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THE MATHEMATICAL STRUCTURE OF
ELEMENTARY PARTICLES

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ABSTRACT

This report consists of the first part of a general theory purporting to describe the mathematical structure of the elementary particles, starting from no preassumed knowledge, but deriving it instead from first principles along the line suggested by Dirac in the 1930's. In particular, quantum mechanics is shown to arise as a consequence of relativity theory and of the theory of generalized curves. In the first part the geometric structure (i.e. the nuclear field) is derived, and one obtains a slightly modified form of the Yukawa potential along with a cylindrical perturbation describing the spin effects. This report gives full details of the results announced in the two previous reports, MRC Technical Summary Report #2067, which appeared in Proc. Ioffe Conf. Imperial College, London (1980) and MRC Technical Summary Report #2317 appearing in J.O.T.A., Sept. 1983.

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SIGNIFICANCE AND EXPLANATION

According to Dirac the two fundamental problems that science faces are: the problem of matter and the problem of life. The present work is a contribution toward the analysis of the first problem, in the sense that the true nature of the elementary particles, their fields, masses, interactions, couplings and transformations can be derived from the basic mathematical structures found to be physically relevant, by a step by step analysis of the intrinsic objects pertaining to these structures. Not only are general results obtained but also numerical ones that compare favourably with the experimental data. The relevance of any progress in this direction needs hardly to be stressed.

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CHAPTER I

THE MATHEMATICAL FRAMEWORK

1.1 THE CONSTRUCTIVE APPROACH

The search for mathematical structures in the background of physical phenomena rests on the postulate that an appropriate mathematical description is possible, in some manner.

It is a logical sequel to this assumption that the appropriate mathematical structures underlying physical phenomena ought to contain, in germinal form, the essential features needed for the description of the fundamental physical entities and processes.

This information ought to be retrievable through the analysis of the essential objects of these mathematical structures, namely its intrinsic objects.

These objects are given, as a rule, as critical points of basic functionals associated with these structures. In this connection, singularities in general must be expected to play a fundamental role.

Of course, more complex processes are then described by various kinds of appropriate superpositions of the intrinsic objects.

The analysis of the intrinsic objects and their superpositions to build more complex objects is what we call constructive approach.

This is the basic method of analysis to be employed in this work.

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A mathematical analysis of physical phenomena requires identifications between the physical and the mathematical objects. This assignment is, a priori, bound to the choice of mathematical structures involved. The value of these choices is measured in terms of the end results and by the new understanding they bring, the object of any theory being to get closer and closer to the physical facts.

These considerations are taken in due regard along this work.

1.2 INTRINSIC OBJECTS

Our starting structure is a 4-dimensional differentiable manifold V_4 on which we shall take as physically relevant the family of Lorentzian metrics g^{ij}, i.e. pseudo-riemannian metrics of signature $(-1,1,1,1)$.

To each such metric are associated, in particular, the invariants

(1.1)
$$ds^2 = g_{ij}\, dx^i\, dx^j,$$

(1.2)
$$(\nabla\varphi)^2 = g^{ij}\, \partial_i\varphi\, \partial_j\varphi, \qquad \partial_i \equiv \frac{\partial}{\partial x^i}\,,$$

and the Laplace-Beltrami operator

(1.3)
$$\Delta_2 = \frac{1}{|g|^{1/2}}\, \frac{\partial}{\partial x^i}\, |g|^{1/2}\, g^{ij}\, \frac{\partial}{\partial x^j}\,,$$

where $g = \det g_{ij}$.

We also define the operators (p.55 [10])

$$\operatorname{grad}\varphi = \nabla\varphi = g^{ij}\, \frac{\partial\varphi}{\partial x^j} \quad \text{on scalar fields, and}$$

$$\operatorname{div} f = \frac{1}{|g|^{1/2}}\, \frac{\partial f^i}{\partial x^i}\, |g|^{1/2}\,,$$

$$(\text{curl } f)_{ik} = \frac{\partial f_k}{\partial x^i} - \frac{\partial f_i}{\partial x^k}, \quad \text{on vector fields.}$$

Other invariants and operators can also be defined on higher-order objects.

When we take as objects curves on V_4 with a given metric, the basic functional is the integral of ds^2 along curves, and the intrinsic objects i.e. the critical points of this functional are the geodesics. The field of Physics relating to these objects is Classical Physics in its primary form of study of point motions.

Instead if we take as basic objects functions on V_4 (with a given metric) the naturally associated functional is the integral of $(\nabla\varphi)^2$ over some open domain of V_4 and the intrinsic objects are the solutions of the wave equation $\Delta_2\varphi = 0$. The corresponding field of Physics is Wave Mechanics, that is, the study of 3-dimensional wave motions. And so on. The enclosed table summarizes these statements.

objects	basic functional	intrinsic objects	field of Physics
curves	$\int_\gamma ds^2$	geodesics	Classical
functions	$\int_\mathcal{D} (\nabla\varphi)^2$	Solutions of wave equation $\Delta_2\varphi = 0$	Wave Mechanics
⋮	⋮	⋮	⋮

In principle one can go further on, for instance, constructing non-commutative field theories along this same line of reasoning, involving higher-order tensor fields.

1.3 CURVES OF INTRINSIC OBJECTS

The motion of a point (i.e. a curve on (V_4, g)) is described in kynematics as a succession of states of motion each of which corresponds to a geodesic in V_4. Therefore the motion of a point, i.e. a general curve on (V_4, g) may be thought of as a <u>curve on the space of geodesics</u>, its concrete representation being their envelope. The deviation from geodesic motion serves then to define the presence of external forces.

Similarly a differentiable function f on V_4 may be described, at least locally, as an envelope function to a succession of solutions of the wave equation. Indeed choose a time reference axis system t on V_4 and consider the corresponding space-like sections Σ_t: $t =$ const. The Cauchy problem $\Delta_2 \varphi = 0$, with $\varphi\big|_{\Sigma_t} = f\big|_{\Sigma_t}$, $\dfrac{\partial \varphi}{\partial t}\Big|_{\Sigma_t} = \dfrac{\partial f}{\partial t}\Big|_{\Sigma_t}$ is locally solvable for each t and therefore f may be thought of as <u>a curve</u> $t \mapsto \varphi_t$ <u>on the space of solutions of the wave equation</u>, of which it is actually the envelope. Of course this representation is tied to the particular choice of space sections, i.e. to the particular observer.

<u>Curves of intrinsic objects are therefore the next natural objects to be considered</u>.

The question now is: what class of curves to consider?

1.4 GENERALIZED CURVES

L.C. Young showed in 1933 that the problems of the Calculus of Variations always have a solution (under minimal assumptions, cf. p.178 [15]) provided we enlarge the usual space of classical curves, to the space of generalized curves, that he introduced

exactly for this purpose.

For example, on the set of absolutely continuous curves in the plane $(x(t), y(t))$, $0 \leq t \leq 1$, with end-points $(0,0)$ and $(1,0)$ we have the strict inequality

$$\int_0^1 (1+y^2)[\,|\dot{x}| + |\,|\dot{x}|-|\dot{y}|\,|\,]dt > \int_0^1 |\dot{x}|dt \geq 1,$$

and since the infimum is 1 (a minimizing sequence is given by saw-tooth like curves with alternating slopes ± 1 and with height tending to zero) it is not attained in the given set. Therefore the minimization problem for the given functional does not have a classical solution, yet it has a generalized one.

1.4.1 <u>A generalized curve is simply a curve given by a Lipschitz-</u> <u>ian vector-valued function</u> $x(t)$, $0 \leq t \leq 1$, <u>plus, for almost</u> <u>every</u> t, <u>a non-negative measure</u> μ_t <u>on the set of unit vectors</u> <u>(identify it to the unit sphere)</u> <u>such that</u> i) <u>the measure of each</u> <u>Borel set of the sphere is a uniformly bounded and measurable</u> <u>function of</u> t, <u>and</u> ii) <u>for almost every</u> t <u>the mean-value vector,</u> <u>with respect to the corresponding measure, coincides with</u> $\dot{x}(t)$.

In the above example the solution is the generalized curve given by $x(t) = t$, $y(t) \equiv 0$, $0 \leq t \leq 1$, plus the measures that assign, for any t, the weights $1/\sqrt{2}$ to the unit vectors forming angles of $\pm 45°$ with the positive x-axis, and zero to all others.

The definition above results from the following construction. If we fix a given compact set K in $\mathbb{R}^n$, which is the closure of its interior say, then any piecewise C^1 curve γ lying in K, given in parametric form by $x(t)$, $0 \leq t \leq 1$, can be thought of as a continuous linear functional on the space of continuous functions f on $K \times S$ (S the unit sphere in $\mathbb{R}^n$)

with the sup-norm, defined by

$$(1.4) \qquad \gamma(f) = \int_0^1 f\left(x(t), \frac{\dot{x}(t)}{|\dot{x}(t)|}\right) |\dot{x}(t)| dt.$$

(at the points where $|\dot{x}(t)| = 0$ the integrand is set equal to zero).

Two curves γ, $\tilde{\gamma}$ are identified if $\gamma(f) = \tilde{\gamma}(f)$ for all f's.

Since

$$|\gamma(f)| \leq \max|f| \int_0^1 |\dot{x}(t)| dt$$

and

$$|\gamma(1)| = \int_0^1 |\dot{x}(t)| dt = \ell(\gamma),$$

where $f = 1$ is the function $f(\cdot, \cdot) \equiv 1$, we see that the norm of γ is its length $\ell(\gamma)$.

1.4.2 <u>A generalized curve is then defined as a weak-* sequential limit of classical curves</u> γ_n, i.e. <u>as those linear functionals g defined by the condition that</u>

$$g(f) = \lim_{n \to \infty} \gamma_n(f) \quad \text{exists for all} \quad f \in C(K \times S).$$

As the unit ball of the space dual to $C(K \times S)$ is weak-* sequentially compact, this definition guarantees the existence of a weak-* converging subsequence out of any bounded minimizing sequence, whose limit is thus a generalized curve, and which need not be a classical curve, as shown in the earlier example.

The fundamental result of L.C. Young (Representation Theorem p.171 [15]) is that this and the previous definition are equivalent, via the Riesz representation theorem.

In particular, from the first definition, the associated linear functional of the second one is found through

$$(1.5) \qquad g(f) = \int_0^1 \left[\int_S f(x(t),v)\mu_t(dv) \right] dt, \qquad f \in C(K \times S),$$

which is clearly well defined and continuous thanks to condition i) in 1.4.1.

Using (1.5) one can check that at the generalized curve described earlier as the solution of the given example, the infimum 1 is indeed attained.

In a time-oriented spacetime manifold (V_4, g), timelike curves represent states of motion of a spatial point Q in three-dimensional space-sections, defined by a given observer. To each direction at a point P in V_4 we can associate locally and biuniquely a single geodesic which, when it is time-like or isotropic (i.e. causal), and, say, positively time oriented, represents the free motion of Q along the spatial projection of that direction. We can thus think of the given measure on the unit sphere at P as a measure on the geodesics through P. When this measure has support on the (positively time oriented) causal geodesics only, the corresponding generalized curve can be thought of as representing a (weighted) statistical superposition of all possible causal geodesic motions of the base point Q, which then actually performs the mean average motion.

Generalized motions occur, in particular, when the conditions of the physical problem described place restrictions on the class of allowed classical motions, leaving as only alternative its statistical superposition. (A classical example is the motion of a relativistic particle cf. p. 35 [7]).

1.4.3 REMARKS ON STOCHASTICITY AND HIDDEN MOTIONS

The basic laws of classical point motion can be phrased in

terms of the Calculus of Variations. In view of the results of L.C. Young and in order not to unduly restrict the range of the phenomena covered, a priori, we see that the natural place to start looking for solutions is in the set of generalized curves. This indicates that, in principle, even classical problems (v.g. example (1.5)) may lead to stochastic solutions, so that, contrarily to widely held belief, probability and indeterminism are not an exclusive characteristic of Quantum Mechanics. Indeed they belong, in principle, in all fields of Physics, having t ᵤ₃ with the Lagrangian character of the laws governing motions r ⌐ than with the nature of the objects that perform the motion.

An important example in the realm of point motion is afforded by <u>dynamical friction</u>: a point moving under the presence of dynamical friction may be thought of as indefinitely alternating between instantaneous free motion and instantaneous rest, both with say equal probabilities. One can get a good feeling of a situation similar to this in trying to hold a car still up a hill in gear, by pressing the clutch conveniently. The curve of the resulting state of motion, in its ideal limit, is precisely the infinitesimal zig-zag generalized curve of the example given before, in (t,x) space.

A generalized non-classical curve provides an example of an extreme form of indeterminism: it has precise space location and absolute randomness in momentum space. It also provides an example of hidden motions building up an otherwise apparently classical motion.

In our opinion L.C. Young's results clarify the whole issue of indeterminism and hidden-variables: under this new light, and at least in the present context, these are no longer philosophical

questions but strict mathematical facts derived from the Lagrangian structure involved. His results also freed us from the burden caused by overly pessimistic interpretations of indeterminism. On the contrary, they actually show Nature has at its disposal new, unsuspected, degrees of freedom, while retaining the essential part of its classical character, namely spatial localization.

In closing, it is noteworthy to call attention to the fact that the idea behind the concept of weak-* topology, namely the description of objects solely on account of their effects on others, corresponds to the same idea that led Heisenberg to describe the observables via Matrix Mechanics, i.e. solely in terms of measurable data resulting from possible experiments performed on the observables. It is remarkable but not totally surprising that, in L.C. Young's context, the stochastic nature results as a mathematical consequence.

1.4.4 CONCLUSION

From now on we shall agree that the motions in our spaces are to be described by generalized curves of intrinsic objects, of which the averaged envelope object, curve or function, is what is observed.

The concept of generalized curve in the general case will be that given by our first Definition 1.4.1. This pressuposes a normed topology on the set of intrinsic objects. Our next task will be to find out the appropriate way to do this in the case of waves.

We now go back to the study of waves on (V_4, ε).

1.5 MONOCHROMATIC WAVES

The solutions of the wave equation obviously constitute a linear space. More precisely, the germ of solutions of the wave equation in a neighbourhood of a point form a linear space. (To start, most of our analysis is to be done locally.)

The algebra generated by a single solution of $\Delta_2\varphi = 0$ consists of solutions of this equation again if and only if φ satisfies in addition $(\nabla\varphi)^2 = 0$. This follows easily from the formula

$$(1.6) \qquad \Delta_2 f(\varphi) = f' \Delta_2\varphi + f''(\nabla\varphi)^2,$$

valid for f twice differentiable, if φ is (essentially) real-valued, or f analytic if φ is complex-valued.

The solutions of the pair of equations

$$(1.7) \qquad \begin{cases} \Delta_2\varphi = 0 & \text{(the wave equation)}, \\ (\nabla\varphi)^2 = 0 & \text{(the eikonal)}, \end{cases}$$

are called <u>monochromatic waves</u>. They represent pure light waves.

We will denote the set of monochromatic waves by $\mathfrak{m}$.

An algebra all of whose elements are monochromatic waves will be called a <u>monochromatic algebra</u>.

1.6 QUANTUM TRANSITIONS

From the identity

$$(1.8) \qquad e^{-i\varphi} \Delta_2 e^{i\varphi} = i\Delta_2\varphi - (\nabla\varphi)^2$$

we obtain, if $\Delta_2\varphi = 0$,

$$(1.9) \qquad (\nabla\varphi)^2 = -e^{-i\varphi}\Delta_2 e^{i\varphi}.$$

Consequently by applying the mapping $\varphi \mapsto \exp(i\varphi) = \psi$, our linear space of solutions of the wave equation goes into a multiplicative group, whereas the Lagrangian $(\nabla\varphi)^2$ goes into $-\dfrac{\Delta_2\psi}{\psi}$.

If the φ's are real the $e^{i\varphi}$'s are bounded and we can embed the above group in the Banach algebra under the sup norm that it generates under pointwise operations and completion. To distinguish between them we call the original linear space _phase space_ $\mathcal{S}$ and the newly introduced algebra, the algebra of wave states G or simply _algebra of states_.

It is easy to see that the critical points associated with the integrand $-\Delta_2\psi/\psi$ are those for which $\Delta_2 \ell n \psi = 0$, i.e. exactly those whose phase function satisfy the wave equation. These intrinsic states will be called _elementary states_.

The new representation has two advantages over the original one. It is richer in structure, and in elements, as $\mathcal{S}$ is mapped into a subset of the set of invertible elements Ω of G, and so by taking logarithm pointwise, on the elements of Ω, we obtain an enlargement of $\mathcal{S}$ by possibly multivalued functions.

The second advantage, that actually justifies the whole construction, is that the integrand of the Lagrangian expressed in the form $-\Delta_2\psi/\psi$, when integrated, exhibits jumps across the boundary $\partial\Omega$. To illustrate this, we first give an example in Minkowski space.

1.6.1 AN EXAMPLE

Consider G to be the algebra of almost-periodic functions in $\mathbb{R}^4$ i.e. the algebra under the sup norm and under the pointwise operations, generated by the exponentials $e^{ik\cdot x}$, $k \in \mathbb{R}^4$ ([1]).

Clearly all such plane waves are elementary states in Minkowski space, and therefore the algebra they generate is the natural one to consider according to the rules given in 1.1.

The mean value integral

$$(1.10) \qquad \oint f := \lim_{n \to \infty} \frac{1}{|K_n|} \int_{K_n} f \, dV,$$

where $dV = dx^1 dx^2 dx^3 dx^4$, and K_n is a sequence of cubes of side n centered at the origin say, and $|K_n|$ its volume, is well defined in $\hat{G}$. In particular the wave operator Δ_2 restricted to those elements of G whose first and second derivatives belong in G again, is symmetric with respect to the scalar product induced by $\oint$ because then integration by parts with zero boundary terms holds for $\oint$. On the elementary states of this subset consider the basic functional

$$(1.11) \qquad \lambda(\psi) = -\oint \frac{\Delta_2 \psi}{\psi}.$$

In particular if $\psi = e^{ik \cdot x}$ we have $\lambda(e^{ik \cdot x}) = k^2$ and if $w = e^{ik_o \cdot x}$ is a plane monochromatic wave, $k_o^2 = 0$.

Assume $k_o \neq 0$. Consider the continuous curve in G

$$(1.12) \qquad \psi_t = \psi((1-t)+tw)), \qquad 0 \leq t \leq 1,$$

connecting the states $\psi_o = \psi$ and $\psi_1 = e^{ik_1 \cdot x}$ where $k_1 = k+k_o$.

All ψ_t are elementary states because these form a multiplicative group and because any analytic function of a monochromatic state is again elementary.

Let us examine the behavior of $\lambda(\psi_t)$. Setting $\epsilon = \frac{t}{1-t}$, then as long as $0 \leq \epsilon < 1$ or equivalently $0 \leq t < \frac{1}{2}$, we may write

$$(1.13) \qquad \lambda(\psi_t) = -\int \frac{\Delta_2 \psi_t}{\psi_t} = -\int \frac{\Delta_2 \psi + \epsilon \Delta_2 \psi_1}{\psi(1+\epsilon w)} .$$

$$= -\int \left(\frac{\Delta_2 \psi}{\psi} + \epsilon \frac{\Delta_2 \psi_1}{\psi} \right)(1 - \epsilon w + \epsilon^2 w^2 - \ldots)$$

$$= \int (k^2 + \epsilon k_1^2 w)(1 - \epsilon w + \epsilon^2 w^2 - \ldots)$$

$$= k^2 + (k_1^2 - k^2) \sum_{n=1}^{\infty} (-1)^{n-1} \epsilon^n \int w^n$$

$$= k^2 ,$$

because if $nk_0 \neq 0$ then $\int e^{ink_0 \cdot x} = 0$.

Hence $\qquad \lambda(\psi_t) \equiv \lambda(\psi) = k^2$ for $0 \leqslant t < \frac{1}{2}$.

Similarly starting at the other end i.e. interchanging the roles of ψ and ψ_1, w goes into w^{-1} and t into 1-t, we get

$$(1.14) \qquad \lambda(\psi_t) = \lambda(\psi_1) = k_1^2 \quad \text{for} \quad \frac{1}{2} < t \leqslant 1.$$

If $k \cdot k_0 \neq 0$ we have

$$k_1^2 = k^2 + 2k \cdot k_0 \neq k^2 .$$

Hence on this continuous curve of elementary states in G the basic functional $\lambda = \int (\nabla \varphi_t)^2$, $t \neq \frac{1}{2}$, (φ_t phase function of ψ_t), has two constant distinct values, exhibiting a jump at $t = \frac{1}{2}$. Clearly the state $\psi_{1/2} = \frac{1}{2} \psi(1+w) \in \partial\Omega$, because $1+w = 1+e^{ik_0 \cdot x} = 0$ along the planes $k_0 \cdot x = (2n+1)\pi$, $n = 0, \pm 1, \ldots$.

1.6.2 QUANTIZATION OF THE BASIC FUNCTIONAL

We now give the general analysis of the quantum behavior exemplified above.

In general let G be a Banach algebra of continuous complex-valued functions on a Lorentz manifold (V_4, g), containing the

constant functions, closed under complex-conjugation, with the algebraic operations defined pointwise and the sup-norm, and containing a dense subset G_2 of twice-differentiable functions which are mapped by the Laplace-Beltrami operator Δ_2 again into G. Assume further that $f \in G$ is invertible with inverse f^{-1} in G again if and only if $\inf_{V_4} |f(x)| > 0$. The set of invertible elements is denoted by Ω.

Furthermore assume a positive linear functional, denoted by $\int$, is defined on G, allowing the usual rules of integration by parts, with no boundary terms.

These conditions are fullfilled by the Banach algebra of almost periodic functions on Minkowski space, of the previous example, with $\int$ defined by the mean-value integral in (1.10), and also by the algebra of weakly amost periodic functions (see [1], [2], [5], [6]).

Consider as before the functional $\lambda: G_2 \cap \Omega \to \mathbb{C}$ defined by

$$(1.15) \qquad \lambda(\varphi) = \int \frac{\Delta_2 \varphi}{\varphi} .$$

The critical points of λ are those u such that

$$(1.16) \qquad \operatorname{div}\left(\frac{\nabla u}{u}\right) = 0 \quad \text{i.e.} \quad \frac{\Delta_2 u}{u} - \left(\frac{\nabla u}{u}\right)^2 = 0.$$

If the linear functional is strictly positive, i.e. $\int |\varphi|^2 = 0$ iff $\varphi = 0$, (1.15) and (1.16) are to hold in G, otherwise in the sense of the inner product space defined by $\int$ on G. This also applies to the subsequent formulas. (The first situation holds for the almost-periodic functions, the second for the weak ones.)

By (1.16) the set C of critical points of λ is clearly a subgroup of Ω.

The monochromatic functions of G are, as before, those $w \in G_2$ such that

$$(1.17) \qquad \Delta_2 w = 0, \qquad (\nabla w)^2 = 0,$$

and their set is denoted $\mathfrak{m}$, again.

By virtue of (1.6) $f(w) \in \mathfrak{m}$ again if f is an analytic function on a neighbourhood of the set of values taken by w on V_4. Since by (1.16) $\mathfrak{m} \cap \Omega \subset C$, we have that $uf(w) \in C$ if $w \in \mathfrak{m}$ and $f(w) \in \Omega$.

The spectrum $\sigma(v)$, for any $v \in G$, is defined by $\sigma(v) = \{z \in \mathbb{C} \mid v-ze \notin \Omega\}$ and therefore, by a previously assumed property, is the closure of the set of values $v(x)$ taken by v on V_4. It is obviously a compact non-void subset of $\mathbb{C}$.

Ω has either one or else infinitely many maximal connected components, of which Ω_o is the one containing the identity e $(e(x) \equiv 1)$.

Two elements f, h belong in the same component of Ω, iff $fh^{-1} \in \Omega_o$. Further $f \in \Omega \setminus \Omega_o$ iff its spectrum $\sigma(f)$ separates 0 and ∞.

The logarithm function, as a mapping from G into G again, is defined only on Ω_o (p.15 [11]).

With these preliminaries we can now show that the quantum jumps arise as a generalized form of the standard argument principle.

1.6.3 THEOREM. <u>Let</u> $u \in C$, $w \in \mathfrak{m} \cap \Omega$. <u>Denote by</u> $H_1, \ldots, H_n, \ldots$ <u>the bounded maximal connected components of the complement of</u> $\sigma(w)$.

<u>Then there exist fixed numbers</u> q_i, $i=1,2,\ldots$, <u>depending on</u> u <u>and</u> w <u>only, such that for any function</u> $f(z)$ <u>analytic in a neighbourhood of</u> $\sigma(w)$ <u>and with no zeros in</u> $\sigma(w)$, <u>we have</u>

(1.18)
$$\lambda(uf(w)) = \lambda(u) + \sum_i (N_i - P_i)q_i \; ,$$

where N_i, P_i are the number of zeros and poles, respectively, of f in H_i, i=1,2,... .

In particular choosing $\alpha_i \in H_i$, the q_i are given by

(1.19)
$$q_i = 2 \oint \frac{\nabla u}{u} \cdot \frac{\nabla w}{w - \alpha_i} \; , \qquad i=1,2,\dots .$$

In the special case of the previous example, when $u = e^{ik \cdot x}$, $w = e^{ik_o \cdot x}$, $k_o^2 = 0$, $k_o \cdot k \neq 0$, and the spectrum $\sigma(w) = S^1$ (S^1 = unit circle), (1.18) becomes

(1.20)
$$-\lambda(e^{ik \cdot x} f(e^{ik_o \cdot x})) = k^2 + 2(k_o \cdot k)(N - P),$$

where N and P are the number of zeros and poles, respectively, of f inside the unit disk. One can then see in (1.20) what happens when one follows along the curve of analytic functions $f_t(z) = (1-t) + tz$, used in that example, or for that matter, any analytic curve joining 1 and z.

Proof of (1.18) Let f = f(w), with f(z) as in the hypothesis.

Direct computation gives, using (1.17)

(1.21)
$$\frac{\Delta_2 uf}{uf} - \frac{\Delta_2 u}{u} = 2 \frac{\nabla u}{u} \cdot \frac{\nabla f}{f} \; ,$$

which, integrated, yields

(1.22)
$$\lambda(uf) - \lambda(u) = 2 \oint \frac{\nabla u}{u} \cdot \frac{\nabla f}{f} \; .$$

In particular this shows that q_i in (1.19) are well-defined. From (1.22) one gets directly

(1.23)
$$\lambda(ufh) - \lambda(u) = [\lambda(uf) - \lambda(u)] + [\lambda(uh) - \lambda(u)],$$

(1.24)
$$\lambda(uf^{-1}) - \lambda(u) = -[\lambda(uf) - \lambda(u)],$$

where $f = f(w)$, $h = h(w)$, here and in the sequel.

Now if $f \in \Omega_o$, then $\ln f \in G$ and $\frac{\nabla f}{f} = \nabla \ln f$, which substituted in (1.22) gives, upon integration,

$$\lambda(uf) - \lambda(u) = 2 \oint \frac{\nabla u}{u} \cdot \nabla \ln f$$

$$= -2 \oint f \operatorname{div} \frac{\nabla u}{u} = 0,$$

by (1.16). Hence $\lambda(uf) = \lambda(u)$ if $f \in \Omega_o$.

If now f, h belong in the same component of Ω we can write $uh = uf \cdot hf^{-1}$, and since $hf^{-1} \in \Omega_o$, the previous result yields

$$(1.25) \qquad\qquad \lambda(uh) = \lambda(uf).$$

This shows that $\lambda(uf(w))$ is locally constant in Ω as f varies in the set of analytic functions.

Let now $f(z) = z-\xi$ with $\xi \in H_i$. Then $z-\xi$ can be changed analytically into $z-\alpha_i$ withoug ξ leaving H_i, which means that $w-\xi e$ and $w-\alpha_i e$ are in the same connected component of Ω. Therefore by (1.25), (1.22) and (1.19)

$$(1.26) \qquad\qquad \lambda(u(w-\xi e)) - \lambda(u) = q_i,$$

and by (1.24)

$$(1.27) \qquad\qquad \lambda\left(u(w-\xi e)^{-1}\right) - \lambda(u) = -q_i.$$

Instead if ξ belongs in the unbounded component of the complement of $\sigma(w)$, we may let $\xi \to \infty$ withoug crossing $\sigma(w)$, so that

$$(1.28) \qquad \lambda(u(w-\xi e)) - \lambda(u) = 2 \oint \frac{\nabla u}{u} \cdot \frac{\nabla w}{w-\xi e}$$

$$= \lim_{\xi \to \infty} 2 \oint \frac{\nabla u}{u} \cdot \frac{\nabla w}{w-\xi e} = 0.$$

Therefore if $f(z) = c_o \prod_{i=1}^{N}(z-a_i) \cdot \prod_{j=1}^{P} \frac{1}{z-b_j}$, $c_o \neq 0$,

$a_i, b_j \notin \sigma(w)$, (1.18) follows from (1.23), (1.26), (1.27) and (1.28).

In the general case, if $f(z)$ is a holomorphic function in a neighbourhood of $\sigma(w)$, without zeros in $\sigma(w)$, we can find a rational function $r(z)$ such that

$$|f(z)-r(z)| < \min_{\sigma(w)} |f(z)| \quad \text{in} \quad \sigma(w),$$

by Runge's theorem (p.256 [13]). Then $r(z)$ has no zeros in $\sigma(w)$ too, and $r(w)$ and $f(w)$ are in the same component of Ω, so that (1.18) holds for $f(w)$ too. The proof is complete.

COMMENTS

1. The quantization formula (1.18) tells us how the basic functional changes when we perturb the elementary state u into $uf(w)$, with f analytic near and on $\sigma(w)$. Changes occur only when zeros or poles of $f(z)$ reach and eventually cross the boundary of $\sigma(w)$, and these changes are integer multiples of fixed quanta q_i, each one attached to the hole H_i whose boundary is reached and crossed (while u, w remain fixed).

2. Let T_u denote the linear operator $h \mapsto uh$, $h \in G$, $u \in \Omega$. The very simple analysis above hinges on the fact that $T_u^{-1} \Delta_2 T_u - T_{\Delta_2 u/u}$ is a derivation on the germ $\mathcal{F}(w)$ of functions of w which are analytic in a neighbourhood of $\sigma(w)$, cf. (1.21), and it could have been performed abstractly without further mention to the special case under consideration.

The general abstract theory relative to the functional λ of the form (1.15) is presented in [9], and it essentially shows, when specialized to 2nd order differential operators like Δ_2,

that the condition $w \in \mathfrak{m}$ is not only sufficient but also necessary in order for the quantum behavior of λ to occur.

1.7 RELATIONS WITH CLASSICAL GAUGE THEORY

The set of linear mappings $T_{f^{-1}}$ of G defined by $h \mapsto f^{-1}h$, $f \in \Omega$, $h \in G$, is a group which maps each connected component of Ω onto another one. In terms of functions on V_4 it changes locally the scale of the functions, i.e. the ratio of the values of any function at two distinct points is changed in a given proportion, and is therefore called a <u>gauge transformation of the first kind</u> (p.234 [12]). Under this transformation we have that

$$(1.29) \qquad \Delta_2 \mapsto T_f^{-1} \Delta_2 T_f = \Delta_2 + 2 \frac{\nabla f}{f} \cdot \nabla + \frac{\Delta_2 f}{f} \quad .$$

On the other hand we can write

$$(1.30) \qquad \Delta_2 = -\nabla^* \nabla ,$$

where $\nabla^* = -\text{div}$ is the operator from vector fields into scalars adjoint to the operator $\nabla = \text{grad}$ from scalar fields into vector fields, with respect to the hermitean form $\int \bar{\psi}^i \cdot \varphi^i |g|^{1/2} \, dV$.

Similarly $\int \bar{\psi}^i \cdot (\varphi A^i) dV = \int \varphi (\bar{A}^i \cdot \psi^i) dV$, so that $(A^i)^* = \bar{A}^i \cdot$. (As before the dot means scalar product in the metric of V_4).

Therefore if $\nabla \mapsto \nabla + \frac{\nabla f}{f}$ then

$$(1.31) \qquad \nabla^* \nabla \mapsto (\nabla + \frac{\nabla f}{f})^* (\nabla + \frac{\nabla f}{f}) = [-\text{div} + (\overline{\frac{\nabla f}{f}} \cdot)] (\nabla + \frac{\nabla f}{f}) \quad .$$

If we assume that

$$(1.32) \qquad (\overline{\frac{\nabla f}{f}}) = - \frac{\nabla f}{f} ,$$

that is to say $|f(x)| \equiv 1$, which means f is a phase factor i.e.

f is a section of a circle bundle over V_4 (p.4 [3]), then (1.31) yields

$$(1.32) \qquad (\nabla + \tfrac{\nabla f}{f})^*(\nabla + \tfrac{\nabla f}{f}) = (-\text{div} - \tfrac{\nabla f}{f} \cdot)(\nabla + \tfrac{\nabla f}{f})$$

$$= -\Delta_2 - 2 \tfrac{\nabla f}{f} \cdot \nabla - (\tfrac{\nabla f}{f})^2 - \tfrac{\Delta_2 f}{f} + (\tfrac{\nabla f}{f})^2$$

$$= -[\Delta_2 + 2 \tfrac{\nabla f}{f} \cdot \nabla + \tfrac{\Delta_2 f}{f}]$$

$$= -T_f^{-1} \Delta_2 T_f \;.$$

Consequently if f is a phase factor on V_4 then under the gauge transformation of the first kind $h \mapsto f^{-1}h$, the change of Δ_2 into $T_f^{-1} \Delta_2 T_f$ can be completely described by the transformation

$$(1.33) \qquad \nabla \mapsto \nabla + \tfrac{\nabla f}{f} \;,$$

in view of (1.29), (1.30), (1.31) and (1.32).

Now if A is a vector field on V_4 the transformation $\nabla \mapsto \nabla + A$ is called a gauge transformation of the second kind (p.232,238 [12]). In electromagnetism A is called electromagnetic vector potential and the antisymmetric tensor $F = \text{curl } A$ is the electromagnetic field tensor. The Maxwell equations are then equivalent to the requirement that F be a critical point of the function $\int F^{\mu\nu}F_{\mu\nu}$ (p.6 [8]). The electromagnetic vector potential A above is restricted by the Lorentz condition

$$(1.34) \qquad \text{div } A = 0.$$

In summary, when f is a phase factor, the gauge transformations of 1st and 2nd kind are equivalent, and give rise to the electromagnetic vector potential $\tfrac{\nabla f}{f}$.

The Lorentz condition (1.34) becomes $\text{div } \tfrac{\nabla f}{f} = 0$, which is simply our previous condition that f be an elementary state, i.e.

a critical point of $\lambda(f)$ given by (1.15).

When $\frac{\nabla f}{f}$ canot be written globally as a gradient $\nabla\varphi$, f is said to be a non-integrable phase factor (p.12 [14], p.118 [3], p.62 [4]) When f belongs to our algebra G, this is equivalent to saying that f does not have a logarithm in G, which means that $f \in \Omega\backslash\Omega_o$.

In any case, however, $F = \text{curl }\frac{\nabla f}{f}$ is always identically zero because $\frac{\nabla f}{f}$ can be written locally as grad Log f, where Log is a pointwise locally defined logarithm determination.

1.7.1 CHOICE OF GAUGE

Consider now all connected components Ω_α of Ω. Any such component can be made into Ω_o by a gauge-transformation of the first kind: it suffices to take $f \in \Omega_\alpha$ and consider $h \mapsto f^{-1}h$. (This is clearly a diffeomorphism of Ω). This choice of such a component to play the role of the principal one is what is called a choice of gauge. In principle different observers may be in different gauges (thanks to their past history, for instance) and it is an experimental task that they first determine what their (relative) gauges are before they start comparing experiments. In other words, there is no preferred component Ω_α, the concept of principal component being a relative one.

Thanks to our previous analysis, that says that the ∇ operator of one becomes the $\nabla + \frac{\nabla f}{f}$ operator of the other, we can interpret the difference of gauges as being perfectly equivalent to the presence of the electromagnetic vector potential $\frac{\nabla f}{f}$ in the second observer's referential. However as the electromagnetic tensor $F \equiv 0$, this is an instance of the Bohm-Aharonov phenomenon: non-null effects associated with identically zero electromagnetic

tensor fields. That there are non-null effects is checked by our previous analysis of the functional $\lambda(uf(w))$, where u is any elementary state and f, besides being a phase factor, is also monochromatic. In this case λ, which is locally constant, depends on which Ω_α f belongs to, that is to say, on the choice of gauge. (Notice that for plane waves $u = e^{ik \cdot x}$, $\lambda(u) = -k^2$ gives minus the square rest mass of the wave.)

1.7.2 ALTERNATIVE INTERPRETATIONS

According to the two ways of interpreting a linear operation (as a mapping on the vector space or as change of referential frames) we have two possibilities.

Indeed let $w \in \mathbb{M}$ and let $f_t(w)$ $t \in [0,1]$, with $f_t(z)$ analytic in a neighbourhood of $\sigma(w)$, be a continuous curve on G. Earlier we considered for any $u \in C$, the curve of elementary states $uf_t(w)$ and described in (1.18) the behavior of $\lambda(uf_t(w))$ along this curve. In particular we considered $uf_t(w)$ as a perturbation, or _excitation_, of u evolving in time.

We can also regard

$$(1.35) \qquad\qquad u \longmapsto T_{f_t(w)} u$$

as a continuous curve of gauge transformations of 1st kind acting on a _fixed_ elementary state u, which, when f_t crosses $\partial\Omega$, determines a change of gauge. When that happens f_t cannot be a phase factor for all t obviously, so that no electromagnetic interpretation can be given all along the evolution in t. However if, say, the initial states f_0 and f_1 are phase factors (i.e. $|f_0| \equiv |f_1| \equiv 1$), this change of gauge is equivalent to the appearance of a (non-trivial) electromagnetic vector potential

between the initial and final states of u.

In any of these interpretations a non-null effect is detected by a jump in λ as given by (1.18): this quantum transition is interpreted in the first case as an excitation of the state u, and in the second case as a change of gauge of u materialized by the appearance of the corresponding electromagnetic vector potential.

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CHAPTER II

LIGHT QUANTA

2.1 SINGULAR SETS AND LIGHT QUANTA

We have seen in the previous chapter that quantum transitions of the basic functional λ occur at the elementary states $uf(w)$, where $f(w)$, an analytic function of the monochromatic wave w, is no longer invertible.

Let $N(f)$ denote the set of points in V_4 where $f(w)$ is not invertible.

On $N(f)$ the corresponding phase function has logarithmic singularity. As this set is responsible for the possible jump of $\lambda(uf(w))$, we may assign to this singularity the role of carrier of the energy difference. (Here we interpret $-\lambda = \int (\nabla \varphi)^2$ as square rest mass, by analogy with the case of plane waves).

For instance in Example 1.6.1 the 4-planes $k_o \cdot x = (2n+1)\pi$ $n = 0, \pm 1, \pm 2, \ldots$ carry this energy difference.

It is possible to exhibit monochromatic waves such that $N(f)$ is a single 4-plane in Minkowski space. Indeed let g be a non-periodic, almost periodic function of a single real variable. The auto-correlation function

$$(2.1) \qquad f(x) = \int \overline{g(y)} g(x+y) \, dy$$

is again almost periodic (p.32, [6]) and satisfies

$$(2.2) \qquad f(0) > |f(x)| \qquad x \neq 0.$$

This follows from Schwarz's inequality in (2.1)

$$|f(x)| \leq \int |g|^2 = f(0).$$

If equality holds for $x_0 \neq 0$ then necessarily $g(x_0+y) = \pm g(y)$ for all y in the present case, and this implies periodicity of g.

A second example is given by any almost periodic function whose Fourier coefficients are all positive and among whose exponents λ_n there are at least three which are linearly independent over the integers. In this case (p.63 [6]) one has, pointwise

$$f(x) = \sum_n A_n e^{i\lambda_n x} .$$

This implies

$$|f(x)| \leq \sum_n A_n = f(0).$$

If $x_0 \neq 0$ and $|f(x_0)| = f(0)$, then $|\sum_n A_n e^{i\lambda_n x_0}| = \sum_n A_n$ holds iff the complex numbers $e^{i\lambda_n x_0}$ are all equal, which requires, in particular, that any three exponents λ_n be linearly dependent over the integers.

Take any such function f of class C^2 and $k_0 \in \mathbb{R}^4$, $k_0 \neq 0$, $k_0^2 = 0$. Then $\varphi(x) := f(0) - f(k_0 \cdot x)$, $x \in \mathbb{R}^4$, is a monochromatic wave whose singular set $N(\varphi)$ is precisely the plane $k_0 \cdot x = 0$.

Clearly in this case we may think of radiation being propagated with the speed of light along a moving 3-plane carrying a quantum of energy.

These types of moving singularities can be properly identified as _light quanta_.

Our next task is to find out all possible dimensions of singular sets $N(f)$ of monochromatic functions.

2.2 DIMENSIONALITY OF SINGULAR SETS

Recall that a C^2 real or complex-valued function f on (V_4, g) is a monochromatic wave, in symbols $f \in \mathbb{M}$, if it satisfies

$$(2.3) \qquad\qquad \Delta_2 \, \varphi(f) = 0$$

for any entire function φ.

In the first case all C^2 functions of f and, in the second case, all analytic or anti-analytic functions of f belong in $\mathbb{M}$ again.

If f is real, smooth and $\nabla f \neq 0$, $N(f)$ is locally three-dimensional. If it is complex and Re f and Im f are functionally independent, $N(f)$ is two-dimensional.

Yet the (Newtonian) picture of a photon as an isolated point like singularity moving with the speed of light, requires a one-dimensional singular set $N(f)$.

Can we achieve this by going over to hypercomplex valued functions, for example quaternion valued ones?

The answer to this basic question is no as we shall now show.

2.2.1 HYPERCOMPLEX FUNCTIONS

A system S of hypercomplex numbers or, in modern language, a finite-dimensional linear algebra over $\mathbb{R}$ (or $\mathbb{C}$), is a finite-dimensional vector space over $\mathbb{R}$ (or $\mathbb{C}$) on which multiplication of any ordered pair of elements is defined, taking into S again, being distributive with respect to vector addition (pp. 10,22 [4], p.5 [3]).

If $e_1, e_2, \ldots, e_n$ form a basis of this vector space we get

$$(2.4) \qquad e_i e_j = \sum_{k=1}^{n} \gamma_{ijk} e_k \qquad (i,j=1,\ldots,n),$$

with $\gamma_{ijk} \in \mathbb{R}$ (or $\mathbb{C}$). The constants γ_{ijk} are called the constants of the multiplication table of S, with respect to the given basis.

They are arbitrary and once given, define the multiplication according to (2.4) thanks to the distributivity with respect to addition.

The product in S is associative if and only if $e_i(e_j e_k) = (e_i e_j)e_k$ for all $i,j,k = 1,\ldots,n$ and this imposes conditions on the γ_{ijk} (p.92 [4]).

In addition S has a principal unit u, i.e. an element such that $ux = xu = x$ for all x, if and only if there are numbers $\alpha_1, \ldots, \alpha_n$ such that $\sum_i \alpha_i \gamma_{ijk} = \delta_{jk}$ $(j,k=1,\ldots,n)$. (p.8 [3]).

In general S need not be commutative.

However the algebra generated by a single S-valued function f on V_4 is always commutative provided S is associative (p.9 [3]).

Since our analysis of the quantum transitions in the previous chapter applied to commutative monochromatic algebras with unit, we therefore assume that S is associative and has a principal unit.

In this case S is isomorphic to a subalgebra of the algebra M_n of $n \times n$ matrices over $\mathbb{R}$ (or $\mathbb{C}$) through the correspondence

$$a = a_1 e_1 + \ldots + a_n e_n \to T_a \in M_n ,$$

which associates with an element $a \in S$ the matrix of the linear operation $x \mapsto ax$ on S, with respect to the basis $(e_1, \ldots, e_n)$ (Thm. 2, p.96 [3]). T_a is given explicitly by

$$(T_a)_{jk} = \sum_{i=1}^{n} a_i \gamma_{ijk} \qquad (j,k=1,\ldots,n).$$

S may contain zero-divisors i.e. non-invertible elements other than zero. An element a is non-invertible if and only if det $T_a = 0$, which means at least one of the eigenvalues λ_i of T_a is zero.

If therefore f is an S-valued function on V_4 then $N(f)$, defined as the set of points x of V_4 where $f(x)$ is not invertible, is the set of points where at least one of the (possibly complex) eigenvalues λ_i of T_f is zero. If the λ_i's are locally smooth functions, $N(f)$ will be locally the (finite) union of the sets $N(\lambda_i)$, each of which is at least two-dimensional. Hence so will be $N(f)$.

This shows that we cannot obtain lower dimensional singular sets $N(f)$ by going over to hypercomplex-valued functions. Therefore the Newtonian picture cannot hold in the simple form proposed.

2.2.2 MONOCHROMATIC HYPERCOMPLEX FUNCTIONS

We shall say that an S-valued function f is locally smooth if its components f_i <u>and</u> its eigenvalues λ_i can be chosen locally as smooth functions on V_4.

Clearly the condition

$$0 = \Delta_2 f = e_1 \Delta_2 f_1 + \ldots + e_n \Delta_2 f_n$$

implies that each $\Delta_2 f_i = 0$.

As the entries of T_f are given by $\sum_{i=1}^{n} f_i \gamma_{ijk}$, $j,k = 1,\ldots,n$, and the γ's are constant, this implies that all the entries of T_f satisfy this equation again.

The hypothesis that $f \in \mathfrak{m}$ means that any entire function φ of f with real (or complex) coefficients, satisfies $\Delta_2 \varphi(f) = 0$. Combining this with the previous remark and with the known fact that

$$\operatorname{tr} T_{\varphi(f)} = \sum_i \varphi(\lambda_i),$$

we get

$$(2.5) \qquad 0 = \Delta_2 \sum_i \varphi(\lambda_i) = \sum_i [\varphi'(\lambda_i)\Delta_2\lambda_i + \varphi''(\lambda_i)(\nabla\varphi)^2].$$

Let n_P be the number of distinct eigenvalues of T_f at the point $P \in V_4$. By taking $\varphi(\lambda) = \lambda^p/p$, $p = 1,2,\ldots,2n_P$, in turn in (2.5), we obtain $2n_P$ linear homogeneous equations with a confluent Vandermonde coefficient matrix

$$\begin{pmatrix} 1 & \cdots\cdots & 1 & 0 & \cdots\cdots\cdots & 0 \\ \lambda_1 & \cdots\cdots & \lambda_{n_P} & 1 & \cdots\cdots\cdots & 1 \\ \vdots & & \vdots & \vdots & & \vdots \\ \lambda_1^{2n_P-1} & \cdots\cdots & \lambda_{n_P}^{2n_P-1} & (2n_P-1)\lambda_1^{2n_P-2} & \cdots & (2n_P-1)\lambda_{n_P}^{2n_P-2} \end{pmatrix}$$

at the point P.

The unknowns are the sums $\sum_i \Delta_2\lambda_i^{(k)}$, $\sum_i(\nabla\lambda_i^{(k)})^2$, $k = 1,2,\ldots,n_P$ where $\lambda_i^{(k)}$ are the original eigenvalues grouped by the condition $\lambda_i^{(k)} = \lambda_k$ at P, $i = 1,\ldots,k = 1,\ldots,n_P$ (so called $\lambda(P)$-groups).

Since the above system has non-zero determinant (p.322 [9]) we get the $2n_P$ conditions holding at P

$$\sum_i \Delta_2 \lambda_i^{(k)} = 0, \qquad \sum_i (\nabla \lambda_i^{(k)})^2 = 0,$$

$k = 1, \ldots, n_p$.

Simple eigenvalues therefore satisfy $\Delta_2 \lambda = 0$, $(\nabla \lambda)^2 = 0$ i.e. $\lambda \in \mathfrak{m}$. So do obviously the multiple eigenvalues of a group of eigenvalues that are coincident in an open set and remain distinct from the others in that set. More general situations arise as limiting combinations of both of these cases.

We may therefore conclude that, generically speaking, the eigenvalues of an S-valued monochromatic function should again be monochromatic. This therefore implies that

$$(2.6) \qquad N(f) = \bigcup_{i=1}^{n} N(\lambda_i), \quad \text{with } \lambda_i \in \mathfrak{m}, \quad i=1,\ldots,n.$$

As a consequence the analysis of singular sets of monochromatic hypercomplex-valued functions reduces to the analysis of those of real or complex-valued monochromatic functions.

2.2.3 DIVISION ALGEBRAS

We will now show that $N(f)$ reduces to a single set $N(\lambda)$, λ real or complex, for all possible S-valued functions if and only if S is a division algebra over $\mathbb{R}$ or $\mathbb{C}$, i.e. S is either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, where $\mathbb{H}$ denotes the quaternions (with real coefficients).

To show this we need the following facts. Any linear associative algebra has a uniquely determined maximal *nilpotent* ideal (its radical R) and is isomorphic to the sum of R with the semisimple algebra S/R (p.125 [4], p.158 [17]). Each semi-

simple algebra is the direct sum of simple algebras, and Cartan's fundamental theorem says that the simple algebras over $\mathbb{R}$ are just the matrix algebras $M_m(\mathbb{R})$, $M_m(\mathbb{C})$ and $M_m(\mathbb{H})$, and over $\mathbb{C}$ just $M_m(\mathbb{C})$, up to isomorphisms (p.56 [3]). In particular from this follows that the only real division algebras, i.e. real algebras with no zero divisors, are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, and the only complex one is $\mathbb{C}$ itself.

To prove the above claim we first remark that

1. If $a \in S$ is invertible then so is $a+r$ for any $r \in R$, and vice-versa, because $(1-r')^{-1}$ exists if $r' \in R$ and is given by $1 + r' + \ldots + r'^n$ and therefore so does

$$(a+r)^{-1} = (1+a^{-1}r)^{-1}a^{-1}.$$

2. If $p(x)$ is a polynomial in the indeterminate x, then

$$p(a+r) = p(a) + r' \qquad r' \in R,$$

and so also for any analytic function f.

3. Take any basis of S formed by a basis $e_1, \ldots, e_p$ of R and a basis $e_{p+1}, \ldots, e_n$ of a linear space C complementary to R in S. Since R is an ideal, we have $e_i e_j \in R$ if not both i,j are $> p$. This means in particular that if $i,j,k > p$ then $e_k e_i e_j$ has the same last $n-p$ coefcients that it would have if we had disregarded in the product $e_i e_j$ its coefficients with respect to $e_1, \ldots, e_p$ in the given basis (by induction this extends to any number of factors). Therefore if we define a new product in C given by the original multiplication table restricted to indices $i,j > p$, leaving the vector addition unchanged, C is then a concrote representation of S/R (p.153 [17]).

Furthermore if p is a polynomial, if $a \in C$ and $r \in R$ then

(*) $\pi p(a+r) = \tilde{p}(a),$

where π is the projection on C along R and $\tilde{p}$ is the same polynomial p but computed on the element $a \in C$ with the restricted multiplication table defined above.

Let therefore q be a smooth monochromatic S-valued function on (V_4, g). Decomposing it according to the subspaces C and R we get $q = a+r$ with smooth functions $a \in C$, $r \in R$. We claim that a is a monochromatic C-valued function under the restricted multiplication table.

Indeed by hypothesis $\Delta_2 p(a+r) = 0$ for any polynomial p, and this holds if and only if each of the coefficients of $p(a+r)$ with respect to the basis $e_1, \ldots, e_n$ satisfies the same equation. But this then implies, in particular, $\Delta_2 \pi p(a+r) = 0$ and by (*) $\Delta_2 \tilde{p}(a) = 0$, which proves the claim.

Combining 1., 2. and 3. we have

$$N(p(a+r)) = N(p(a)) = N(\pi p(a)) = N(\tilde{p}(a)),$$

for polynomials and so for analytic functions too.

This means that passing from S into C (with the restricted multiplication) preserves the monochromatic functions and their singular sets.

Since C with the new multiplication is semi-simple, the claim in 2.2.3 now follows from the fact that it is then a direct sum of the matrix algebras given by Cartan's results.

In the case of division algebras, as $R, C \subset H$, it suffices that we obtain the general form for the quaternion valued monochromatic functions,

because the real or complex ones are then obtained by restriction (and complexification). Further the knowledge of $N(\lambda)$ for real and complex monochromatic λ's gives $N(f)$ for general hyper-complex functions f, according to (2.6).

2.3 MONOCHROMATIC QUATERNION-VALUED FUNCTIONS

Let us introduce the quaternionic units i_1, i_2, i_3 given by the multiplication rules

$$i_1 i_2 = i_3, \qquad i_2 i_3 = i_1, \qquad i_3 i_1 = i_2$$

$$i_j i_k = -i_k i_j, \qquad k \neq j,$$

$$i_k^2 = -1, \qquad j,k = 1,2,3.$$

We shall also introduce the notation $(\varphi,\psi) \in \mathfrak{m}$ to mean that $\varphi \in \mathfrak{m}$, $\psi \in \mathfrak{m}$ __and__ further $\nabla\varphi \cdot \nabla\psi \equiv 0$, which is the requirement that any algebraic combination of φ, ψ belong in $\mathfrak{m}$ again. It will also be assumed that φ and ψ are functionally independent, to rule out trivial cases.

We then have the following theorem

2.3.1 THEOREM. __Any monochromatic quaternion valued function__ F __on__ (V_4,g) __is determined by a triple of real valued functions__ (φ,f,ρ) __such that__

$$(\varphi,f+i\rho) \in \mathfrak{m},$$

__in the form__

$$F = f + \rho[i_1 G(\varphi) + i_2 H(\varphi) + i_3 P(\varphi)],$$

__where__ G, H, P __are real valued functions satisfying__ $G^2 + H^2 + P^2 \equiv 1$.

More precisely, $\mathcal{J} := i_1 G + i_2 H + i_3 P$ may be described as a section of a sphere-bundle on V_4 (see also p.258 [11]).

__Proof__ Let $F = f + i_1 g + i_2 h + i_3 p$ be a smooth quaternion-valued function $\in \mathfrak{m}$ with $g^2 + h^2 + p^2 \neq 0$.

The condition $\Delta_2 F = 0$ requires

$$(2.7) \qquad \Delta_2 f = \Delta_2 g = \Delta_2 h = \Delta_2 p = 0.$$

Since

$$(\nabla F)^2 = (\nabla f)^2 - (\nabla g)^2 - (\nabla h)^2 - (\nabla p)^2 + 2 i_1 \nabla f \cdot \nabla g + 2 i_2 \nabla f \cdot \nabla h + 2 i_3 \nabla f \cdot \nabla p,$$

the condtion $(\nabla F)^2 = 0$ implies

$$(2.8) \qquad (\nabla f)^2 = (\nabla g)^2 + (\nabla h)^2 + (\nabla p)^2 ,$$

$$(2.9) \qquad \nabla f \cdot \nabla g = \nabla f \cdot \nabla h = \nabla f \cdot \nabla p = 0.$$

The eigenvalues of a quaternion, namely the real or complex λ such that $f + i_1 g + i_2 h + i_3 p - \lambda$ is not invertible, or equivalently, such that $(f-\lambda)^2 + g^2 + h^2 + p^2 = 0$, are obviously given by $\lambda_\pm = f \pm i\rho$, where

$$(2.10) \qquad \rho = \sqrt{g^2 + h^2 + p^2} > 0.$$

From our previous analysis $\lambda_\pm \in \mathfrak{m}$, which implies, in addition to (2.7),

$$(2.11) \qquad \Delta_2 \rho = 0,$$

$$(2.12) \qquad (\nabla f)^2 = (\nabla \rho)^2,$$

$$(2.13) \qquad \nabla f \cdot \nabla \rho = 0,$$

as one obtains by specializing (2.7), (2.8) and (2.9) to the complex case.

We now consider the algebra over the reals generated by F (we have to restrict the coefficients to $\mathbb{R}$ because H is a division algebra over $\mathbb{R}$, but over $\mathbb{C}$ it is not).

The analytic functions in the complex plane generated by polynomials with real coefficients are those whose domain is symmetric about the real axis and which satisfy $\Phi(z) = \overline{\Phi(\bar{z})}$ (called __intrinsic__ functions on $\mathbb{C}$ by Rinehart p.5 [16]). If $\Phi(x+iy) = u(x,y) + iv(x,y)$ is an intrinsic entire function, and u and v its real and complex part, respectively, then, if x_0, x_1, x_2, x_3 are real,

$$(2.14) \qquad \Phi(x_0+i_1 x_1+i_2 x_2+i_3 x_3) = u(x_0,q) + \mathcal{J}v(x_0,q),$$

where $q = \sqrt{x_1^2+x_2^2+x_3^2}$ and

$$\mathcal{J} = i_1 \, x_1/q + i_2 \, x_2/q + i_3 \, x_3/q.$$

This follows directly from the observation that in $x_0 + i_1 x_1 + i_2 x_2 + i_3 x_3 = x_0 + \mathcal{J}q$ the powers of $\mathcal{J}$ behave like those of i, i.e. $\mathcal{J}^2 = -1$, $\mathcal{J}^3 = -\mathcal{J},\ldots$, plus the fact that the coefficients in the power series of Φ are real. (This is also shown in general by Rinehart (Thm.7.1 p.14 [16]) as a consequence of the fact that the intrinsic functions over H may be characterized as those that are invariant under the automorphisms or anti-isomorphisms of H.)

Using (2.14) we get

$$(2.15) \qquad \Phi(f+i_1 g+i_2 h+i_3 p) = u(f,\rho) + v(f,\rho)(i_1 G+i_2 H+i_3 P),$$

where we have set

$$(2.16) \qquad\qquad G = g/\rho, \quad H = h/\rho, \quad P = p/\rho.$$

We now show that

(2.17) $$\Delta_2 G = \Delta_2 H = \Delta_2 P = 0$$

and

(2.18) $$(\nabla G)^2 + (\nabla H)^2 + (\nabla P)^2 = 0.$$

First notice that, in view of (2.9) and (2.13),

(2.19) $$\nabla f \cdot \nabla G = \nabla f \cdot \nabla H = \nabla f \cdot \nabla P = 0.$$

Now $\Delta_2 \Phi(F) = 0$ implies

$$\Delta_2(vG) = \Delta_2(vH) = \Delta_2(vP) = 0.$$

Further $\nabla v = v_x \nabla f + v_y \nabla \rho$ and so

$$\Delta_2 v = \operatorname{div} \nabla v = v_x \Delta_2 f + v_y \Delta_2 \rho +$$
$$+ v_{xx}(\nabla f)^2 + 2 v_{xy} \nabla f \cdot \nabla \rho + v_{yy}(\nabla \rho)^2$$
$$= (v_{xx} + v_{yy})(\nabla f)^2 = 0$$

where we used (2.7), (2.11), (2.13), (2.12) and the fact that v is harmonic in (x,y).

Therefore

(2.20) $$0 = \Delta_2(vG) = v\Delta_2 G + 2\nabla v \cdot \nabla G + G\Delta_2 v$$
$$= v\Delta_2 G + 2v_x \nabla f \cdot \nabla G + 2v_y \nabla \rho \cdot \nabla G$$
$$= v\Delta_2 G + 2v_y \nabla \rho \cdot \nabla G,$$

by (2.19).

Since v is at our disposal this equation implies that $\Delta_2 G = \nabla \rho \cdot \nabla G = 0$. Indeed take the intrinsic function $\Phi(x+iy) = e^{\lambda(x+iy)} = e^{\lambda x} \cos \lambda y + i e^{\lambda x} \sin \lambda y$, $\lambda \in \mathbb{R}$. Then $v = e^{\lambda x} \sin \lambda y$ and (2.20) becomes

$$(\Delta_2 G)\sin \lambda \rho = -2\lambda(\nabla \rho \cdot \nabla G)\cos \lambda \rho.$$

Since $\rho \neq 0$ we can choose $\lambda \neq 0$ so that $\cos \lambda \rho = 1$,

which gives $\nabla\rho\cdot\nabla G = 0$, and then $\sin\lambda\rho = 1$ which gives $\Delta_2 G = 0$. Similarly for H and P, so that (2.17) is proved, together with

(2.21) $$\nabla\rho\cdot\nabla G = \nabla\rho\cdot\nabla H = \nabla\rho\cdot\nabla P = 0.$$

To prove (2.18) we first apply (2.8) to the function in (2.15), obtaining

(2.22) $$(\nabla u)^2 = (\nabla(vG))^2 + (\nabla(vH))^2 + (\nabla(vP))^2.$$

Now in view of (2.13) and (2.12)

$$(\nabla u)^2 = (u_x\nabla f + u_y\nabla\rho)^2 = (u_x^2 + u_y^2)(\nabla f)^2.$$

Similarly

$$(\nabla(vG))^2 = v^2(\nabla G)^2 + 2vG\nabla v\cdot\nabla G + G^2(\nabla v)^2$$
$$= v^2(\nabla G)^2 + G^2(v_x^2 + v_y^2)(\nabla f)^2,$$

because $\nabla v\cdot\nabla G = 0$ in view of (2.19) and (2.21).

Analogous expressions hold for H and P so that, since $u_x^2 + u_y^2 = v_x^2 + v_y^2$ by the Cauchy-Riemann equations, (2.22) becomes

$$(u_x^2+u_y^2)(\nabla f)^2 = v^2[(\nabla G)^2+(\nabla H)^2+(\nabla P)^2] + (u_x^2+u_y^2)(\nabla f)^2$$

because $G^2 + H^2 + P^2 = 1$. This gives (2.18).

We now show that there is a real valued function φ such that G, H and P are functions of φ. For this purpose we will show that necessarily

(2.23) $$(\nabla G)^2 = (\nabla H)^2 = (\nabla P)^2 = 0,$$

(2.24) $$\nabla G\cdot\nabla H = \nabla G\cdot\nabla P = \nabla H\cdot\nabla P = 0,$$

everywhere.

First we note that the group of automorphisms and anti-

automorphisms of $\mathbb{H}$, which are precisely the rotations that leave the real unit unchanged, possibly combined with reflections, preserve (2.7), (2.8), (2.9) and (2.10) and so also (2.11), (2.12) and (2.13) as well as the condition $G^2 + H^2 + P^2 = 1$. Further the intrinsic functions on $\mathbb{H}$ are invariant under this group.

Therefore we may always apply a (constant) rotation on the space of i_1, i_2, i_3 to make $G,H,P \neq 0$ at a particular point $p \in V_4$. Then it is clear that the new G, H, P will satisfy (2.23) and (2.24) if and only if so did the old ones. Suppose therefore that $G,H,P \neq 0$ at p.

Since $P = (1-G^2-H^2)^{1/2} > 0$ at p, and so in a neighbourhood of p, and $\Delta_2 P = 0$, we get by differentiation

$$2(1-G^2-H^2)(-\Delta_2 G^2 - \Delta_2 H^2) = (\nabla(G^2+H^2))^2$$

i.e.

$$2(G^2+H^2-1)[2G\Delta_2 G + 2(\nabla G)^2 + 2H\Delta_2 H + 2(\nabla H)^2] =$$
$$= 4G^2(\nabla G)^2 + 4H^2(\nabla H)^2 + 8HG\nabla H \cdot \nabla G.$$

Using (2.7) and simplifying

(2.25) $$(1-G^2)(\nabla H)^2 + (1-H^2)(\nabla G)^2 = -2HG\nabla H \cdot \nabla G.$$

Now if ∇H and ∇G are space-like, Schwarz's inequality $|\nabla H \cdot \nabla G| \leq |\nabla H| \cdot |\nabla G|$ applies. Therefore taking absolute values in (2.25) we get, since $(\nabla H)^2$ and $(\nabla G)^2$ have same sign and $1-G^2 > 0$, $1-H^2 > 0$,

$$(1-G^2)|\nabla H|^2 + (1-H^2)|\nabla G|^2 \leq 2|HG| \cdot |\nabla H| \cdot |\nabla G|$$

i.e.

(2.26) $$(1-G^2)|\nabla H|^2 - 2|HG| \cdot |\nabla H| \cdot |\nabla G| + (1-H^2)|\nabla G|^2 \leq 0.$$

The determinant of the matrix of this quadratic form in

$(|\nabla G|, |\nabla H|)$ is $(1-G^2)(1-H^2) - H^2 G^2 = 1 - G^2 - H^2 = P^2 > 0$ and its trace is

$$2 - G^2 - H^2 = 1 + P^2 > 1,$$

so its eigenvalues are positive. This implies in (2.26) that $|\nabla H| = |\nabla G| = 0$, i.e., $(\nabla H)^2 = (\nabla G)^2 = 0$ and so also $(\nabla P)^2 = 0$ by (2.18) and $\nabla H \cdot \nabla G = 0$ by (2.25). Interchanging the roles of G, H, P in (2.25) we get now the remaining equations in (2.24).

Therefore (2.23) and (2.24) hold when two of the vectors ∇G, ∇H, ∇P are space-like, since this property is preserved under the small rotation that may be needed to make $G, H, P \neq 0$ at a given $p \in V_4$.

We need consider therefore only the case when just one of them is space-like, say ∇H, one is time-like, say ∇P, and the third one ∇G is time-like or isotropic. Now the small rotation that may be necessary to make $G, H, P \neq 0$ at a given $p \in V_4$, may change the character of ∇G if it is isotropic. If it becomes space-like, we are back in the previous case, so we need consider only the remaining case.

Clearly then the subspace determined by ∇G and ∇H cuts the light-cone and so we may rotate ∇H and ∇G by the above procedure till ∇H touches the light cone, at the point p, while leaving P and ∇P unchanged. Since ∇H becomes isotropic, ∇G must then get space-like so as to compensate $(\nabla P)^2$ in (2.18). Therefore by continuity, shortly before ∇H touches the light-cone both ∇H and ∇G will be space-like and since $P \neq 0$, this then reduces the problem to the previous case. This proves (2.23) and (2.24) hold everywhere.

To complete the proof we only need observe that any two real isotropic vectors which are orthogonal in (V_4, g) are necessarily

paralel. Hence if $\nabla H \neq 0$, necessarily $\nabla P = \mu \nabla H$, $\nabla G = \lambda \nabla H$, with μ, λ real functions, and therefore $P = P(H)$, $G = G(H)$ locally, as claimed. In view of (2.19), (2.21) and (2.23), $(H, f+i\rho) \in \mathfrak{m}$ and this ends the proof of Theorem 2.3.1.

2.3.1 MAXIMAL MONOCHROMATIC ALGEBRAS

A monochromatic algebra is called <u>maximal monochromatic</u> if it is not a proper subalgebra of a monochromatic algebra.

The importance of maximal monochromatic algebras in our context is obvious, in particular with respect to the question of singular sets.

The main result in this respect is

2.3.2 THEOREM. <u>The maximal monochromatic C^2 algebras in (V_4, g) are precisely those generated by a single pair $(\varphi, f+i\rho) \in \mathfrak{m}$, with φ, f, ρ real, and are constituted by C^2-functions of the form</u>

$$(2.27) \qquad \zeta(f,\rho,\varphi) + \eta(f,\rho,\varphi)\{i_1 G(\varphi) + i_2 H(\varphi) + i_3 P(\varphi)\}$$

<u>in the quaternionic case, and</u>

$$(2.28) \qquad \zeta(f,\rho,\varphi) + i\eta(f,\rho,\varphi),$$

<u>in the complex case, where, for each fixed φ, $\zeta + i\eta$ is an intrinsic analytic or (anti-analytic) function of $f + i\rho$, the C^2-dependence on φ is arbitrary and $G^2 + H^2 + P^2 \equiv 1$, with $G, H, P \in C^2$ real, but otherwise arbitrary. In the complex case, non-intrinsic functions are allowed.</u>

<u>Proof</u>.

1. We first prove that the most general quaternion-valued mono-
   chromatic C^2 function of a pair $(\varphi, f+i\rho) \in \mathfrak{m}$, has the
form (2.27). Indeed let $F(f,\rho,\varphi) \in \mathfrak{m}$ be a C^2 quaternion valued
function.

By Theorem 2.3.1 it has the expression $F = \zeta + \eta\{i_1\Gamma_2(\Phi) +$
$+ i_3\Gamma_3(\Phi)\}$ with $(\Phi, \zeta+i\eta) \in \mathfrak{m}$, $\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2 \equiv 1$, Γ_i real valued.

By assumption ζ, η and Φ are functions of (f,ρ,φ). Since
$\zeta + i\eta \in \mathfrak{m}$ and $\Phi \in \mathfrak{m}$, it suffices therefore that we analyze the
problem for these particular functions, and this reduces the problem
to the case when F is a real or complex-valued function of the
pair $(\varphi, f+i\rho) \in \mathfrak{m}$.

In this case, using the properties of this pair, we get
from $\nabla F = F_\varphi \nabla\varphi + F_f \nabla f + F_\rho \nabla\rho$, $(\nabla F)^2 = (F_f^2 + F_\rho^2)(\nabla f)^2$.

As f is independent on φ then $(\nabla f)^2 \neq 0$, as remarked
earlier. Therefore necessarily $F_f^2 + F_\rho^2 = (F_f + iF_\rho)(F_f - iF_\rho) = 0$
i.e. F must be an analytic or anti-analytic function of $f + i\rho$.
No additional restriction is placed on F as a function of φ.

The condition $\Delta_2 F = 0$ is then automatically satisfied
because

$$\Delta_2 F = F_\varphi \Delta_2\varphi + F_\rho \Delta_2\rho + F_f \Delta_2 f + 2F_{\varphi\rho}\nabla\varphi\cdot\nabla\rho +$$

$$+ 2F_{f\varphi}\nabla\varphi\cdot\nabla f + 2F_{\rho f}\nabla f\cdot\nabla\rho + F_{\varphi\varphi}(\nabla\varphi)^2 +$$

$$+ F_{\rho\rho}(\nabla\rho)^2 + F_{ff}(\nabla f)^2 =$$

$$= (F_{ff} + F_{\rho\rho})(\nabla f)^2 = 0,$$

as F is harmonic in (f,ρ).

Therefore $\zeta(f,\rho,\varphi)$ and $\eta(f,\rho,\varphi)$ satisfy the stated con-
ditions, and so do the Γ_i's. Since Γ_i are real-valued they are

therefore constant as functions of $f+i\rho$, i.e. depend on φ only, as claimed.

The complex case is obtained by specializing $H \equiv P \equiv 0$, $G \equiv 1$, and, by complexification, non-intrinsic functions are obtained.

2. It is easy to check that (2.27) belongs in the real algebra generated by φ and $f+i\rho$ (or by φ and $f-i\rho$ if it is anti-analytic in $f+i\rho$). Similarly (2.28) belongs in the complex algebra generated by φ and $f+i\rho$ (or by φ and $f-i\rho$, as before). The same applies trivially to functions of φ only.

3. We prove now the maximality property. In any of the two cases above let a monochromatic algebra contain $(\varphi, f+i\rho) \in \mathfrak{m}$ and a third function F. By Theorem 2.3.1 this function is given in terms of a pair $(\tilde{\varphi}, \tilde{f}+i\tilde{\rho}) \in \mathfrak{m}$. If the function is trivial, it is expressible as a function of $(\varphi, f+i\rho)$ too. If not, it depends non-trivially on at least one of $\tilde{\varphi}$, $\tilde{f}+i\tilde{\rho}$. In that case since the functions lie in a monochromatic algebra, the corresponding $\nabla\tilde{\varphi}$ and/or $\nabla(\tilde{f}+i\tilde{\rho})$ must be ortoghonal to both $\nabla\varphi$ and $\nabla(f+i\rho)$. (Notice the later commute with ∇F, in the scalar product).

If $\nabla\tilde{\varphi}\cdot\nabla\varphi = 0$ locally, then necessarily $\tilde{\varphi} = \tilde{\varphi}(\varphi)$, as both are real monochromatic and so $\tilde{\varphi}$ lies in the algebra of φ.

If $\nabla(\tilde{f}+i\tilde{\rho})\cdot\nabla(f+i\rho) = 0$ and $\nabla(\tilde{f}+i\tilde{\rho})\cdot\nabla\varphi = 0$ then we have $\nabla(\tilde{f}+i\tilde{\rho}) = \alpha(x)\nabla\varphi + \beta(x)\nabla(f+i\rho)$ with α, β complex functions on V_4, by the lemma below. This implies that $\tilde{f} + i\tilde{\rho} \in \mathfrak{m}$ is (locally) a function of (f, ρ, φ) and so by Theorem 2.31 and item 2., belongs in the algebra generated by $(\varphi, f+i\rho)$.

The theorem is proved once we show the

2.3.3 <u>Lemma</u>. If two isotropic vectors v_1, v_2 are orthogonal to a real isotropic vector v_3 in Minkowski space then either v_1, v_2, v_3 or v_1, $\bar{v}_2$, v_3 are linearly dependent. If v_2 is orthogonal to v_1 then the first case holds.

<u>Proof</u>. We may assume the real vector is $(1,1,0,0)$, the signature being $(-1,1,1,1)$. Since linear combinations of v_1, v_2 with v_3 preserves their stated properties we can make the first components of v_1, v_2 into zero by adding a convenient multiple of v_3. But then since they are orthogonal to the real vector also their second components are zero. So they are of the form $(0,0,a,b)$ $(0,0,c,d)$, and by isotropy $a^2+b^2 = c^2+d^2 = 0$ i.e. $b = \pm ia$, $d = \pm ic$. Hence they are multiples of $(0,0,1,\pm i)$ and $(0,0,1,\pm i)$. For any choice of sign either these vectors are equal or one is equal to the complex-conjugate of the other. They can be orthogonal only in the first case. The result follows.

Clearly the result holds pointwise in a (V_4,g) because we can always make $g_{ij} = \text{diag}(-1,1,1,1)$ at a fixed point p.

2.3.4 REMARKS. 1. In view of Theorem 2.3.2 all functions of $f+i\rho$ in the same algebra must be simultaneously analytic or anti-analytic in the same connected regions. For simplicity we refer to them as analytic, bearing in mind these two possibilities.

2. It is clear that $(\varphi, f+i\rho) \in \mathfrak{m}$ implies that also $(\varphi, f-i\rho) \in \mathfrak{m}$. However since the analytic functions in $f-i\rho$ are precisely the anti-analytic functions in $f+i\rho$ and vice-versa, we will not consider those two pairs as distinct, because they generate the same maximal monochromatic algebras, according to Theorem 2.3.2.

2.4 GENERAL FORM OF SINGULAR SETS. PHYSICAL IDENTIFICATIONS

According to the above results the most general form for singular sets N of monochromatic complex or quaternionic functions is given by the equations

$$(2.29) \quad \begin{aligned} \xi(f, \rho, \varphi) &= 0 \\ \eta(f, \rho, \varphi) &= 0, \qquad (\varphi, f+i\rho) \in \mathfrak{m}. \end{aligned}$$

Although N is locally at least 2-dimensional we have now the possiblity to locate a <u>higher-order zero on a bicharacteristic line</u>.

For instance the singular set of $\varphi \cdot (f+i\rho)$ is the union of the 2-dimensional set $f = \rho = 0$ and the 3-dimensional set $\varphi = 0$, and, since $(\nabla \varphi)^2 = 0$, their intersection $f = \rho = \varphi = 0$ is a bicharacteristic line carrying an isolated zero of higher-order. The corresponding phase function has a higher order singularity located at a single point in three space, moving with the speed of light along the singular line $f = \rho = 0$, acompanied by the wave front singularity $\varphi = 0$.

2.4.1 PHOTON, NODAL LINES, MONOPOLES

We shall see later on that a typical case in cartesian coordinates (x, y, z, t) in $\mathbb{R}^4$ is given by $f+i\rho = y+iz$ and $\varphi = f(r)-t$, where $r = (x^2+y^2+z^2)^{1/2}$ and f is a monotonic function of the radius r.

In this case, for the function

$$(2.30) \qquad (y+iz)(f(r)-t)$$

the singular set consists of a spherical wave front in 3-space moving with the speed of light and cutting the singular x axis $y = z = 0$ at a single point in the positive semi-axis $x \geq 0$, where therefore lies a higher order singularity.

This higher order singular point, piloting a lower order singular spherical wave, along a lower order singular line is now liable to represent the <u>photon</u>, conceived as a moving point singularity carrying energy, in agreement with the experimentally observed corpuscular behavior of the photon (One can experimentally detect the arrival of a single photon at a metallic plate, and obtain indirect pictures of its trajectory in cloud chambers).

On the other hand the weaker singularity carried by the spherical wave front $f(r)-t = 0$ is responsible for the diffraction patterns in the typical slit experiments, according to Huyghens' law of propagation of singularities (eikonal equation), and so accounts for the experimentally observed wave nature of the photon.

In this way, the purely analytical characterization of the maximal monochromatic algebras leads us, unequivocally, to the correct conclusions as regards the physical nature of the photon and expresses its dual wave-corpuscular nature as a simple mathematical fact.

The line $y = z = 0$ in 3-space carries a singularity too but this is a standing one, independent on time, and therefore its presence is detected through different effects.

Actually this line is a so called <u>nodal line</u> of the wave function $\Psi \equiv y+iz$ (p.67 [5], [7]) or a <u>dislocation line</u> of the planes of constant phase of Ψ ([1], [10]).

Around this line occur vortices of the flux $\nabla \ell n \Psi$ of the phase function (when the circulation of this flux around the nodal

line is non-zero), described in detail by Hirschfelder ([7]). Al-
ternatively Dirac found these nodal lines when considering singu-
larities of wave functions, upon imposing the only requirement
that the complex-valued function Ψ (in our example equal to
y+iz) be single-valued and smooth, but not necessarily with
single-valued argument, and then quantized them in terms of the
winding number of the vector field (Re Ψ, Im Ψ) along a closed
curve around the line. He then found that one could remove the
non-0 circulation by means of a gauge transformation of the second
kind, and that the electromagnetic vector potential associated
with this transformation (cf. 1.7.1, Chapter I) was precisely the
same electromagnetic potential produced by a string of magnetic
dipoles starting at a given point and going to infinity. When
removed, after the gauge transformation, remains only the electro-
magnetic potential produced by a magnetic monopole at the initial
point. He then equated the effect of the circulation around the
nodal line in the original gauge to the effect of a monopole in
the new gauge.

His quantization by the winding number is actually just a
special case of our general quantization theorem in Chapter I,
and his gauge interpretation is thus a concrete examplification
of the meaning of the analysis given in 1.7.1.

We shall explain in detail the results of Dirac regarding
strings of manetic dipoles and monopoles in Chapter VII where we
shall need to make specific computations with them.

We can however advance that the effect of the time in-
dependent nodal line is thus a purely magnetic one. This fits
well with the strictly complex character of the monochromatic
wave $f+i\rho$ that defines the nodal line $(f = \rho = 0)$.

2.4.2 NEUTRINO, SCREW NODAL LINES

The variety of types of singular sets defined by (2.29) is very great, as exemplified by Nye and Berry ([10]).

Besides the singular set that we previously identified with the photon (spherical wave front plus a nodal line) there is also the remarkable singular set of the monochromatic wave constructed out of $\varphi = f(r)-t$ and $f+i\rho = y+iz$ in $\mathbb{R}^4$, by the following sum

$$(2.31) \qquad \epsilon e^{i\omega[f(r)-t]} - (y+iz), \qquad \epsilon > 0.$$

Its singular set is given by

$$(2.32) \qquad \begin{array}{l} y = \epsilon \cos \omega[f(r)-t] \\ \\ z = \epsilon \sin \omega[f(r)-t] \end{array}$$

which represents an helicoidal line lying on the cylinder $y^2+z^2=\epsilon^2$, and moving with (variable) speed of light along its tangent direction at each of its points. (For commodity the reader may assume $f(r) = r$ just to get a better visualization: the speed is then constant and the helicoid has constant step.)

Taking $y-iz$ instead, we get a screw motion with opposite handedness.

The singular set is thus a moving screw in 3-space that can be right or left handed, and may carry the energy associated with a quantum jump, as shown in Chapter I.

It seems therefore that a monochromatic wave like this can represent appropriately a right or left-handed _neutrino_, concretely identified with its singular set.

It has then quite distinct properties from those associated

with a photon. For it is given by an infinitely long moving right
or left-handed helicoidal line in 3-space while the photon is given
by a point piloting a spherical wave.

In particular if the singular screw line (2.32) is asso-
ciated with an elementary state u and carries energy E in the
manner described in Chapter I (Theorem 1.6.3), it also carries the
angular momentum $\epsilon^2 E\omega/c^2$ directed along the x-axis, (in the given
referential). Hence the neutrino carries angular momentum while
the photon does not.

On the other hand, according to this description, the neu-
trino should not have (primary) diffraction patterns as the photon
does, which would explain why it is so difficult to detect.

The infinitely long screw line seems to agree, in principle,
with the experimentally estimated fact that the neutrino has an
extremely long absorption path, of the order of 100 light years in
ordinary condensed media (p.58 [8]) or 1000 light years in water
(pp.295,298 [2]).

2.5 SPINOR AND TWISTOR DESCRIPTION

2.5.1 SPINORS

If (x^o,x^1,x^2,x^3) is a vector in Minkowski space we may
associate with it the 2x2 hermitean matrix

$$(2.33) \qquad x = (x^o,x^1,x^2,x^3) \to X = \frac{1}{\sqrt{2}}\begin{pmatrix} x^o+x^1 & x^2+ix^3 \\ x^2-ix^3 & x^o-x^1 \end{pmatrix}.$$

This is obviously a linear isomorphism of $\mathbb{R}^4$ with the

(real) space of 2x2 hermitean matrices. Direct computation shows then that when the vector is acted upon by a proper Lorentz transformation L, the associated hermitean matrix undergoes multiplication by a 2x2 complex unimodular matrix on the left and by its transpose conjugate on the right. The unimodular matrix is uniquely determined by L, except for sign of course. This correspondence gives an isomorphism between the group SL(2,$\mathbb{C}$) of 2x2 unimodular complex matrices and the twofold universal covering of the connected subgroup of the Lorentz group O(1,3).

A spin vector is then defined as an element of $\mathbb{C}^2$, which is acted upon by the unimodular matrix associated with L, while L acts on $\mathbb{R}^4$.

A spin vector field on (V_4, g) is defined locally by taking a smooth moving orthonormal frame, i.e., such that on each point the metric has the standard form $(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$, and assigning, in a smooth way, a spin vector at each point of the corresponding Minkowskian tangent space.

Since in the correspondence (2.33) we have

$$(2.34) \qquad (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = \tfrac{1}{2} \det X,$$

then any (real) isotropic vector corresponds to an hermitean matrix given by the tensor product of a spin vector ω^A with its complex conjugate $\bar{\omega}^{A'}$ (the vector is then called future-pointing), or with $-\bar{\omega}^{A'}$ (the vector is called past-pointing), (A,A'=1,2).

The spin vector ω^A is determined by the vector, up to a factor $e^{i\theta}$, θ real, obviously. This extra degree of freedom relates to a possible _polarization_ of the objects involved (see [11], p.152 [13]).

Clearly complex isotropic vectors are given by the tensor product $\omega^A \pi^{A'}$ of spin vectors.

2.5.2 Finally we remark that two isotropic vectors are orthogonal if and only if either their first or their second associated spin vectors are parallel.

Indeed let, in matrix form,

$$u = \begin{pmatrix} a \\ b \end{pmatrix} (c,d) \qquad , \qquad v = \begin{pmatrix} e \\ f \end{pmatrix} (g,h) \ .$$

Using polarization of bilinear forms and (2.34), we get

$$
\begin{aligned}
u^i v_i &= \tfrac{1}{4} \left[(u^i + v^i)(u_i + v_i) - (u^i - v^i)(u_i - v_i) \right] \\
&= \tfrac{1}{4} \det \left[\begin{pmatrix} a \\ b \end{pmatrix} (c,d) + \begin{pmatrix} e \\ f \end{pmatrix} (g,h) \right] \\
&\quad - \tfrac{1}{4} \det \left[\begin{pmatrix} a \\ b \end{pmatrix} (c,d) - \begin{pmatrix} e \\ f \end{pmatrix} (g,h) \right] \\
&= \tfrac{1}{2} \ (af - be)(ch - dg).
\end{aligned}
$$

Hence $u^i v_i = 0$ iff either $af = be$ or $ch = dg$, as claimed.

2.5.3 TWISTORS AND MAXIMAL MONOCHROMATIC ALGEBRAS

Consider the generators $(\varphi, f + i\rho) \in \mathfrak{m}$ of a maximal monochromatic algebra.

The vector fields $\nabla\varphi$ and $\nabla(f + i\rho)$ are, respectively, real and complex isotropic fields, mutually orthogonal on (V_4, g). By the previous analysis $\nabla\varphi$ is given by a spin vector field ω^A in the spinor form

$$(2.35) \qquad \nabla\varphi = \omega^A \bar{\omega}^{A'} \qquad (\text{or} \quad -\omega^A \bar{\omega}^{A'})$$

and, since $\nabla(f+i\rho)$ is isotropic and orthogonal to $\nabla\varphi$,

(2.36) $$\nabla(f+i\rho) = \omega^A \bar\pi^{A'} \qquad (\text{or } \pi^A \bar\omega^{A'}),$$

where π^A is another spin vector field.

Consequently the pair $(\nabla\varphi, \nabla(f+i\rho))$ of vector fields is completely determined by the ordered pair of spin vector fields

(2.37) $$(\omega^A, \pi^{A'}),$$

but we have a fourfold map here since we have altogether 4 different ways of building the vector fields according to (2.35) and (2.36), out of the ordered pair (2.37). (See in this connection p.45 [14].)

The correspondence (2.33), extended to complex vectors x, shows that the second choice in (2.35) reverses $\nabla\varphi$ from, say, a future-pointing into a past-pointing isotropic vector while in (2.36) it chooses the complex-conjugate $\nabla(f-i\rho)$ instead of $\nabla(f+i\rho)$, reversing the roles of analytic and anti-analytic functions, which means inversion of handedness.

Choosing locally a given time orientation and a given handedness, corresponds to a particular choice of the assignements in (2.35) and (2.36).

The ordered pair (2.37) of spin vectors at a point in (V_4, g) is called a (local) <u>twistor</u> and the corresponding field a (local) <u>twistor field</u> (pp. 373,374 [15]).

From the twistor field we determine the real and complex vector fields by (2.35) and (2.36) and (2.33), which, upon integration yield an (equivalent) pair $(\varphi, f+i\rho)$.

This means <u>we can completely characterize a maximal mono-chromatic algebra</u> (<u>and consequently the light quanta it represents</u>) <u>in terms of a twistor field with divergence free associated vector fields</u>. (The new representation is even richer as it has built in an extra degree of freedom, namely, <u>polarization</u>, due to the factor $e^{i\theta}$ mentioned before.)

This result, showing that what we have identified as light quanta are indeed given by twistor fields, consubstantiates the belief of Penrose that twistors are the appropriate tool to describe zero rest mass particles and to effect the connection of classical general relativity with quantum mechanics (p.403 [15]).

2.5.4 CLASSICAL INTERPRETATION, HELICITY AND SPIN

Penrose's definition of twistors in Minkowski space (p.278 [15]) starts with the fact that if a zero rest mass particle has momentum p^a and angular momentum M^{ab} ($= -M^{ba}$), ($a,b = 0,1,2,3$) with respect to some origin then, say, in spinor form

$$(2.38) \qquad p_{AA'} = \bar{\pi}_A \, \pi_{A'} \, ,$$

$$(2.39) \qquad M^{AA'BB'} = i\omega^{(A}\bar{\pi}^{B)} \, \epsilon^{A'B'} - i\epsilon^{AB} \, \bar{\omega}^{(A'}\bar{\pi}^{B')} \, ,$$

where $Z^{\alpha} = (\omega^A, \pi_{A'})$ is a twistor. (Here brackets mean symmetrization and $\epsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the spinor index raising operator.) As before, the twistor is only defined up to a phase change $e^{i\theta}$, θ real.

The vector p_a is an eigenvector of

$$(2.40) \qquad \frac{1}{2} \, e_{abcd} \, p^b \, M^{cd} = sp_a \, .$$

(Here e_{abcd} is antisymmetric in all indices and $e_{0123} = 1$.)

The eigenvalue s is the <u>helicity</u> of the particle and $|s|$ its spin. It is also given by

$$(2.41) \qquad 2s = Z^{\alpha} \bar{Z}_{\alpha} = \omega^{A} \bar{\pi}_{A} + \pi_{A'} \bar{\omega}^{A'} \, ,$$

where, by definition, $\bar{Z}_{\alpha} = (\bar{\pi}_{A}, \bar{\omega}^{A'})$ is the complex conjugate of Z^{α} and the above expression their scalar product.

If $Z^{\alpha} \bar{Z}_{\alpha} = 0$ and $Z^{\alpha} \neq 0$ there is a single line Z of points with respect to which $M^{ab} = 0$, and it is parallel to p^{a}, therefore isotropic. If X^{α} is another null twistor (i.e. such that $X^{\alpha} \bar{X}_{\alpha} = 0$) and $X^{\alpha} \neq 0$, with associated isotropic line X, then X and Z meet if and only if $X^{\alpha} \bar{Z}_{\alpha} = 0$ (p.350 [12]). The isotropic line Z, which describes the twistor Z^{α} up to a factor, is thus completely characterized by the congruence constituted of isotropic lines that meet Z, i.e. by the family

$$(2.42) \qquad \{X \mid X^{\alpha} \bar{X}_{\alpha} = 0, \ X^{\alpha} \bar{Z}_{\alpha} = 0\}.$$

If $Z^{\alpha} \bar{Z}_{\alpha} \neq 0$ we can again describe Z^{α} by the congruence of isotropic lines that satisfy (2.42), but now there is no isotropic line associated with Z^{α}, and the lines associated with X^{α} twist about one another (right-handedly when s > 0, left-handedly when s < 0) and never quite meet. A complete analytical description is given on p.351 [12] and a nice pictorial description is given on p.291 [15]. The computation is given for $Z_{\alpha} = (\varepsilon, 0, 1, 0)$, ε real $\neq 0$, which is the generic case up to a general Poincaré transformation.

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CHAPTER III

MASSIVE QUANTA

3.1 REPRESENTATION OF MASSIVE QUANTA

The trajectory γ of a massive quanta is, by definition, a time-like path. A single time-like trajectory cannot carry an isolated singularity of a distributional solution of the wave equation $\Delta_2 f = 0$. (p. 639 [3]).

Therefore we cannot expect to represent massive quanta in a way similar to the way we represent light quanta, namely by means of the singular set of an elementary state.

Nevertheless it is clear that we can describe a time-like curve as the locus of a generalized curve constructed out of families of bicharacteristic curves. This fact will provide us with the appropriate way to achieve the representation of massive quanta.

For that consider two distinct maximal monochromatic algebras h and h' in (V_4, g). Let S and S' denote all possible singular sets of functions in h and h', respectively.

Let γ be a time-like continuous curve on a two-dimensional submanifold spanned by bicharacteristic curves which are the loci of isolated higher-order singularities lying in elements of S and S'.

The curve γ can then be locally approximated in the C^o topology by continuous broken lines $\beta^{(n)}$ each of whose segments $\beta_i^{(n)}$ are segments of the above bicharacteristics.

If we restrict γ to a compact set K, which is the closure of an open set in V_4, then since all causal curves lying in K are uniformly Lipschitz continuous, the C^o-converging sequence $\beta^{(n)} \to \gamma$ in K will be uniformly bounded in length, hence in norm. Therefore a subsequence will be weak-* convergent to a generalized curve having γ as its geometrical locus.

As there are only two possible directions v, v' at each point of γ we end up with a probability measure with two values p, p' associated with these directions, at each point of γ. Let γ be given by $x = x(\tau)$, τ real.

This generalized curve in V_4 induces in turn, a generalized curve Γ on $h \cup h' \subset C$, as follows. Let $f_\tau \in h$, $f'_\tau \in h'$, be the monochromatic states whose singular set define the higher-order singularity on the bicharacteristic curves through $x(\tau) \in \gamma$, mentioned before. Γ is then defined as the mapping

$$(3.1) \qquad \tau \longmapsto (f_\tau, p; f'_\tau, p'),$$

where $p = p(x(\tau))$, $p' = p'(x(\tau))$ are the probabilities originally associated with v and v'.

This generalized curve is thus a statistical superposition of elementary states describing light-quanta, and whose higher-order singular set intersect along γ, and thus yield a singularity moving with speed less than the speed of light. This singularity represents thus a massive quanta.

In summary the presence of a massive quanta in (V_4, g) is described by means of two distinct maximal monochromatic algebras.

The _envelope state_ ϕ of this generalized curve is then defined by the equations

(3.2)
$$\Phi\Big|_{t=\tau} = pf_t + p'f'_t\Big|_{t=\tau} \ ,$$

(3.3)
$$\nabla\Phi\Big|_{t=\tau} = \left(p\nabla f_t + p'\nabla f'_t\right)\Big|_{t=\tau} \ ,$$

provided such a solution exists.

If we had originally an elementary state u superposed to the f's, the envelope state $u\Phi$ would be defined instead by

$$u\Phi\Big|_{t=\tau} = puf_t + p'uf'_t\Big|_{t=\tau} \ ,$$

$$\nabla(u\Phi)\Big|_{t=\tau} = \left(p\nabla(uf_t) + p'\nabla(uf'_\tau)\right)\Big|_{t=\tau} \ .$$

However since $u \neq 0$ these equations are equivalent to the previous one as one sees expanding the gradient.

We can think that, just as the locus γ is what is seen *on the average* of the generalized curve, so Φ is what is seen on the average as the result of the statistical superposition of the light quanta in Γ (compare with the description in §4.7 p.120 [4]).

Each separate curve

(3.4)
$$\tau \xmapsto{\ \psi\ } f_\tau \in h \ ,$$

(3.5)
$$\tau' \xmapsto{\ \psi'\ } f'_\tau \in h' \ ,$$

defines a classical state, according to 1.3, and, when p and p' independ on τ, the generalized curve Γ may be thought of as a statistical superposition of these classical states with probabilities p and p'. These are then the hidden classical states building up the observed state $\Phi = p\psi + p'\psi'$.

3.2   QUANTUM-MECHANICAL POTENTIAL

Assume now that we have an elementary state u , with ∇u
defining a time-like geodesic flow and that Φ is constant along
any of the corresponding geodesics (i.e. Φ is a stationary state
with respect to the time defined by ∇u).

Take local Gaussian coordinates so that $u = u(t)$ and
$ds^2 = dt^2 - \tilde{g}_{ij} \, dx^i dx^j$, $(i,j = 1,2,3)$.

As u is elementary we have

$$0 = \text{div} \ \frac{\nabla u}{u} = \left(\frac{u_t}{u}\right)_t$$

i.e. $u = c \, e^{imt}$.

From
$$\Delta_2 u = u\left(\frac{\nabla u}{u}\right)^2 = -m^2 u ,$$

$$\nabla u \cdot \nabla \Phi = 0 ,$$

$$\Delta_2 \Phi = -\tilde{\Delta}_2 \Phi ,$$

and
$$\Delta_2(u\Phi) = \Phi \Delta_2 u + 2\nabla u \cdot \nabla \Phi + u\Delta_2 \Phi ,$$

we get

(3.6)
$$(\Delta_2 + m^2 + Q)u\Phi = 0,$$

where

(3.7)
$$Q = \frac{\tilde{\Delta}_2 \Phi}{\Phi} .$$

(Here $\tilde{\Delta}_2$ is the Laplace-Beltrami operator corresponding to the
metric $\tilde{g}_{ij}$ in the space sections $t = \text{const.}$)

Equation (3.6) shows therefore that the envelope function
$u\Phi$ satisfies a Klein-Gordon equation with the quantum-mechanical
potential Q (p.50 [2], [1], [5]).

3.3 EXAMPLE

For example consider for simplicity two-dimensional Minkowski space (t,x), and on it a generalized curve with the t-axis as locus, and the measures $p = p' = 1/2$ attached to the unit vectors $1/\sqrt{2}(1,1)$ and $1/\sqrt{2}(1,-1)$.

Take

$$f_\tau(t,x) = e^{i(t-\tau-x)} - 1,$$

$$f'_\tau(t,x) = e^{-i(t-\tau+x)} - 1,$$

which are curves of monochromatic states, that have zeros on the t-axis at the time $t = \tau$.

Their envelope function is

$$\Phi(t,x) = e^{-ix} - 1.$$

Indeed Φ satisfies (3.2) by construction, and satisfies (3.3) because $\nabla\Phi\big|_{t=\tau} = (0, -ie^{-ix})$, which agrees with

$$\nabla \frac{1}{2} (e^{i(t-\tau-x)} + e^{-i(t-\tau+x)} - 2) = (-e^{-ix}\sin(t-\tau), -ie^{-ix}\cos(t-\tau),$$

when we put $t = \tau$.

The *envelope* function has a line of zeros on the t-axis (the other zeros could be removed by eliminating periodicity by the procedure shown in 2.1, Chapter II).

The original family of monochromatic functions represent a light quanta moving forward and backward, respectively, both located at $x = 0$. Their statistical superposition produces the envelope state Φ which has a standing singularity at $x = 0$ and represents therefore a stationary massive quanta at the origin.

The corresponding quantum-mechanical potential is

$$Q = 1/(1-e^{ix}).$$

More concrete examples will arise naturally along our analysis. Nevertheless, despite its simplicity, this example is typical.

3.4 COMMENTS 1. It is clear that just as forces arise as consequence of non-geodesic motion in point motion, so the quantum-mechanical forces, as expressed by the potential Q, arise in the envelope procedure because by construction this procedure preserves the value of the functions and their gradients, but not necessarily their second derivatives.

2. The analysis in this chapter, although simple, illustrates the usefulness of the constructive approach. In particular it clarifies the meaning of statistical superpositions as a natural consequence of L.C. Young's result on the need of generalized curves, via the envelope procedure, and explains, in principle, the origin of quantum-mechanical forces.

3.5 DISTINCTION OF MAXIMAL MONOCHROMATIC ALGEBRAS

LEMMA. The maximal monochromatic algebras h and h', with generators $(f, \varphi+i\psi)$ and $(\tilde{f}, \tilde{\varphi}+i\tilde{\psi})$ respectively, are distinct if and only if

$$(3.8) \qquad\qquad \nabla f \cdot \nabla \tilde{f} \not\equiv 0.$$

Proof Indeed if $\nabla f \cdot \nabla \tilde{f} \equiv 0$ then necessarily f is a function of $\tilde{f}$ and therefore

$$\nabla f \cdot \nabla (\tilde{\varphi}+i\tilde{\psi}) = 0,$$

besides $\nabla f \cdot \nabla (\varphi+i\psi) = 0$.

But then Lemma 2.3.3 tells us that either ∇f, $\nabla(\varphi+i\psi)$,

$\nabla(\widetilde{\varphi}+i\widetilde{\psi})$ or ∇f, $\nabla(\varphi+i\psi)$, $\nabla(\widetilde{\varphi}-i\widetilde{\psi})$ are linearly dependent. In any case $\widetilde{\varphi}+i\widetilde{\psi}$ is a function of $(f, \varphi+i\psi)$ and since so is $\widetilde{f}$, the two algebras are the same (here again we are including the possibility of having to consider analytic functions in one and anti-analytic in the other).

Conversely assume the algebras are not distinct. Then by Theorem 2.3.2 $\widetilde{f}$ is a function of $(f, \varphi+i\psi)$, analytic or anti-analytic in $\varphi+i\psi$, and so $\nabla f \cdot \nabla \widetilde{f} \equiv 0$.

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CHAPTER IV

ELEMENTARY GEOMETRIES

4.1 FORMULATION OF THE PROBLEM

We are now ready to start our attack on the fundamental problem of identification of the nature of particles.

As mentioned in the introduction, the presence of a particle in the differentiable manifold V_4 is characterized by the fact that light quanta can be emitted and can be absorbed (by the particle).

According to our analysis in Chapter II, light quanta are described by maximal monochromatic algebras with respect to some Lorentz metric in V_4. As emission and absorption correspond to distinct algebras, we see that the presence of a particle in V_4 is characterized by a Lorentz metric g in V_4 and at least two monochromatic pairs $(f, \varphi + i\psi)$, $(\tilde{f}, \tilde{\varphi} + i\tilde{\psi})$.

This same conclusion is reached if we look for Lorentz metrics on V_4 which admit massive quanta.

If we assume that the metric is given, we have to solve the set of non-linear partial differential equations that define the pair $(f, \varphi + i\psi) \in \mathfrak{m}$. From the very way our problem is formulated however, it is natural to assume instead that the generators $(f, \varphi + i\psi) \in \mathfrak{m}$ and $(\tilde{f}, \tilde{\varphi} + i\tilde{\psi}) \in \mathfrak{m}$ are known and that the metric is to be determined. We then get a system of nonlinear partial differential equations in the g^{ij}'s.

In this way we obtain the following fundamental result

4.2 THEOREM. <u>A Lorentzian space</u> (V_4, g) <u>admits two distinct</u> <u>maximal monochromatic algebras if and only if it is the riemannian</u> <u>product of two two-dimensional geometries, one definite and the</u> <u>other indefinite.</u>

<u>If it admits more than two such algebras, it actually admits</u> <u>infinitely many and is locally Minkowski.</u>

<u>In any of these cases, the complex monochromatic generators</u> <u>of any two maximal monochromatic algebras can be taken as common.</u>

Specifically there are coordinate charts such that

$$(4.1) \qquad ds^2 = a^{-1}(x^1, x^4)[(dx^1)^2 - (dx^4)^2] + \alpha^{-1}(x^2, x^3)[(dx^2)^2 + (dx^3)^2].$$

Geometries of this form will be called <u>elementary geometries</u>.

This result has far-reaching consequences that we now describe.

4.3 CONSEQUENCES

4.3.1. Locally Minkowski spaces are spaces devoid of masses and fields and therefore represent the vacuum in the given differentiable manifold.

4.3.2. If (V_4, g) carries a particle, it cannot carry another one, otherwise the space would admit more than two distinct maximal monochromatic algebras, and by the theorem, would be locally Minkowski, hence vacuous.

This result therefore establishes a (kind of) correspondence between non-flat metrics of the form (4.1) and particles (while the flat metrics correspond to the vacuum).

In other words, <u>a particle</u> (up to what we have found until now) <u>is a geometry</u> of a special kind.

4.3.3. By the previous item we can only describe classically a one-particle universe or the vacuum, at a time. Thus in order to describe simultaneously more than one particle in the given differentiable manifold we must resort to a non-classical procedure. Just as before we had to use generalized curves to represent massive quanta out of light quanta, now we must consider a generalized Lorentz manifold, defined as a statistical distribution on the set of elementary geometries in V_4, each corresponding to a particle present in V_4 and to the vaccum.

As before, we define the corresponding average metric and agree that this is the metric observed. The real picture consists however of the statistical superposition of the hidden elementary geometries, that combine through averaging to give the observed Lorentz geometry. This generalized metric construction is thus a simpler version of the many-world picture.

4.3.4. Although we can figure out what a generalized metric should be, the problem of how to construct it explicitly is highly non-trivial and corresponds essentially to the complete mathematical description of the interaction between particles. This involves conceptual and technical difficulties that we are not yet ready to tackle, at this stage. Luckily, approximate analyses are possible after we get more information on the structure of the elementary particles themselves, that is to say, on one-particle universes. After that, this problem can then be reconsidered.

4.3.5. However some further considerations are convenient here. Recall that Einstein's equation $G_{\alpha\beta} = 0$ can be recovered as the extremal of Hilbert's Lagrangian $L = R$, where R is the scalar curvature of the unknown metric g^{ij}, integrated between two space

sections Σ, Σ' of V_4 with given initial and final (induced) metrics (pp. 485,491 [2]). This still holds true if the Lagrangian of an electromagnetic field, namely $F^{\mu\nu}F_{\mu\nu}$, is added to L (p. 504 [2], p.24 [3]) in which case the equation becomes $G_{\alpha\beta} = T_{\alpha\beta}$.

Assuming that topologically $V_4 = \Sigma \times R$, one can introduce the superspace $\mathcal{S}$ on Σ, consisting of all positive definite metrics on Σ and put an appropriate metric on $\mathcal{S}$ (de Witt metric, p.212 [1]). One can then consider a curve on $\mathcal{S}$ and associate to this curve a Lorentz metric in $\Sigma \times R$, in such a way that the induced 3-dimensional sections of this metric are the elements of the curve (p.224 [1]). Furthermore this allows one to express the scalar curvature R in V_4, appearing in Hilbert's Lagrangian, in terms of the intrinsic and extrinsic curvatures of the elements of the given curve and so to formulate Einstein's equation entirely in terms of curves in $\mathcal{S}$.

When the solution is an ordinary curve this analysis is called classical geometrodynamics. In our case, however, we must restrict the 3-geometries to be spatial sections of the elementary geometries (4.1). Therefore if more than one particle is present, no classical solution exists. However a generalized curve built out of spatial sections of the elementary geometries (4.1) may solve the corresponding variational problem, and this is at the root of quantum geometrodynamics. In fact the envelope procedure, that defines the average classical curve resulting from a generaliz-ed curve, corresponds to first quantization in the case of light waves, and to second quantization in the present case.

In this case, however, two new difficulties arise. First the given Lagrangian involves second derivatives of the unknown g^{ij}'s. Second, to define statistical superposition of geometries

we must find out how do they look in a common chart, to start.
The first problem is bypassed by the Palatini device (pp.491-502 [2],
p.45 [3]) in the case of classical solutions; what happens when the
solution is a generalized curve remains to be seen. The really
difficult problem is the second one, because it reflects the real
issue at stake, namely, the mathematical description of the
interaction between the particles.

4.3.6. PRESERVATION OF ANALITICITY

Suppose we build a massive quanta, in the way described
in Chapter III, out of functions in the two given maximal mono-
chromatic algebras h, h'. By the theorem they may be taken with
a common complex generator. If the given functions in h, h'
are both analytic, or both anti-analytic, in the complex generator,
then so will be their envelope, because the envelope procedure
preserves first derivatives and linear relations, and so also the
Cauchy-Riemann equations. This fact will have importan conse-
quences.

4.3.7. TWISTOR DESCRIPTION OF PARTICLES AND MASSIVE QUANTA

Consider the generators $(f, \varphi + i\psi)$ and $(\tilde{f}, \varphi + i\psi)$ of the
two distinct monochromatic algebras h, h'.

Let the spinor representation of ∇f and $\nabla \tilde{f}$ be $\omega^A \bar{\omega}^{A'}$,
say, and $\pi^A \bar{\pi}^{A'}$, say, respectively. Since $\nabla(\varphi + i\psi)$ is a strictly
complex isotropic vector orthogonal to both ∇f and $\nabla \tilde{f}$ then,
by 2.5.2, necessarily it has the spinor representation

$$\gamma \omega^A \bar{\pi}^{A'} ,$$

(or $\bar{\gamma} \pi^A \bar{\omega}^{A'}$, its complex-conjugate), with γ a scalar field.

Therefore $(\nabla f, \nabla \tilde{f}, \nabla(\varphi + i\psi))$ is described by the <u>twistor
plus a scalar</u> - field

$$(\omega^A, \pi_{A'}; \gamma).$$

The twistor fields associated to the monochromatic algebras h and h' are, respectively,

$$Z^\beta = (\omega^A, \bar\gamma\pi_{A'})$$

and

$$Y^\beta = (\pi^A, \gamma\omega_{A'}) ,$$

according to (2.36).

Computing explicitly with respect to the basis where $g^{ij} = \text{diag}(a,-a,-\alpha,-\alpha)$, $i,j = 0,1,2,3$, we get

$$\omega^A = a^{1/4} e_1, \qquad \pi^A = a^{1/4} e_2, \qquad \gamma = \left(\frac{\alpha}{a}\right)^{1/2},$$

where e_1, e_2 are the unit vectors in $\mathbb{C}^2$.

As a consequence

$$Z^\beta \bar Z_\beta = Y^\beta \bar Y_\beta = 2\,\text{Re}(\bar\gamma\omega^A\pi_A) = 2\sqrt\alpha.$$

In Chapter V we will consider the case when $\alpha \equiv 1$ and $a^{-1/2} = \text{sech}\,\sqrt\lambda(x^0+x^1)$. In that case $2s := Z^\beta \bar Z_\beta$ is constant everywhere and $s \equiv 1$, so we can speak of helicity and spin. Further the scalar field $\gamma = \text{sech}\,\sqrt\lambda(x^0+x^1)$ is known as soon as the parameter λ is given, and this characterizes the kind of particle under consideration. In this case therefore the two monochromatic algebras h and h' are completely characterized by the single twistor field $(\omega^A, \pi_{A'})$ plus the numerical parameter λ. In other words the elementary geometry associated to the particle, and so also its massive quanta, is characterized by a _single_ twistor field, just as the light quanta were in 2.5.3.

Therefore in this respect, i.e., in terms of their classification by twistor fields, light quanta, massive quanta and the elementary particles themselves are on an equal footing. Yet we know that they are all quite distinct objects, and that the given

twistor fields are used in different ways to build them.

This result may explain why many methods of dealing with quanta, light or massive, and with elementary particles, work just as if they all were objects of the same kind.

4.3.8. Finally we remark that the twistor representation shows that there may be an interaction between the independent spaces entering in the riemannian product (4.1). Indeed if π is substituted by $e^{i\theta}\pi$ and ω by $e^{i\beta}\omega$, θ and β real constants, we know that ∇f and $\nabla \tilde{f}$ do not change but $\nabla(\varphi+i\psi)$ goes into $\nabla(e^{i(\beta-\theta)}(\varphi+i\psi))$. This represents a rotation by an angle $\beta-\theta$ of the original axes in (x^2,x^3) space. Hence the polarization of the isotropic real vectors may produce rotation of the (x^2,x^3) plane, and vice-versa.

We now proceed to the proof of the theorem.

4.4 PROOF OF THE THEOREM

1. Let $(f,\varphi+i\psi)$ and $(\tilde{f},\tilde{\varphi}+i\tilde{\psi})$ he two distinct monochromatic pairs. This means, in particular, that

$$(4.2) \qquad (\nabla f)^2 = 0, \qquad \nabla f \neq 0,$$

$$(4.3) \qquad (\nabla\varphi)^2 = (\nabla\psi)^2 \equiv \beta \neq 0$$

$$(4.4) \qquad \nabla f \cdot \nabla\varphi = 0,$$

$$(4.5) \qquad \nabla f \cdot \nabla\psi = 0,$$

$$(4.6) \qquad \nabla\varphi \cdot \nabla\psi = 0,$$

with similar equations holding for $\tilde{f}$, $\tilde{\varphi}$, $\tilde{\psi}$. Furthermore $\nabla f \cdot \nabla \tilde{f} \neq 0$, by Lemma 3.5.

In view of these relations we can choose local coordinates defined by

$$(4.7) \qquad x^1 = \tilde{f}, \quad x^2 = \tilde{\varphi}, \quad x^3 = \tilde{\psi}, \quad x^4 = f.$$

From (4.2) applied to $\tilde{f}$ and f we get $g^{11} = g^{44} = 0$.

From the remaining relations we get similarly

$$g^{12} = g^{13} = g^{23} = 0$$

$$g^{22} = g^{33} \equiv \alpha \neq 0,$$

so that g^{ij} has the expression

$$(4.8) \qquad g^{ij} = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & \alpha & 0 & b \\ 0 & 0 & \alpha & c \\ a & b & c & 0 \end{pmatrix}.$$

Here

$$(4.9) \qquad \begin{aligned} a &= \nabla f \cdot \nabla \tilde{f} \neq 0 \\ b &= \nabla f \cdot \nabla \tilde{\varphi}, \\ c &= \nabla f \cdot \nabla \tilde{\psi}, \\ \alpha &= (\nabla \tilde{\varphi})^2 = (\nabla \tilde{\psi})^2 \neq 0 \end{aligned}$$

We may assume, without loss of generality, that $a > 0$, $\alpha > 0$. (This simply amounts to the choice of the canonical form $(1,1,1,-1)$).

Using the notation $\dfrac{\partial}{\partial x^i} \varphi = \varphi_i$, we get from (4.3)

$$(4.10) \qquad \begin{aligned} \beta &= 2\varphi_4(a\varphi_1 + b\varphi_2 + c\varphi_3) - \alpha(\varphi_2^2 + \varphi_3^2) \\ &= 2\psi_4(a\psi_1 + b\psi_2 + c\psi_3) + \alpha(\psi_2^2 + \psi_3^2). \end{aligned}$$

From (4.4) and (4.5) we get

(4.11)
$$\nabla\varphi \cdot \nabla f = a\varphi_1 + b\varphi_2 + c\varphi_3 = 0,$$

(4.12)
$$\nabla\psi \cdot \nabla f = a\psi_1 + b\psi_2 + c\psi_3 = 0.$$

Hence (4.10) gives, as $\alpha \neq 0$,

(4.13)
$$\varphi_2^2 + \varphi_3^2 = \psi_2^2 + \psi_3^2 = \frac{\beta}{\alpha}.$$

Finally from (4.6) follows

(4.14)
$$\nabla\varphi \cdot \nabla\psi = a(\varphi_1\psi_4 + \psi_1\varphi_4) + b(\varphi_4\psi_2 + \psi_4\varphi_2)$$
$$+ c(\varphi_4\psi_3 + \psi_4\varphi_3) + \alpha(\varphi_2\psi_2 + \varphi_3\psi_3)$$
$$= 0,$$

which becomes, after rearrangement,

$$\psi_4(a\varphi_1 + b\varphi_2 + c\varphi_3) + \varphi_4(a\psi_1 + b\psi_2 + c\psi_3) +$$
$$+ \alpha(\varphi_2\psi_2 + \varphi_3\psi_3) = 0.$$

In view of (4.11) and (4.12) this gives

(4.15)
$$\varphi_2\psi_2 + \varphi_3\psi_3 = 0.$$

The solutions of (4.13) and (4.15) are the Cauchy-Riemann equations

$$\varphi_3 = \pm\psi_2, \qquad \varphi_2 = \mp\psi_3.$$

We may choose one of the determinations, say

(4.16)
$$\varphi_2 = \psi_3, \qquad \varphi_3 = -\psi_2.$$

In particular

(4.16)'
$$\varphi_{33} + \varphi_{22} = \psi_{33} + \psi_{22} = 0.$$

2. Since $|g|^{1/2} = |\det g_{ij}|^{1/2} = \frac{1}{a\alpha}$, the Laplace-Beltrami equation $\Delta_2 \Phi = 0$ in these coordinates has the expression

$$(4.17) \qquad \partial_1\left(\frac{\Phi_4}{\alpha}\right) + \partial_2\left(\frac{\Phi_2}{a} + \frac{b}{a\alpha}\Phi_4\right) + \partial_3\left(\frac{\Phi_3}{a} + \frac{c}{a\alpha}\Phi_4\right) +$$
$$+ \partial_4\left(\frac{\Phi_1}{\alpha} + \frac{b}{a\alpha}\Phi_2 + \frac{c}{a\alpha}\Phi_3\right) = 0.$$

Replacing Φ by x^1, x^2, x^3, x^4 gives, respectively

$$(4.18) \qquad \partial_4\left(\frac{1}{\alpha}\right) = 0,$$

$$(4.19) \qquad \partial_2\left(\frac{1}{a}\right) + \partial_4\left(\frac{b}{a\alpha}\right) = 0,$$

$$(4.20) \qquad \partial_3\left(\frac{1}{a}\right) + \partial_4\left(\frac{c}{a\alpha}\right) = 0,$$

$$(4.21) \qquad \partial_1\left(\frac{1}{\alpha}\right) + \partial_2\left(\frac{b}{a\alpha}\right) + \partial_3\left(\frac{c}{a\alpha}\right) = 0.$$

Now we remark that in (4.17) the coefficient of φ_4, upon expansion, is just the right-hand side of (4.21), hence it vanishes. If $\Phi = \varphi$ then it satisfies (4.11), and so, using (4.18) and (4.21), (4.17) becomes

$$(4.22) \qquad \frac{1}{\alpha}\varphi_{41} + \frac{1}{a}\varphi_{22} + \varphi_2\,\partial_2\left(\frac{1}{a}\right) + \frac{b}{a\alpha}\varphi_{42} + \frac{1}{a}\varphi_{33} + \varphi_3\,\partial_3\left(\frac{1}{a}\right) +$$
$$+ \frac{c}{a\alpha}\varphi_{43} = 0.$$

But

$$\frac{1}{\alpha}\varphi_{41} + \frac{b}{a\alpha}\varphi_{42} + \frac{c}{a\alpha}\varphi_{43} = \partial_4\left(\frac{1}{\alpha}\varphi_1 + \frac{b}{a\alpha}\varphi_2 + \frac{c}{a\alpha}\varphi_3\right) -$$
$$- \varphi_1\,\partial_4\left(\frac{1}{\alpha}\right) - \varphi_2\,\partial_4\left(\frac{b}{a\alpha}\right) - \varphi_3\,\partial_4\left(\frac{c}{a\alpha}\right)$$
$$= \varphi_2\,\partial_2\left(\frac{1}{a}\right) + \varphi_3\,\partial_3\left(\frac{1}{a}\right),$$

in view of (4.11), (4.18), (4.19) and (4.20).

In view of this and of (4.16)', (4.22) becomes

$$(4.23) \qquad \varphi_2\,\partial_2\left(\frac{1}{a}\right) + \varphi_3\,\partial_3\left(\frac{1}{a}\right) = 0.$$

Similarly for ψ

(4.24)
$$\psi_2 \, \partial_2(\tfrac{1}{a}) + \psi_3 \, \partial_3(\tfrac{1}{a}) = 0.$$

But (4.23) and (4.24) are a homogeneous linear system in $\partial_2(\tfrac{1}{a})$ and $\partial_3(\tfrac{1}{a})$ with determinant $\varphi_2\psi_3 - \psi_2\varphi_3 = \varphi_2^2 + \varphi_3^2 = \tfrac{\beta}{\alpha} \neq 0$, by (4.16) and (4.13). Hence necessarily

(4.25)
$$\partial_2(\tfrac{1}{a}) = \partial_3(\tfrac{1}{a}) = 0,$$

i.e. $a = a(x^1, x^4)$, as claimed.

In particular (4.19) and (4.20) become

(4.26)
$$\partial_4(\tfrac{b}{a\alpha}) = \partial_4(\tfrac{c}{a\alpha}) = 0, \quad \text{or using (4.18),}$$

$$\partial_4(\tfrac{b}{a}) = \partial_4(\tfrac{c}{a}) = 0.$$

3. Introduce now the new set of coordinates

(4.27) $\quad \tilde{x}^1 = x^4 = f, \quad \tilde{x}^2 = \varphi, \quad \tilde{x}_3 = \psi, \quad \tilde{x}^4 = x^1 = \tilde{f},$

which amounts to the interchange of the roles of the given mono-chromatic pairs.

We have

(4.28)
$$\frac{\partial \tilde{x}^i}{\partial x^P} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 \\ \psi_1 & \psi_2 & \psi_3 & \psi_4 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

The metric in the new coordinates takes the form

(4.29)
$$\tilde{g}^{ij} = \frac{\partial \tilde{x}^i}{\partial x^p} \cdot \frac{\partial \tilde{x}^j}{\partial x^q} \, g^{pq} \; =$$

$$= \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & \alpha(\varphi_2^2+\varphi_3^2) & 0 & a\varphi_4 \\ 0 & 0 & \alpha(\varphi_2^2+\varphi_3^2) & a\psi_4 \\ a & a\varphi_4 & a\psi_4 & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & \beta & 0 & \tilde{b} \\ 0 & 0 & \beta & \tilde{c} \\ a & \tilde{b} & \tilde{c} & 0 \end{pmatrix} ,$$

where use is made of (4.11), (4.12) and (4.15).

From (4.27) it is clear that $a \simeq a(x^1,x^4) = \tilde{a}(\tilde{x}^1,\tilde{x}^4)$.

Notice also that $\dfrac{\partial}{\partial \tilde{x}^i} = \dfrac{\partial x^j}{\partial \tilde{x}^i}\dfrac{\partial}{\partial x^j}$, gives, by (4.27)

$$\frac{\partial}{\partial \tilde{x}^1} = \frac{\partial}{\partial x^4}, \quad \frac{\partial}{\partial \tilde{x}^4} = \frac{\partial}{\partial x^1}.$$

Writing $\dfrac{\partial}{\partial \tilde{x}^i} \equiv \partial_{\tilde{i}}$, the equations (4.18) and (4.26) become therefore

$$\partial_{\tilde{4}}\left(\frac{1}{\beta}\right) = 0 \qquad \text{i.e.} \qquad \partial_1\left(\frac{1}{\alpha(\varphi_2^2+\varphi_3^2)}\right) = 0,$$

$$\partial_{\tilde{4}}\left(\frac{\tilde{b}}{a\beta}\right) = 0 \qquad \text{i.e.} \qquad \partial_1\left(\frac{\varphi_4}{\alpha(\varphi_2^2+\varphi_3^2)}\right) = 0,$$

$$\partial_{\tilde{4}}\left(\frac{\tilde{c}}{a\beta}\right) = 0 \qquad \text{i.e.} \qquad \partial_1\left(\frac{\psi_4}{\alpha(\varphi_2^2+\varphi_3^2)}\right) = 0.$$

From these equations follows

(4.30)
$$\varphi_{41} = \psi_{41} = 0.$$

Writing (4.11) in the form

$$\varphi_1 + \frac{b}{a}\varphi_2 + \frac{c}{a}\psi_3 = 0,$$

applying ∂_4 and using (4.26) and (4.30) we get

$$\frac{b}{a}\varphi_{24} + \frac{c}{a}\varphi_{34} = 0.$$

Doing the same with (4.12) and collecting the two results we get

(4.31)
$$b\,\varphi_{24} + c\,\varphi_{34} = 0,$$
$$b\,\psi_{24} + c\,\psi_{24} = 0.$$

This last equation becomes, in view of (4.16)

(4.32)
$$c\,\varphi_{24} - b\,\varphi_{34} = 0.$$

Now if $b^2 + c^2 \equiv 0$ in a neighbourhood of a given point, the metric (4.8) has the required form in that neighbourhood because (4.18) and (4.21) yield $\alpha = \alpha(x^2, x^3)$. Otherwise if there is a neighbourhood of that point where $b^2 + c^2 > 0$ (4.31) and (4.32) imply

$$\varphi_{24} = \varphi_{34} = \psi_{24} = \psi_{34} = 0$$

there, which, together with (4.30) show that φ_4 and ψ_4 are functions of x^4 alone. Hence, in particular, φ splits into a sum of a function $h(x^4)$ and a function of (x^1, x^2, x^3). Introducing $\varphi - h(x^4)$ instead of φ, which does not change the properties of the original monochromatic pair, we obtain $\varphi_4 = 0$ for the new φ. By the analogous procedure we obtain $\psi_4 = 0$.

Hence in (4.29) the metric becomes

(4.32)
$$\tilde{g}^{ij} = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ a & 0 & 0 & 0 \end{pmatrix},$$

with $a = a(\tilde{x}^1, \tilde{x}^4)$, and by (4.18) and (4.21) as applied to this case, $\beta = \beta(\tilde{x}^2, \tilde{x}^3)$. Hence in any case the metric can be put in the form claimed.

4. From the symmetry of (4.32) with respect to x^1 and x^4 se see that both $(f, \varphi+i\psi)$ and $(\tilde{f}, \varphi+i\psi)$ generate maximal monochromatic algebras, and these algebras are distinct by Lemma 3.5. This same lemma tells us that $(\tilde{f}, \varphi+i\psi)$ and $(\tilde{f}, \tilde{\varphi}+i\tilde{\psi})$ generate the same maximal monochromatic algebra. Hence, without loss of generality, the complex generators may be taken to be common.

5. If the metric has the form (4.32) it is seen directly that (x^1, x^2+ix^3) and (x^4, x^2+ix^3) generate distinct maximal monochromatic algebras.

6. Assume now that there is a third independent maximal monochromatic algebra with generators $(\hat{f}, \hat{\varphi}+i\hat{\psi})$.

According to the previous steps we may assume that $b = c = 0$ in the expression (4.8) of g^{ij}.

Letting $\partial_i \hat{f} \equiv w_i$ and introducing $\hat{f}$ in place of x^4 (i.e. of f) in (4.7) we obtain

$$(4.33) \quad \hat{g}^{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ a & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & w_1 \\ 0 & 1 & 0 & w_2 \\ 0 & 0 & 1 & w_3 \\ 0 & 0 & 0 & w_4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & aw_4 \\ 0 & \alpha & 0 & \alpha w_2 \\ 0 & 0 & \alpha & \alpha w_3 \\ aw_4 & \alpha w_2 & \alpha w_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \hat{a} \\ 0 & \alpha & 0 & \hat{b} \\ 0 & 0 & \alpha & \hat{c} \\ \hat{a} & \hat{b} & \hat{c} & 0 \end{pmatrix} ,$$

where use is made of

$$(4.34) \qquad (\nabla \hat{f})^2 = 2aw_1 w_4 + \alpha(w_2^2 + w_3^2) = 0,$$

as computed in the metric (4.8).

In the new coordinates $\hat{x}^1 = x^1$, $\hat{x}^2 = x^2$, $\hat{x}^3 = x^3$, $\hat{x}^4 = \hat{f}$, we have

$$\partial_{\hat{1}} = \partial_1, \quad \partial_{\hat{2}} = \partial_2, \quad \partial_{\hat{3}} = \partial_3,$$

$$\partial_{\hat{4}} = -\frac{w_1}{w_4}\partial_1 - \frac{w_2}{w_4}\partial_2 - \frac{w_3}{w_4}\partial_3 + \frac{\partial_4}{w_4}.$$

From the analogue of (4.25), namely

$$\partial_{\hat{2}}\left(\frac{1}{\hat{a}}\right) = \partial_{\hat{3}}\left(\frac{1}{\hat{a}}\right) = 0,$$

we get

$$\partial_2(aw_4) = \partial_3(aw_4) = 0,$$

i.e., in view of (4.25),

(4.35) $$w_{42} = w_{43} = 0.$$

Had we substituted $\hat{f}$ in place of x^1 instead, we would have gotten analogously

(4.36) $$w_{12} = w_{13} = 0,$$

i.e. w_1 and w_4 independ on x^2, x^3 and w_2, w_3 independ on x^1, x^4.

From the analogue of (4.21) we now get

$$\partial_1\left(\frac{1}{\alpha}\right) + \partial_2\left(\frac{\hat{b}}{\hat{a}\alpha}\right) + \partial_3\left(\frac{\hat{c}}{\hat{a}\alpha}\right) = 0,$$

i.e., as α independs on x^1,

$$\partial_2\left(\frac{\alpha w_2}{\alpha aw_4}\right) + \partial_3\left(\frac{\alpha w_3}{\alpha aw_4}\right) = 0,$$

and, in view of (4.35), (4.36) and (4.25), this becomes

(4.37) $$w_{22} + w_{33} = 0.$$

On the other hand, in the metric g^{ij},

$$\Delta_2 \hat{f} = 2aw_{14} + \alpha(w_{22}+w_{33}) = 0$$

implies now

(4.38)
$$w_{14} = 0.$$

From (4.34), as α, w_2, w_3 independ on x^1, x^4 we get by differentiation

$$\partial_1(aw_1w_4) = \partial_4(aw_1w_4) = 0,$$

and since aw_1w_4 also independs on x^2, x^3 we see that

(4.39)
$$2aw_1w_4 = K, \quad K \quad \text{constant.}$$

Hence from (4.34)

(4.40)
$$\alpha = \frac{-K}{w_2^2+w_3^2}.$$

From (4.35), (4.36), (4.38) and (4.37) we have that

$$w = \omega(x^2,x^3) + p(x^1) + q(x^4),$$

with ω harmonic in x^2, x^3.

Since $w_1 = p'(x^1)$ and $w_4 = q'(x_4)$, (4.39) yields

(4.41)
$$a = \frac{K}{2p'(x^1)q'(x^4)}.$$

Let $\chi(x^2+ix^3)$ be an analytic function having ω as its real part. Then the change of coordinates

$$x^2 \longmapsto \text{Re } \chi, \qquad x^3 \longmapsto \text{Im } \chi,$$
$$x^1 \longmapsto p(x^1), \qquad x^4 \longmapsto q(x^4),$$

makes g^{ij} a constant matrix, in view of (4.40) and (4.41), which proves the metric is locally Minkowski.

7. In the Minkowski case let $g^{ij} = \text{diag}(1,1,1,-1)$. Take the real and the complex orthogonal isotropic vectors

$$k_o = (0,0,1,1), \quad k_c = (1,i,0,0).$$

Let L be a proper Lorentz transformation.

Then $f(x) = (Lk_o) \cdot x$ and $\varphi + i\psi = (Lk_c) \cdot x$ are generators of maximal monochromatic algebras, for whichever L one chooses. As Lk_o describes the light cone when L varies, this gives us infinitely many distinct monochromatic algebras, by Lemma 3.5.

The theorem is proved.

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CHAPTER V

PARTICLES

5.1 INTRODUCTION

In this chapter we complete the ∧ geometric description of particles by exhibiting the metric of the corresponding elementary geometry in a concrete referential in $\mathbb{R}^4$, namely

$$(5.0) \qquad ds^2 = \frac{dr^2}{e^{2\mu|r-r_o|}-1} - (1-e^{-2\mu|r-r_o|})dt^2 + dy^2 + dz^2.$$

Here x,y,z are cartesian coordinates, $r = (x^2+y^2+z^2)^{1/2}$, t is time and $\mu > 0$, $r_o > 0$ are constants.

We start by assuming that, due to its elementary character, the energy-momentum tensor associated with the elementary geometry describing a particle should be that of a pure electromagnetic field. It is only when the metrics of various particles are superposed that more general energy-momentum tensors arise. (In this connection see the considerations on pp.45,46 [7] and on p.107 [10].)

In that case, since elementary geometries are locally the riemannian product of two 2-dimensional geometries, we are precisely under the hypothesis of [1], and therefore the metric is a Bertotti-Robinson metric, which means each of the two-geometries has constant curvature.

Further natural assumptions are used to arrive at the final form of the metric above, and will be explained along the analysis. In principle other explicit geometries are possible if different assumptions are made.

5.2 PURE ELECTROMAGNETIC FIELD

Introducing in (4.1) the new coordinates $x^1 = \sqrt{2}(X^1 + X^4)$, $x^4 = \sqrt{2}(X^1 - X^4)$, the metric of an elementary geometry acquires the diagonal form

$$(5.1) \qquad g_{ij} = \begin{pmatrix} a^{-1} & & & \bigcirc \\ & \alpha^{-1} & & \\ & & \alpha^{-1} & \\ \bigcirc & & & -a^{-1} \end{pmatrix}$$

We assume $a = a(X^1, X^4) > 0$, $\alpha = \alpha(X^2, X^3) > 0$.

We now compute the energy-momentum tensor T^{ij} defined by

$$(5.2) \qquad T^{ij} = G^{ij} + \Lambda\, g^{ij},$$

where Λ is a cosmological constant and G^{ij} is Einstein's tensor

$$(5.3) \qquad G^{ij} = R^{ij} - \frac{1}{2} R\, g^{ij}.$$

Here R^{ij} is Ricci's tensor and $R = g^{ij} R_{ij}$ the scalar curvature, where

$$R_{i\ell} = \partial_k \Gamma^k_{i\ell} - \partial_i \Gamma^k_{k\ell} + \Gamma^k_{i\ell} \Gamma^h_{kh} - \Gamma^k_{ih} \Gamma^h_{\ell k},$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\ell} [\partial_j g_{k\ell} + \partial_k g_{\ell j} - \partial_\ell g_{jk}]$$

$$\partial_i \equiv \frac{\partial}{\partial x^i}.$$

Using Dingle's formulae (p.70 [8]) one obtains

$$G^{ij} = 0, \qquad i \neq j,$$

$$G^{11} = -G^{44} = -\frac{a\alpha}{2} (\partial_2^2 + \partial_3^2)\log \alpha^{-1},$$

$$G^{22} = G^{33} = -\frac{a\alpha}{2} (\partial_1^2 - \partial_4^2)\log a^{-1}.$$

Hence $T^i_j = G^i_j + \Lambda$ yields

$$
(5.4) \quad T^i_j = \begin{pmatrix} -\frac{\alpha}{2}(\partial_2^2 + \partial_3^2)\log \alpha^{-1} + \Lambda & & & \\ & -\frac{a}{2}(\partial_1^2 - \partial_4^2)\log a^{-1} + \Lambda & & \\ & & -\frac{a}{2}(\partial_1^2 - \partial_4^2)\log a^{-1} + \Lambda & \\ & & & -\frac{\alpha}{2}(\partial_2^2 + \partial_3^2)\log \alpha^{-1} + \Lambda \end{pmatrix}
$$

By assumption there should exist a vector potential A_i such that the antisymmetric electromagnetic tensor

$$
(5.5) \qquad F_{ij} = \partial_i A_j - \partial_j A_i
$$

satisfies

$$
(5.6) \qquad T^i_j = \frac{1}{4\pi}\left(-F^i_k F^k_j + \frac{1}{4} F^\ell_k F^k_\ell \, \delta^i_j\right) .
$$

In particular this implies

$$
(5.6)' \qquad T^i_i = 0.
$$

One identifies F_{ij} with the usual quantities in 3-space by taking an orthonormal tetrad $\lambda^i_{(a)}$, $(a=1,2,3,4))$, i.e. four vectors such that

$$
\mathcal{E}_{ij} \, \lambda^i_{(a)} \, \lambda^j_{(b)} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} ,
$$

and forming the matrix

$$F_{(ab)} = F_{ij} \lambda^i_{(a)} \lambda^j_{(b)} = \begin{pmatrix} 0 & H_3 & -H_2 & E_1 \\ -H_3 & 0 & H_1 & E_2 \\ H_2 & -H_1 & 0 & E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{pmatrix} .$$

(E_1, E_2, E_3) is then the electric field and (H_1, H_2, H_3) the magnetic field with respect to the given basis.

Similarly from the expression of the 4-current vector

$$(5.7) \qquad J^i = \frac{1}{4\pi} \frac{1}{|g|^{1/2}} \partial_j (|g|^{1/2} F^{ij}),$$

we obtain the charge density $J_{(4)} = J_i \lambda^i_{(4)}$, and the current density $J_{(\alpha)} = J_i \lambda^i_{(\alpha)}$, $\alpha = 1, 2, 3$, in that basis.

Now $(5.6)'$ combined with (5.4) plus separation of variables yields the equations

$$(5.8) \qquad -\frac{\alpha}{2} (\partial_{22} + \partial_{33}) \log \alpha^{-1} = -\Lambda - q,$$

$$(5.9) \qquad -\frac{a}{2} (\partial_{11} - \partial_{44}) \log a^{-1} = -\Lambda + q,$$

where q is some constant.

In this case

$$(5.10) \qquad T^i_j = q \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} .$$

The general solution of (5.6) with T^i_j given by (5.10) and F_{ij} antisymmetric is

$$(5.11) \quad F_j^i = \sqrt{8\pi q}\begin{pmatrix} & & & \cos\theta \\ & & \sin\theta & \\ & -\sin\theta & & \\ \cos\theta & & & \end{pmatrix}, \quad \theta \text{ real}, \ q \geq 0.$$

This corresponds to the canonical form of a non-null field when $q > 0$ (p.481 [9], [1]).

At this point we have two alternatives. We may follow Rainich (pp.131, 132 [10]) and set $\theta = 0$ or π. In this case taking an orthonormal tetrad with vectors along the axes X^1, X^2, X^3, X^4 the magnetic field H_i is zero and the only non-zero component of the electric field is E_1. This means no magnetic only electric charge exists. Instead in case $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, only H_1 would be non-zero.

Or else we may make the assumption that the particle may behave in any of the extreme ways ($\theta = 0, \pi$ - electric charge only or $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ - magnetic charge only) and as a statistical super-position of both, presenting one or the other aspect according to the kind of stimulus imposed.

This interpretation is more in accordance with our method of analysis as explained in Chapter I. In fact the first alternative is related to the idea that electric and magnetic charges, as objects, are independent on the geometry and perhaps produce it, whereas our approach shows that, at this scale, they are just a feature of the geometry itself, not independent objects. In this connection it is instructive to read section 8 p. 13 in [7].

In this chapter we will only consider the case of pure electric charge ($\theta = 0, \pi$) postponing the case of pure magnetic field to Chapter X.

First let us remark from (5.10) that

$$v_i v^j T^i_j = q[-(v_1 v^1 + v_4 v^4) + v_2 v^2 + v_3 v^3]$$
$$= q[-v_i v^i + 2(v_2 v^2 + v_3 v^3)].$$

Since $v_2 v^2 + v_3 v^3 = \alpha^{-1}[(v^2)^2 + (v^3)^2] \geq 0$, the energy-momentum tensor T^i_j is non-negative on causal vectors, i.e. on those satisfying $v_i v^i \leq 0$, as it should.

Setting $\theta = 0$, π in (5.11) we get

$$(5.12) \qquad F^i_j = \pm \sqrt{8\pi q} \begin{pmatrix} \bigcirc & & & 1 \\ & 0 & & \\ & & 0 & \\ 1 & & & \bigcirc \end{pmatrix}, \qquad q \geq 0.$$

The 4-current is then

$$(5.13) \qquad J^i = \pm \sqrt{\tfrac{a\alpha q}{2\pi}} \left(\partial_4(\tfrac{1}{\alpha}), 0, 0, -\partial_1(\tfrac{1}{\alpha}) \right) = 0,$$

as $\alpha = \alpha(x^2, x^3)$.

Regardless of the particular expression of $a = a(x^1, x^4)$, there is always a vector potential satisfying (5.5) for the electromagnetic tensor

$$(5.12)' \qquad F_{ij} = \pm \sqrt{8\pi q} \begin{pmatrix} \bigcirc & & & a^{-1} \\ & & 0 & \\ & 0 & & \\ -a^{-1} & & & \bigcirc \end{pmatrix}.$$

It suffices to take for instance

$$(5.14) \qquad A_i = \pm \sqrt{8\pi q} \left(0, 0, 0, \int a^{-1} dx^1 \right),$$

because then

$$\partial_1 A_4 - \partial_4 A_1 = a^{-1},$$

and the other terms not involving both indices 1,4 are zero.

Therefore conditions (5.8) and (5.9) with $q \geq 0$ are necessary and sufficient in order for the elementary geometry to correspond to a pure electromagnetic field.

As remarked earlier, conditions (5.8) and (5.9) mean that the definite metric $\alpha^{-1}[(dX^2)^2 + (dX^3)^2]$ has constant curvature $-(\Lambda+q)$ and the indefinite metric $a^{-1}[(dX^1)^2 - (dX^4)^2]$ has constant curvature $q - \Lambda$. These are the Bertotti-Robinson metrics.

We therefore have proved that the Bertotti-Robinson metrics are the only ones admitting absorption and emission of quanta and corresponding to a pure electromagnetic field.

A typical example is given by the following conformal Minkowski space

$$(5.15) \qquad \frac{dr^2 - dt^2}{r^2} + d\Omega^2, \qquad r > 0.$$

Here $d\Omega^2$ is the usual metric on the unit sphere, which is to be thought of as isometrically immersed in a spatial section $t = \text{const.}$ of Minkowski space.

5.3 CONCRETE CHART IDENTIFICATION

The elementary geometries (5.1) were defined abstractly in terms of monochromatic waves taken as sets of coordinates. We now have to identify these coordinates globally in terms of concrete referentials, with respect to which measurements can be made by a given observer. (This is an instance of the chart-fitting problem referred to in 4.3.5: here we are trying to fit the geometries of the vacuum and of one particle.)

To start with we must fix a choice of differentiable manifold.

1. Recall that our final geometry will be a statistical superposition of elementary geometries defined on a common differentiable manifold. On this manifold the vacuum is represented by locally Minkowski geometries. <u>We choose Minkowski space as the elementary geometry of the vacuum, hence R^4 as background</u>.

This is then thought of as *the geometry of the vacuum laboratory frame*, i.e. the one with respect to which the measurements are assumed to be made.

2. Suppose now a single particle is brought into play, say, at rest with respect to the vacuum laboratory frame. Consider its geometry just by itself, not statistically superposed to the *vacuum geometry*. Suppose the time-axis of the observer and of the particle's geometry as given by X^4 in (5.1), coincide. The two-dimensional space (X^2, X^3) with respect to which the particle's geometry has the form (5.1) provide a local foliation of the spatial sections $X^4 = $ const. We assume that some leaf of this foliation yields an isometric immersion of the definite geometry $\alpha^{-1}[(dX^2)^2 + (dX^3)^2]$ into the above spatial section of Minkowski space.

3. If we want the particles to have finite spatial extent, we must require singularities of the geometry to have bounded spatial sections because these singularities define the boundaries of what one usually pictures as a particle.

4. Finally we require that the maximal monochromatic algebras associated with the particle be sufficiently general so

as to admit non-trivial regular analytic functions, for all our constructions leading to the description of quanta were based on this assumption.

These requirements imply that $\Lambda + q = 0$, i.e. that the two-dimensional definite geometry be euclidean.

Indeed the representation of the (x^2, x^3) geometry of the riemannian product as an isometric immersion in $\mathbb{R}^3$ requires the corresponding image to be regular, otherwise upon product with the remaining coordinate parametrizing the foliation one would get an unbounded singularity, contradicting requirement 3. But then Hilbert's theorem (p.446 [3]) on non-existence of complete regular surfaces of negative constant curvature isometrically immersed in $\mathbb{R}^3$, excludes the case $-(\Lambda+q) < 0$.

On the other hand in case $-(\Lambda+q) > 0$, the geometry is that of an euclidean 2-sphere. The only regular analytic functions on a 2-sphere are the constant functions and this contradicts requirement 4.

Hence $\Lambda + q = 0$, as claimed.

Furthermore, by a corollary to a theorem of Hartman-Nirenberg (p.408, [3]) the complete regular surface is then a cylinder or a plane. Now foliations of $\mathbb{R}^3$ by cylinders with closed directrix curve necessarily have a singular straight line axis and this is a singularity of the (x^1, x^2, x^3) geometry because all the leaves would have to be isometric. This contradicts requirement 3. Hence the foliation is by planes or plane-like cylinders.

By (5.8) $\log \alpha$ is then harmonic in (x^2, x^3) so if we denote by $h(x^2 + ix^3)$ an analytic function having $\log \alpha^{-\frac{1}{2}}$ as its real part then, with respect to the new coordinates

$$Y^2 + iY^3 = \int e^{h(z)} dz, \qquad z = X^2 + iX^3,$$

the definite metric has the euclidean form

(5.16)
$$(dY^2)^2 + (dY^3)^2.$$

We can then identify the coordinates Y^2, Y^3 with cartesian coordinates in the vacuum laboratory frame (say, by appealing to symmetry considerations). As the analytic change of coordinates preserves the properties of the associated maximal monochromatic algebras, we may as well assume that (X^2, X^3) themselves are cartesian coordinates in $\mathbb{R}^3$ and that $\alpha \equiv 1$.

5.4 CHART IDENTIFICATION FOR THE INDEFINITE METRIC

Since $q \geq 0$ and, by 5.3, $\Lambda + q = 0$, the cosmological constant satisfies

(5.17)
$$\Lambda \leq 0.$$

Setting $\lambda = |\Lambda| = q$, for convenience, equation (5.9) becomes

(5.18)
$$(\partial_1^2 - \partial_4^2) \log a^{-1} = -4\lambda a^{-1}.$$

We now assume that the geometry does not depend on the time coordinate X^4, which means the particle is static in time. This particular time axis, with respect to which it is static, may be called the particle _natural_ time.

In this case the solution of (5.18) with $a = a(X^1)$ is given by

(5.19)
$$a^{-1} = B \operatorname{sech}^2 \sqrt{2\lambda B} \, (X^1 + C),$$
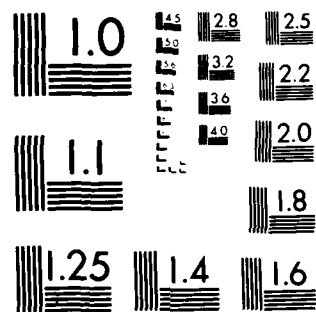
B, C constants.

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Clearly $B > 0$, otherwise a^{-1} is negative, X^1 becomes a time-coordinate and the particle is transient, not static. Without loss of generality we may take $C = 0$.

The metric for a (static) particle is then

$$(5.20) \quad ds^2 = B \operatorname{sech}^2 \sqrt{2\lambda B} \, X^1 [(dX^1)^2 - (dX^4)^2] + (dX^2)^2 + (dX^3)^2,$$

with $B > 0$.

5.4.1 Now we must identify the remaining coordinate X^1 with some coordinate complementing the cartesian coordinates X^2, X^3 locally in $\mathbb{R}^3$.

1. In first place if singularities arise they will be level surfaces of X^1 in $\mathbb{R}^3$, as the metric in (X^2, X^3) is regular and the final metric is a product metric. By our earlier requirement 2. in 5.3 these must be bounded, hence closed (coordinate) surfaces.

For convenience let us change notation setting $X^1 = \rho$, $X^4 = t$, $X^2 = y$, $X^3 = z$ and letting x denote the third cartesian coordinate. In this case we rewrite (5.20) as

$$(5.21) \quad ds^2 = B \operatorname{sech}^2 \sqrt{2\lambda B} \rho \, (d\rho^2 - dt^2) + dy^2 + dz^2,$$

where now the upper 2 is an exponent, not an index.

2. The usual spherically-symmetric geometries (Schwarzschild, Reissner-Nordstrøm, etc.) are given in the form

$$(5.22) \quad \frac{dr^2}{\varphi(r)} - \varphi(r)dt^2 + r^2 \, d\Omega^2.$$

The Eddington-Finkelstein coordinates for these geometries (p.828 [9]) are esentially obtained by replacing r by

$$r^* = \int \frac{dr}{\varphi(r)} \, ,$$

in which case (5.22) becomes

(5.23) $$\varphi(r(r^*))(dr^{*2} - dt^2) + r^2(r^*)d\Omega^2.$$

Although (5.21) lacks the term dx^2 and is not spherically symmetric, its indefinite part has an analogous expression to that of (5.23). Combined with the considerations in 1. this suggests that we take $\rho = \rho(r)$ (hence the closed surfaces $\rho = $ const. are spheres), and that <u>we define the function</u> $\rho(r)$ <u>by the condition that</u> (5.21) <u>acquires an expression analogous to</u> (5.22) <u>in terms of</u> r <u>and</u> t.

Setting $\rho = \rho(r)$ and

(5.24) $$dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\Omega^2,$$

we get in (5.22)

(5.25) $$ds^2 = [B(\text{sech}^2 \sqrt{2\lambda B} \,\rho)\rho'^2 + 1]dr^2 - B \,\text{sech}^2 \sqrt{2\lambda B} \,\rho \, dt^2 +$$
$$+ r^2 \, d\Omega^2 - dx^2.$$

Now apply the requirement above to get

(5.26) $$B(\text{sech}^2 \sqrt{2\lambda B} \,\rho)\rho'^2 + 1 = \frac{1}{B \,\text{sech}^2 \sqrt{2\lambda B} \,\rho} \, .$$

Setting $\zeta = \text{sech}^2 \sqrt{2\lambda B} \,\rho$ this becomes

(5.27) $$\frac{d\zeta}{\sqrt{1-\zeta} \sqrt{1-B\zeta}} = \pm 2 \sqrt{\frac{2\lambda}{B}} \, dr.$$

Now we show that <u>necessarily</u> $B = 1$.

Indeed assuming $B \neq 1$, integration of (5.27) gives

$$\tanh^{-1} \sqrt{\frac{1-\xi}{1/B-\xi}} = \pm\sqrt{2\lambda}\ (r-r_0),$$

with r_0 a constant of integration.

Solving for ξ we get

(5.28)
$$\text{sech}^2 \sqrt{2\lambda B}\ \rho = \xi = \frac{1-\frac{\eta}{B}}{1-\eta},$$

where $\eta = \tanh^2 \sqrt{2\lambda}\ (r-r_0).$

Clearly $0 \le \eta \le 1$. Since $0 \le \xi \le 1$, we get from (5.28)

$$\frac{1 - \eta/B}{1-\eta} \le 1,$$

i.e. $B \le 1.$

If $B < 1$ then for the values of η in the range $1 > \eta > B$, we get $\xi = \frac{1 - \eta/B}{1 - \eta} < 0$. This means that for $r > r_0 + \frac{1}{\sqrt{2\lambda}} \tanh^{-1} \sqrt{B}$ there is no (real) ρ satisfying (5.28).

Hence in order for (5.26) to have a real solution $\rho = \rho(r)$ over the whole range $(0,\infty)$ of r it is necessary that $B = 1$, as claimed.

In this case (5.27) yields

$$\frac{d\xi}{1-\xi} = \pm 2 \sqrt{2\lambda}\ dr,$$

i.e. in conjunction with (5.28),

(5.29)
$$\xi = \text{sech}^2 \sqrt{2\lambda}\ \rho = 1-e^{\mp 2\sqrt{2\lambda}\ (r-r_0)},$$

with $r_0 \ge 0$ a constant of integration.

For simplicity let us put

(5.30)
$$\mu = \sqrt{2\lambda} > 0.$$

The metric (5.25) then becomes

$$(5.31) \qquad ds^2 = \frac{dr^2}{1-e^{\pm 2\mu(r-r_o)}} - (1-e^{\pm 2\mu(r-r_o)})dt^2 +$$

$$+ r^2 \, d\Omega^2 - dx^2.$$

In order for t to retain its time character in (5.31) over the whole range $(0,\infty)$ of r, the coefficient $1-e^{\pm 2\mu(r-r_o)}$ must remain positive so that finally

$$(5.32) \qquad ds^2 = \frac{dr^2}{1-e^{-2\mu|r-r_o|}} - (1-e^{-2\mu|r-r_o|})dt^2 +$$

$$+ r^2 \, d\Omega^2 - dx^2.$$

Observe that, except for the correction term $-dx^2$ that changes spherical symmetry into cylindrical symmetry around the x-axis, the elementary geometry (5.32) has the desired form (5.22). This correction will be shown later to be responsible for the spin and nuclear magnetic moment of the particle, so we will call the x-axis through the origin, the spin-axis.

5.4.2 COMMENT. The representation (5.32) has the following useful property. If a large number of particles with randomly distributed spin axis x lie in a given bounded region, then on any observer far from the region, they will act as if they were spherically symmetric, because the terms $-dx^2$ will contribute to the average metric a term of the form $-p(dx^2+dy^2+dz^2)$, with $1 > p > 0$.

5.4.3 TRANSFORMATION FORMULAE

From (5.19) and (5.29) we get

$$(5.33) \qquad a^{-1} = \text{sech}^2 \mu\rho = 1-e^{-2\mu|r-r_o|} \qquad \text{i.e.}$$

$$\tanh \mu\rho = \pm e^{-\mu|r-r_o|} .$$

Hence

(5.34)
$$\rho = \pm \frac{1}{2\mu} \log \frac{1+e^{-\mu|r-r_o|}}{1-e^{-\mu|r-r_o|}} \, .$$

Similarly from

$$-\mu|r-r_o| = \log|\tanh \mu\rho|$$

we get

(5.35)
$$r = r_o + \frac{1}{\mu} \log|\tanh \mu\rho| \quad , (r < r_o)$$

(5.36)
$$r = r_o - \frac{1}{\mu} \log|\tanh \mu\rho| \quad (r \geq r_o).$$

Finally
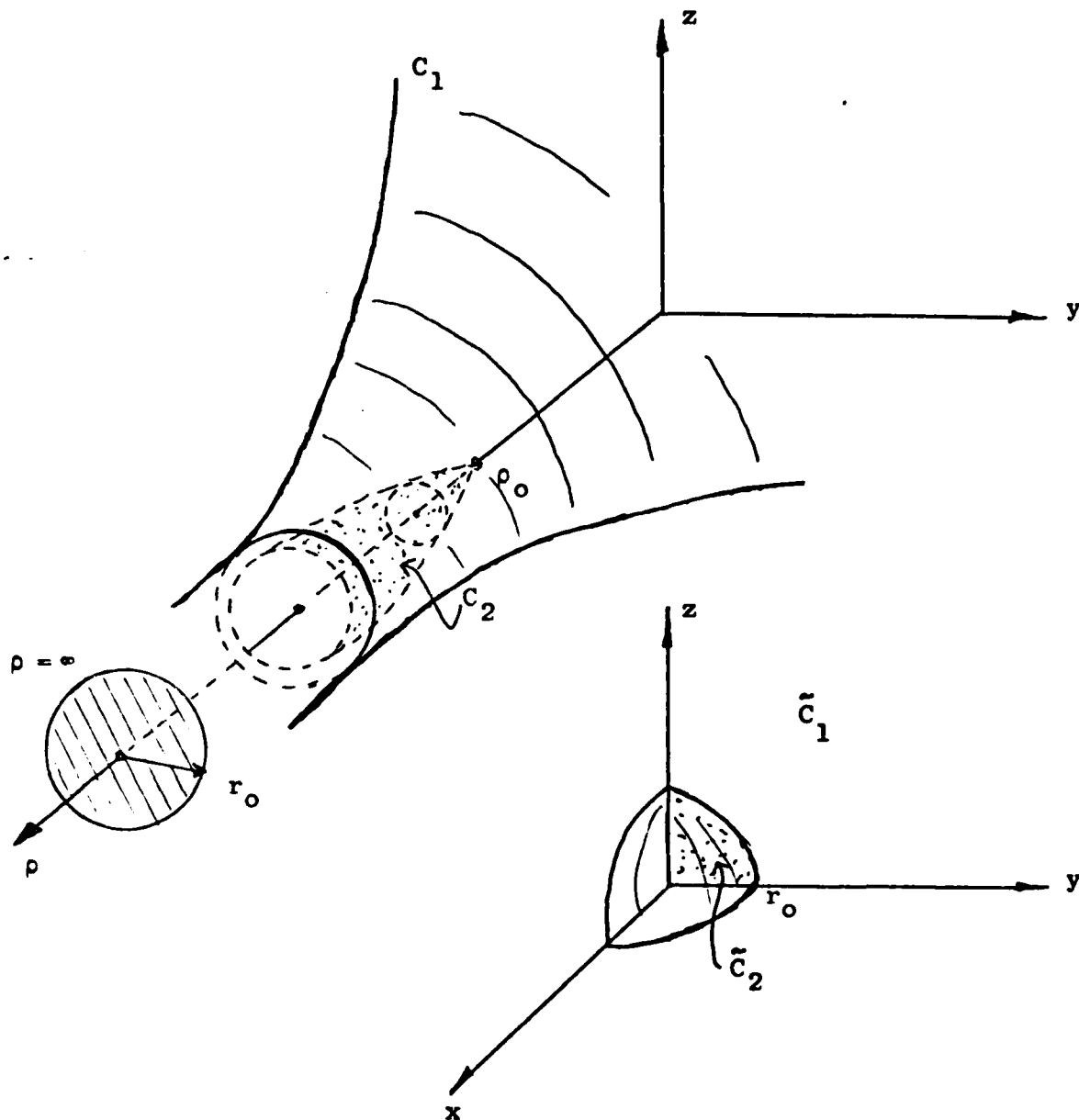
(5.37)
$$x = \pm[r_o + \frac{1}{\mu} \operatorname{sgn}(r_o-r)\log|\tanh \mu\rho| - y^2 - z^2]^{1/2}.$$

5.5 GLOBAL GEOMETRICAL CORRESPONDENCE

5.5.1 It is clear that the set of coordinates (r,y,z) in $\mathbb{R}^3$ are bona-fide coordinates only on the half-spaces $x > 0$ or $x < 0$.

Consider the $(\rho = X^1, \quad y = X^2, \quad z = X^3)$-space with the variables represented as cartesian coordinates in $\mathbb{R}^3$, and $\rho > 0$.

In the figure above the conoids C_1 and C_2 correspond to $x = 0$ i.e. to $r = (y^2+z^2)^{1/2}$ in (5.36) and (5.35), respectively.

From $y^2+z^2 \leq r^2$ combined with (5.35) and (5.36) we see that the interior of the conoids C_1 and C_2 in the figure are mapped, respectively, in the exterior and the interior of the sphere of radius r_0 in the $x > 0$ region of (x,y,z) space,

(assuming we choose the $+$ sign in (5.37)). The origin $r = 0$ corresponds to

(5.38)
$$\rho_o = \frac{1}{2\mu} \log \frac{1+e^{-\mu r_o}}{1-e^{-\mu r_o}} \, ,$$

while $r = r_o$ corresponds to $\rho = \infty$.

In (x,y,z) space $C_1 \cup C_2$ becomes the plane $x = 0$ whereas the plane $\rho = 0$ corresponds to $r = \infty$. C_1 becomes the region $\tilde{C}_1$ outside the circle of radius r_o' in the (y,z)-plane and C_2 its interior. The infinitely far disk $\rho = \infty$, $y^2+z^2 \le r_o$ becomes the half-sphere $r = r_o$, $x > 0$, and the whole region interior to $C_1 [C_2]$ becomes the half-space $x > 0$ outside [inside] the sphere $r = r_o$.

Observe that in (ρ,y,z) space the shell of the particle $(r=r_o)$ is located at ∞.

The alternative choice of $-$ sign in (5.37) covers the other half $x < 0$ of (x,y,z)-space.

Therefore the mapping of the interior of the conoid C_1 in the $\rho > 0$ region of (ρ,y,z)-space into (x,y,z) space as defined by (5.37), is two-to-one. Similarly for the interior of C_2, lying in $\rho > 0$.

Despite the fact that (5.34) gives us the choice of $\pm$ sign for ρ, we cannot encompass the mapping of the above solid conoids for both $\rho > 0$ and $\rho < 0$ continuously into (x,y,z)-space, trying to make it one-to-one, say.

Indeed, consider the outside of the particle for instance. In that region $\rho-t$ with ρ given by (5.34) is an incoming spherical wave for the $+$ choice of sign and an outgoing one for the $-$ sign. No continuous local choice of coordinates in (x,y,z)-

space will change one into the other (as far as one single particle is considered) because an outgoing wave sent by an isolated particle in R^4 remains outgoing forever.

5.5.2 CHARGE CONJUGATION

This physically obligatory dichotomy in the choice of ± signs in (5.34) classifies the particles (considered as geometries) into <u>two disjoint classes</u>: the <u>positively</u> and the <u>negatively charged particles</u> (see (5.47)). In particular the abstract operator C of <u>charge-conjugation</u> is represented in (ρ,y,z,t)-space by the concrete operation of <u>reflection</u> on the plane $\rho = 0$, namely $\rho \xrightarrow{C} -\rho$.

So, in last instance, the duality of electric charges is simply an expression of the duality existing in R^3 between outgoing and incoming light waves sent by an (isolated) particle. It is only when these intrinsic objects are used as coordinates to represent the particle geometry that this dichotomy becomes clear.

It is also clear now that the operators of <u>parity inversion</u> $(y,z) \xrightarrow{P} (z,y)$, <u>time-inversion</u> $t \xrightarrow{T} -t$ and <u>charge conjugation</u> $\rho \xrightarrow{C} -\rho$ <u>are all geometrical operations in the same</u> (ρ,y,z,t)- <u>space</u>. Their combinations acquire thus a purely geometrical meaning.

The important fact that $\rho-t$ represents incoming waves for the positively charged geometries and, at the same time, outgoing waves for the negatively charged ones, will serve to resolve the paradox of causality involved in the problem of interaction of particles through (apparently) advanced potentials (p. [11]), as we will see in Chapter X.

5.6 SINGULARITIES

Although (5.32) is a useful representation of the particle metric, it has one too many variables. In terms of the independent coordinates (r,y,z,t) the metric (5.32) becomes

$$(5.39) \qquad ds^2 = \frac{dr^2}{e^{2\mu|r-r_0|} - 1} - (1 - e^{-2\mu|r-r_0|})dt^2 + dy^2 + dz^2.$$

Clearly the sphere $r = r_0$, which we shall call the <u>shell</u> of the particle, is a singularity of the immersion.

As mentioned before (r,y,z) are bona-fide coordinates only for the semi-spaces $x > 0$ or $x < 0$. As we glue these two charts along the plane $x = 0$, in the natural way, we obtain a singularity of the so obtained geometry along this plane.

Indeed the element of volume induced by the elementary geometry (5.39) in the space-sections $t = $ const. is $\frac{1}{e^{2\mu|r-r_0|} - 1}$ drdydz which, for $r \neq r_0$, is well-behaved, whereas the set of unit vectors $e_y = \frac{\partial}{\partial y}$, $e_z = \frac{\partial}{\partial z}$ and $e_r = \frac{\partial}{\partial r}$ is degenerate (i.e. linearly dependent) on $x = 0$.

In particular a unit vector $e_x = \frac{\partial}{\partial x}$ lying on the plane $z = 0$ may be expressed as
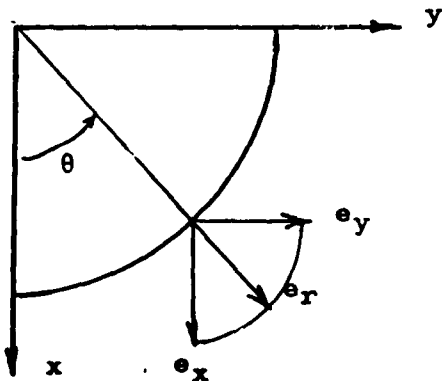
$$e_x = \frac{e_r}{\cos \theta} - e_y \, tg \, \theta.$$

The squared norm of e_x in the metric (5.39) is then

$$e_x^2 = (e^{2\mu|r-r_o|} - 1)\frac{1}{\cos^2\theta} + tg^2\theta$$

and clearly $e_x^2 \to \infty$ as $\theta \to \frac{\pi}{2}$, if r lies outside a neighbourhood of r_o. Consequently, in order for continuous vector fields defined in the vacuum background geometry to be continuous in the metric (5.39) too, they must be tangent to the (y,z)-plane at $x=0$, just as e_y, e_z and e_r $(r \neq 0)$ are. (Hence this is also the case for the gradient of the functions in the associated monochromatic algebras.)

From (5.39) we also see that as $r \to \infty$ the metric has the asymptotic degenerate expression $dy^2 + dz^2 - dt^2$, singling out again the spin direction x, along which the speed of light becomes infinite as $r \to \infty$. These singularities, (plane $x = 0$ and x-direction at ∞) reflect in part the remarkable physical properties, related to nuclear magnetic moment and ferromagnetism, associated with the spin, as we will see in Chapter X.

COMMENT. These unusual types of singularities rather than being unphysical, serve instead to explain the origin of some interaction forces. Indeed if the elementary geometries of various particles and of the vacuum are statistically superposed to produce an average metric which is to be asymptotically euclidean, or as close to it, at infinity, then mutual orientation forces will arise between the spin-axes of the various particles so as to conform to this condition at infinity. As for the metric degeneracy at infinity, a single particle will in fact be always statistically superposed to the vacuum, so that the speed of light, in the average metric, will in fact remain bounded as $r \to \infty$.

Recall that our elementary geometries will play, among geometries, a role somehow similar to that played by the Dirac delta among functions. It should not be surprising that they be singular in many respects.

5.7 ELECTRIC CHARGE

Since the jacobian matrix of the mapping $(\rho,y,z,t) \longmapsto$
$\longmapsto (r,y,z,t)$ is $\text{diag}(\rho',1,1,1)$, the electromagnetic tensor
$(5.12)'$ becomes in the new coordinates, noting that $q = \lambda$,

$$(5.40) \qquad F_{ij} = \pm \sqrt{8\pi\lambda}\; a^{-1}\rho' \begin{pmatrix} \bigcirc & & 1 \\ & 0 & \\ & 0 & \\ -1 & & \bigcirc \end{pmatrix}.$$

From (5.19), (5.26) and (5.33) we have

$$(a^{-1}\rho')^2 = 1-a^{-1} = e^{-2\mu|r-r_0|}, \qquad \text{i.e.}$$

$$(5.41) \qquad a^{-1}\rho' = \pm\, \text{sgn}(r_0-r)e^{-\mu|r-r_0|}, \qquad \text{for } r \neq r_0,$$

with the choice of sign being the same as in (5.34).

We choose once and for all the minus sign in (5.40), i.e.
$\theta = \pi$ in (5.11), so that for $r \neq r_0$, and for the $+$ sign in
(5.34)

$$(5.40)' \qquad F_{ij} = \sqrt{8\pi\lambda}\; \text{sgn}(r-r_0)e^{-\mu|r-r_0|} \begin{pmatrix} \bigcirc & & 1 \\ & 0 & \\ & 0 & \\ -1 & & \bigcirc \end{pmatrix}.$$

As in (5.14) the corresponding vector potential is

$$(5.42) \qquad A_i = -\sqrt{8\pi}\; e^{-\mu|r-r_0|}(0,0,0,1).$$

Consider now the orthonormal tetrad determined by the
(r,y,z,t)-axes

$$(5.43) \quad \begin{cases} \lambda^i_{(1)} = ([e^{2\mu|r-r_o|}-1]^{1/2}, 0,0,0), \\ \lambda^i_{(2)} = (0,1,0,0) \\ \lambda^i_{(3)} = (0,0,1,0) \\ \lambda^i_{(4)} = (0,0,0,[1-e^{-2\mu|r-r_o|}]^{-1/2}). \end{cases}$$

Clearly with respect to this orthonormal tetrad

$$E_2 = F_{(24)} = 0, \qquad E_3 = F_{(34)} = 0,$$

$$H_1 = F_{(23)} = 0, \qquad H_2 = F_{(31)} = 0, \qquad H_3 = F_{(12)} = 0,$$

and

$$E_1 = F_{(1,4)} = F_{ij}\,\lambda^i_{(1)}\,\lambda^j_{(4)} = \sqrt{8\pi\lambda}\ \mathrm{sgn}(r-r_o).$$

The magnetic field is null and the electric field is radial and constant in absolute value, oriented positively outside the shell and negatively inside it. On the shell of the particle the total resultant electric field is zero, which means no self-force. On the other hand the constant non-zero radial electric field inside the particle indicates a wormhole topology is more appropriate (p.837 [9]). See also section 5.13.

Since $|g|^{1/2} = e^{-\mu|r-r_o|}$ we have from (5.40)'

$$|g|^{1/2}\,F^{ij} = \sqrt{8\pi\lambda}\ \mathrm{sgn}(r-r_o) \begin{pmatrix} \bigcirc & & & 1 \\ & 0 & 0 & \\ & 0 & 0 & \\ -1 & & & \bigcirc \end{pmatrix}.$$

Hence by (5.7)

$$J^i = \frac{1}{4\pi}\,e^{\mu|r-r_o|}\,\sqrt{8\pi\lambda}\ 2\delta_{r_o}(r)(0,0,0,-1),$$

i.e.

$$(5.44) \qquad J^i = -\sqrt{\frac{2\lambda}{\pi}}\,(0,0,0,\delta_{r_o}(r)),$$

where $\delta_{r_o}(r)$ is Dirac's measure at $r = r_o$.

Formula (5.44) gives us the classical picture of a homogeneously charged spherical shell, but with non-classical metric and with zero self-force.

The electric charge enclosed in a region V_3 of the spatial section $t = $ const. is given by

$$(5.45) \qquad Q = \int_{V_3} J_{(4)} \, d_3 v,$$

where $J_{(4)} = J_i \lambda^i_{(4)}$, and $d_3 v$ is the element of volume in $t = $ const. derived from the given metric. As the metric representation (5.39) is singular at $r = r_o$, we work with volume forms. Thus

$$(5.46) \qquad J_{(4)} d_3 v = J_i \lambda^i_{(4)} \frac{e^{-\mu|r-r_o|}}{[1-e^{-2\mu|r-r_o|}]^{1/2}} \, dr\,dy\,dz =$$

$$= \sqrt{\frac{2\lambda}{\pi}} \, \delta_{r_o} \, dr\,dy\,dz.$$

Therefore integrating over any region V_3 which contains the particle we get for its charge, in the general case,

$$(5.47) \qquad Q = \pm \sqrt{8\pi\lambda} \; r_o^2 \, ,$$

with the choice of sign being the same as in (5.34).

5.8 INTRODUCTION OF UNITS

If we work in the C.G.S. system, Einstein's equations for the gravitational field are

$$(5.48) \qquad R_i^j - \frac{1}{2} R \, \delta_i^j + \Lambda \, \delta_i^j = \frac{8\pi G}{c^4} \, T_i^j \, ,$$

where $G = 6.67 \times 10^{-8} \; \text{cm}^3 \; \text{g}^{-1} \; \text{s}^{-2}$ (p.343 [6]).

Here G is determined by the condition that in the Newtonian approximation $(|v| \ll c)$ with $\Lambda = 0$, (5.48) yield Poisson's equation $\Delta\varphi = 4\pi G\rho$ and hence Newton's law of attraction $F = -G\dfrac{mm'}{r^2}$ (p.356 [6]).

If we are dealing with the curvature of space produced by fields of different nature, like for instance of electromagnetic or nuclear origin, we must, a priori, allow for a different constant in (5.4) (compare pp.191-192 [4], pp.123,203 [5]). We therefore introduce a factor $1/K$ in (5.48) so that its right-hand side becomes $\dfrac{8\pi G}{Kc^4} \; T^i_j$.

The factor K measures then the relative strength of the curvature produced by the classical gravitational field of masses with respect to that produced by the field in consideration. Comparing the right-hand side of (5.48) (with the above factor) with the left-hand side of (5.2) and using (5.10), we get

$$(5.49) \qquad \tau^j_i = \frac{Kc^4}{8\pi G}\, T^j_i = \frac{\lambda Kc^4}{8\pi G} \begin{pmatrix} -1 & & & \bigcirc \\ & 1 & & \\ & & 1 & \\ \bigcirc & & & -1 \end{pmatrix},$$

so that expressions of the energy become multiplied by the scalar factor

$$(5.50) \qquad \beta^2 = \frac{Kc^4}{8\pi G}\, ,$$

whereas the previous expressions for the electromagnetic tensor F_{ij}, the 4-current vector J^i, the vector potential A^i and the total charge Q are multiplied by β. So in C.G.S. units

$$(5.51) \qquad Q = \pm c^2 \sqrt{\frac{\lambda K}{G}}\, r_o^2\, .$$

Here the unknown parameters are λ, K, r_o.

5.9 ENERGY OF THE FIELD

The energy of the field for an observer whose unit time-like vector field is X is given by the integral of the energy-momentum tensor $\tau(X,X)$ evaluated at X, taken over the space sections determined by X.

It is convenient for us to start with (ρ,z,y,t)-space where the metric is given by

$$(5.52) \qquad\qquad ds^2 = a^{-1}(d\rho^2 - dt^2) + dy^2 + dz^2,$$

with $a^{-1} = \operatorname{sech}^2 \sqrt{2\lambda}\, \rho$.

We evaluate the energy in the rest frame of reference of the particle, namely that given by the unit vector field $X = (0,0,0,\sqrt{a})$.

From (5.49)

$$(5.53) \qquad\qquad \tau_{ij} = \lambda\beta^2 \begin{pmatrix} -a^{-1} & & & \bigcirc \\ & 1 & & \\ & & 1 & \\ \bigcirc & & & a^{-1} \end{pmatrix},$$

so that $\tau(X,X) = a\tau_{44} = \lambda\beta^2$.

Therefore the total energy on any spatial section $t = $ const. is given by

$$(5.54) \qquad\qquad E = \lambda\beta^2 V_3 \,,$$

where V_3 is the volume of this section in the induced metric.

We now switch to (r,y,z,t)-space where this induced metric is $\tilde{g}_{ij} = \operatorname{diag}\left(\dfrac{e^{-2\mu|r-r_o|}}{1-e^{-2\mu|r-r_o|}},\ 1,\ 1\right)$ and we get

$$(5.55) \qquad V_3 = 2 \int |\tilde{g}_{ij}|^{1/2} \, dr\,dy\,dz =$$

$$= 2 \int_0^\infty \left(\int_{y^2+z^2 \le r^2} \frac{e^{-\mu|r-r_o|}}{[1-e^{-2\mu|r-r_o|}]^{1/2}} \, dy\,dz \right) dr$$

$$= 2\pi \int_0^\infty \frac{r^2 \, e^{-\mu|r-r_o|}}{[1-e^{-2\mu|r-r_o|}]^{1/2}} \, dr.$$

The factor 2 is due to the fact that we are integrating over two charts $(x > 0$ and $x < 0)$.

One gets explicitly

$$(5.56) \qquad V_3 = \frac{\pi^2}{\mu^3} \Theta(R),$$

where

$$(5.57) \qquad R = \mu r_o = \sqrt{2\lambda} \, r_o,$$

and

$$(5.58) \qquad \Theta(R) = 2R^2 + (\ln 2)^2 + \frac{\pi^2}{12} +$$

$$+ \frac{4}{\pi} \sum_{n=1}^\infty \frac{1.3\ldots(2n-1)}{2.4\ldots(2n)} \cdot \frac{1-e^{-(2n+1)R}}{(2n+1)^3} .$$

To four significant digits in the coefficients

$$(5.58)' \qquad \Theta(R) = 2R^2 + 2.6047 - 1.2732 \, e^{-R} - 0.0236 \, e^{-3R}$$

$$- 0.0038 \, e^{-5R} - 0.0012 \, e^{-7R} + \epsilon,$$

$$0 \le \epsilon < 0.00136$$

Combining (5.54), (5.55) and (5.56) we get

$$(5.54)' \qquad E = \frac{\pi}{16} \frac{Kc^4}{G\sqrt{2\lambda}} \Theta(R).$$

Identifying the energy of the electromagnetic field to the rest mass m_o of the particle we get therefore

(5.59)
$$\boxed{m_o = \frac{\pi}{16} \frac{Kc^2}{G\sqrt{2\lambda}} \Theta(R)} \ ,$$

with R given by (5.57).

5.10 MASS OF THE ASSOCIATED MASSIVE QUANTA

The Laplace-Beltrami operator corresponding to the metric (5.52) is

(5.60)
$$\Delta_2 = a(\partial_1^2 - \partial_4^2) + \partial_2^2 + \partial_3^2 \ ,$$

where $\partial_i \equiv \frac{\partial}{\partial X^i}$.

As shown in Chapter IV the monochromatic algebras $\mathfrak{h}$ and $\mathfrak{h}'$ associated with this elementary geometry are given by C^2 functions of $X^1 - X^4$ and $X^1 + X^4$ respectively, analytic in $X^2 + iX^3$. For definitness we reserve $\mathfrak{h}, \mathfrak{h}'$ to denote those analytic and $\bar{\mathfrak{h}}, \bar{\mathfrak{h}}'$ to denote the anti-analytic ones.

As seen in 4.3.6, the massive quanta described by the envelope space $e[\bar{e}]$ generated by $\mathfrak{h}$ and $\mathfrak{h}'$ [$\bar{\mathfrak{h}}$ and $\bar{\mathfrak{h}}'$, resp.] are again analytic [anti-analytic] in $X^2 + iX^3$. Consequently Δ_2 acting on e or $\bar{e}$ reduces to $a(\partial_1^2 - \partial_4^2)$. Hence the elementary states in e or $\bar{e}$ are <u>solutions of the string equation</u>, i.e. linear combinations of functions of $X^1 - X^4$ and $X^1 + X^4$ analytic [resp. anti-analytic] in $X^2 + iX^3$. In other words the subspace of elementary states in $e[\bar{e}]$ coincides with the linear spann of $\mathfrak{h}$ and $\mathfrak{h}'$ [$\bar{\mathfrak{h}}$ and $\bar{\mathfrak{h}}'$, resp.].

5.10.1 STATIONARY STATES

If we look for time-periodic solutions of $\Delta_2 \psi = 0$ in $\mathcal{C}$ of the form

$$\psi = \varphi(X^1, X^2 + iX^3) e^{imX^4}, \quad m \in \mathbf{R},$$

we get

(5.61)
$$\partial_1^2 \varphi = -m^2 \varphi$$

i.e.

(5.62)
$$\varphi = A e^{imX^1} + B e^{-imX^1} = C \sin(mX^1 + \gamma_m),$$

where A, B, C and γ_m are analytic functions of $X^2 + iX^3$.

If we look at ψ in the concrete referential (r, y, z, t) we see that in order for it to be smooth at the origin $r = 0$ we must impose the boundary condition

(5.63)
$$\left. \frac{\partial \varphi}{\partial r} \right|_{r=0} = 0, \qquad \text{i.e.,}$$

$$\frac{\partial \varphi}{\partial X^1} = 0$$

at $X_0^1 \equiv \rho_0 = \frac{1}{2\mu} \log \frac{1 + e^{-\mu r_0}}{1 - e^{-\mu r_0}}$, for a positively charged particle, and at $X_0^1 = -\rho_0$ for a negative one.

At $r = +\infty$ we impose the boundary condition (to be justified later)

(5.64)
$$\left. \varphi \right|_{r=+\infty} = 0, \qquad \text{i.e.,}$$

$$\left. \varphi \right|_{X^1=0} = 0,$$

which implies in (5.62) that $\gamma_m = 0 \pmod{\pi}$, i.e. that $\varphi = C \sin mX^1$.

Together with the other boundary condition (5.63), this yields $\cos m \rho_0 = 0$, which implies $m\rho_0 = (2n+1)\frac{\pi}{2}$, $n = 0,1,2,\ldots$ i.e.

(5.65) $$m = \left(n + \frac{1}{2}\right)\frac{\pi}{\rho_0} \qquad n = 0,1,2,\ldots$$

If we set $X^4 = ct$ in the expression e^{imX^4}, so that this t is expressed in time units (seconds), we get e^{imct}, which compared with the usual expression $e^{i\frac{E}{\hbar}t}$ yields $E = \hbar cm$. Consequently the energy levels of the stationary elementary states in $\mathcal{E}$ are given by

(5.66) $$E_n = \left(n + \frac{1}{2}\right)\hbar c\,\frac{\pi}{\rho_0} = \left(n + \frac{1}{2}\right)\frac{hc}{2\rho_0}, \qquad n = 0,1,\ldots$$

In particular the mass m_0' of the associated massive quanta, corresponding to the unit jump of the energy levels is

$$m_0' = \frac{1}{c^2}\cdot\frac{hc}{2\rho_0} = \frac{h}{c}\cdot\frac{1}{2\rho_0},$$

i.e. using (5.30) and (5.38)

(5.67) $$m_0' = \frac{h}{c}\,\frac{\sqrt{2\lambda}}{\log\dfrac{1+e^{-\mu r_0}}{1-e^{-\mu r_0}}}.$$

5.11   BOUND STATES AND THE BOUNDARY CONDITION AT INFINITY

From (5.62) we have

$$\varphi = (A+B)\cos mX^1 + i(A-B)\sin mX^1.$$

For large r we get from (5.34)

$$X^1 = \rho \sim \mu^{-1} e^{-\mu|r-r_0|},$$

so that

(5.68) $\qquad \varphi \sim A+B + im(A-B)\mu^{-1} e^{-\mu|r-r_o|}$ as $r \to \infty$.

If A and B are analytic functions of X^2+iX^3 or order less than one, or of order one and type less than μ, then $(A-B)e^{-\mu|r-r_o|} \in L^2(\mathbb{R}^3)$. In order for φ to belong in $L^2(\mathbb{R}^3)$ it is then necessary and sufficient that $\int_{\mathbb{R}^2} |A+B|^2 \, dX^2 dX^3 = 0$ i.e. that $A+B \equiv 0$, which is equivalent to (5.64).

In other words, under the above assumptions on A and B, the boundary condition $\varphi\big|_{r=+\infty} = 0$ is a necessary and sufficient condition in order for the elementary stationary states in e or $\bar{e}$ to belong in $L^2(\mathbb{R}^3)$ i.e. for them to correspond to <u>bound states</u> of the particle, with respect to the vacuum backgrond metric.

In particular it also implies that the bound states in e and e' are <u>antisymmetric functions</u> of ρ and that the charge conjugation operator C preserves the bound states of same energy level.

We remark that the almost periodic functions $e^{im(X^1+X^4)} \in h$ and $e^{im(-X^1+X^4)} \in h'$, produce, by linear super-position, the stationary states $e^{i(n+\frac{1}{2})m'_o X^4} \sin(n+\frac{1}{2})m'_o X^1$, which are again almost periodic and represent massive quanta of the particle, just as in Example 3.3. These in turn are transformed in concrete (r,y,z,t)-space into stationary bound states, i.e. into time-periodic and Lebesgue square-integrable functions in the $\mathbb{R}^3$ space section of background space.

This change of structure (valid only for the subspace ge-nerated by the above stationary states) from almost-periodic functions into $\mathbb{R}^3$ square-integrable ones, is accomplished by the requirement of zero boundary condition in concrete space or, equivalently, by the requirement of antisymmetry in ρ in iso-

tropic (ρ, t)-space.

5.12 COMMENTS

The stationary states are given by

$$(5.69) \qquad e^{i(n+\frac{1}{2})m_0' x^4} \sin(n+\tfrac{1}{2})m_0' x^1, \qquad n = 0,1,2,\ldots,$$

$$(5.70) \qquad x^1 = \rho = \pm \frac{1}{2\mu} \log \frac{1+e^{-\mu|r-r_0|}}{1-e^{-\mu|r-r_0|}} .$$

As functions of r they are symmetric about $r = r_0$ in the intervals $[0, r_0]$ and $[r_0, 2r_0]$ and oscillate infinitely often near $r = r_0$, singling out again the shell of the particle as a singular region.

For $r > 2r_0$ they oscillate only a finite number of times and become asymptotic to $m e^{\mu r_0} \mu^{-1} e^{-\mu r}$, for all $m = (n+\tfrac{1}{2})m_0'$, as $r \to \infty$. In particular the asymptotic bound states $m e^{\mu r_0} \mu^{-1} e^{-\mu r} (y \pm iz)$ are equal, except for the numerical value of the coefficients, to the eigenfunctions of the hydrogen atom corresponding to the parameters $N = 2$, $\ell = \pm 1$ on p.229 [12], namely $(y \pm iz) e^{-\frac{r}{2r_0}}$.

Here r_0 is the radius of the first Bohr orbit so that $1/2r_0 \simeq 10^8$ cm^{-1}. As we will see, for the electron $\mu \simeq 10^{12}$ cm^{-1}.

As remarked in 5.11 the states (5.69) arise as statistical superpositions, at each instant of time, of an outgoing and an incoming spherical (monochromatic) light wave. If multiplied, or added to $(y \pm iz)$, say, they yield standing massive quanta.

5.13 WORMHOLE TOPOLOGY

Since the metric and the functions in h and h' are symmetric about $r = r_0$ in the intervals $[0, 2r_0]$, we may as well represent the region $[0, r_0]$ as a second copy of the region $[r_0, 2r_0]$ and therefore consider only the solid conoid C_1 in 5.5, in the $\rho > 0$ region. If we change the topology of R^3 by excluding from it the open ball of radius r_0 and consider another copy of the remaining region having the sphere $r = r_0$ in common with the first copy, we can now map the conoid C_1 in the region $\rho < 0$ into the second copy.
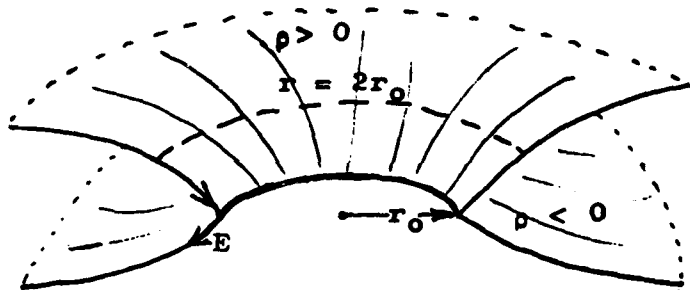
In this new topology our functions are defined for all real values of ρ, the stationary states are antisymmetric in ρ, and the non-zero radial constant electric field is continuous everywhere. The electric charge is then represented by the topological effect of the trapping of electric lines of force by the hole (p.1200, [9]).

Further we may think the second copy as continued into the first until the sphere $r = 2r_0$ is reached and consider this added region as the interior of the shell as seen by an observer in the second copy. And vice-versa. Yet since the jacobian ρ' in these two copies have opposite signs, the electric vector field must be taken as reversed (according to (5.40) and sequel).

We could call the copy $\rho < 0$ the anti-space of the copy $\rho > 0$, for observers in $\rho < 0$ see the particle with a negative charge, and vice-versa. This is of course only a mathematical construct, but gives a place where to situate anti-matter. Similarly we call the sphere $r = 2r_o$ the anti-center of the particle (for it is the center of the corresponding anti-particle).

5.14 DETERMINATION OF THE PARAMETERS

We recollect the relevant formulae

(5.51)
$$Q = e = c^2 \sqrt{\frac{\lambda K}{G}}\, r_o^2 \; ,$$

(5.59)
$$m_o = \frac{\pi}{16}\,\frac{Kc^2}{G\sqrt{2\lambda}}\,\Theta(R),$$

(5.67)
$$m_o' = \frac{h}{c}\,\frac{\sqrt{2\lambda}}{\log \coth \frac{R}{2}} \; ,$$

where

(5.71)
$$R = \sqrt{2\lambda}\, r_o = \mu r_o \; ,$$

hence

(5.72)
$$\lambda\, r_o^2 = \frac{R^2}{2} \; ,$$

and $\Theta(R)$ is given by (5.58).

We explicit K in (5.51), using (5.72)

(5.73)
$$K = \frac{e^2 G}{c^4 \lambda r_o^4} = \frac{2e^2 G}{c^4} \cdot \frac{1}{r_o^2 R^2} \; ,$$

and then use it in (5.59), obtaining

$$m_o = \frac{\pi}{8} \cdot \frac{e^2}{c^2} \cdot \frac{1}{r_o} \cdot \frac{\Theta(R)}{R^3}, \qquad \text{i.e.,}$$

(5.74)
$$m_o r_o = \frac{\pi}{8} \cdot \frac{e^2}{c^2} \cdot \frac{\Theta(R)}{R^3} \; .$$

Next we multiply (5.67) by r_0 and use (5.71) to get

$$(5.75) \qquad m_0' r_0 = \frac{h}{c} \frac{R}{\log \coth \frac{R}{2}} .$$

From (5.75) and (5.74) we get

$$(5.76) \qquad \frac{m_0}{m_0'} = \frac{1}{16} \cdot \frac{e^2}{\hbar c} \cdot \frac{\Theta(R)}{R^4} \log \coth \frac{R}{2} .$$

Finally

$$(5.77) \qquad \log \coth \frac{R}{2} = \frac{16}{\alpha} \cdot \frac{m_0}{m_0'} \cdot \frac{R^4}{\Theta(R)} ,$$

where $\alpha = \frac{e^2}{\hbar c} \simeq \frac{1}{137.036}$ is the fine structure constant.

For a given ratio m_0/m_0' there is one and just one root $R = R(m_0/m_0')$ of (5.77). Indeed we have from (5.55) and (5.56), setting $s = r/r_0$,

$$\frac{\Theta(R)}{R^4} = \frac{2}{\pi} \cdot \frac{1}{R} \int_0^\infty s^2 \frac{e^{-R|s-1|}}{[1-e^{-2R|s-1|}]^{1/2}} ds .$$

This shows $\Theta(R)/R^4$ is monotonically decreasing with $R \geq 0$, from $+\infty$ at $R = 0$ to 0 as $R \to +\infty$. Hence $R^4/\Theta(R)$ increases monotonically from 0 to $+\infty$ for $R \geq 0$. On the other hand $\log \coth \frac{R}{2}$ decreases monotonically from $+\infty$ at $R = 0$ to 0 at $R = +\infty$.

This shows that, for a given ratio m_0/m_0', there is just one solution of (5.77), and that R is a decreasing function of this ratio.

From (5.75) we then have

$$(5.78) \qquad r_0 = \frac{h}{c} \cdot \frac{R}{\log \coth \frac{R}{2}} \cdot \frac{1}{m_0'} ,$$

whereas λ and K are then given by (5.72) and (5.73), respectively.

5.14 PROTON

For the proton $m_0 = 1.672614 \times 10^{-24}$ g and for its quanta (pion $\pi_0 \to 2\gamma$), $m_0' = 2.40616 \times 10^{-25}$ g.

Hence $m_0/m_0' = 6.95138$ so that (5.77) becomes

$$\log \coth \frac{R}{2} = 15{,}241.44 \frac{R^4}{\Theta(R)} \ .$$

The root of this equation was computed with double precision using the expression (5.58)' first with $\epsilon \doteq .0014$ and then with $\epsilon = 0$, and then taking as upper bound the upper bound of the numerical results for the first case and as lower bound, the lower bound of the second case. This assures that the root of the above equation lies in

(5.79) $0.12822 < R < 0.12826.$

However for reasons to be explained in Chapter VI we take for R the value

(5.80) $R = 0.128172 \simeq 0.1282,$

in which case

(5.80)' $\log \coth \frac{R}{2} = 2.7488936.$

Since $h = 6.626197 \times 10^{-27}$ gcm^2/sec and $c = 2.997925 \times 10^{10}$ cm/sec, (5.78) gives

(5.81) $r_0 = 0.4283 \times 10^{-13}$ cm.

From (5.72) we get

$$\lambda = 4.478 \times 10^{24} \text{ cm}^{-2},$$

and from (5.73)

$$K = 1.264 \times 10^{-39},$$

where we used the values

$$e = 4.803 \times 10^{-10} \ (gcm^3)^{1/2}/sec$$

$$G = 6.673 \times 10^{-8} \ cm^3/g \ sec^2 \ .$$

We see that the computed value for the radius r_o of the proton agrees well with the estimated value for the hard core of a nucleon $(\sim 0.5 \times 10^{-13} \ cm)$ (p.59, [2]).

5.16 ELECTRON

From the decay of the muon

$$\mu^- \rightarrow e^- + \nu + \bar{\nu},$$

and from the fact that no other particle decays into the electron emitting two light quanta, we may take it for granted that the pair $\nu + \bar{\nu}$ plays the role of the electron quanta.

Since the ratio of the mass of the negative muon to the mass of the electron is 206.77 we have, for the electron, $m_o'/m_o = 206.77-1 = 205.77$.

Therefore (5.77) becomes

$$\log \coth \frac{R}{2} = 10.6556 \ \frac{R^4}{\Theta(R)} \ .$$

Repeating the same procedure as for the proton, we obtain

(5.82) $0.74077 < R < 0.74086.$

Again, for reasons to be explained in Chapter VI, we take

(5.83) $R = 0.732985,$

for which

(5.84) $$\log \coth \frac{R}{2} = 1.047198.$$

As $m_o = 9.10956 \times 10^{-28}$ g the above ratio gives $m'_o = 1.8745 \times 10^{-25}$ g, so that from (5.78) we get

(5.85) $$r_o = 8.2532 \times 10^{-13} \text{ cm}.$$

As before, (5.72) and (5.73) yield now

$$\lambda = 3.944 \times 10^{23} \text{ cm}^{-2}$$

$$K = 1.0415 \times 10^{-43}.$$

The above computed radius r_o for the electron is about three times as large as the classical electron radius conventionally defined as $e^2/m_o c^2 \simeq 2.818 \times 10^{-13}$ cm. Anyway, contrarily to the nucleons, electrons are a very soft particle for they behave as if they were point-like (p.23, [2]). This behavior makes unpracticable the experimental determination of its radius. This matter will be further analyzed and explained in Chapter X.

The cosmological constants λ have the appropriate order of magnitude and so also the coupling constants $1/K$ (see [5]).

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generalized curves, maximal monochromatic algebras, light quanta, massive quanta, constant curvature, Bertotti-Robinson metrics, electron, photon, neutrons, protons, gauge theory, quantum transitions, quaternions

20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

This report consists of the first part of a general theory purporting to describe the mathematical structure of the elementary particles, starting from no preassumed knowledge, but deriving it instead from first principles along the line suggested by Dirac in the 1930's. In particular, quantum mechanics is shown to arise as a consequence of relativity theory and of the
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ABSTRACT (continued)

theory of generalized curves. In the first part the geometric structure
(i.e. the nuclear field) is derived, and one obtains a slightly modified
form of the Yukawa potential along with a cylindrical perturbation
describing the spin effects. This report gives full details of the
results announced in the two previous reports, MRC Technical Summary
Report #2067, which appeared in Proc. Ioffe Conf. Imperial College, London
(1980) and MRC Technical Summary Report #2317 appearing in J.O.T.A., Sept.
1983.