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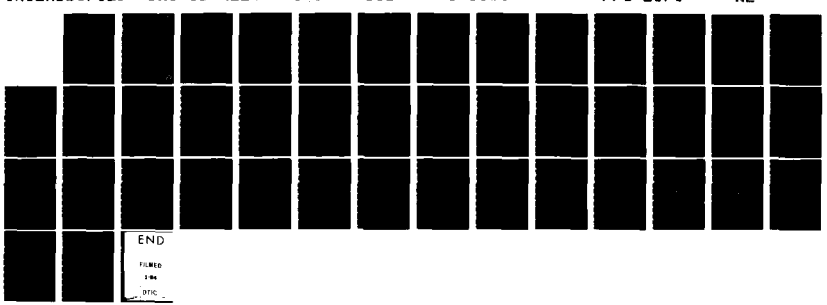

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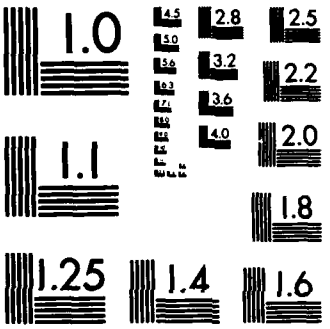

SAI-83-1124/PRI-66 October 1983

NONLINEAR ORBITS IN FREE-ELECTRON LASERS WITH A LINEAR MAGNETIC WIGGLER AND A STRONG AXIAL MAGNETIC GUIDE FIELD

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Abstract

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER SAI-83-1124/PRI-66	2. GOVT ACCESSION NO. AD-A135 984	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Nonlinear orbits in free-electron lasers with a linear magnetic wiggler and a strong axial magnetic guide field		5. TYPE OF REPORT & PERIOD COVERED 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) B. Hafizi and R. E. Aamodt		8. CONTRACT OR GRANT NUMBER(s) N00014-79-C-0555
9. PERFORMING ORGANIZATION NAME AND ADDRESS Science Applications, Inc. Plasma Research Institute 934 Pearl Street, Boulder, Colorado 80302		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research (Code 412) 800 N. Quincy Street Arlington, VA 22217		12. REPORT DATE October 1983 13. NUMBER OF PAGES 39
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

free-electron lasers
nonlinear orbits
orbital stability
directed-beam quality
stochasticity

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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S/N 0102-LF-014-6601

unclassified
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

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the drift chamber experience limited excursions in gyroradius and in axial velocity are determined. The mean drift of the guiding center off axis is used to determine the allowable length of the wiggler and of the drift tube for beam propagation. With increasing wiggler field bounded motion is eliminated, leading to transition between resonances, chaotic motion and significant spread in axial velocities of electrons. Comparison with experimental results is presented.

I. INTRODUCTION

With a sufficiently high current, the ponderomotive force arising from the interaction of the directed momentum of particles with the magnetic wiggler in a free electron laser (FEL) drives a space charge wave in the beam. Although this increases the efficiency of the device, one drawback in the use of a high-current, nonneutral beam--typically produced by a Marx-type generator--is that the beam can rapidly expand due to its own space charge effects. To remedy this, it is natural to introduce an axial magnetic guide field in addition to the magnetic wiggler field necessary for laser operation. However, the combined influence of these fields may inhibit beam propagation and degrade the amplification properties of the system. For example, the collective cyclotron modes of the beam due to the presence of the guide field can compete with the FEL mode, and, on the level of single particle motion, interaction between the gyromotion and the periodicity in the axial momentum due to the presence of the wiggler can induce a resonant transfer of the axial momentum for that in the perpendicular direction. Since this leads to dispersal of electrons within a beam, an increase in spectral width and a reduction in gain is inevitable. For efficient operation it is therefore imperative that the orbits of the electrons be free of large amplitude variations in axial velocity and that the off-axis drifts be limited in extent so that the electrons do not reach the walls of the drift tube.

The present paper is addressed to the single particle aspect of these problems, which might be referred to as issues concerned with orbital stability, including beam propagation and directed-beam quality. Specifically, a perturbative technique of sufficient generality to study the nonlinear stability properties of wiggler fields of arbitrary geometry superimposed on a strong

uniform axial guide field is developed and applied to the case of a periodic wiggler.

It should be emphasized from the outset that the formalism presented herein is applicable to electric and magnetic fields satisfying the full set of Maxwell's equations including, in principle, all self fields. However, for simplicity in this study only vacuum fields will be considered. The use of the correct functional form for a solenoidal (div $\vec{H} = 0$) and irrotational (rot $\vec{H} = 0$) magnetic field, aside form satisfying the constraints imposed by Maxwell's equations,[1] is particularly important in the case of a linear wiggler since the net drift of guiding centers induced by the azimuthal asymmetry of the field dictates that the off-axis variation of the magnetic field be included.

For sufficiently small wiggler fields nonlinear orbital stability is assured for certain classes of particles. Put differently, these classes are characterized by bounded (but not necessarily small) oscillations of gyroradius and of axial velocity. The parameters characterizing these classes are determined and the optimal values for maximizing the number of particles undergoing limited excursions in axial velocity may be deduced. While boundedly oscillating in axial velocity, the guiding centers of these particles experience a mean drift off axis which restricts the maximal interaction length of the drift tube. This propagation length is also determinable from the analysis.

With increased wiggler strength, bounded motion is no longer allowed above some threshold value which depends on the region of phase space. In other words, as the wiggler amplitude increases, the "confined" classes alluded to in the foregoing are eliminated one after the other until no region of phase space contains within itself electrons whose motions are bounded. When a given confined class is eliminated, the electrons therein experience large changes in

axial velocity as they jump from one resonance to the next in a haphazard manner, increasing the effective parallel temperature of the beam and thereby reducing the gain. The analysis which follows furnishes numerical values for the wiggler field strength below which such undesirable effects are avoided.

The organization of this paper is as follows. In Sec. II a simple generalization of the usual definitions of magnetic moment, gyrophase, and of guiding-center coordinates is employed to rewrite the single particle Hamiltonian in a form suitable for analysis of the effects of resonance between the gyromotion and the periodicity induced in the axial motion by the wiggler. In Sec. III it is shown that, provided the motion is dominated by a single resonance, the system possesses a hidden symmetry, which is manifested by the presence of a conserved quantity that is independent of the Hamiltonian. This fact allows a complete qualitative description of all possible motions of a particle propagating through the drift tube, as detailed in Sec. IV. Section V considers the possibility of stochastic motion under the influence of multiple resonances. The analysis presented in this work is put into perspective by examining the results of a free electron laser experiment in Sec. VI.

II. HAMILTONIAN FORMALISM

In principle, starting from Newton's equations of motion for an electron under the combined influence of a magnetic wiggler and an axial magnetic guide field application of perturbation theory permits the determination of many of the orbit characteristics needed for most FEL applications. Such an approach has already been successfully applied to the case of a longitudinal magnetic wiggler.[2] It turns out that a significant generalization and simplification can be simultaneously effected in problems of this nature by use of powerful Hamiltonian techniques.

A vector potential representing a linear magnetic wiggler of the form (0, δH coskz ch ky , $-\delta H$ sinkz sh ky) with period $2\pi k^{-1}$ along the z-axis superimposed on a guide field (0, 0, H_0) may be written as

$$\vec{A} = (-H_0 y + \delta H k^{-1} \text{ sinkz ch ky}, \ 0, \ 0) \ ,$$

where ch ky and sh ky are the hyperbolic cosine and sine of ky, respectively. The corresponding single-particle (relativistic) Hamiltonian $\mathscr{H}$ is given by

$$\mathscr{H}^2/c^2 - m^2c^2 = p_z^2 + p_y^2 + (P_x + m\Omega_0 y - m\delta\Omega k^{-1} \text{ sinkz ch ky})^2,$$

$$\equiv \hat{\mathscr{H}} \ ,$$

wherein $\Omega_0 = qH_0/mc$ and $\delta\Omega = q\delta H/mc$ are the gyrofrequencies of a particle of charge q and (rest-) mass m in magnetic fields of intensity H_0 and δH, respectively. The momenta p_z, p_y, and P_x are related to the velocities v_i (i = x,y,z) in the usual way:[3]

$$p_{z,y} = \gamma m v_{z,y} \quad ,$$

$$p_x = \gamma m v_x + (q/c)A_x \quad ,$$

with

$$\gamma = [1 - (v_x^2 + v_y^2 + v_z^2)/c^2]^{-\frac{1}{2}} \quad ,$$

the relativistic factor, being related to the Hamiltonian (or energy) via $\mathscr{H} = \gamma mc^2$. In the present analysis, in which self fields are neglected and the only external fields are magnetic, the energy $\mathscr{H}$ and the relativistic factor γ are constants of the motion. The gyrofrequencies corrected for the relativistic mass variation are $\Omega_o^{rel} = \Omega_o/\gamma$ and $\delta\Omega^{rel} = \delta\Omega/\gamma$.

It is usual practice in studying particle motion in, say, a magnetic trap or in the earth's magnetic field to use, in lieu of the momenta p_i, certain adiabatic invariants which are constants of the motion to great accuracy for slowly varying fields.[4] One such variable is the magnetic moment $\mu = |\vec{v}_\perp|^2/2|\vec{H}|$, with $\vec{v}_\perp$ being the component of velocity perpendicular to the magnetic field $\vec{H}$. Introduction of a (generalized) magnetic moment is expedient in the present case, too, since it is a strict constant of the motion in the absence of, and undergoes slow evolution in the presence of, a small amplitude wiggler.

Effecting a transformation $(x,P_x;\ y,p_y;\ z,p_z) \rightarrow (\theta,\mu;\ Y,m\Omega_o X;\ z,p_z)$ via

$$\mu = (2m\Omega_o)^{-1}[(P_x + m\Omega_o y)^2 + p_y^2] \quad ,$$

$$\theta = \text{arctg}[(P_x + m\Omega_o y)/p_y] \quad ,$$

$$Y = -(m\Omega_o)^{-1}P_x \quad ,$$

$$X = x + (m\Omega_o)^{-1}p_y \quad ,$$

the Hamiltonian may be rewritten as

$$\hat{\mathcal{H}} - \left(\frac{m\delta\Omega}{2k}\right)^2 = p_z^2 + 2m\Omega_o\mu + k^{-1}2m\delta\Omega m\Omega_o\rho \sin\theta \sin kz \cosh k(Y - \rho\sin\theta)$$

$$+ \left(\frac{m\delta\Omega}{2k}\right)^2[\cosh 2k(Y - \rho\sin\theta) - \cos 2kz] - \left(\frac{m\delta\Omega}{2k}\right)^2 \cosh 2k(Y - \rho\sin\theta)\cos 2kz$$

wherein

$$\rho = \left(\frac{2\mu}{m\Omega_o}\right)^{\frac{1}{2}} \quad , \tag{1}$$

and

$$X = x - \rho\cos\theta \quad ,$$

$$Y = y + \rho\sin\theta \quad .$$

Defining

$$\varepsilon = \delta H/H_o = \delta\Omega/\Omega_o \quad ,$$

$$\alpha = (m\Omega_o/2k)^2 \quad ,$$

and

$$\zeta = k\rho \quad , \tag{2}$$

and noting that $\partial_\zeta \sinh(kY - \zeta\sin\theta) = -\sin\theta \cosh(kY - \zeta\sin\theta)$, the Hamiltonian can be expressed as

$$\hat{\mathcal{H}} - \varepsilon^2\alpha = p_z^2 + 2m\Omega_o\mu - 8\varepsilon\alpha\zeta \sin kz\,\partial_\zeta \sinh(kY - \zeta\sin\theta) +$$

$$+ \varepsilon^2\alpha[\cosh 2(kY - \zeta\sin\theta) - \cos 2kz] - \varepsilon^2\alpha \cosh 2(kY - \zeta\sin\theta)\cos 2kz \; .$$

The crucial point that must be emphasized is that in the absence of the wiggler ($\delta H = \varepsilon = 0$) the foregoing definitions for μ, θ, X, Y, and ρ correspond

to the usual definitions of magnetic moment, of gyroangle, of the x-, and y-components of the guiding-center, and of gyroradius of the particle in the (uniform) guide field H_0, Fig. 1. This correspondence does not persist with a non-zero wiggler field, in part due to the appearance of $P_x + m\Omega_0 y$ -- which equals $\gamma m v_x$ plus a correction term proportional to the vector potential of the wiggler -- in the definitions. Notwithstanding, the generalized transformation effected in this paper is exact and designed with a view to constructing a simple perturbation theory in powers of ε.

Let it be noted, too, that the change of variables u[illegible] belongs to the class of canonical transformations which in essence means [illegible] the equations of motion for the new variables are given by the usual formulas; thus

$$\dot{\mu} = -\partial_\theta \mathcal{K}$$

$$\dot{\theta} = \partial_\mu \mathcal{K} = \Omega^{rel} + o(\varepsilon) \quad , \tag{3}$$

$$\dot{z} = \partial_{p_z} \mathcal{K} = \frac{p_z}{\gamma m} = v_z \quad , \tag{4}$$

etc. In particular, since the expression for $\mathcal{K}$ is not an explicit function of the x component of the guiding-center, X, the y component of the guiding-center, Y, which is conjugate to $m\Omega_0 X$, is a constant of the motion. This is merely a reflection of the fact that, in terms of the original variables, $\vec{A}$ and hence $\mathcal{K}$ are independent of x, which implies that P_x is a constant of motion.

III. RESONANT INTERACTION

In the context of the present analysis, resonance refers to the matching between some integer multiple ℓ of the gyrofrequency and the effective frequency of the modulated motion along the z axis; i.e., $\ell\dot{\theta} + k\dot{z} = 0$, which may be re-expressed as $\ell\Omega_o^{rel} + kv_z \simeq 0$ on using Eqs. (3) and (4). Such a resonance, commonly referred to as a coupling resonance, induces a vigorous exchange of energy between the longitudinal and perpendicular degrees of freedom of a particle traversing the device. As is well-known, gyrotrons operate by transforming the perpendicular energy of the particles into electromagnetic radiation at the cyclotron frequency Ω/γ. Free electron lasers, with their very favorable wavelength scaling ($\lambda_{radiation} \simeq \frac{1}{2}\lambda_{wiggler}/\gamma^2$), depend for successful operation on the emission of radiation originating from the directed energy of the particles. For this reason and also to avoid the possibility of the radiation level being reduced by Landau damping, in FEL experiments a considerable amount of emphasis is attached to the generation of beams with as little energy spread as possible. However, even if the electrons entering the drift tube were almost entirely monoenergetic, coupling resonances would greatly augment the energy spread and seriously degrade the gain. It is precisely for this reason that the study and control of coupling resonances is of paramount importance to the success of free electron lasers.

To perform an analysis correct to $o(\varepsilon^2)$, the hyperbolic functions are expanded in terms of modified Bessel functions of the first kind, I_n, and all but the rapidly varying second-order terms retained in the expression for the Hamiltonian:

$$\hat{\mathcal{H}} - \varepsilon^2\alpha \simeq p_z^2 + 2m\Omega_o\mu + 4\varepsilon\alpha\sum_{n=-\infty}^{\infty} i^{n+1}\left[\exp(i\xi_n) - (-1)^n \exp(-i\xi_n)\right] F_n(k\gamma)\zeta I_n'(\zeta)$$

$$+ \varepsilon^2\alpha I_o(2\zeta)\mathrm{ch}2k\gamma - (-1)^{\ell}\varepsilon^2\alpha I_{2\ell}(2\zeta)\mathrm{ch}2k\gamma \cos 2\xi_{\ell} \quad , \qquad (5)$$

wherein

$$\zeta_n \equiv n\theta + kz \quad , \tag{6}$$

$$F_n(kY) \equiv \tfrac{1}{2}[\exp(kY) - (-1)^n \exp(-kY)] \quad , \tag{7}$$

and a prime denotes differentiation with respect to the argument. Now, the equation of motion for θ is given by

$$\dot{\theta} = \partial_\mu \mathscr{H} = \Omega_o^{rel} + \frac{2\varepsilon\alpha}{\gamma m} \sum i^{n+1}[\exp(i\xi_n) - (-1)^n \exp(-i\xi_n)] F_n(kY) \partial_\mu [\zeta I_n'(\zeta)] + o(\varepsilon^2).$$

For a particle that is resonant at, say, the second harmonic, i.e., $\xi_2 = 2\theta + kz$ is the slowly varying phase, the terms involving $\zeta_{n\neq 2} = (n\theta + kz)_{n\neq 2}$ will be rapidly oscillating and are not expected, on intuitive grounds, to have a substantial effect on the motion. The only exception to this being the $n = 1$ harmonic since $\partial_\mu [\zeta I_1'(\zeta)] \sim 1/\zeta$ and for sufficiently small gyroradii it overwhelms the slowly varying second harmonic (i.e., resonant) term. In the Appendix the ponderomotive potential due to the rapdily oscillating terms is calculated and shown to be nondivergent and its effect on the mean motion to be uniformly on the order of ε^2. Accordingly, all the rapidly varying first order terms and the slowly varying second order terms may be safely dropped, and the resonant Hamiltonian written as

$$\hat{\mathscr{H}} = p_z^2 + 2m\Omega_o\mu + 4\varepsilon\alpha i^{\ell+1}[\exp(i\xi_\ell) - (-1)^\ell \exp(-i\xi_\ell)] F_\ell(kY) \zeta I_\ell'(\zeta) + \ldots \tag{8}$$

A significant feature of the single resonance Hamiltonian (8) is that it pertains to a system with two degrees of freedom $[(p_z, z), (\mu, \theta)]$ and yet the variables z and θ appear in the linear combination $kz + \ell\theta$ only. The presence of this hidden symmetry is quite common in nonlinear mechanics and there is a standard procedure in identifying the constant of motion associated with it.

The equations of motion for the axial momentum and for the magnetic moment are given by

$$\dot{p}_z = -\partial_z \mathcal{H} = \frac{2\varepsilon\alpha}{\gamma m} k i^{\ell}[\exp(i\xi_{\ell}) + (-1)^{\ell} \exp(-i\xi_{\ell})]F_{\ell}\zeta I'_{\ell} \quad ,$$

$$\dot{\mu} = -\partial_{\theta} \mathcal{H} = \frac{2\varepsilon\alpha}{\gamma m} \ell i^{\ell}[\exp(i\xi_{\ell}) + (-1)^{\ell} \exp(-i\xi_{\ell})]F_{\ell}\zeta I'_{\xi} \quad .$$

Dividing the first expression by the second,

$$\frac{dp_z}{d\mu} = \frac{dp_z/dt}{d\mu/dt} = k/\ell \quad ,$$

hence it is clear that as an electron moves under the influence of a single resonance its direction of motion in (p_z,μ)-space is a straight line whose gradient is given by the quotient of the constants k and ℓ, whence, on integration

$$\ell p_z - k\mu = \text{constant.}$$

Having identified this constant of the motion, it is now a simple matter to use it in order to reduce the dynamical system to one with a single degree of freedom. This may be accomplished in a number of ways, the most expeditious being, perhaps, through the transformation $(p_z,z;\ \mu,\theta) \rightarrow (I,\xi_I;\ J,\xi_J)$ given by the generating function

$$S = (\ell\theta + kz)\frac{\ell m\Omega_o}{k^2} I - \frac{\ell m\Omega_o}{k} z + \frac{\ell^2 m\Omega_o}{k^2} J\theta \quad , \qquad \ell \neq 0 \ ,$$

which is chosen in such a way as to belong to the class of canonical transformations [cf. statements preceding Eq. (3)]. The connection formulas between the old $(p_z,z;\ \mu,\theta)$ and new $(I,\xi_I;\ J,\xi_J)$ variables are furnished by[5]

$$\mu \equiv \partial_\theta S = \frac{\ell^2 m\Omega_o}{k^2}(I + J) \quad , \tag{9}$$

$$p_z \equiv \partial_z S = \frac{\ell m\Omega_o}{k}(I - 1) \quad , \tag{10}$$

$$\xi_I \equiv \partial_I S = \frac{\ell m\Omega_o}{k^2}(\ell\theta + kz)$$

$$= \frac{\ell m\Omega_o}{k^2}\xi_\ell \quad ,$$

$$\xi_J \equiv \partial_J S = \frac{\ell^2 m\Omega_o}{k^2}\theta \quad .$$

Noting that $\dot{\xi}_I = 0$ corresponds to resonance between the modulated axial motion and the ℓth harmonic of the gyromotion, and referring to Eqs. (3) and (4), it is evident that the (dimensionless) variable I measures the departure of the axial momentum from the resonant value $-\ell m\Omega_o/k$. Next, defining

$$H_\ell \equiv \left(\frac{k}{\ell m\Omega_o}\right)^2 \hat{\mathcal{H}} - 1 - 2J \quad ,$$

Eq. (8) may be rewritten as

$$H_\ell = I^2 + \frac{\varepsilon}{\ell^2} i^{\ell+1}[\exp(i\xi_\ell) - (-1)^\ell \exp(-i\xi_\ell)]F_\ell(kY)\zeta I_\ell'(\zeta) + \ldots \quad , \tag{11}$$

from which it immediately follows that J (being conjugate to ξ_J) is conserved. That J is conserved is entirely an artifact of the single resonance approximation. This is clearly brought out by deriving the equations of motion for p_z and μ not from the single resonance Hamiltonian given by Eq. (8) but from Eq. (5). It is then apparent that the direction of motion $dp_z/d\mu$ is no longer constant. Another perspective of this phenomenon is gained by noting that there are only

two global symmetries associated with this problem as manifested by the absence of any explicit dependence on time or on the x-coordinate variable in the Hamiltonian. Conservation of J, however, is related to an imposed symmetry, which is unbroken as long as the nonresonant interaction terms in the Hamiltonian are ignorable. The importance attached to conservation of J insofar as it relates to constraining the spread in energy of a beam traversing the drift tube is discussed in the following sections.

It is worth pointing out that since Y is a constant of the motion, the effective coupling constant in the expression for the Hamiltonian, Eq. (11) is $\varepsilon F_\ell(kY)$. Referring to Eq. (7) it is manifest that this effective coupling constant for even ℓ is much smaller than that for odd ℓ for those particles whose guiding centers are close to the axis ($kY \ll 1$), indicating a much weaker interchange between axial and perpendicular degrees of freedom in the former case.

The equations of motion are given by

$$\dot{I} = -\partial_{\xi_I} \mathcal{H} = \frac{\ell\Omega_o^{rel}}{2} \frac{\varepsilon}{\ell^2} i^\ell [\exp(i\xi_\ell) + (-1)^\ell \exp(-i\xi_\ell)] F_\ell \zeta I'_\ell \quad , \tag{12}$$

$$\dot{\xi}_\ell = \frac{\ell\Omega_o^{rel}}{2} \left\{ 2I + \frac{\varepsilon}{\ell^2} i^{\ell+1} [\exp(i\xi_\ell) - (-1)^\ell \exp(-i\xi_\ell)] F_\ell \partial_I (\zeta I'_\ell) \right\} \quad , \tag{13}$$

and

$$\dot{X} = \frac{-\Omega_o^{rel}}{2k} \varepsilon i^{\ell+1} [\exp(i\xi_\ell) - (-1)^\ell \exp(-i\xi_\ell)] F_{\ell+1} \zeta I'_\ell \quad . \tag{14}$$

IV. MOTION IN A SINGLE RESONANCE

Since the Hamiltonian given by Eq. (11) is conserved, the dynamical system (or flow) generated by it is integrable; in other words, it is possible, in principle, to solve Eqs. (12)-(14) to express the development of I, ξ_ℓ and X as functions of time. It turns out that, not unlike many other problems, it is far more expeditious and convenient to use the energy conservation theorem to determine, and describe the salient features of, the possible orbits than to attempt to unravel, and extract useful information from, the algebraically complicated solutions of the equations of motion.

Upon squaring and combining Eqs. (11) and (12) in such a way as to eliminate the dependence on the angle ξ_ℓ, there results the following energy conservation theorem:

$$(\frac{2\gamma}{\ell\Omega_o}\dot{I})^2 + 'V'(I) = 0 \quad , \tag{15}$$

wherein the effective potential is given by

$$'V'(I) = (H_\ell - I^2)^2 - [\frac{2\epsilon}{\ell^2}F_\ell(k\gamma)\zeta I'_\ell]^2 \quad , \tag{16}$$

with

$$\zeta = \ell[2(I + J)]^{\frac{1}{2}} \quad . \qquad [\text{Using Eqs. (1), (2) and (9)}]$$

Note that since $\zeta \geq 0$, the last expression imposes the constraint $I \geq -J$. One further constraint is obtained by eliminating I from Eqs. (9) and (10), squaring the resulting expression for J, and combining with the expression for H_ℓ given before Eq. (11):

$$J^2 - H_\ell = (\frac{-kc}{\ell\Omega_o})(\frac{\zeta}{\ell})^2[\gamma^2 - 1 - (\frac{\gamma v_\perp}{c})^2]^{\frac{1}{2}} - \left(\frac{kv_\perp}{\ell\Omega_o^{rel}}\right)^2 + [\tfrac{1}{2}(\frac{\zeta}{\ell})^2]^2$$

$$\approx (\frac{-kc}{\ell\Omega_o})(\frac{\zeta}{\ell})^2[\gamma^2 - 1 - (\frac{\gamma v_\perp}{c})^2]^{\frac{1}{2}} - (\frac{\zeta}{\ell})^2 + [\tfrac{1}{2}(\frac{\zeta}{\ell})^2]^2 \quad , \tag{17}$$

where the last expression is derived for $\varepsilon \ll 1$. Recalling that $\Omega_o < 0$ for electrons it follows that $J^2 > H_\ell$ for $(kc/|\Omega_o|)[\gamma^2 - 1 - (\gamma v_\perp/c)^2]^{\frac{1}{2}} \geq 1$, which is generally the case. However, for completeness the case $J^2 < H_\ell$ will be considered, too. It is now possible to classify the motion according as the four sign combinations for J and H_ℓ, and as sketched in Fig. 2. In plotting the effective potential it is useful to note that the first term in Eq. (16) is a biquadratic with zeros at $\pm H_\ell^{\frac{1}{2}}$ for $H_\ell > 0$, and of parabolic form with a (single) minimum above the abscissa at $I = 0$ for $H_\ell < 0$. The second term in Eq. (16) is a monotonically decreasing function of ζ (and of I) with a zero at $I = -J$. Additionally, it is apparent from Eq. (15) that allowed motion is confined to $'V'(I) \leq 0$ with the sum of the kinetic and potential energies being zero, i.e., lying on the abscissa.

It is simple to show that, within the context of the single resonance approximation, bounded motion for $J < 0$ and $J^2 > H_\ell$ [Figs. 2(e) and 2(g)] is impossible; in other words the form of the effective potential is always as shown in Figs. 2(e) and 2(g)--with one, and only one, intercept on the abscissa. $'V'(I) = 0$ implies that

$$I^2 - H_\ell = \pm \frac{2\varepsilon}{\ell^2} |F_\ell| \zeta I_\ell' \quad .$$

Being interested in $I > 0$, $I^2 > J^2$, and noting that $J^2 > H_\ell$ for Figs. 2(e) and 2(g), it follows that

$$I = (H_\ell + \frac{2\varepsilon}{\ell^2} |F_\ell| \zeta I_\ell')^{\frac{1}{2}} \quad .$$

Since the left member of this equation has a constant gradient, whereas the right member has a monotonically increasing gradient, and also since the right member always starts below the left member, there can only be one root, as indicated in Fig. 3.

Corresponding to Fig. 2(a) ($H_\ell < 0$, $J > 0$), the roots of $'V'(I) = 0$ are

given by

$$I = \pm(H_\ell + \frac{2\varepsilon}{\ell^2}|F_\ell|\zeta I_\ell')^{\frac{1}{2}} \quad , \tag{18}$$

the left and right members of which are shown in Fig. 4. The root denoted by (1) in this figure is always present, whereas the roots denoted by 2 and 3 disappear for sufficiently large ε, as indicated by the short-dash curves in Fig. 4, thereby eliminating the possibility of bounded motion in this case. The critical value of ε may be estimated as follows. Assuming that $\zeta I_\ell'$ may be approximated by

$$\zeta I_\ell' = a_\ell + b_\ell I \ , \quad a_\ell, b_\ell > 0 \ , \tag{19}$$

in the vicinity of the roots (2) and (3), substitution of this into the upper branch of Eq. (18) and the requirement of the equality of the roots of the resulting quadratic equation yields the critical wiggler strength above which bounded motion is not possible:

$$\varepsilon_c \simeq \frac{-\ell^2 H_\ell}{2|F_\ell| a_\ell} \ . \tag{20}$$

Although this formula indicates the scaling of ε_c, to calculate its value in particular cases requires an estimate of a_ℓ which, due to the transcendental nature of the Bessel function $I_\ell(\zeta)$, necessitates a numerical root determination of Eq. (18).

For the case represented by Fig. 2(c), the roots of 'V'(I) = 0 are determined by

$$I = \pm(H_\ell + \frac{2\varepsilon}{\ell^2}|F_\ell|\zeta I_\ell')^{\frac{1}{2}} \quad , \qquad \text{for } |I| > H_\ell^{\frac{1}{2}} \ ,$$

$$I = \pm(H_\ell - \frac{2\varepsilon}{\ell^2}|F_\ell|\zeta I')^{\frac{1}{2}} \quad , \qquad \text{for } |I| < H_\ell^{\frac{1}{2}} \ .$$

The points marked (1) through (5) in Fig. 5 show all the possible roots in the general case. The root denoted by (1) always exists, whereas the pair (2) and (3) and the pair (4) and (5) coalesce, respectively, and disappear as ε increases. This is indicated by the short-dash curves in Fig. 5. (In general, either pair can merge and disappear first with increasing ε.) Employing a linear approximation of the form (19) to the right of $-H_\ell^{1/2}$ in Fig. 2(c) yields the following solutions for 'V'(I) = 0

$$I \simeq \frac{-\varepsilon}{\ell^2}|F_\ell|b_\ell \pm (H_\ell - \frac{2\varepsilon}{\ell^2}|F_\ell|a_\ell)^{1/2} \quad ,$$

where a term of order ε^2 in the parentheses has been dropped by virtue of the fact that H_ℓ is correct to order ε. Root (2) corresponds to the lower sign in this expression:

$$I^{>} = \frac{-\varepsilon}{\ell^2}|F_\ell|b_\ell - (H_\ell - \frac{2\varepsilon}{\ell^2}|F_\ell|a_\ell)^{1/2} \quad , \tag{21}$$

and the condition for the reality of this root furnishes an estimate of the threshold value for the wiggler strength above which the local maximum in the vicinity of I = 0 in Fig. 2(c) does not constrain the particles to the left or to the right of the ordinate:

$$\varepsilon_c \simeq \frac{\ell^2 H_\ell}{2|F_\ell|a_\ell} \quad . \tag{22}$$

Using the same linear approximation [with the same values for a_ℓ and b_ℓ that appear in Eq. (21)], the intercept to the left of $I = -H_\ell^{1/2}$ in Fig. 2(c) is given by

$$I^{<} = \frac{\varepsilon}{\ell^2}|F_\ell|b_\ell - (H_\ell + \frac{2\varepsilon}{\ell^2}|F_\ell|a_\ell)^{1/2} \quad . \tag{23}$$

The following cases may now be distinguished. For very small values of the wiggler strength, bounded oscillations of gyroradius and of axial velocity

of the electrons are possible in between the roots (1) and (2), and in between (3) and (4)- with the two regions being disconnected. With increasing ε, either the roots (2) and (3), or the roots (4) and (5) disappear first. In the former case, $\varepsilon > \varepsilon_c$ with ε_c given by Eq. (22), and the particles oscillate back and forth between the roots (1) and (4). In the latter case, bounded motion between the roots 1 and 2 is allowed only. [Since $I < 0$ in this interval, it follows from Eq. (10) that only those particles whose axial momenta exceed the resonant value are confined.] Now for ε greater than the value indicated by Eq. (22), bounded motion is completely eliminated.

It is worth pointing out the analogy between the motions for the particular cases sketched in Figs. 2(a) and 2(c) and that of a pendulum. Figure 2(a) corresponds to the case in which the pendulum bob oscillates back and forth about the equilibrium, with limited excursion in both the momentum and the angular displacement about zero. This is usually referred to as libration. For motion corresponding to that in either of the minima in Fig. 2(c), the bob rotates clockwise or counterclockwise, with the angular displacement changing by 2π on each complete revolution.

While for confined motion the gyroradius and axial velocity oscillate about certain constant values, the guiding center coordinate X oscillates about a mean value which itself drifts away from the axis. This mean drift may be determined in the following way. The fixed points $(\hat{I},\hat{\xi}_\ell)$ of the equations of motion (12) and (13) corresponding to the equilibria depicted in Fig. 2(a) are determined from $\dot{I} = \dot{\xi}_\ell = 0$:

$$\exp(i\hat{\xi}_\ell) + (-1)^\ell \exp(-i\hat{\xi}_\ell) = 0 \quad ,$$

$$2\hat{I} + \frac{\varepsilon}{\ell^2} i^{\ell+1}[\exp(i\hat{\xi}_\ell) - (-1)^\ell \exp(-i\hat{\xi}_\ell)]F_\ell \, \partial_{\hat{I}}(\hat{\zeta}\hat{I}_\ell') = 0 \quad .$$

Linearizing Eqs. (12) and (13) about $(\hat{I},\hat{\xi}_\ell)$ and combining, there results the following equation of motion for $\delta\xi_\ell = \xi_\ell - \hat{\xi}_\ell$

$$\delta\ddot{\xi}_\ell = \pm(\ell\Omega_o^{rel})^2 \frac{\varepsilon}{\ell^2} F_\ell \hat{\zeta}\hat{I}'_\ell (1 \pm \frac{\varepsilon}{\ell^2} F_\ell \partial^2_{\hat{I}} \zeta I'_\ell)\delta\xi_\ell$$

$$\simeq \pm(\ell\Omega_o^{rel})^2 \frac{\varepsilon}{\ell^2} F_\ell \hat{\zeta}\hat{I}'_\ell \delta\xi_\ell \quad , \quad \varepsilon \ll 1 \quad ,$$

whence the characteristic exponent [for $\delta\xi_\ell \sim \exp(\lambda t)$] for <u>oscillatory</u> motion in a trough is given

$$\lambda = \pm i\varepsilon^{\frac{1}{2}} |\Omega_o^{rel}| (|F_\ell| \hat{\zeta}\hat{I}'_\ell)^{\frac{1}{2}} \quad . \tag{24}$$

In the terminology of charged particle accelerators $|i\lambda|$ is referred to as the frequency of <u>phase oscillations</u> (the oscillations of the resonance phase ξ_ℓ). One remarkable feature of the formula for the frequency of phase oscillations is its scaling with ε.

The position of stable equilibrium ($\delta\ddot{\xi}_\ell \sim -\delta\xi_\ell$) for phase oscillations due to even-ℓ coupling resonances corresponds to the condition that the gyrophase and the effective "axial phase" be in quadrature, i.e., $\ell\theta + kz = \pi/2$ or $3\pi/2$, and is dependent on whether Y is greater than or less than zero. The position of stable equilibrium for phase oscillations for odd-ℓ coupling resonances corresponds to the phases being in phase or antiphase ($\ell\theta + kz = 0$ or π), irrespective of the y component of the guiding center.

The general solution for I close to a stable fixed point is given by $I = \hat{I}\cos i\lambda(t - t_o)$. Eliminating the angle dependence between Eqs. (11) and (14), using this solution for I, and integrating the resulting equation for X, there obtains

$$X = X_o - \frac{\ell^2\Omega_o^{rel}}{2k} \frac{F_{\ell+1}}{F_\ell} [(H_\ell - \hat{I}^2/2)t - \frac{1}{i\lambda}(\hat{I}/2)^2 \sin 2i\lambda(t - t_o)] \ ,$$

so that the mean drift of the guiding center is given by

$$\bar{X} = X_o - \frac{\ell^2}{2k}\frac{F_{\ell+1}}{F_\ell}(H_\ell - \hat{I}^2/2)\Omega_o^{rel}t \quad . \tag{25}$$

This formula may be used to determine the length of the drift tube compatible with propagation of a beam of electrons with given values for the constants of motion, H_ℓ, Y, etc.

The characteristic exponent (24) is derived for stable oscillation (or libration) around the resonance value I = 0, corresponding to the potential minimum of Fig. 2(a). The rotational states of Figs. 2(c), 2(d), and 2(f) are not determined by the fixed points of the equations of motion. The frequencies of oscillations in these minima are now determined. For a minimum located at $\hat{I}$,

$$'V'(I) = 'V'(\hat{I}) + \tfrac{1}{2}\left.\frac{d^2{}'V'(I)}{dI^2}\right|_{\hat{I}}(I - \hat{I})^2 + \ldots \text{, for } |I - \hat{I}| \ll 1 \quad ,$$

so that the energy conservation theorem (16) may be used to obtain

$$\int \frac{dI}{\left[-'V'(\hat{I}) - \tfrac{1}{2}\left.\frac{d^2{}'V'}{dI^2}\right|_{\hat{I}}(I - \hat{I})^2 + \ldots\right]^{\frac{1}{2}}} = \pm\frac{\ell|\Omega_o^{rel}|}{2}t + \text{constant} \quad .$$

Setting $I - \hat{I} = \left[-2'V'/(d^2{}'V'/dI^2)\right]^{\frac{1}{2}}_{\hat{I}}\sin\phi$, this expression becomes

$$\left(\frac{2}{d^2{}'V'/dI^2}\right)^{\frac{1}{2}}_{I}\int d\phi\,\frac{\cos\phi}{(\cos^2\phi + \ldots)^{\frac{1}{2}}} = \pm\frac{\ell|\Omega_o^{rel}|}{2}t + \text{constant} \quad ,$$

whence

$$I - \hat{I} \simeq \text{constant} \times \sin\left[\frac{\ell}{2}\left|\tfrac{1}{2}\left.\frac{d^2{}'V'}{dI^2}\right|_{\hat{I}}\right|^{\frac{1}{2}}|\Omega_o^{rel}|t + \text{constant}\right] \quad .$$

V. CHAOTIC MOTION

The discussion of the preceding section is predicated upon the assumption that the motion of an electron is influenced by a single resonance, as expressed by the expression for the Hamiltonian, Eq. (11). This premise is justified so long as the particle remains close to resonance; in other words $\dot{\xi}_n = n\dot{\theta} + k\dot{z}$ is close to zero for $n = \ell$ and far from zero for all other n. For wigglers of sufficiently small strength a "trapped" electron executes small amplitude oscillations in the vicinity of the resonant value for p_z. As the amplitude of these oscillations increases with ε, there comes a point where the neighboring resonances ($n = \ell \pm 1$) perturb the motion to such a degree that the motion may no longer be considered to be locked to a single resonance ($n = \ell$). Under these circumstances a given electron can wander from one resonance to the next, with significant interchange of energy between the parallel and perpendicular degrees of freedom. This, patently, is an undesirable mode of operation for a free-electron laser. In addition to particles with large excursions in axial velocity, the particles whose motion, as discussed in the previous section, is unbounded belong to the class of particles wandering between neighboring resonances.

It may be recalled that, according to the discussion following Eq. (8), the appearance of the variables θ and z in the combination $\ell\theta + kz$ in the expression for the Hamiltonian implies a hidden symmetry in consequence of which a conserved quantity, independent of the Hamiltonian, could be shown to exist. This constant of the motion, denoted by J, is strictly conserved so long as the single-resonance approximation is valid. With $\mathcal{H}$ and J being conserved, the two-degree-of-freedom system represented by (8) is integrable according to well-known classical theorems.[6] If the single-resonance approxiamtion is not valid,

J is no longer conserved and evolves in time at a rate determined by the intermixing of the neighboring resonances with the fundamental resonance ($n = \ell$).

Adapting Chirikov's[7] overlap criterion, an estimate of the threshold to stochastic behavior is easily obtained. This criterion requires that the wiggler strength be such that the sum of the half-widths of two neighboring resonances exceed the separation between them, i.e.

$$\tfrac{1}{2}(\Delta p_z)_\ell + \tfrac{1}{2}(\Delta p_z)_{\ell\pm1} \gtrsim \frac{m|\Omega_o|}{k} \quad ,$$

where Δp_z denotes the resonance width. Using Eq. (10) this may be rewritten in terms of ΔI:

$$\ell\Delta I|_\ell + (\ell \pm 1)\Delta I|_{\ell\pm1} \gtrsim 2 \quad . \tag{26}$$

As an example, for the case discussed in the preceding section, where a particle oscillates between the roots (1) and (2) of Fig. 2(c), $\Delta I_\ell = I^> - I^<$, i.e.,

$$\Delta I_\ell = \frac{-2\varepsilon}{\ell^2}|F_\ell|b_\ell + \left(H_\ell + \frac{2\varepsilon}{\ell^2}|F_\ell|a_\ell\right)^{\frac{1}{2}} - \left(H_\ell - \frac{2\varepsilon}{\ell^2}|F_\ell|a_\ell\right)^{\frac{1}{2}} \quad ,$$

or

$$\Delta I_\ell = \frac{2\varepsilon}{\ell^2}|F_\ell|a_\ell\left(\frac{1}{\sigma} - \frac{1}{J}\right) \quad , \tag{27}$$

where $2\sigma = \left[H_\ell + (2\varepsilon/\ell^2)|F_\ell|a_\ell\right]^{\frac{1}{2}} + \left[H_\ell - (2\varepsilon/\ell^2)|F_\ell|a_\ell\right]^{\frac{1}{2}}$, and assuming that the interpolation given by Eq. (19) is approximately valid down to $\zeta = 0$, so that $J \simeq a_\ell/b_\ell$. Substitution of Eq. (27) into the left-member of (26) will determine if for a given value of ε the motion is integrable or not. Alternatively, one may use the inequality (26) to calculate the threshold wiggler strength for stochastic motion. A crude estimate of the time τ_{tr} taken by an electron to traverse a single resonance is obtained from Eq. (24) to be

$$\tau_{tr} \gtrsim \frac{1}{\epsilon^{\frac{1}{2}}\Omega_0^{rel}(|F_\ell|\hat{\zeta}\hat{I}'_\ell)^{\frac{1}{2}}} \quad .$$

It must be borne in mind, however, that due to the random nature of resonance-crossing a given electron may or may not move onto an adjacent resonance after a time on the order of τ_{tr}.

Finally, let it be noted that the threshold wiggler strength required for stochastic behavior, determined from Eq. (26), has no a priori relation to the threshold value required for unbounded motion under the influence of a single resonance, Eq. (20) or Eq. (22).

VI. COMPARISON WITH EXPERIMENT

To begin with, it should be emphasized that the analysis presented in this paper of the problem of beam propagation in a drift tube in the presence of a linear magnetic wiggler and an axial magnetic guide field is predicated upon the validity of the perturbation scheme based on the smallness of ε (the ratio of the wiggler field strength to that of the axial guide field). With this proviso, it is interesting to examine some of the particle dynamics relevant to a free electron laser experiment carried out by Roberson et al.[8] at the Naval Research Laboratory (NRL).

In the experiments an electron beam from a hot cathode induction linac is used to drive a free electron laser. The beam is accelerated through a potential difference of about 350 volts ($\gamma \simeq 1.68$) before impinging on the anode of roughly 1 cm in diameter and entering the drift tube. The linear wiggler has a period of 3 cm and the guide field strength is 2.2 kOe, so that the particles are ostensibly near the $\ell = 2$ resonance harmonic. Measurements indicate a spread in the initial perpendicular speed of $v_\perp/c \simeq 0.08\text{-}0.24$. Fig. 6 shows the effective potentials 'V'(I) over a limited range of I for the $\ell = 2$ and $\ell = 1$ resonances in the case of a particle whose y component of guiding center coordinate is at 0.1 cm and for $\varepsilon = 0.1$. For the $\ell = 2$ resonance the initial value of the gyroradius is $\rho = 0.35$ cm. For both resonances it is seen that oscillatory motion about $I = 0$ (corresponding to exact resonance) is possible. Fig. 6(c) shows, very schematically, the "islands" surrounding each resonance wherein a particle oscillates as its perpendicular and parallel energies are interchanged due to a coupling resonance. Examination of Fig. 6(c) evinces that for a particle initially locked in the $\ell = 2$ resonance, the $\ell = 1$

resonance induces a mild perturbation in its motion. A similar picture emerges for the case where the initial gyroradius is somewhat smaller. For example, for $\rho = 0.1$ cm the width of the $\ell = 2$ island is about three times smaller than that for $\rho = 0.35$ cm, whereas the $\ell = 1$ island is not modified to any great extent. The case depicted in Fig. 7 (also for $\epsilon = 0.1$) corresponds to a guiding center of 0.3 cm. Examination of Fig. 7(c) indicates that for a particle initially locked in either of the resonances, the other resonance induces a strong perturbation in its motion, if not completely dislodging and causing it to wander from one resonance to the other. According to the discussion preceding Eq. (12) the example of Fig. 7 is roughly equivalent to the case for which $Y = 0.1$ (as in Fig. 6) but with $\epsilon = 0.32$. Figure 8 shows the effective potentials for a particle on the extreme edge of beam as it enters the wiggler and whose y component of the guiding center is at 0.5 cm. In this case the initial gyroradius is chosen to be $\rho = 0.1$ cm, thus allowing passage of the particle through the anode (whose diameter is roughly 1 cm). The relevant features to note in Fig. 8(c) as compared to Figs. 6(c) and 7(c) are the increased width of the $\ell = 2$ resonant island (due to its dpendence on sh kY) and the slight shrinkage of the $\ell = 1$ resonant island (due to its dependence on $k\rho$). Overall, it is manifest that resonance interaction (between the $\ell = 1$ and $\ell = 2$ harmonic, at least) is much weaker than the earlier cases with the larger gyroradius. The example in Fig. 8 is roughly equivalent to one with guiding center $Y = 0.3$ cm and $\epsilon \simeq 0.2$

Two remarks of relevance to all the cases just described are in order. First, the $\ell = 3$ harmonic is expected to have a negligible influence on the particle motion in this parameter regime since it is not an energetically allowed resonance. Second, the location and the width of the $\ell = 1$ resonance

island in Figs. 6-8 is such that it is always intertwined with the $\ell = 0$ resonance, so that if a particle enters the $\ell = 1$ resonance the influence of the $\ell = 0$ harmonic is likely to be significant.

A quantity that should be quite relevant in an experiment whose aim is to propagate a beam down a drift tube is the mean drift of the x component of the guiding center $\bar{X}$, as given by Eq. (25), assuming that a particle remains trapped in a single resonance while traversing the tube. If the length of the tube is L cm, $\bar{X}$ may be approximated by

$$\bar{X} \simeq X_0 - \frac{\ell^2}{2} \frac{F_{\ell+1}}{F_\ell} (H_\ell - \hat{I}^2/2) \frac{\Omega_0^{rel}}{ck} \frac{c}{v_z} L$$

whence $\bar{X} - X_0 \sim -1.4$ cm for the particle locked on the $\ell = 2$ resonance in Fig. 7(a) and propagating through a tube of length 60 cm.

One striking aspect of the free electron laser experiment at NRL[8] is that as the wiggler field is increased, with ε taking on the values 0, 0.23, 0.34 and 0.46, no significant reduction in the beam intensity, determined from the x-ray exposures generated in a tantalum target, is detected as the beam propagates the entire length of the device. This is in contrast to the results obtained by numerical simulations of the experiment.[9] With the limitations imposed by the perturbative analysis presented in this paper, it appears that, in the light of the conclusions drawn from Figs. 6-8, a possible explanation of the experimental results is the following. Given the very narrow aperture of the anode the particles with a small scatter in velocity (namely, small $v_\perp/c$) tend to be locked onto the $\ell = 2$ resonance throughout their passage through the wiggler and undergo very limited excursions in parallel velocity. Moreover, the guiding centers of these particles follow trajectories whose y components are constant and whose

x components drift by small amounts through the tube. However, as indicated by Fig. 7 there are some classes of particles--those with relatively large gyro-radii and close to the edge of beam--whose motion may be quite eratic, wandering from one resonance to the next and transforming a substantial fraction of their directed parallel energy into the perpendicular degree of freedom. The experimental observation may therefore be a reflection of the fact that the electron beams used in the NRL experiments are narrow enough and of sufficiently small emittance that the fraction of particles whose motion is Brownian is quite insignificant.

VII. CONCLUSIONS

Starting with a Hamiltonian formulation of the dynamics of a single particle traversing a drift tube under the combined influence of a linear magnetic wiggler and an axial magnetic guide field, it is shown that the all-important issues of nonlinear orbital stability, propagation and directed-beam quality may be examined in a very simple manner by using effective-potential theory. Within the regime of validity of this perturbation theory, it is concluded that there is a limit on the strength of the wiggler field above which the propagation of the beam is seriously hampered, leading to a significant increase in its energy spread and an inevitable reduction in the gain of the laser.

The efficacy of the Hamiltonian approach, applicable to wigglers of arbitrary geometry, may be gleaned from the simple way in which a given set of parameters may be appraised as regards orbital stability and beam quality, and has therefore much to commend it to those who contemplate such experiments.

ACKNOWLEDGMENT

This work was supported by the Office of Naval Research.

APPENDIX: EFFECT OF HIGH-FREQUENCY TERMS ON THE MOTION CLOSE TO A RESONANCE

The effect of the non-resonant, high-frequency terms on the motion in proximity of a single resonance may be determined by a standard ponderomotive-potential analysis.[10] Assuming that the first-order non-resonant terms in the Hamiltonian induce small changes $\delta\mu$, δp_z, $\delta\theta$, and δz in the variables μ, p_z, θ, and z, perturbation of the expression for the Hamiltonian (5) yields

$$\frac{\mathcal{H}^2}{c^2} + \frac{2\mathcal{H}}{c^2}\delta\mathcal{H} - m^2c^2 - \varepsilon^2\alpha$$

$$= p_z^2 + 2p_z\delta p_z + (\delta p_z)^2 + 2m\Omega_o\mu + 2m\Omega_o\delta\mu$$

$$+ 4\varepsilon\alpha i^{\ell+1}[\exp(i\xi_\ell) - (-1)^\ell \exp(-i\xi_\ell)]F_\ell\zeta I_\ell' + 4\varepsilon\alpha\sum' i^{n+1}[\exp(i\xi_n) - (-1)^n \exp(-i\xi_n)]F_n\zeta$$

$$+ 4\varepsilon\alpha\sum' i^{n+1}[\exp(i\xi_n) - (-1)^n \exp(-i\xi_n)]F_n\frac{\partial\zeta I_n'}{\partial\mu}\delta\mu + 4\varepsilon\alpha\sum' i^{n+1}[\exp(i\xi_n) + (-1)^n \exp(-i\xi_n$$

$$\times F_n\zeta I_n'(n\delta\theta + k\delta z) + \varepsilon^2\alpha I_o(2\zeta)\mathrm{ch}2kY - (-1)^\ell\varepsilon^2\alpha I_{2\ell}(2\zeta)\mathrm{ch}2kY\cos 2\xi_\ell \quad , \qquad \text{(A1)}$$

whence

$$\delta\mathcal{H} = \frac{2\varepsilon\alpha}{\gamma m}\sum' i^{n+1}[\exp(i\xi_n) - (-1)^n \exp(-i\xi_n)]F_n\zeta I_n' + \frac{p_z}{\gamma m}\delta p_z + \frac{\Omega_o}{\gamma}\delta\mu \quad ,$$

wherein $\sum'$ indicates that $n = \ell$ is excluded. Using

$$\delta\dot{\theta} = \partial_\mu\delta\mathcal{H} = \frac{2\varepsilon\alpha}{\gamma m}\sum' i^{n+1}[\exp(i\xi_n) - (-1)^n \exp(-i\xi_n)]F_n\frac{\partial\zeta I_n'}{\partial\mu} \quad ,$$

the lowest order solution for $\delta\theta$ is given by

$$\delta\theta = \frac{2\varepsilon\alpha}{\gamma m} \sum{}' i^n [\exp(i\xi_n) + (-1)^n \exp(-i\xi_n)] F_n \frac{\partial \zeta I'_n/\partial\mu}{(n\Omega_o/\gamma) + kv_z} \quad ;$$

similar expressions may be obtained for $\delta\mu$, δp_z, and δz. These resulting expressions are then substituted into Eq. (A1) and the slowly varying terms retained, leading to the final expression for the Hamiltonian:

$$\frac{\mathcal{H}^2}{c^2} - m^2c^2 - \varepsilon^2\alpha$$

$$= p_z^2 + 2m\Omega_o\mu + 4\varepsilon\alpha i^{\ell+1}[\exp(i\xi_\ell) - (-1)^\ell \exp(-i\xi_\ell)] F_\ell(kY)\zeta I'_\ell(\zeta)$$

$$+ \varepsilon^2\alpha I_o(2\zeta)\mathrm{ch}2kY - (-1)^\ell\varepsilon^2\alpha I_{2\ell}(2\zeta)\mathrm{ch}2kY \cos 2\xi_\ell - \frac{(4\varepsilon\alpha)^2}{\gamma m}\sum_n{}' \frac{F_n^2}{(n\Omega_o/\gamma) + kv_z}$$

$$\times \left[n\frac{\partial(\zeta I'_n)^2}{\partial\mu} - \frac{3}{2}\frac{k^2}{\gamma m}\frac{(\zeta I'_n)^2}{(n\Omega_o/\gamma) + kv_z}\right] + (-1)^\ell \frac{(4\varepsilon\alpha)^2}{\gamma m}\cos 2\xi_\ell \sum_n{}' \frac{F_n^2}{(2\ell - n)\Omega_o/\gamma + kv_z}$$

$$\times \left[2\ell\zeta I'_{2\ell-n}\frac{\partial\zeta I'_n}{\partial\mu} - n\frac{\partial}{\partial\mu}(\zeta I'_{2\ell-n}\,\zeta I'_n) + \frac{k^2}{\gamma m}\frac{\zeta I'_{2\ell-n}\zeta I'_n}{(2\ell-n)\Omega_o/\gamma + kv_z} - \frac{k^2}{2\gamma m}\frac{\zeta I'_{2\ell-n}\zeta I'_n}{(n\Omega_o/\gamma) + kv_z}\right].$$

Using

$$I_\ell(\zeta) \sim \frac{(\frac{1}{2}\zeta)^\ell}{\ell!} \quad , \qquad \zeta \to 0 \ ,$$

it is simple to check that none of the terms in the ponderomotive potential leads to divergent behavior for the equations of motion as $\zeta \to 0$.

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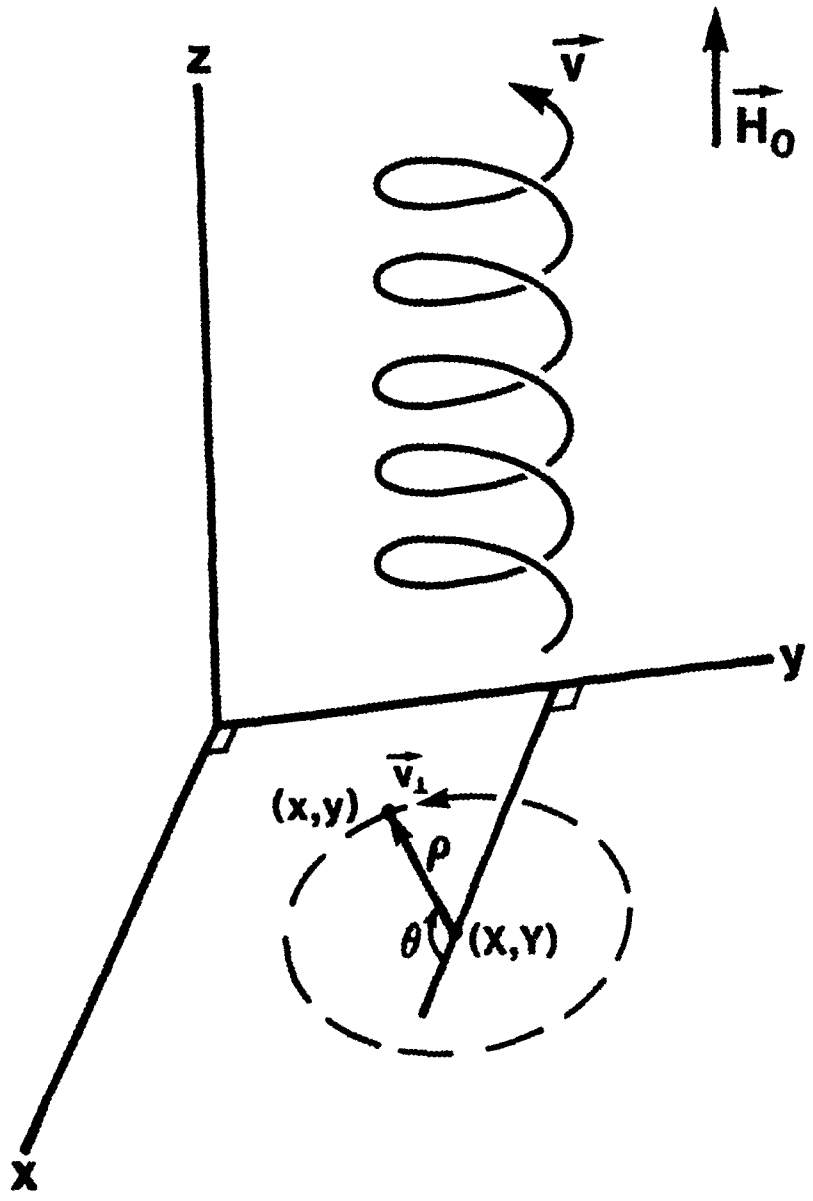

Fig. 1 Helical trajectory of a particle in a magnetic field $\vec{H}_0$. Dashed oval is the projection of the trajectory on z = 0 plane, indicating guiding center coordinates (X,Y), gyroradius ρ, and gyroangle θ.

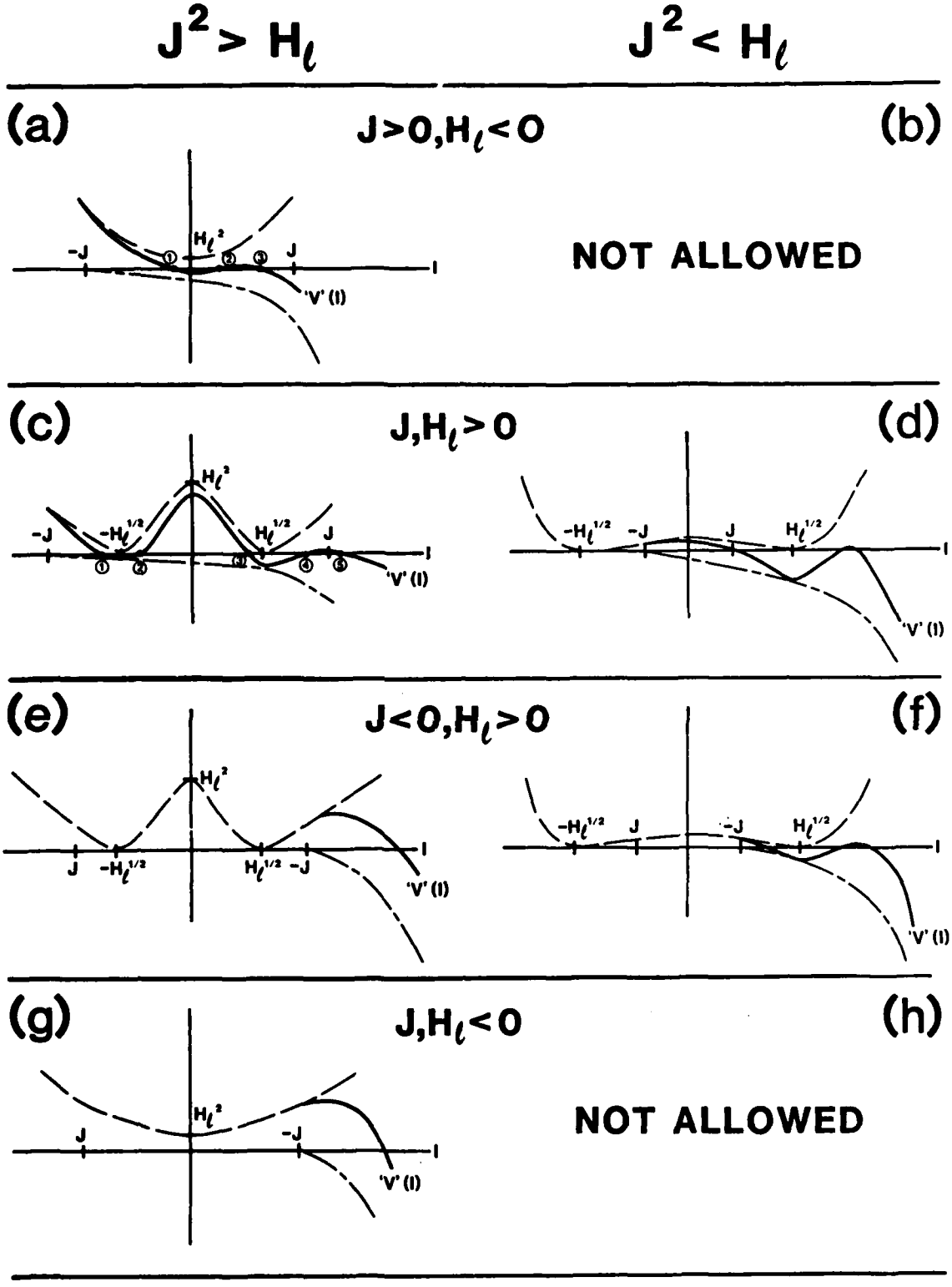

Fig. 2 $(I^2 - H_\ell)^2$(—— ——), $-[2\epsilon F_\ell(k\gamma)\zeta I'_\ell(\zeta)/\ell^2]^2$ (—— – ——), and effective potential 'V'(I)(——) versus I.

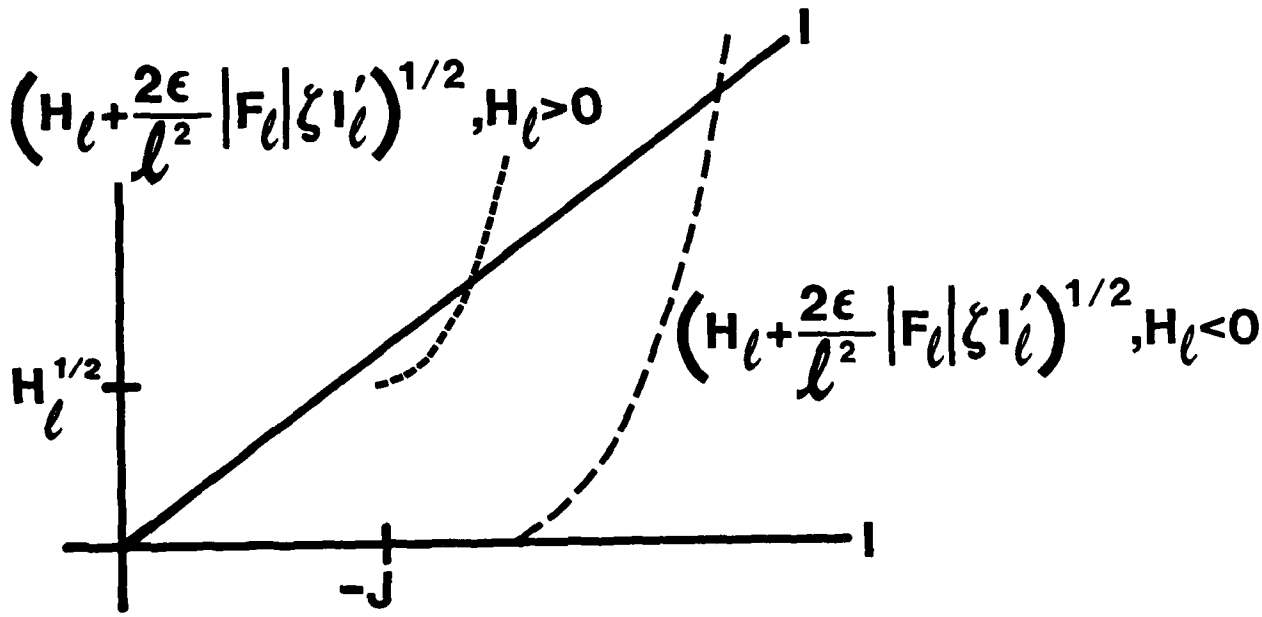

Fig. 3 Sketch used to prove that for $J^2 > H_\ell$, $J < 0$ does not allow bounded motion in the single resonance approximation.

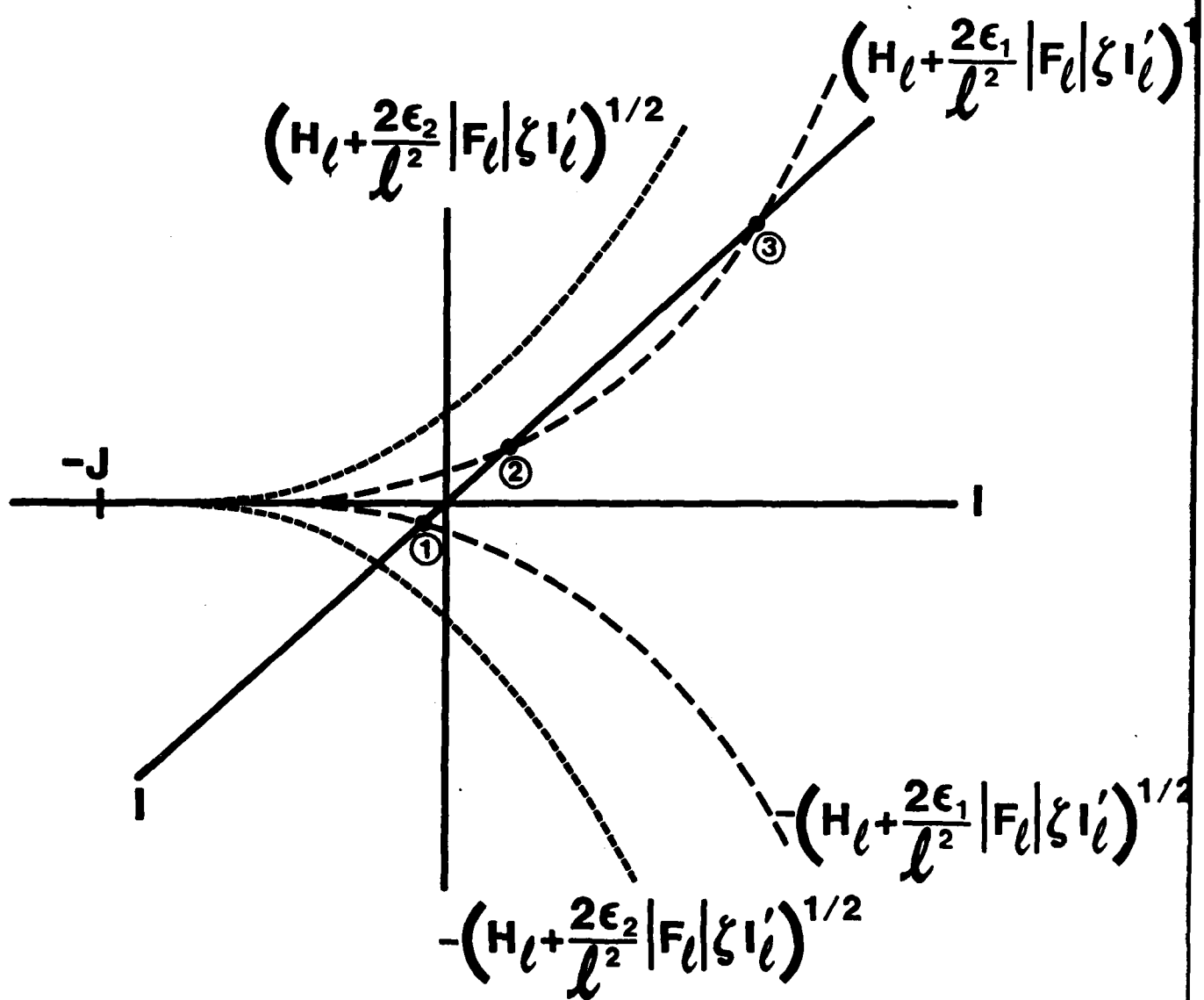

Fig. 4 Sketch indicating elimination, for sufficiently large ϵ, of bounded motion, corresponding to disappearance of roots ② and ③, for $J > 0$, $H_\ell < 0$ [cf. Fig. 2(a)]; $\epsilon_2 > \epsilon_1$.

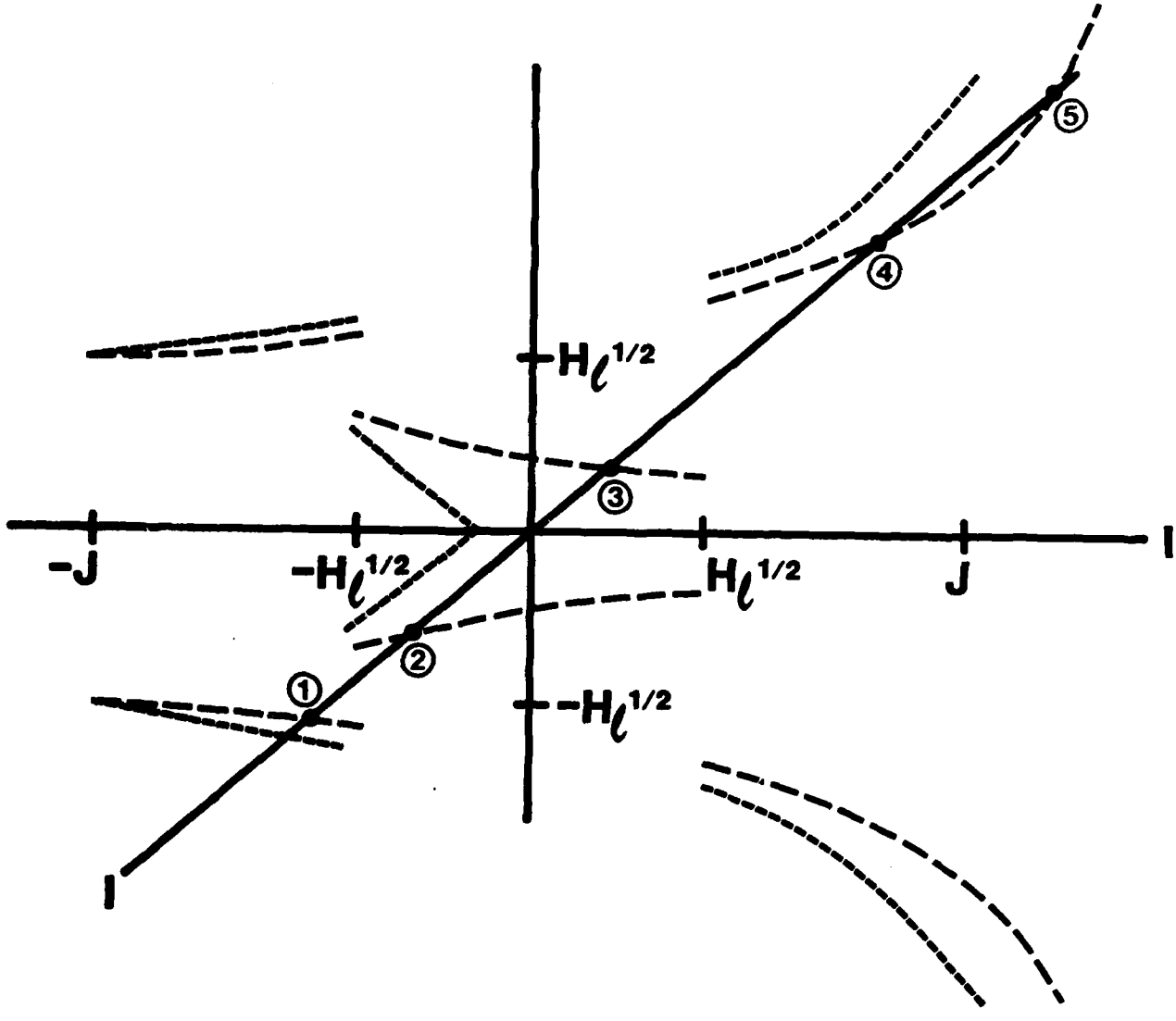

Fig. 5 Sketch indicating disappearance of the pair of roots (2) and (3) and the pair of roots (4) and (5) with increasing ϵ for J, $H_\ell > 0$. Short-dash curves correspond to a stronger wiggler field than long-dash curves.

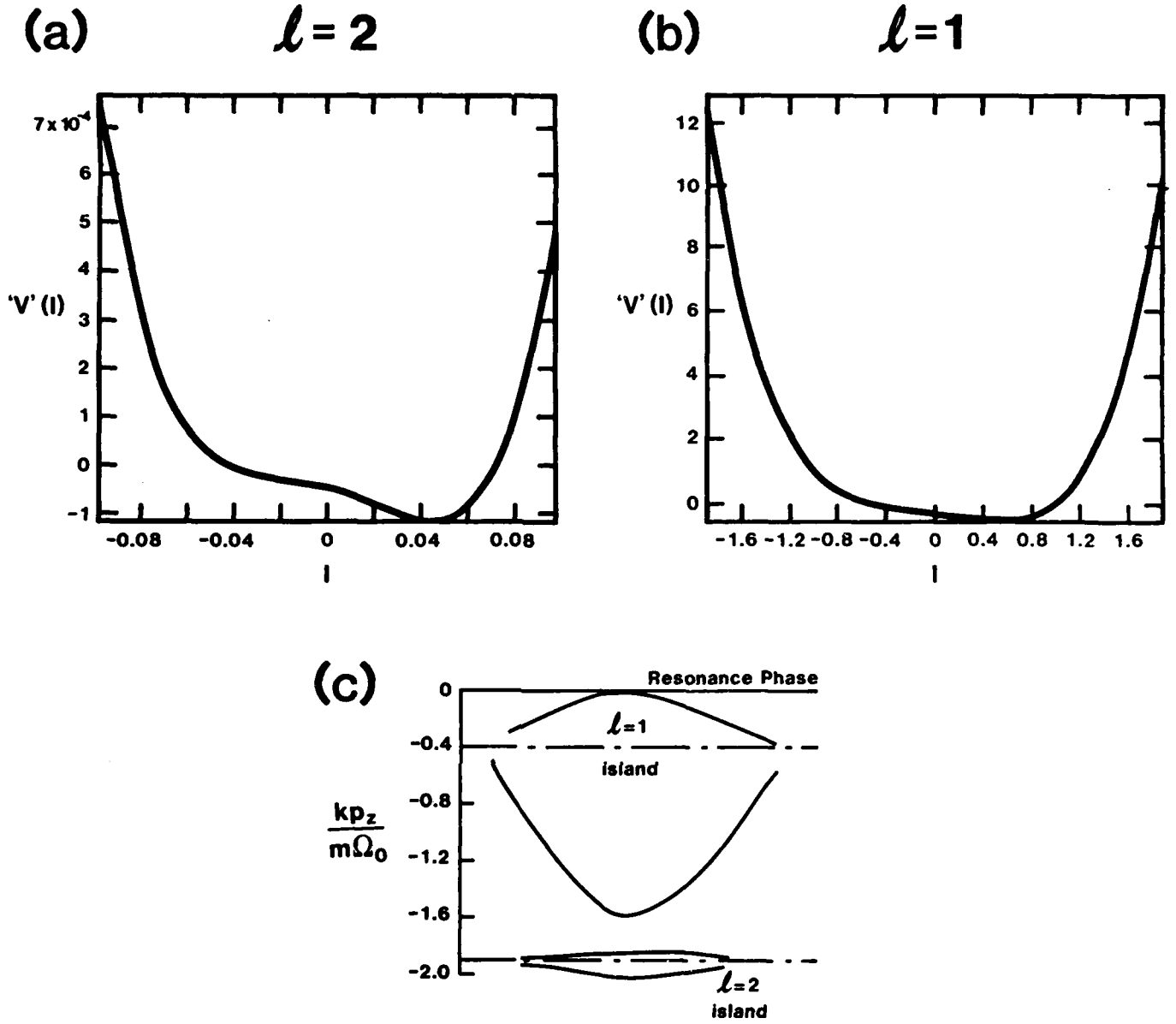

Fig. 6 (a) and (b) Effective potential 'V'(I) versus I. H_0 = 2.2 kOe, wiggler wavelength $2\pi k^{-1}$ = 3 cm, γ = 1.68, ϵ = 0.1, y component of guiding center Y = 0.1 cm; initial gyroradii are 0.35 cm and 0.95 cm, respectively.

(c) Schematic of ℓ = 1 and ℓ = 2 islands determined from the effective potentials in (a) and (b). Ordinate is $(m\Omega_0)^{-1}kp_z$, abscissa is the resonance phase $\xi_\ell = \ell\theta + kz$ for ℓ = 1 or 2.

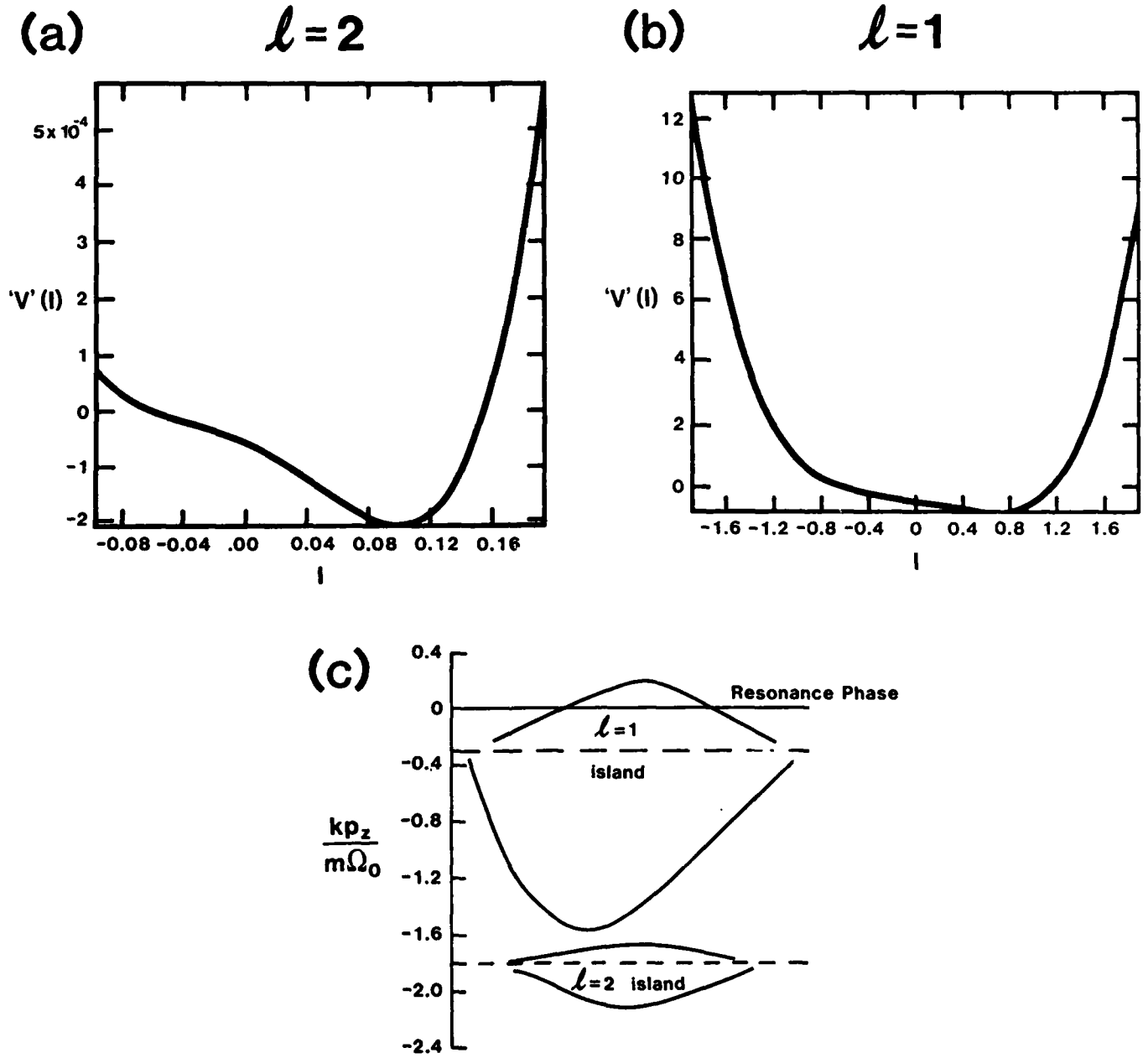

Fig. 7 Same as Fig. 6, except that y component of guiding center Y = 0.3 cm.

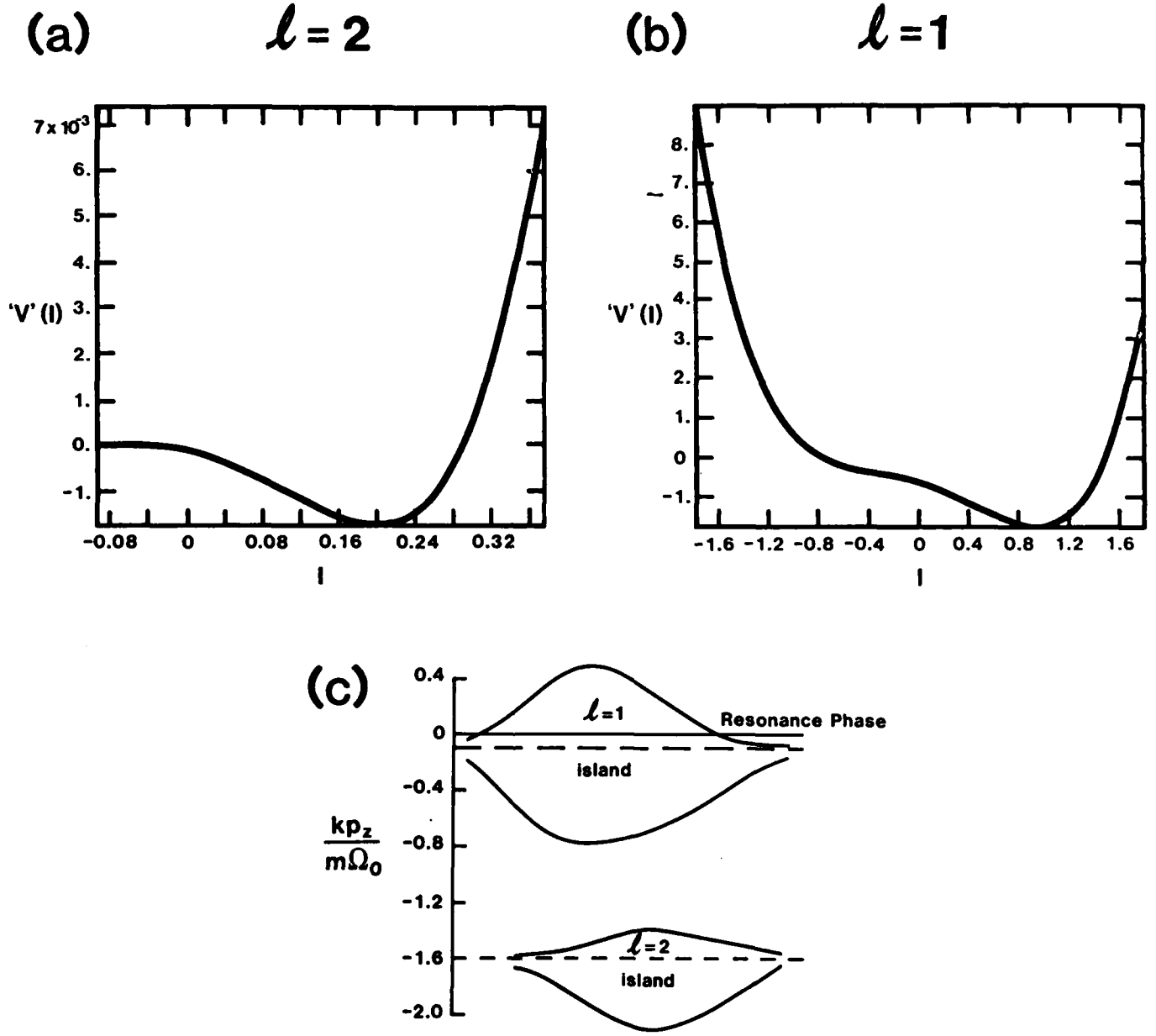

Fig. 8 Same as Fig. 6, except that y component of guiding center Y = 0.5 cm and initial gyroradii are 0.1 (a) and 0.8 cm (b).

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