(0,\cdot) \equiv 0$.

Generally the first order term can only approximate the solution outside a small sphere (with radius at least $O(\lambda|\ln\lambda|)$) around S.

Theorem 4.4 can be used to compute an estimate for the threshold-gate-voltage, thatis the gate-voltage at which strong inversion (the minority carrier density at the interface is at least as large as the absolute value of the doping profile) starts.

To demonstrate this we take the n-channel MOS transistor as shown in Figure 1. n is the minority carrier density in the p-region $\Omega_1$, therefore strong inversion occurs if

$$n^3(0,q) \geq |D^3(0,q)|$$

holds for some $q \in \partial\Omega_{OS}$. Away from $\Gamma_1$, $\Gamma_2$ we get (up to $O(\lambda)$-terms) $n^3(0,q) \approx$ $\approx \bar{n}^3(0,q)e^{\tilde{\psi}(0,q)}$ such that the strong inversion condition is

$$\tilde{\psi}(0,q) \geq \ln\left(\frac{|D^3(0,q)|}{\bar{n}^3(0,q)}\right) := \tilde{\psi}_{INV}(0,q) .$$

(4.36), (4.37) yields

$$(\tilde{\psi}_b(0,q))_{INV}^2 = 2G(\tilde{\psi}_{INV}(0,q),q) \leq 2\,\frac{\delta^4 e^{\frac{|V_{max}|}{U_T}}}{|D^3(0,q)|}$$

$$+ 2|D^3(0,q)|\ln\frac{|D^3(0,q)|}{\delta^4} + 2|D^3(0,q)|\frac{|V_{max}|}{U_T}$$

and from (3.26), (2.8)(b) we derive that strong inversion occurs if

$$\frac{U_G}{U_T} \geq \frac{U_F}{U_T} + \frac{d\varepsilon_S}{\lambda\varepsilon_O}\sqrt{2\,\frac{\delta^4 e^{\frac{|V_{max}|}{U_T}}}{\min_{\Omega_1}|D|} + 2\max_{\Omega_1}|D|\ln\frac{\max_{\Omega_1}|D|}{\delta^4} + 2\max_{\Omega_1}|D|\,\frac{|V_{max}|}{U_T}}$$

holds.

## 5. ASYMPTOTIC REPRESENTATION OF SOLUTIONS

To demonstrate the validity of the expansions we show that if the device is in thermal equilibrium, then the solution of the singularly perturbed problem is close (in $L^\infty(\Omega)$) to the zeroth order term in the asymptotic expansion (3.1) assuming that $\lambda$ is small. For simplicity we assume that there is only one $C^\infty$-junction $\Gamma$ ($\Omega = \Omega_0 \cup \Gamma \cup \Omega_1$). $\Gamma$ intersects with $\partial\Omega$ in two points $S_1, S_2$ which have positive distance from $\partial\Omega_C$. Also there are spheres with radius $\omega$ centered at $S_1$ and $S_2$ within which $\Gamma$ and $\partial\Omega$ are perpendicular lines. $\partial\Omega$ is assumed to be as in Theorem 2.2. Moreover no semiconductor-oxide interface occurs, i.e. $\Phi = \{\}$.

We have to overcome one technical difficulty, namely that the layer solution $\hat{\psi}(\frac{t}{\lambda}, s)$ is only defined in an open $\omega$-strip $S_\omega(\Gamma)$ about $\Gamma$. In order to extend this function to $\Omega$ we choose $\vartheta \in C^\infty(\mathbb{R})$ with

$$\vartheta(t) = 1 \quad \text{for} \quad t \in [-\tfrac{\omega}{2}, \tfrac{\omega}{2}], \quad \vartheta(t) = 0 \quad \text{for} \quad |t| \geq \omega$$

and set

$$\hat{\psi}_e\left(\frac{t(x,y)}{\lambda}, s(x,y)\right) = \begin{cases} \hat{\psi}\left(\frac{t(x,y)}{\lambda}, s(x,y)\right) \cdot \vartheta(t(x,y)), & (x,y) \in S_\omega(\Gamma) \\ 0, & (x,y) \notin S_\omega(\Gamma) \end{cases} \tag{5.1}$$

(see Fife (1974).

We prove:

Theorem 5.1 Let $\Gamma$ and $\Omega$ fulfill the above stated assumptions. Also $D$ is piecewise constant, that means

$D = \begin{cases} D_0 \text{ in } \Omega_0 \\ D_1 \text{ in } \Omega_1 \end{cases}$ where $D_0, D_1$ are constants with $D_0 \neq D_1$. Then

$$\psi(x,y,\lambda) = \bar{\psi}(x,y) + \psi_e\left(\frac{t(x,y)}{\lambda}, s(x,y)\right) + O(\lambda |\ln\delta|^{\frac{3}{2}}). \tag{5.2}$$

holds if $\lambda$, $\lambda|\ln\delta|^{\frac{5}{2}}$ are sufficiently small and if the device is in thermal equilibrium (i.e. $U_k = 0$)

Proof: In thermal equilibrium $u = v = 1$ holds and (2.4) simplifies to

$$\lambda^2 \Delta\psi = 2\delta^2 \cdot \sinh\psi - D \text{ in } \Omega \tag{5.3)(a}$$

subject to the boundary conditions

$$\nabla\psi \cdot \vec{n}\big|_{\partial\Omega_S} = 0, \qquad \psi\big|_{\partial\Omega_C} = \ln\left[\frac{D+\sqrt{D^2+4\delta^4}}{2\delta^2}\right]_{\partial\Omega_C}. \tag{5.3)(b}$$

We set

(5.4) $\psi = \bar{\psi} + \hat{\psi}_e + \varphi$

where the reduced solution $\bar{\psi}$ is given by (4.6)

(5.5) $\bar{\psi} = \ln\left[\frac{D+\sqrt{D^2+4\delta^4}}{2\delta^2}\right]$

since $\bar{u} = \bar{v} = 1$ holds. $\hat{\psi}_e$ is given by (5.1) ($\hat{\psi}(\tau,s)$ solves (3.5)).

*Assuming* that $\omega$ is less than the distance of $S_1, S_2$ from $\partial\Omega_C$ we get

(5.6)(a) $\varphi|_{\partial\Omega_C} = 0$

and because of (5.1)

(5.6)(b) $\nabla\varphi\cdot\vec{n}|_{\partial\Omega_{is}} = 0$

(note that $\nabla\bar{\psi}\cdot\vec{n}|_{\partial\Omega_{is}} = 0$ because $D$ is piecewise constant and $\vec{r}$ is perpendicular to $\partial\Omega$).

Inserting (5.4) into (5.3)(a) gives

(5.7) $\hat{\psi}_{\tau\tau} + \lambda\hat{\psi}_\tau\Delta t + \lambda^2\Delta\varphi = 2\delta^2\cdot\sinh(\bar{\psi}+\hat{\psi}+\varphi) - D$ in $S_{\frac{\omega}{2}}(\Gamma)$.

(note that $\Delta(\bar{\psi}+\hat{\psi}_e)$ (in the weak sense) is in $L^\infty(\Omega)$ and equals $\frac{1}{\lambda^2}\hat{\psi}_{e\tau\tau} + \frac{1}{\lambda}\hat{\psi}_{e\tau}\Delta t$ since $\bar{\psi}, \hat{\psi}$ *do not depend on* $s$ and since the interface conditions (3.5)(c),(d) hold). Also

(5.8) $\lambda^2\Delta\hat{\psi}_e + \lambda^2\Delta\varphi = 2\delta^2\cdot\sinh(\bar{\psi}+\hat{\psi}_e+\varphi) - D$ in $S_\omega(\Gamma) - S_{\frac{\omega}{2}}(\Gamma)$

and

(5.9) $\lambda^2\Delta\varphi = 2\delta^2\cdot\sinh(\bar{\psi}+\varphi) - D$ in $\Omega - S_\omega(\Gamma)$

holds.

By using the internal layer equations (3.5)(a),(b) we rewrite (5.7) as

(5.10) $\lambda^2\Delta\varphi = 2\delta^2\cosh(\bar{\psi}+\hat{\psi})\varphi + \delta^2 e^{\bar{\psi}+\hat{\psi}}(e^{\varphi}-\varphi-1)$

$- \delta^2 e^{-\bar{\psi}-\hat{\psi}}(e^{-\varphi}+\varphi-1) - \lambda\hat{\psi}_\tau\Delta t$ in $S_{\frac{\omega}{2}}(\Gamma)$

and by using (3.2)(a) (with $\bar{u} = \bar{v} = 1$) we obtain from (5.8), (5.9)

$$(5.11)\qquad \lambda^2\Delta\varphi = 2\delta^2\cosh(\bar{\psi}+\hat{\psi}_e)\varphi + \delta^2 e^{\bar{\psi}+\hat{\psi}_e}(e^{\varphi}-\varphi-1)$$
$$- \delta^2 e^{-\bar{\psi}-\hat{\psi}_e}(e^{-\varphi}+\varphi-1)$$
$$- \delta^2 e^{-\bar{\psi}}(e^{-\hat{\psi}_e}-1) + \delta^2 e^{\bar{\psi}}(e^{\hat{\psi}_e}-1) - \lambda^2\Delta\hat{\psi}_e \text{ in } S_{\omega}(\Gamma) - S_{\frac{\omega}{2}}(\Gamma)$$

$$(5.12)\qquad \lambda^2\Delta\varphi = 2\delta^2\cosh(\bar{\psi})\varphi + \delta^2 e^{\bar{\psi}}(e^{\varphi}-\varphi-1) - \delta^2 e^{-\bar{\psi}}(e^{-\varphi}+\varphi-1) \quad \text{in } \Omega - S_{\omega}(\Gamma).$$

Therefore we have a problem of the form

$$(5.13)(a)\qquad \lambda^2\Delta\varphi = 2\delta^2\cosh(\bar{\psi}+\hat{\psi}_e)\varphi + F(\varphi,x,y,\lambda,\delta) \text{ in } \Omega$$

$$(5.13)(b)\qquad \varphi|_{\partial\Omega_C} = \nabla\varphi\cdot\vec{n}|_{\partial\Omega_{is}} = 0$$

where F fulfills the estimate

$$(5.14)\qquad |F(\varphi,x,y,\lambda,\delta)| \le C(k)(\varphi^2 + \|\hat{\psi}_e\|_{\infty,S_{\omega}(\Gamma)-S_{\frac{\omega}{2}}(\Gamma)} + \lambda\|\psi_z\|_{\infty,S_{\frac{\omega}{2}}(\Gamma)}$$
$$+ \lambda^2\|\Delta\psi_e\|_{\infty,S_{\omega}(\Gamma)-S_{\frac{\omega}{2}}(\Gamma)}), \quad (x,y)\in\Omega$$

for $|\varphi|\le k$ with $C(k)$ independent of $\delta$ and $\lambda$.

From Markowich, Ringhofer, Selberherr and Langer (1982b) we get

$$\|\hat{\psi}_e\|_{\infty,S_{\frac{\omega}{2}}(\Gamma)} \le \max(|\psi_z(0+)|,|\psi_z(0-)|) \le D_1\sqrt{|\ln\delta|}$$

and (4.26), (5.1) imply

$$\|\hat{\psi}_e\|_{\infty,S_{\omega}(\Gamma)-S_{\frac{\omega}{2}}(\Gamma)} \le D_2\frac{1}{\delta^4}\exp(-\frac{E}{\lambda})$$

$$\|\Delta\psi_e\|_{\infty,S_{\omega}(\Gamma)-S_{\frac{\omega}{2}}(\Gamma)} \le D_3\frac{1}{\lambda^2\delta^4}\exp(-\frac{E}{\lambda})$$

where $D_1,D_2,D_3,E > 0$ are independent of $\lambda$ and $\delta$ as $\lambda\to 0+$, $\delta\to 0+$.

This yields

$$(5.14)\qquad |F(\varphi,x,y,\lambda,\delta)| \le C_1(k)(\varphi^2 + \lambda\sqrt{|\ln\delta|})$$

for $|\varphi|\le k$, $\lambda,\delta$ and $|\ln\delta|$ sufficiently small.

We now prove an $L^{\infty}$-estimate for the solution of the problem

(5.15)(a) $L_{\lambda,\delta} w = \lambda^2 \Delta w - 2\delta^2 \cosh(\vec{\psi}+\hat{\psi}_e) w = f(x,y)$ in $\Omega$

(5.15)(b) $w|_{\partial\Omega_C} = \nabla w \cdot \vec{n}|_{\partial\Omega_{is}} = 0$

with $f \in L^\infty(\Omega)$.

Therefore we define the function

(5.16) $$V_b(x,y) = L + \exp\left(-\frac{\beta}{\lambda^2 |\ln\delta|} t^2(x,y)\right) \theta(t(x,y))$$

with $L > 0$, $\beta > 0$.

A straight-forward calculation (analogous to the proof of Theorem 4.3 in Markowich, Ringhofer, Selberherr and Langer (1982b)) which heavily relies on the estimate (4.26) shows that the constants $L, \beta$ can be chosen independently of $\lambda, \delta$ such that

$$L_{\lambda,\delta} V_b(x,y) \leq -\frac{\text{const}}{|\ln\delta|}, \quad (x,y) \in \Omega$$

$$V_b|_{\partial\Omega_C} = L, \quad \nabla V_b \cdot \vec{n}|_{\partial\Omega_{is}} = 0$$

holds. This implies that $V_b$ can be used as comparism function and the maximum principle yields

(5.17) $$\|w\|_{\infty,\Omega} \leq F|\ln\delta| \, \|f\|_{\infty,\Omega}$$

for the solution $w$ of (5.15) ($F > 0$ is independent of $\lambda, \delta$).

Now we define the operator $M: A_{\lambda,\delta} \to L^2(\Omega)$ where $A_{\lambda,\delta} = \{\varphi \in L^2(\Omega) \mid \|\varphi\|_{\infty,\Omega} \leq G\lambda^{\frac{2}{3}} |\ln\delta|^{\frac{2}{3}}\}$

with $G > 0$ independent of $\lambda$ and $\delta$ such that $\varphi = M(\sigma)$ is the solution of

$$\lambda^2 \Delta\varphi = 2 \cosh(\vec{\psi}+\hat{\psi}_e)\varphi + F(\sigma,s,y,\lambda,\delta) \text{ in } \Omega$$

$$\varphi|_{\partial\Omega_O} = \nabla\varphi \cdot \vec{n}|_{\partial\Omega_{is}} = 0.$$

A fixed-point $\varphi^*$ of $M$ is a weak solution of (5.13), since Range $(M) \subset H^1(\Omega)$. $M$ is continuous and therefore Ascoli's Theorem implies that it is compact. From the estimates (5.14), (5.17) we conclude that $M$ map $A_{\lambda,\delta}$ into itself if $G > C_1(k)F$ and $\lambda^{\frac{2}{3}} |\ln\delta|^{\frac{2}{3}}$ is sufficiently small. Since $A_{\lambda,\delta}$ is closed and convex in $L^2(\Omega)$ Schauder's fixed point theorem yields the existence of a solution of (5.13) and (5.2) follows. As mentioned before this solution is unique (in $H^1(\Omega)$). $\square$

Theorem 5.1 can easily be extended to doping profiles which are not piecewise constant. The proof proceeds along the same lines except that $\Delta(\vec{\psi}+\hat{\psi}_e) \in L^\infty(\Omega)$ if $\vec{\psi}$, $\hat{\psi}_e$ depend on $s$. This can be overcome by modifying the interface condition (3.5)(d)

appropriately(see Markowich, Ringhofer, Selberherr and Langer (1982b) for the one-dimensional problem). The extension to more than one junction is also straightforward.

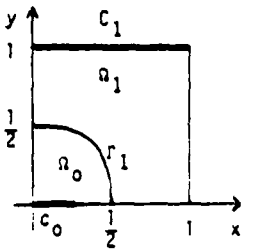
A main assumption of the Theorem is that $\lambda|\ln\lambda|^{\frac{5}{2}}$ is small. (2.6) implies that

$$\delta^2 = \gamma^2\lambda^2, \quad \gamma^2 = \frac{l^2 q n_i}{\varepsilon_s U_T} \tag{5.18}$$

holds. For realistic devices $\gamma^2 \gtrsim 0.1$, therefore $\delta$ can be regarded to be at least proportional to $\lambda$ and $\lambda|\ln\delta|^{\frac{5}{2}} \sim \lambda|\ln\lambda|^{\frac{5}{2}}$ as $\lambda \to 0+$.

## 6. NUMERICAL RESULTS AND EXTENSIONS

We present numerical results for a two-dimensional diode. The geometry of the device is depicted in Figure 2.




$$C = \begin{cases} 10^{17}\ cm^{-3} & \text{in } \Omega_1 \\ -10^{17}\ cm^{-3} & \text{in } \Omega_0 \end{cases}$$

$$l = 5 \times 10^{-3} cm$$

Figure 2. Diode

(2.6) gives $\lambda^2 = \delta^2 = 10^{-7}$.

The computations described in the sequel were performed using a specialised finite difference discretisation method completely based on the singular perturbation approach (see A.Franz et al (1982)).

Figure 3 shows the potential in thermal equilibrium ($U_0=U_1=0V$), Figure 4 shows the electron density n and Figure 5 the hole density p. As proven in Section 4 the n-region $\Omega_1$ is (almost) depleted of holes and the p-region $\Omega_0$ is (almost) depleted of electrons. The internal layer at the pn-junction $\Gamma_1$ is clearly visible. Figure 6 shows the grid used for the discretisation (which was generated automatically by the code). An accumulation of grid points around the pn-junction $\Gamma_1$ (in order to 'balance' the large derivatives of the solutions in perpendicular direction to $\Gamma_1$) is evident.

Figure 7 shows the potential distribution for -10V applied bias (i.e. $U=U_0-U_1=-10V$). The width of the depletion layer increased (compared to the equilibrium problem) (see (4.32)).

The electron density n for U=0.75V applied bias is depicted in Figure 8. $\delta^4 \exp(|U|/U_T) \approx 10^{-1}$ (we are dealing with a low injection case) and n visibly 'lifts off' from zero in the p region (the theory predicts $\tilde{n}=O(\delta^4 \exp(|U|/U_T))$ in $\Omega_0$).

Figure 9 shows the hole density p for U=1V applied bias. Now $\delta^4 \exp(|U|/U_T) \approx 2.35 \times 10^3$ holds (high-injection). $p \gg |D|$ away from the contact in the p-region $\Omega_0$ and the n-region $\Omega_1$ is also flooded with holes. The absolute value of $J_p$, that is $|J_p| = \sqrt{(J_p^x)^2+(J_p^y)^2}$, is depicted in Figure 10. The 'peak' at the edge of the contact $C_0$ represents the singularity at intersections of $\partial\Omega_C$ and $\overline{\partial\Omega}_{1S}$ as discussed in Theorems 2.1, 4.1. This phenomenon is physically interpreted as current crowding at contact edges.

Figure 3

# POTENTIAL DIODE 0V

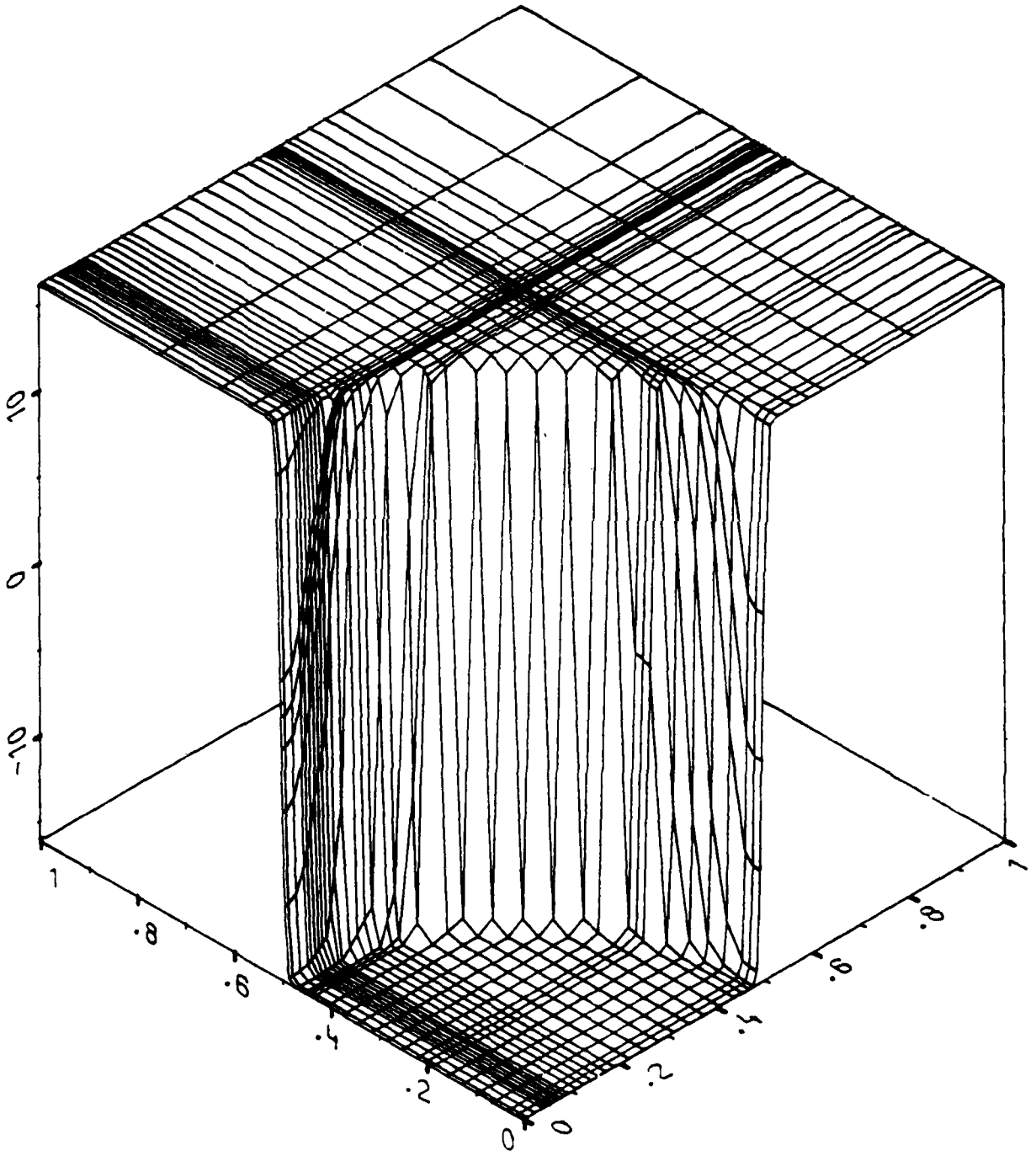


Figure 4

# ELECTRONS DIODE 0V

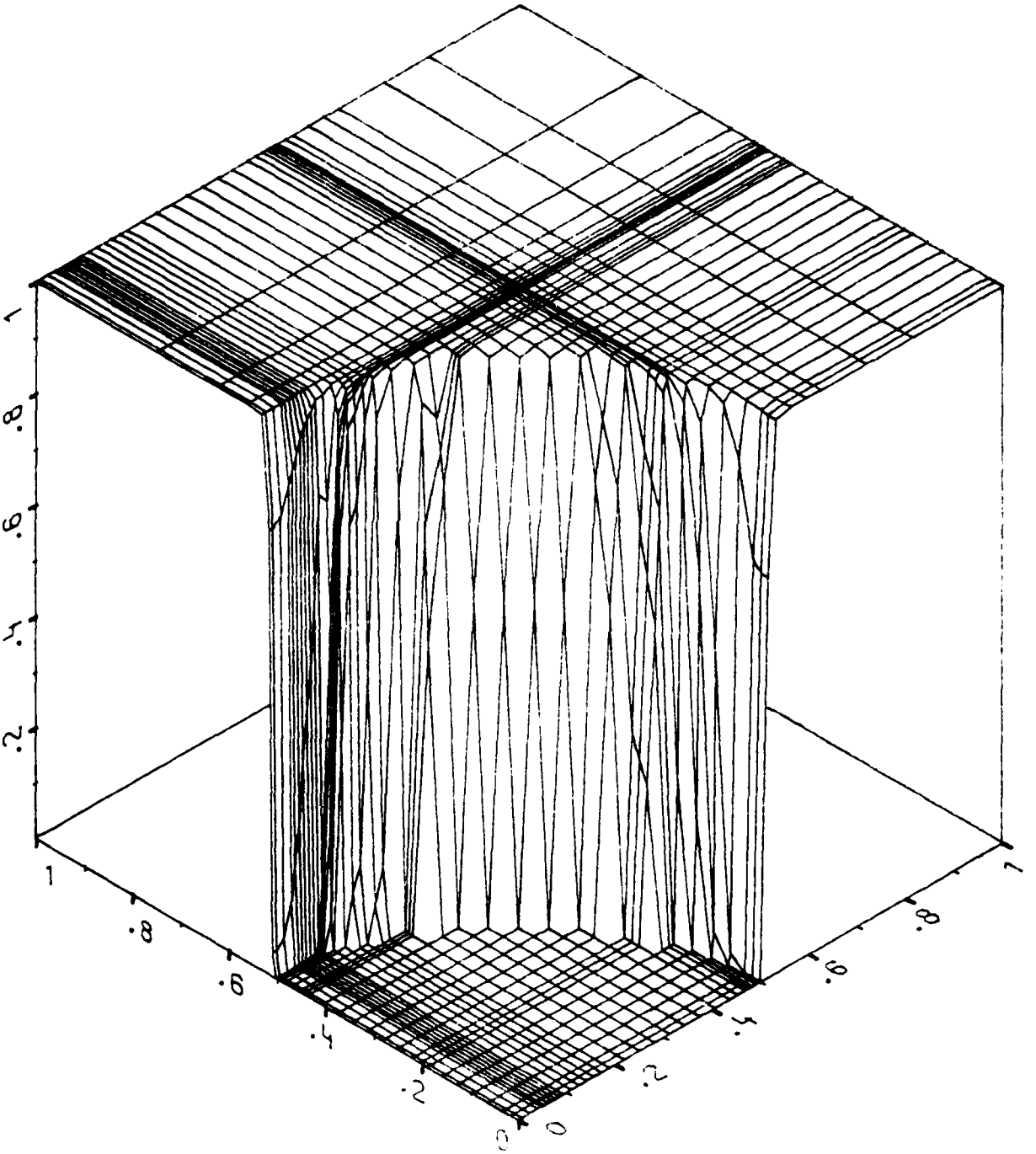


Figure 5

# HOLES DIODE 0V

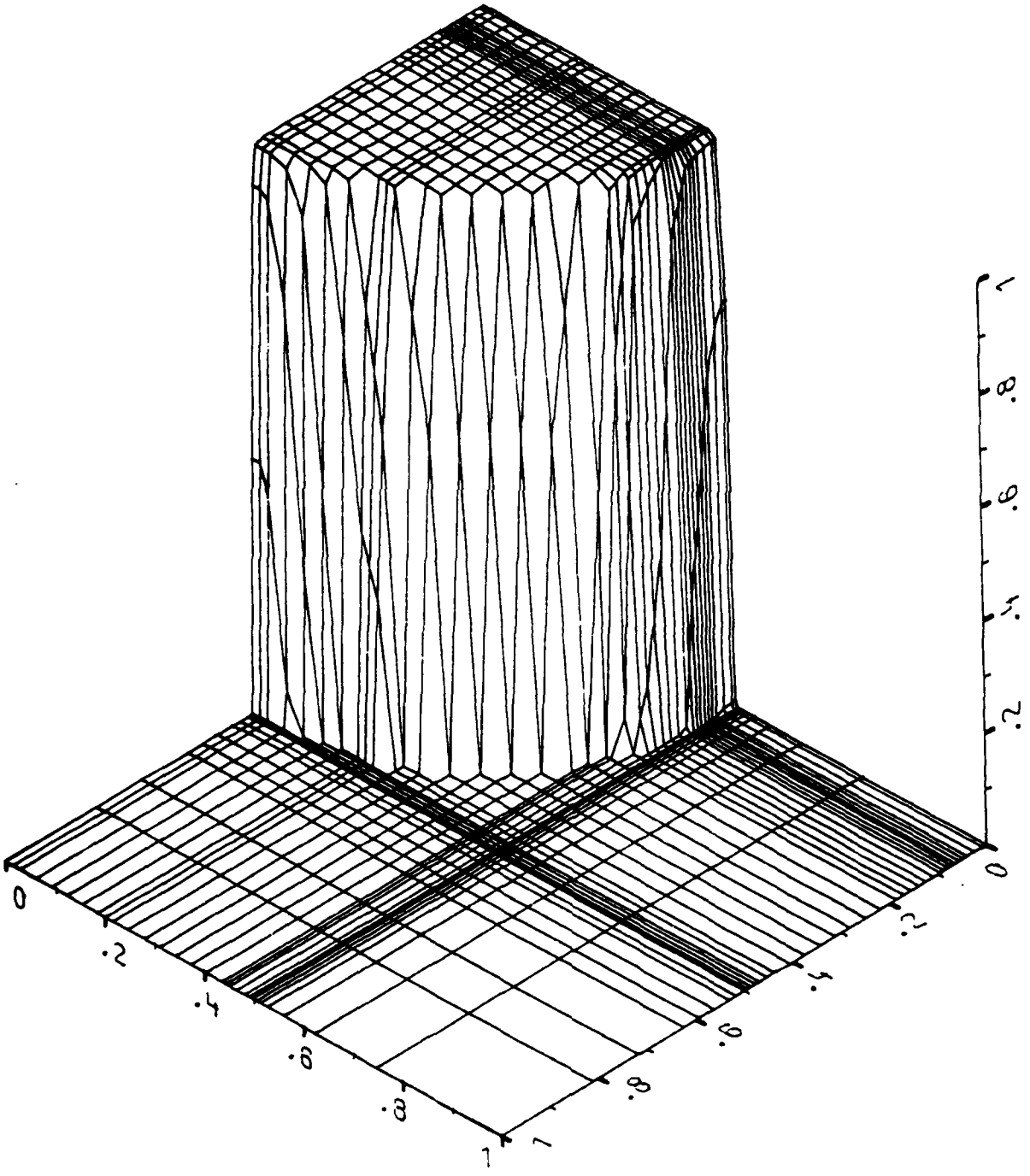


Figure 6

GRID

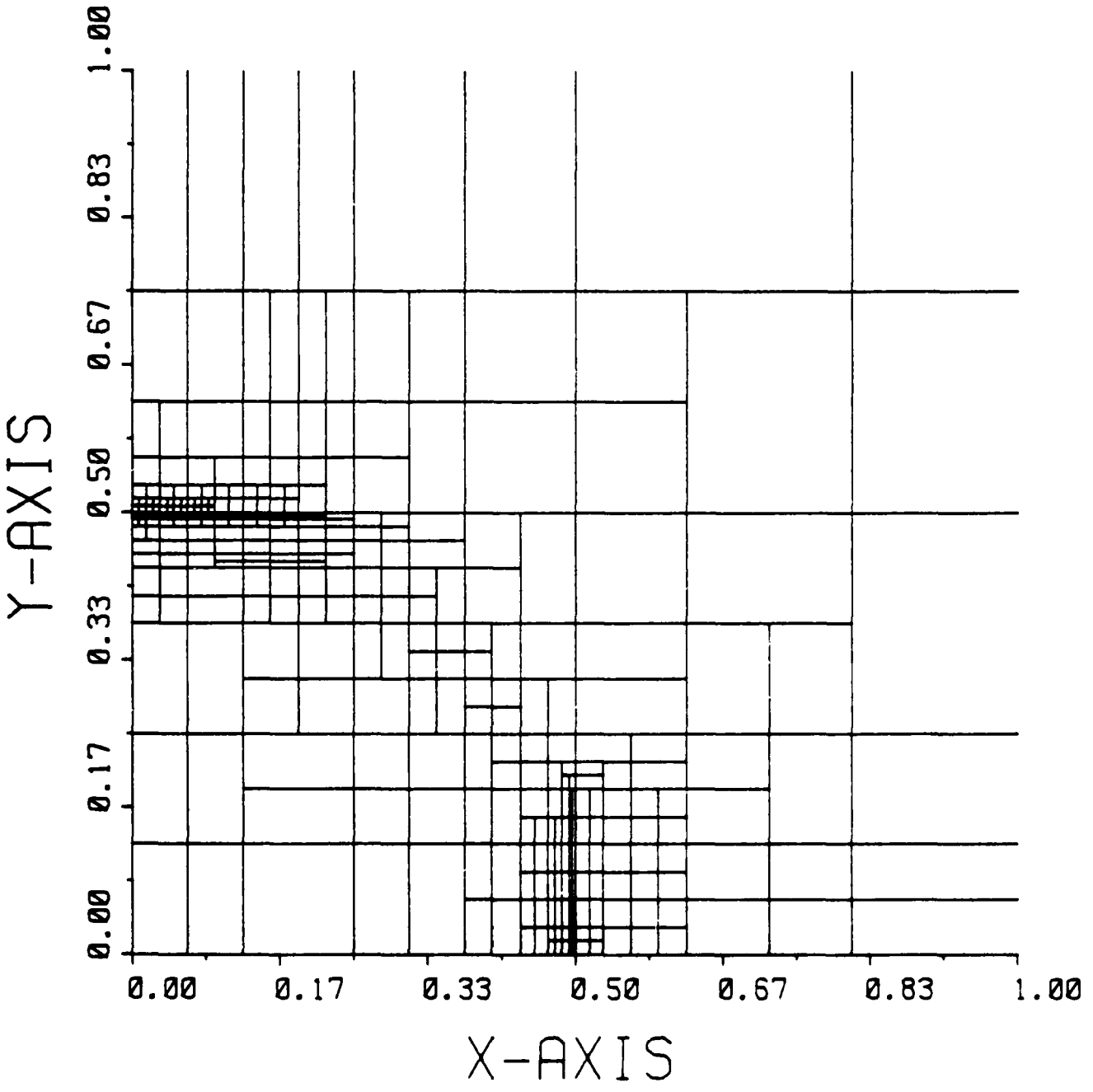


Figure 7

# POTENTIAL DIODE -10V

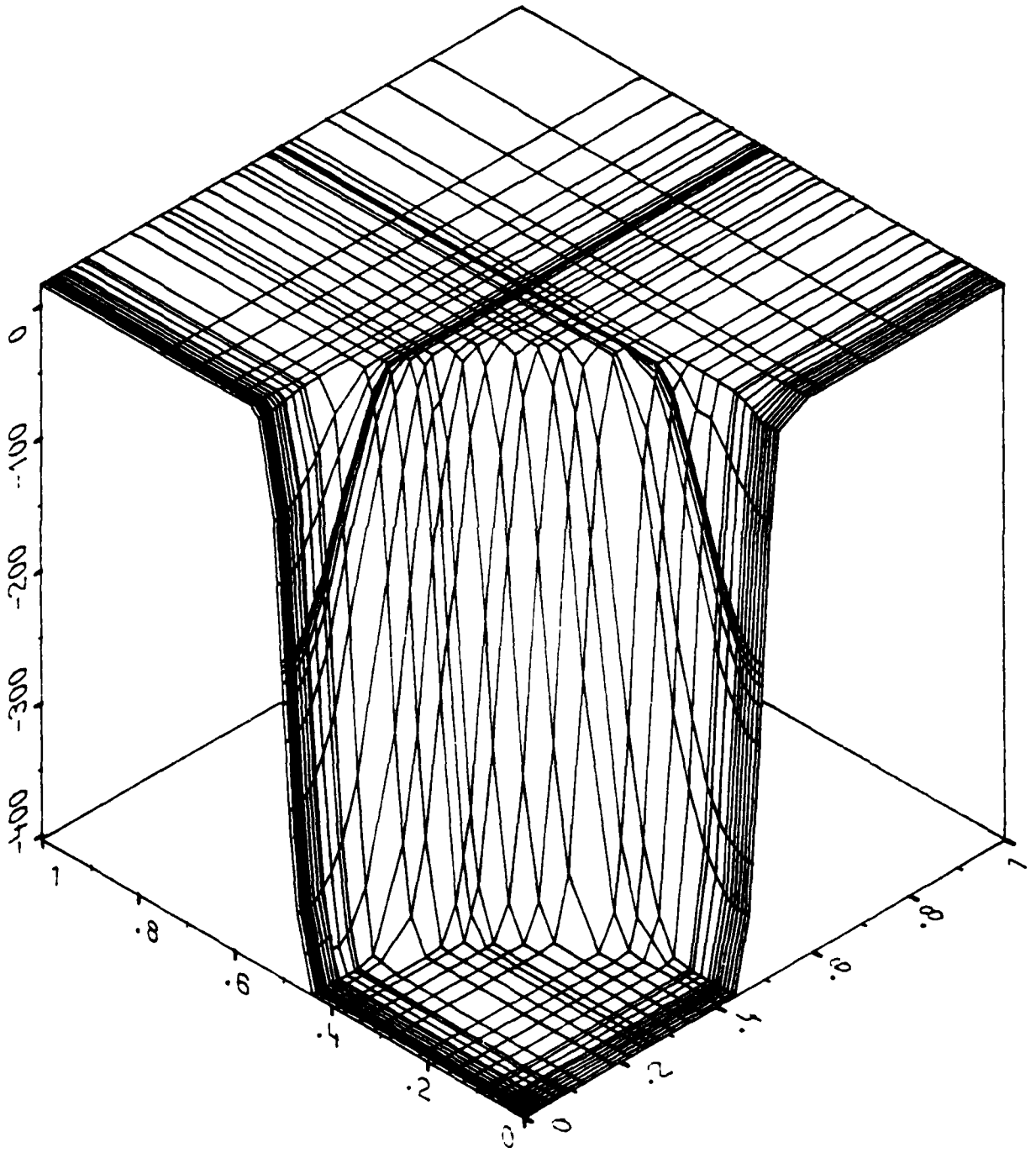


Figure 8

# ELECTRONS DIODE 0.75V

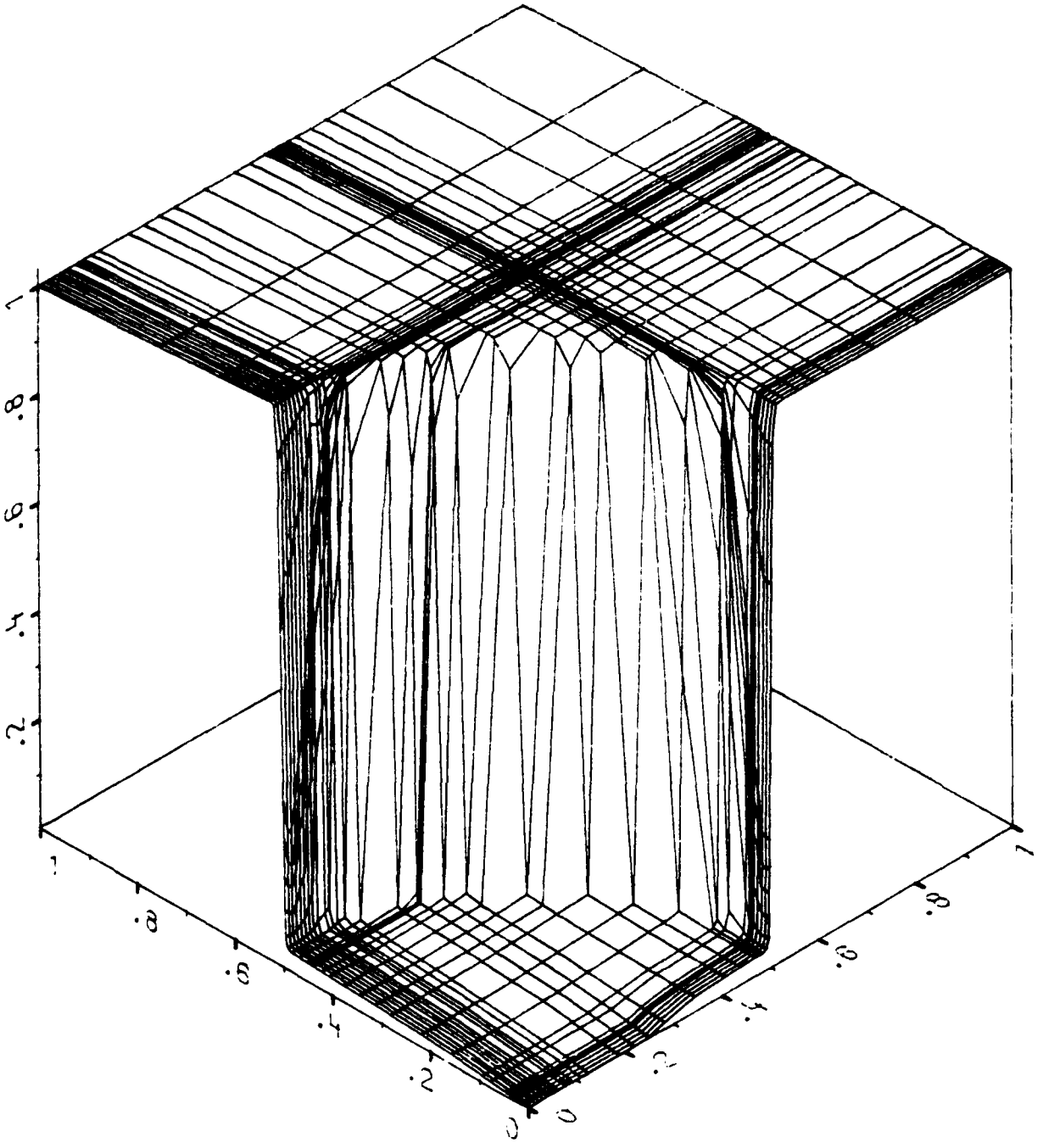


Figure 9

HOLES DIODE 1V

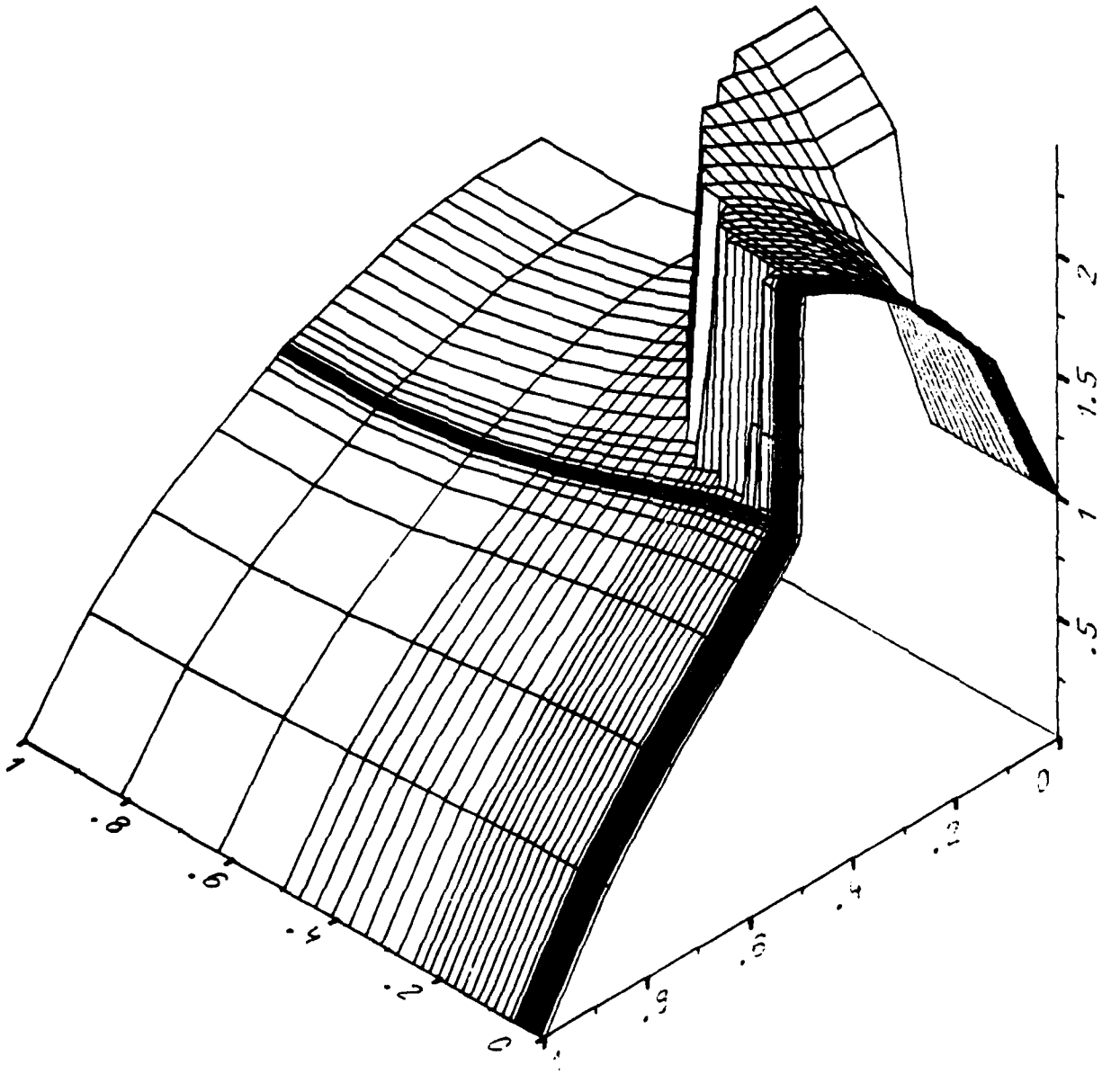
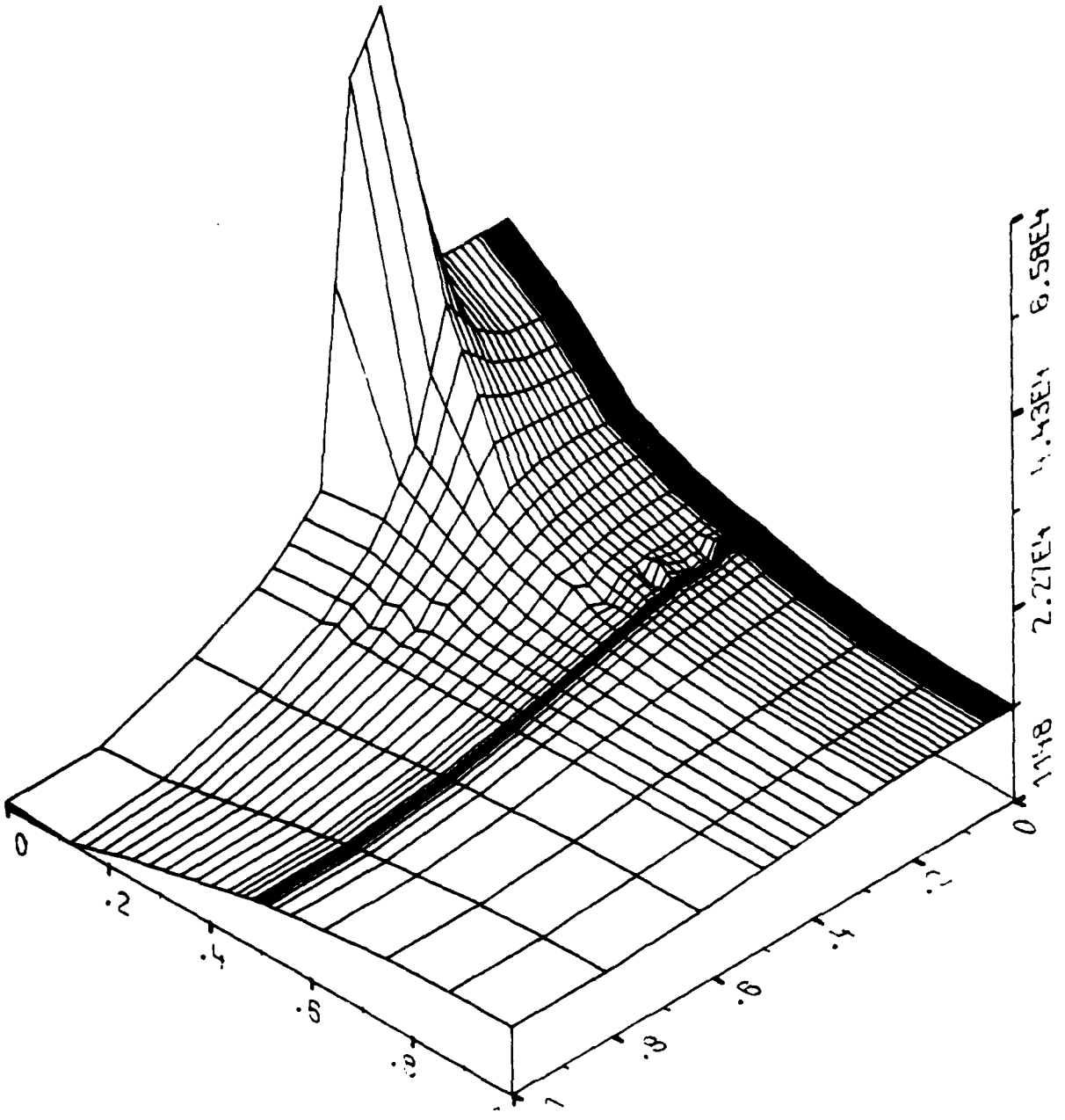


Figure 10



In this paper we only dealt with abrupt doping profiles (that means D has discontinuities accross junctions). This assumption is actually a simplification since 'physical' doping profiles are exponentially graded near junctions. The singular perturbation approach can be modified to accomodate this more realistic situation by setting

$$(6.1) \qquad D(x,y,\lambda) = \bar{D}(x,y) + \hat{D}\left(\frac{t,x,y}{\lambda}, s(x,y)\right)$$

(close to a particular junction $\Gamma$) where $\bar{D}$ is abrupt and $|\hat{D}(\tau,s)| \leq E(s)\exp(-F(s)\tau)$ holds with $E,F > 0$. $D(.,.,\lambda)$ is continuous along $\Gamma$ for $\lambda > 0$.

Then $\hat{D}$ appears as 'inhomogeneity' to the internal layer equation (3.5)(a),(b). Otherwise the preceding theory remains unchanged when $\bar{D}$ is substituted for D.

As mentioned in the introduction we neglected recombination-generation of carriers in our model-equations (1.1). The recombination-generation rate R in (1.1)(b),(c) is (in the most general setting) a nonlinear function of $n,p,J_n,J_p$ and $\nabla\psi$ (see Sze (1981) for details).

The existence proofs (Theorems 2.1 and 4.1) are affected by the introduction of recombination (since they use the linearity of the continuity equations in u and v). The qualitative behaviour of solutions is pretty much unaffected in low-injection conditions since $R \equiv 0$ in thermal equilibrium. In high-injection conditions however recombination generation has a decisive impact on the solutions and cannot be neglected (see Schütz, Selberherr and Pötzl (1982) for a numerical study of 'avalanche' effects in MOS-transistors).

The validity proof for the asymptotic expansions in non-equilibrium is an unsettled issue. For the one-dimensional semiconductor problem Markowich, Ringhofer, Selberherr and Langer (1982 b) gave a proof (with estimate of the remainder term) for a diode close to thermal equilibrium. There is numerical evidence that the asymptotic expansions 'represent' a solution even for large applied voltages (see Markowich, Ringnofer, Selberherr and Langer (1982a,b)), but no estimate of the remainder term for arbitrary bias is known so far.

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In this paper we present a singular perturbation analysis of the fundamental semiconductor device equations which form a system of three second order elliptic differential equations subject to mixed Neumann-Dirichlet boundary conditions. The system consists of Poisson's equation and the continuity equations and describes potential and carrier distributions in an arbitrary semiconductor device.

The singular perturbation parameter is the minimal Debye-length of the device under consideration.

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ABSTRACT (continued)

Using matched asymptotic expansions we demonstrate the occurrence of internal layers at surfaces across which the impurity distribution which appears as an inhomogeneity of Poisson's equation has a jump discontinuity (these surfaces are called 'junctions') and the occurrence of boundary layers at semiconductor-oxide interfaces. We derive the layer-equations and the reduced problem (charge-neutral-approximation) and give existence proofs for these problems. The layer solutions which characterize the solution of the singularly perturbed problem close to junctions and interfaces resp. are shown to decay exponentially away from the junctions and interfaces resp.

We show that, if the device is in thermal equilibrium, then the solution of the semiconductor problem is close to the sum of the reduced solution and the layer solution assuming that the singular perturbation parameter is small. Numerical results for a two-dimensional diode are presented.