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Verification of arms control agreements could be facilitated by deploying ICBMs in a number of identical areas, one or more of which could be chosen for inspection. The paper treats the attack and defense of ICBMs deceptively based and defended by interceptors. A given number of interceptors can be concentrated in selected areas or can be assigned uniformly across areas. The interceptors can be constrained to defend specific missiles or can be preferentially assigned to attackers after the attack is observed.

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ABSTRACT

On-site verification of ICBMs in the context of an arms control agreement might involve a situation where an inspector would choose one or more of a number of identical areas to inspect and would have confidence that the other areas had the same characteristics.

The paper considers optimal attack and defense of missiles deceptively based in a number of identical areas. The attacker may allocate warheads across areas as he desires, and uniformly within areas. The effect of allowing the defender to allocate interceptors non-uniformly across areas or of limiting him to uniform allocations across areas is studied. Both restricting interceptors to defending missiles uniformly within areas, and allowing interceptors to defend missiles preferentially within areas, are studied. Robustness of surviving missiles to the number of attacking warheads is studied. Results are presented for a wide range of cases.

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A. INTRODUCTION

The principal motivation for this paper is that on-site verification of ICBMs in the context of a strategic arms agreement may be a practical possibility in the not-too-distant future. In a situation where on-site inspection would be permitted, it would be likely that both sides would prefer not to allow total inspection but rather to permit inspection of a subset of the ICBMs. If the ICBMs were deceptively based in a number of identical areas, the inspector could choose one or more areas to inspect, and would have some confidence that the area or areas would be representative of the entire force. In this paper the force of ICBMs is allocated across a number of areas such that each area contains an identical number of missiles and shelters.[1]

With respect to interceptor defense, an arms control agreement would ideally (from an inspection point of view) require that inteceptors also be allocated such that each area contains an identical number of interceptors. This would permit verification of the total number of interceptors much easier than would a scheme which monitors the total number before their deployment to the areas or a scheme which estimates the total number based on sample observations from inspections. The present paper addresses the interceptor allocation problem by allowing a total number of interceptors to be allocated across areas and observing under what conditions being limited to uniform allocation results in fewer surviving missiles.

[1]Allocations across areas refer to assignments to areas while allocations within areas refer to assignments to shelters.

In the basic game studied here the attacker and defender know the number of warheads of the attacker and the number of missiles, shelters and interceptors of the defender. The attacker allocates warheads across areas as he desires, and within areas uniformly to shelters. The defender has already allocated missiles and shelters uniformly across areas. He now, effectively simultaneously with the attacker (or, equivalently, not known in advance to the attacker), allocates interceptors across areas as he desires. Two interceptor assignment procedures within areas are investigated. First, the defender assigns interceptors uniformly within areas to defend missiles. Second, the defender observes the attack and then assigns interceptors preferentially within areas to attempt to maximize the number of surviving missiles.

The number of warheads in the attack is not actually known to the defender for planning purposes since the attacker may expend a subset of his total warhead inventory in the attack. Therefore, the robustness of the number of surviving missiles to the attacker's choice of number of warheads is an important issue. The basic framework of the paper facilitates examination of robustness of surviving missiles of the defender to warhead expenditure of the attacker.

One limitation of the paper should be highlighted. Interceptors are assumed not to be vulnerable to attack before a main attack on shelters. There are several plausible situations under which this might be a reasonable assumption, as follows. First, interceptors could be in missile shelters and be assumed to be able to fire at warheads aimed at missiles before warheads hit the shelters in which the interceptors reside. Second, interceptors could be mobile and not targetable. Third, interceptors could be deceptively based in enough of their own shelters that the attacker would prefer not to expend warheads on attacking them. However, if interceptors are vulnerable to a precursor attack a completely different analysis of the problem, including use of interceptors to defend other interceptors, is necessary.

B. ATTACKER AND DEFENDER ALLOCATIONS ACROSS AREAS

A set of allocation options is defined for the attacker and a set of allocation options is defined for the defender. These options are chosen so as to span a wide spectrum of possibilities. Optimal attack and defense options are determined from among these options.

Attacking warheads can be allocated uniformly across all areas or according to one of two possible non-uniform allocations: 70 percent to shelters in half of areas and 30 percent to shelters in half of areas (called 70-30 allocation), and 90 percent to shelters in half of areas and 10 percent to shelters in half of areas (called 90-10 allocation). For instance, if there were 1000 attacking warheads and ten areas, uniform allocation would result in 100 warheads per area in all ten areas; 90-10 allocation would result in 900 warheads in five areas (180 warheads per area) and 100 warheads in five areas (20 warheads per area).

Defending interceptors can be allocated uniformly across all areas or according to one of two possible non-uniform allocations: 70 percent to missiles in half of areas and 30 percent to missiles in half of areas (called 70-30 allocation), and 90 percent to missiles in half of areas and 10 percent to missiles in half of areas (called 90-10 allocation).

Figure 1 represents attacker and defender allocations across areas and associated outcomes. When attacker allocation is uniform there is a particular expected number of surviving missiles resulting from the interaction (denoted by X_{11}, X_{12}, or X_{13}). When defender allocation is uniform there is a particular

	Defender Allocation[1/]		
Attacker Allocation	Uniform	70% to missiles in half of areas 30% to missiles in half of areas	90% to missiles in half of areas 10% to missiles in half of areas
Uniform	X_{11}	X_{12}	X_{13}
70% to shelters in half of areas, 30% to shelters in half of areas	X_{21}	X_{22} = midpoint of matched and mis-matched	X_{23} = midpoint of matched and mis-matched
90% to shelters in half of areas, 10% to shelters in half of areas	X_{31}	X_{32} = midpoint of matched and mis-matched	X_{33} = midpoint of matched and mis-matched

1/ X_{ij} denotes expected number of surviving missiles with attack i and defense j.

Figure 1. ATTACKER AND DEFENDER ALLOCATIONS ACROSS AREAS AND ASSOCIATED OUTCOMES

expected number of surviving missiles resulting from the interaction (denoted by X_{11}, X_{21}, or X_{31}). But when both allocations are non-uniform the interaction can range from perfectly matched to perfectly mis-matched. For instance, if there were four areas attacked by 400 warheads and defended by 100 interceptors then a perfect match of a 90-10 attack against a 70-30 defense would be attack 180, 180, 20, 20 versus defense 35, 35, 15, 15 (where these numbers refer to the allocations across corresponding areas); a perfect mismatch would be attack 180, 180, 20, 20 versus defense 15, 15, 35, 35. The Appendix contains a proof that the expected number of survivors considering all possible matches of offense and defense is the midpoint of the range from perfectly matched to perfectly mis-matched.

Henceforth we will drop the word perfectly and refer to matched and mis-matched combinations. The outcome of matched may sometimes favor the attacker and sometimes favor the defender, as will be explored below, and similarly for the outcome of mis-matched.

It should be noted that when a 90-10 allocation is optimal for the attacker or defender a 100-0 allocation (defined similarly) might yield better payoff. A more complete analysis would increase the number of allocations available to both sides.

C. RESOURCES AND PROBABILITIES OF KILL

The analysis considers the following resources and parameters:

(1) 200 missiles

(2) 1000 and 2000 shelters

(3) 200, 400 and 800 interceptors

(4) 1000, 2000, 4000 and 8000 warheads

(5) kill probabilities:

Attacker	Defender
.7	.7
.95	.7
.7	.95
.95	.95

Missiles and shelters are allocated identically to ten areas.

As mentioned previously, the attacker and defender can allocate warheads and interceptors to areas as follows:

(1) uniform

(2) 70% to half of areas and 30% to half of areas

(3) 90% to half of areas and 10% to half of areas.

Finally, the interceptors within an area can be limited to defending the missile to which they are assigned, or the interceptors within an area can be allocated to warheads preferentially after the attack is observed (defending the missiles from least-attacked to most-attacked, thus attempting to obtain the most surviving missiles for a given number of interceptors).

D. MONTE CARLO SIMULATION

The Monte Carlo simulation addresses the problem of estimating expected numbers of surviving missiles, together with variances. For uniform defense within areas an analytical expression for computing this quantity is available. However, for preferential defense within areas no analytical expression is known to us. There is no specific preferential defense procedure known to us to be best for the defender, so it is useful to experiment with various schemes. The Monte Carlo simulation enables uniform defense within areas and preferential defense within areas to be studied with one internally consistent model.

For the analyses discussed in this paper, 30 sample trials are run for each case of a particular size of attack and defense.

The present paper is limited to interceptors defending only the areas to which they are assigned. Also of interest are layered defenses which include longer-range interceptors capable of defending more than one area. For problems involving layered defenses analytical approaches are intractable except in special cases. The analyses of References [1] and [2] employ a layered defense model known to compute expected numbers of survivors incorrectly in some cases. The Monte Carlo simulation utilized for the present analysis is structured to treat layered defenses, but has not yet been applied to analyses of layered defense problems. We summarize below the model's functions when interceptors defend only the area to which they are allocated.

Steps

The steps of the Monte Carlo simulation are as follows, for the case of uniform defense of missiles within areas:

1. Allocate missiles and shelters uniformly to areas.
2. Allocate interceptors to areas in proportions desired.
3. Allocate warheads to areas in proportions desired.
4. In each area, assign missiles randomly among shelters. Assign interceptors to missiles until interceptors are exhausted (e.g., if there are 20 missiles and 30 interceptors, assign 2 interceptors each to first 10 missiles and 1 interceptor each to second 10 missiles.)
5. In each area, assign warheads to shelters as uniformly as possible.
6. For each missile, note how many warheads are arriving and how many interceptors are defending. Allocate interceptors as uniformly as possible across warheads.
7. Compute surviving warheads after interceptor/warhead engagement using random numbers and interceptor kill probabilities.
8. Compute surviving missiles after warhead/missile engagement using random numbers and warhead kill probabilities.

For the case of preferential defense of missiles within areas, steps 4 and 6 are changed. In step 4, interceptors are not assigned to missiles beforehand. In step 6, after the warhead assignments are observed, interceptors are assigned to defend missiles in such a way that warheads are matched one-for-one. The missiles receiving fewest warheads are defended first, until interceptors are exhausted. If there are extra interceptors, they are added singly to the previously-assigned interceptors in the same order as before.

Example

Consider a case with 200 missiles in 1000 shelters in 10 identical areas, defended by 200 interceptors allocated uniformly across the 10 areas. Let the attack be performed by 1000 warheads assigned 90 percent to half of the areas and 10 percent to half of the areas. Assume that the interceptors will defend missiles uniformly within the areas.

The steps are as follows:

1. Allocate 20 missiles and 100 shelters to each area.

2. Allocate 20 interceptors to each area.

3. Allocate 180 warheads to each of five areas and 20 warheads to each of five areas.

4. In each area, assign the 20 missiles one by one to shelters. For the first missile, draw a random number to determine which of shelters one through 100 it occupies, and so on through the 20th missile. Assign the 20 interceptors one by one to the 20 missiles.

5. In the first five areas, assign one warhead each to the 100 shelters. Then assign one more warhead each to the first 80 shelters. In the second five areas, assign one warhead each to the first 20 shelters.

6. For each missile, note how many warheads are arriving and how many interceptors are defending. In the first five areas there will be two warheads arriving on some shelters and one warhead arriving on the remaining shelters. All 20 missiles will be defended by one interceptor each.

7. In each of the 10 areas, compute the surviving warheads aimed at each missile, after the interceptor/warhead engagement, the result of which is determined by drawing a random number in the interval 0 to 1 and comparing it with the interceptor kill probability.

8. In each of the 10 areas, for each missile, determine if that missile survives after being attacked by a warhead (if there is a surviving warhead aimed at it) by drawing a random number in the interval 0 to 1 and comparing it with the warhead kill probability.

E. MATRIX GAMES

For each combination of attacker and defender resources and parameters, a three-by-three matrix game for attacker and defender allocations is generated. The tableau of Figure 1, previously presented, provides the row and column descriptions for the three attacker and the three defender allocations.

Define the attacker advantage

$$\alpha = \frac{\text{warheads}}{\text{shelters}} - \frac{\text{interceptors}}{\text{missiles}} .$$

The parameter α can be interpreted as the average number of unopposed warheads per missile. In the special case of uniform attack and defense allocations, if α is a positive integer there are exactly α unopposed warheads per missile.

As an example, consider an attack by 2000 warheads on 200 missiles in 1000 shelters, with attacker kill probability = .7 and defender kill probability = .7. Let there be uniform defense within areas.

Figure 2 presents matrix games for 200, 400 and 800 interceptors. Surviving missiles constitute the entries. The game values, denoted by V, are given in the bottom right corners.

Since the attack involves 2000 warheads against 1000 shelters there are 2 warheads per missile. Defenses by 200, 400 and 800 yield 1, 2 and 4 interceptors per missile. Thus α = 1, 0 and -2 in the three cases.

For each matrix game, along the right side are given the worst outcome for the attacker for each allocation and along the bottom are given the worst outcome for the defender

200 Interceptors

$\alpha = 1$

47	59	67*	67*
78	76 (68,84)	73 (59,88)	78
95	93 (89,97)	92 (86,98)	95
47	59	67*	

V=67

400 Interceptors

$\alpha = 0$

123	101	98	123	.12
115	113 (120,106)	104 (105,103)	115	
108*	111 (118,105)	110 (118,102)	111*	.88
108*	101	98		
.39	.61			

V=110

800 Interceptors

$\alpha = -2$

175	162	120	175
167	153 (17C,136)	128 (148,109)	167
146*	140 (161,119)	133 (162,104)	146*
146*	140	120	

V=146

Figure 2. MATRIX GAMES FOR THREE LEVELS OF INTERCEPTORS AND ATTACKS BY 2000 WARHEADS; 200 MISSILES, 1000 SHELTERS, ATTACKER KILL PROBABILITY = .7, DEFENDER KILL PROBABILITY = .7, UNIFORM DEFENSE WITHIN AREAS

for each allocation. The midpoint of the (matched, mis-matched) pair is used in evaluating the matrix games.

In these games, the row player (the attacker) is attempting to minimize surviving missiles, while the column player (the defender) is attempting to maximize surviving missiles. The attacker is guaranteed that survivors will be no more than the minmax value (denoted by an asterisk) and the defender is guaranteed that survivors will be no less than the maxmin value (denoted by an asterisk). The minmax value and maxmin value are equal in games with pure strategy solutions and different in games with mixed strategy solutions.

The first game, with attacker superiority, has a pure strategy solution. The attacker allocates his warheads uniformly. The defender uses a 90-10 allocation with 180 defenders in half the areas (36 in each area, for an average of 1.8 per missile) and 20 defenders in half the areas (4 in each area, for an average of .2 per missile). If the defender were to know that the attacker allocation was surely uniform, he could not benefit by this knowledge but would still use the 90-10 allocation (within the three allocations permitted here; if allowed, he would move toward a 100-0 allocation). If, on the other hand, the attacker knew the defender was surely using 90-10, and if the attacker could also achieve a perfect match, he could use 70-30 to reduce the survivors from 67 to 59. Presenting the information on (matched, mis-matched) in the matrix games gives a measure of both the range of outcomes and the value of information about the opponent's allocation.

The second game, with attacker and defender in parity (α=0), has a mixed strategy solution. The value of the game (110) is between the values associated with the attacker's minmax strategy (111) and the defender's maxmin stragegy (108). For the two allocations of both sides in the mixed strategy solution the outcomes range from 101 to 123.

The third game, with defender superiority, has a pure strategy solution. The attacker chooses a 90-10 allocation and the defender defends uniformly. If the defender were to know the exact attacker 90-10 allocation and could achieve a perfect 90-10 match, he could raise the payoff from 146 to 162.

F. RESULTS FOR 200 MISSILES AND 1000 SHELTERS WITH UNIFORM DEFENSE WITHIN AREAS

Figures 3, 4, 5 and 6 present matrix games for 200 missiles, 1000 shelters, and four combinations of attacker and defender kill probabilities. Interceptor defense within areas is uniform. Attacking warheads are varied from 1000 to 8000 and defending interceptors are varied from 200 to 800. The cases where the attacker and defender are in parity (α=0) are denoted by ovals.

For d = .7 (Figures 3 and 4) mixed strategies obtain for α = 0 and 200 or 400 interceptors. For d = .95 (Figures 5 and 6) mixed strategies obtain for $0 \leq \alpha \leq 2$ and all three interceptor levels. The change to mixed strategies occurs because the uniform attack no longer dominates the other two attacks for all three defenses in those cases when d changes from .7 to .95.

Throughout Figures 3, 4, 5 and 6 there are many cases in which the attacker is superior (α>0) and chooses a uniform allocation. If he were to know of the defender's non-uniform allocation and could match it, he could improve his payoff substantially.

Robustness of defender allocations can be illustrated by considering the middle column of Figure 3, specifically the change in attacking warheads from 2,000 to 4,000. If the defender is optimizing against 2,000 warheads his expected payoff is 110, resulting from randomizing between the first two options. If he is concerned about an attack of 4,000 warheads he should choose the third option, which is much better against 4,000 warheads (yielding at least 30 survivors as compared to at least 11 or 16) but much worse against 2,000 warheads (yielding at least 98 survivors as compared to at least 108 or 101).

Interceptors

Warheads

1,000

200

α = 0

158.2	143.3	131.3	158.2	.02
143.7	139.4 (142.6,136.2)	132.4 (140.8,124.0)	143.7	
130.3	131.7* (140.6,122.8)	132.0 (149.3,114.7)	132.0*	.98
130.3	131.7*	131.3		
	.06	.94		

V=132.0

400

α = -1

187	180	149	187
178	168 (176,159)	148 (162,134)	178
167*	156 (173,140)	148 (176,121)	167*
167*	156	148	

V=167

800

α = -3

199	196	170	199
193	189 (195,184)	164 (180,149)	193
190*	184 (195,174)	162 (193,131)	190*
190*	184	162	

V=190

2,000

α = 1

47	59	67*	67*
78	76 (68,84)	73 (59,88)	78
95	93 (89,97)	92 (86,98)	95
47	59	67*	

V=67

α = 0

123	101	98	123	.12
115	113 (120,106)	104 (105,103)	115	
108*	111 (118,105)	110 (118,102)	111*	.88
108*	101	98		
.39	.61			

V=110

α = -2

175	162	120	175
167	153 (170,136)	128 (148,109)	167
146*	140 (161,119)	133 (162,104)	146*
146*	140	120	

V=146

4,000

α = 3

4	5	6*	6*
16	22 (14,29)	25 (10,40)	25
83	78 (68,88)	72 (51,92)	83
4	5	6*	

V=6

α = 2

11	16	30*	30*
47	42 (24,59)	44 (16,72)	47
95	93 (87,98)	79 (60,99)	95
11	16	30*	

V=30

α = 0

78*	61	68	78*
82	82 (75,90)	72 (51,93)	82
101	101 (102,100)	96 (93,100)	101
78*	61	68	

V=78

8,000

α = 7

0*	0*	0*	0*
1	1 (0,2)	1 (0,2)	1
45	45 (34,57)	46 (24,68)	45
0*	0*	0*	

V=0

α = 6

0*	0*	0*	0*
2	3 (1,5)	7 (0,13)	2
74	67 (52,83)	56 (28,88)	58
0*	0*	0*	

V=0

α = 4

0	1	7*	7*
17	21 (9,38)	25 (1,49)	25
93	87 (79,96)	70 (41,98)	93
0	1	7*	

V=7

Figure 3. MATRIX GAMES FOR 200 MISSILES, 1000 SHELTERS, ATTACKER KILL PROBABILITY = .7, DEFENDER KILL PROBABILITY = .7, UNIFORM DEFENSE WITHIN AREAS

Interceptors

Warheads 1,000

200 — α = 0

144	124	107	144	.16
126	120 (124,116)	113 (122,105)	126	
111	114* (122,106)	118 (134,102)	118*	.84
111	114*	107		
.25		.75		V=116

400 — α = -1

182	173	132	182
168	158 (169,148)	134 (149,119)	168
156*	145 (165,125)	137 (167,108)	156*
156*	145	132	V=156

800 — α = -3

199	193	158	199
192	187 (194,181)	155 (173,136)	192
189*	178 (192,164)	152 (189,114)	189*
189*	178	152	V=189

Warheads 2,000

200 — α = 1

7	24	42	42*
60	55 (41,70)	51 (25,78)	60
89	85 (79,92)	81 (68,94)	89
7	24	42*	V=42

400 — α = 0

101	76	78	101	.002
96*	96 (99,92)	84 (74,94)	96*	.998
98	101 (103,99)	99 (99,99)	101	
96*	76	78		
.79	.21			V=96

800 — α = -2

168	149	100	164
151	142 (16C,124)	116 (133,100)	151
133*	128 (149,107)	125 (150,100)	133*
133*	128	100	V=133

Warheads 4,000

200 — α = 3

0*	0*	0*	0*
2	7 (2,13)	13 (1,26)	13
77	70 (56,85)	62 (34,90)	77
0*	0*	0*	V=0

400 — α = 2

0	2	16*	16*
32	30 (8,51)	32 (1,62)	32
94	88 (80,97)	72 (44,99)	94
0	2	16*	V=16

800 — α = 0

52	39	58*	58*
69	69 (54,85)	59 (26,91)	69
94	98 (96,100)	88 (75,100)	98
52	39	58*	V=58

Warheads 8,000

200 — α = 7

0*	0*	0*	0*
0*	0* (0,0)	0* (0,0)	0*
31	32 (19,45)	34 (8,60)	34
0*	0*	0*	V=0

400 — α = 6

0*	0*	0*	0*
0	0 (0,0)	2 (0,4)	2
67	57 (37,77)	49 (13,86)	67
0*	0*	0*	V=0

800 — α = 4

0	0	2*	2*
6	13 (0,26)	19 (0,38)	19
90	84 (72,96)	61 (24,98)	90
0	0	2*	V=2

Figure 4. MATRIX GAMES FOR 200 MISSILES, 1000 SHELTERS, ATTACKER KILL PROBABILITY = .95, DEFENDER KILL PROBABILITY = .7, UNIFORM DEFENSE WITHIN AREAS

Interceptors

Warheads 1,000

200 Interceptors — α = 0

193	167	142	193	
167	155 (161,150)	144 (157,130)	167	.10
142	143* (156,130)	145 (172,118)	145*	.90
142	143*	142		
.06		.94		V=145

400 Interceptors — α = -1

200	197	156	200	
197	186 (196,175)	156 (171,140)	197	
194*	175 (196,153)	158 (189,128)	194*	
194*	175	156		
				V=194

800 Interceptors — α = -3

200*	200	184	200*	
200*	199 (200,198)	175 (191,160)	200*	
200*	197 (200,195)	166 (196,136)	200*	
200*	197	166		
				V=200

Warheads 2,000

200 Interceptors — α = 1

56	75	95*	95*	.22
96	92 (85,99)	86 (72,101)	96	.78
104	100 (97,103)	97 (92,102)	104	
56	75	86*		
	.35	.65		V=88

400 Interceptors — α = 0

187	137	116	187	.17
139	142 (168,117)	126 (143,108)	142	
116	126* (144,108)	130 (157,104)	130*	.83
116	126*	116		
.16		.84		V=128

800 Interceptors — α = -2

200	195	125	200	
194	179 (197,161)	142 (171,112)	194	
190*	162 (193,130)	148 (191,105)	190*	
190*	162	125		
				V=190

Warheads 4,000

200 Interceptors — α = 3

5	7	8*	8*	
22	28 (16,40)	34 (11,58)	34	
97	87 (76,98)	77 (54,99)	97	
5	7	8*		
				V=8

400 Interceptors — α = 2

16	28	66*	66*	.17
69	61 (34,89)	59 (22,96)	69	.83
100	99 (98,100)	83 (66,100)	100	
16	28	59*		
.11		.89		V=60

800 Interceptors — α = 0

174	107	99	174	.12
111	124* (146,102)	103 (105,101)	124*	.41
102	107 (114,100)	128 (156,100)	128	.47
102	107*	99		
.15	.47	.38		V=114

Warheads 8,000

200 Interceptors — α = 7

0*	0*	0*	0*	
1	1 (1,2)	2 (1,3)	2	
55	57 (41,72)	56 (24,88)	57	
0*	0*	0*		
				V=0

400 Interceptors — α = 6

0	0	1*	1*	
4	6 (2,10)	14 (1,28)	14	
95	80 (63,98)	66 (34,99)	95	
0	0	1*		
				V=1

800 Interceptors — α = 4

1	5	33*	33*	
37	46 (6,86)	47 (2,92)	47	
100	98 (96,100)	74 (48,100)	100	
1	5	33*		
				V=33

Figure 5. MATRIX GAMES FOR 200 MISSILES, 1000 SHELTERS, ATTACKER KILL PROBABILITY = .7, DEFENDER KILL PROBABILITY = .95 UNIFORM DEFENSE WITHIN AREAS

Interceptors: 200, 400, 800 — **Warheads**: 1,000, 2,000, 4,000, 8,000

Warheads 1,000 — Interceptors 200 — α = 0

190	159	121	190	
155	142 (150,134)	128 (143,113)	155	.31
121	128* (142,114)	134 (164,105)	134*	.69
121	128*	121		
	.33	.67		V=133

Warheads 1,000 — Interceptors 400 — α = -1

199	196	142	199	
195	181 (195,166)	145 (164,125)	195	
193*	165 (194,136)	148 (186,110)	193*	
193*	165	142		
				V=193

Warheads 1,000 — Interceptors 800 — α = -3

200*	200	177	200*	
200*	198 (200,196)	167 (186,147)	200*	
200*	197 (200,194)	157 (195,120)	200*	
200*	197	157		
				V=200

Warheads 2,000 — Interceptors 200 — α = 1

10	42	75*	75*	.16
78	72 (58,85)	65 (36,94)	78	.84
97	92 (84,99)	85 (70,100)	97	
10	42	65*		
	.25	.75		V=67

Warheads 2,000 — Interceptors 400 — α = 0

182	116	99	182	.17
120	130 (156,104)	112 (123,100)	130	.11
102	116* (130,101)	122 (145,100)	122*	.72
102	116*	99		
.17	.23	.60		V-117

Warheads 2,000 — Interceptors 800 — α = -2

199	192	104	199	
192	172 (197,148)	132 (163,101)	192	
187*	154 (192,116)	144 (189,100)	187*	
187*	154	104		
				V=187

Warheads 4,000 — Interceptors 200 — α = 3

0*	0*	0*	0*	
3	13 (2,24)	23 (1,46)	23	
97	82 (67,98)	69 (39,99)	97	
0*	0*	0*		
				V=0

Warheads 4,000 — Interceptors 400 — α = 2

0	3	51*	51*	.14
56	49 (14,84)	48 (3,94)	56	.86
100	98 (97.100)	76 (53,100)	100	
0	3	48*		
.04		.96		V=49

Warheads 4,000 — Interceptors 800 — α = 0

165	90	95	165	.10
98*	114 (129,99)	93 (85,100)	114*	.45
100	101 (102,100)	121 (142,100)	121	.45
98*	90	93		
.19	.57	.24		V=105

Warheads 8,000 — Interceptors 200 — α = 7

0*	0*	0*	0*	
0*	0* (0,0)	0* (0,0)	0*	
41	44 (26,62)	47 (10,84)	47	
0*	0*	0*		
				V=0

Warheads 8,000 — Interceptors 400 — α = 6

0*	0*	0*	0*	
0	0 (0,1)	6 (0,12)	6	
95	74 (52,96)	57 (16,99)	95	
0*	0*	0*		
				V=0

Warheads 8,000 — Interceptors 800 — α = 4

0	0	17*	17*	
20	41 (0,82)	45 (0,89)	45	
100	98 (95,100)	66 (33,100)	100	
0	0	17*		
				V=17

Figure 6. MATRIX GAMES FOR 200 MISSILES, 1000 SHELTERS, ATTACKER KILL PROBABILITY = .95, DEFENDER KILL PROBABILITY = .95, UNIFORM DEFENSE WITHIN AREAS

The values of the games are summarized in Table 1. Results for $\alpha = 0$ are indicated by ovals. Attacker kill probability is denoted by a and defender kill probability is denoted by d. Comparing the first and second groups, increasing a from .7 to .95 decreases survivors substantially. Comparing the first and third groups, increasing d from .7 to .95 increases survivors substantially. Comparing the first and fourth groups, where the increases in kill probabilities are symmetric, the effect is most significant for 4000 warheads and 800 interceptors (78 surviving missiles in the first group and 105 surviving missiles in the fourth group). The significant difference is due to the fact that when d = .7 the expected number of warheads getting through the defense is more than one, while when d = .95 the expected number of missiles getting through the defenses is far less than one; the latter case thus yields significantly more surviving missiles.

Table 1. EXPECTED SURVIVING MISSILES FOR 200 MISSILES AND 1000 SHELTERS, UNIFORM DEFENSE WITHIN AREAS

Kill Probabilities	Warheads	Interceptors 200	Interceptors 400	Interceptors 800
a=.7, d=.7	1000	132	167	190
	2000	67	110	146
	4000	6	30	78
	8000	0	0	7
a=.95, d=.7	1000	116	156	189
	2000	42	96	133
	4000	0	16	58
	8000	0	0	2
a=.7, d=.95	1000	145	194	200
	2000	88	128	190
	4000	8	60	114
	8000	0	1	33
a=.95, d=.95	1000	133	193	200
	2000	67	117	187
	4000	0	49	105
	8000	0	0	17

G. RESULTS FOR 200 MISSILES AND 1000 SHELTERS WITH PREFERENTIAL DEFENSE WITHIN AREAS

Figures 7, 8, 9 and 10 present matrix games for the same parameters as in Figures 3, 4, 5 and 6, but for preferential defense within areas rather than uniform defense within areas. Table 2 summarizes the game values from Figures 7, 8, 9 and 10, and compares the game values for uniform defense and preferential defense.

Figures 7, 8, 9 and 10 reveal that mixed strategies are not employed when there is preferential defense (except for two cases in which numerical properties result in mixed strategies, but in these two cases the minmax and maxmin values are essentially identical.) There are significant changes in optimal allocations when defense is changed from uniform defense within areas to preferential defense within areas. In particular, when the attacker is superior the payoff from all three types of defender allocation is essentially identical, rather than the defender obtaining much higher payoff from non-uniform allocations. Thus uniform interceptor allocations to areas become satisfactory, and verification of arms control agreements becomes much easier. Furthermore, uniform interceptor allocations are robust against increases in attack sizes.

Table 2 shows that preferential defense improves results in almost all cases where $\alpha \geq 0$. When d = .95 the improvements are greater than when d = .7. When d = .95 and $\alpha > 0$ surviving missiles can increase dramatically. For instance, for a = .7, d = .95, 4000 warheads and 200 interceptors, survivors increase from 8 to 45.

Interceptors

Warheads

1,000

200

α = 0

158	145	131	158
153	152 (156,148)	138 (146,131)	153
144*	144* (154,134)	140 (159,121)	144*
144*	144*	131	

V=144

400

α = -1

187	180	150	187
180	175 (184,166)	157 (172,141)	180
168*	163 (177,149)	157 (185,128)	168*
168*	163	150	

V=168

800

α = -3

199	197	170	199
194	192 (198,185)	172 (189,156)	194
190	185 (196,174)	168 (198,138)	190*
190*	185	168	

V=190

2,000

α = 1

72*	70	71	72*
93	90 (84,96)	84 (73,94)	93
115	114 (117,110)	105 (106,104)	115
72*	70	71	

V=72

α = 0

123*	112	103	123*
126	122 (126,119)	109 (107,111)	126
128	127 (135,119)	119 (132,107)	128
123*	112	103	

V=123

α = -2

175	162	129	175
163	158 (172,144)	136 (152,121)	163
146*	146* (161,132)	143 (173,113)	146*
146*	146*	129	

V=146

4,000

α = 3

22*	20	21	22*
38	36 (29,44)	35 (22,47)	38
89	85 (78,93)	77 (58,96)	89
22*	20	21	

V=22

α = 2

38	40*	39	40*
63	61 (52,70)	55 (37,72)	63
102	100 (97,102)	86 (71,101)	102
38	40*	39	

V=40

α = 0

78*	77	76	78*
95	92 (83,101)	81 (64,99)	95
111	110 (113,106)	101 (100,102)	111
78*	77	76	

V=78

8,000

α = 7

3*	3*	3*	3*
9	9 (7,12)	9 (3,15)	9
53	53 (43,64)	49 (27,72)	53
3*	3*	3*	

V=3

α = 6

7*	7*	7*	7*
16	16 (12,21)	15 (6,24)	16
75	72 (60,84)	65 (38,92)	75
7*	7*	7*	

V=7

α = 4

15*	15*	15*	15*
30	30 (21,39)	30 (12,48)	30
94	90 (82,98)	74 (50,99)	94
15*	15*	15*	

V=15

Figure 7. MATRIX GAMES FOR 200 MISSILES, 1000 SHELTERS, ATTACKER KILL PROBABILITY = .7, DEFENDER KILL PROBABILITY = .7, PREFERENTIAL DEFENSE WITHIN AREAS

Interceptors

Warheads 1,000

200 — α = 0

144	123	106	144
141	140 (144,137)	123 (131,114)	141
135*	134 (143,124)	131 (148,113)	135*
135*	123	106	

V=135

400 — α = -1

182	171	131	182
173	167 (178,157)	145 (161,128)	173
159*	155 (170,140)	149 (180,119)	159*
159*	155	131	

V=159

800 — α = -3

199	194	158	199
193	189 (197,181)	168 (187,149)	193
188*	180 (194,165)	163 (197,129)	188*
188*	180	158	

V=188

Warheads 2,000

200 — α = 1

52*	51	52*	52*
80	76 (66,85)	65 (45,85)	80
109	107 (108,106)	98 (95,102)	109
52*	51	52*	

V=52

400 — α = 0

101*	95	88	101*
112	107 (105,109)	91 (78,104)	112
118	118 (124,113)	110 (116,103)	118
101*	95	88	

V=101

800 — α = -2

167.6	148.6	117.2	167.6	.03
151.2	147.4 (162.8,132.0)	127.0 (140.9,113.1)	151.2	
132.3	135.6* (148.6,122.6)	136.5 (163.8,109.1)	136.5*	.97
132.3	135.6*	117.2		
	.60	.40		

V=136.0

Warheads 4,000

200 — α = 3

12	12	14*	14*
28	27 (21,34)	25 (12,38)	28
84	78 (66,90)	68 (41,94)	84
12	12	14*	

V=14

400 — α = 2

26*	25	25	26*
46	47 (37,56)	41 (20,63)	47
98	94 (89,100)	77 (54,100)	98
26*	25	25	

V=26

800 — α = 0

52	54	64*	64*
80	77 (60,94)	70 (45,95)	80
104	104 (106,103)	94 (86,101)	104
52	54	64*	

V=64

Warheads 8,000

200 — α = 7

1	2*	2*	2*
5	5 (4,7)	5 (2,8)	5
43	44 (33,55)	40 (16,64)	44
1	2*	2*	

V=2

400 — α = 6

3*	3*	3*	3*
9	10 (7,12)	9 (3,15)	10
68	63 (49,77)	57 (27,88)	68
3*	3*	3*	

V=3

800 — α = 4

8*	7	6	8*
17	19 (13,25)	22 (6,37)	22
91	85 (74,96)	68 (38,98)	91
8*	7	6	

V=8

Figure 8. MATRIX GAMES FOR 200 MISSILES, 1000 SHELTERS, ATTACKER KILL PROBABILITY = .95, DEFENDER KILL PROBABILITY = .7, PREFERENTIAL DEFENSE WITHIN AREAS

Interceptors

Warheads

1,000 Warheads, 200 Interceptors

α = 0

193	167	143	193
177	175 (188,161)	151 (168,135)	177
159*	159* (176,143)	158 (189,126)	159*
159*	159*	143	

V=159

1,000 Warheads, 400 Interceptors

α = -1

200	197	157	200
197	192 (198,187)	166 (184,148)	197
194*	182 (197,167)	167 (198,136)	194*
194*	182	157	

V=194

1,000 Warheads, 800 Interceptors

α = -3

200*	200	184	200*
200*	199 (200,198)	184 (199,169)	200*
200*	198 (200,195)	176 (200,152)	200*
200*	198	176	

V=200

2,000 Warheads, 200 Interceptors

α = 1

101	102*	101	102*
123	118 (114,122)	105 (100,110)	123
129	129 (139,119)	119 (132,106)	129
[illegible]	102*	101	

V=102

2,000 Warheads, 400 Interceptors

α = 0

187	155	124	187
164	162 (182,142)	133 (146,120)	164
153*	151 (169,134)	147 (183,111)	153*
153*	151	124	

V=153

2,000 Warheads, 800 Interceptors

α = -2

200	195	143	200
194	188 (198,178)	155 (178,131)	194
190	177 (194,161)	161 (198,123)	190*
190*	170	143	

V=190

4,000 Warheads, 200 Interceptors

α = 3

45*	44	45*	45*
65	61 (52,70)	60 (45,75)	65
107	102 (100,103)	88 (77,100)	107
45*	44	45*	

V=45

4,000 Warheads, 400 Interceptors

α = 2

87	88*	88*	88*
112	105 (100,110)	88 (76,101)	112
120	120 (129,112)	106 (108,104)	120
87	88*	88*	

V=88

4,000 Warheads, 800 Interceptors

α = 0

174	143	115	174
156	154 (171,137)	120 (127,113)	156
143*	143* (161,125)	139 (170,108)	143*
143*	143*	115	

V=143

8,000 Warheads, 200 Interceptors

α = 7

15*	15*	15*	15*
23	25 (20,30)	24 (15,33)	25
72	72 (59,86)	66 (37,95)	72
15*	15*	15*	

V=15

8,000 Warheads, 400 Interceptors

α = 6

37	37	39*	39*
46	48 (40,56)	47 (30,63)	48
103	96 (90,101)	80 (60,100)	103
37	37	39*	

V=39

8,000 Warheads, 800 Interceptors

α = 4

76*	76*	75	76*
93	88 (77,99)	78 (61,94)	93
115	115 (121,109)	98 (92,103)	115
76*	76*	75	

V=76

Figure 9. MATRIX GAMES FOR 200 MISSILES, 1000 SHELTERS, ATTACKER KILL PROBABILITY = .7, DEFENDER KILL PROBABILITY = .95, PREFERENTIAL DEFENSE WITHIN AREAS

Interceptors

Warheads 1,000

200 — α = 0

190	157	121	190
173	170 (186,155)	139 (157,121)	173
155*	154 (173,136)	152 (186,117)	155*
155*	154	121	

V=155

400 — α = -1

199	197	141	199
195	190 (197,183)	159 (179,139)	195
193*	179 (195,163)	163 (198,127)	193*
193*	179	141	

V=193

800 — α = -3

200*	200	177	200*
200*	199 (200,197)	181 (198,165)	200*
200*	197 (200,194)	173 (200,145)	200*
200*	197	173	

V=200

Warheads 2,000

200 — α = 1

90	91*	90	91*
119	111 (104,117)	93 (79,106)	119
126	127 (136,117)	115 (125,104)	127
90	91*	90	

V=91

400 — α = 0

181.7	149.3	116.2	181.7	.01
161.3	157.9 (178.6,137.1)	124.2 (132.3,116.1)	161.3	
149.2	149.7* (166.1,133.3)	143.3 (177.6,109.0)	149.7*	.99
149.2	149.3*	116.2		
.01	.99			

V=149.7

800 — α = -2

199	192	137	199
192	185 (197,174)	152 (175,129)	192
187*	175 (192,157)	160 (198,122)	187*
187*	175	137	

V=187

Warheads 4,000

200 — α = 3

41	41	42*	42*
62	57 (47,67)	53 (36,70)	62
105	97 (92,103)	80 (60,100)	105
41	41	42*	

V=42

400 — α = 2

83*	81	82	83*
106	99 (92,106)	83 (69,97)	106
118	118 (126,111)	99 (94,104)	118
83*	81	82	

V=83

800 — α = 0

165	137	112	165
150	147 (160,134)	115 (119,111)	150
140*	138 (154,122)	135 (162,107)	140*
140*	137	112	

V=140

Warheads 8,000

200 — α = 7

14*	14*	13	14*
20	23 (18,28)	22 (13,31)	23
67	68 (53,83)	60 (26,93)	68
14*	14*	13	

V=14

400 — α = 6

33	34*	34*	34*
41	45 (36,53)	42 (25,59)	45
99	92 (84,100)	74 (49,100)	99
33	34*	34*	

V=34

800 — α = 4

68*	68*	68*	68*
84	81 (70,93)	72 (52,91)	84
113	112 (116,108)	92 (81,103)	113
68*	68*	68*	

V=68

Figure 10. MATRIX GAMES FOR 200 MISSILES, 1000 SHELTERS, ATTACKER KILL PROBABILITY = .95, DEFENDER KILL PROBABILITY = .95, PREFERENTIAL DEFENSE WITHIN AREAS

Table 2. EXPECTED SURVIVING MISSILES FOR 200 MISSILES AND 1000 SHELTERS, UNIFORM DEFENSE AND PREFERENTIAL DEFENSE WITHIN AREAS

		Uniform De			Preferential Defense		
		Interceptors					
Kill Probabilities	Warheads	200	400	800	200	400	800
a=.7, d=.7	1000	132	167	190	144	168	190
	2000	67	110	146	72	123	146
	4000	6	30	78	22	40	78
	8000	0	0	7	3	7	15
a=.95, d=.7	1000	116	156	189	135	159	188
	2000	42	96	133	52	101	136
	4000	0	16	58	14	26	64
	8000	0	0	2	2	3	8
a=.7, d=.95	1000	145	194	200	159	194	200
	2000	88	128	190	102	153	190
	4000	8	60	114	45	88	143
	8000	0	1	33	15	39	76
a=.95, d=.95	1000	133	193	200	155	193	200
	2000	67	117	187	91	150	187
	4000	0	49	105	42	83	140
	8000	0	0	17	14	34	68

H. RESULTS FOR 200 MISSILES AND 2000 SHELTERS WITH UNIFORM DEFENSE WITHIN AREAS

Figures 11, 12, 13 and 14 present matrix games for 2000 shelters. Table 3 summarizes the game values from these figures, together with results for 1000 shelters.

Table 3 shows that when both warheads and shelters are doubled the number of surviving missiles is constant. Doubling the number of shelters simply shifts the results of Table 2. For uniform attack and defense within areas results are entirely dependent on the average engagement at each defended missile, as represented by α.

Figures 11, 12, 13 and 14 are included for documentation purposes. They contain more defense-favorable cases and fewer offense-favorable cases than do Figures 7, 8, 9 and 10. However, the interesting cases where α is fairly close to zero are the same in the two sets of matrix games except for Monte Carlo effects.

Interceptors: 200, 400, 800 — Warheads: 1,000, 2,000, 4,000, 8,000

Warheads 1,000

Interceptors 200 — α = -.5

179*	173	166	179*
180	173 (178,168)	164 (174,154)	180
179*	173 (183,163)	166 (186,146)	179*
179*	173	164	

V=179

Interceptors 400 — α = -1.5

195	190	173	195
194*	190 (193,188)	175 (184,165)	194*
194*	190 (196,183)	175 (194,155)	194*
194*	190	173	

V=194

Interceptors 800 — α = -3.5

199*	198	184	199*
199*	198 (199,197)	184 (150,178)	199*
200	198 (199,196)	184 (198,172)	200
199*	198	184	

V=199

Warheads 2,000

Interceptors 200 — α = 0

157	144	131	157	.05
146	138 (142,133)	132 (141,123)	146	
131	131 (139,124)	132* (150,115)	132*	.95
131	131	132*		
.07		.93		

V=132

Interceptors 400 — α = -1

188	180	149	188
177	168 (175,161)	148 (160,135)	177
167*	158 (174,142)	149 (176,122)	167*
167*	158	148	

V=167

Interceptors 800 — α = -3

199	196	168	199
195	190 (196,185)	165 (181,149)	195
190*	184 (195,173)	161 (192,131)	190*
190*	184	161	

V=190

Warheads 4,000

Interceptors 200 — α = 1

47	58	66	66*
79	76 (68,84)	74 (59,89)	79
96	95 (91,98)	91 (85,98)	96
47	58	66*	

V=66

Interceptors 400 — α = 0

125	104	99	125	.14
114	113 (121,106)	104 (105,102)	114	
108*	112 (119,105)	110 (118,102)	112*	.86
108*	104	99		
.31	.69			

V=111

Interceptors 800 — α = -2

173	161	120	173
162	153 (169,137)	128 (148,109)	162
144*	138 (159,118)	133 (163,103)	144*
144*	138	120	

V=144

Warheads 8,000

Interceptors 200 — α = 3

4	5	6*	6*
18	21 (13,29)	25 (10,40)	25
83	78 (68,88)	72 (51,93)	83
4	5	6*	

V=6

Interceptors 400 — α = 2

12	17	30*	30*
46	43 (25,61)	43 (15,71)	46
95	92 (87,98)	80 (61,99)	95
12	17	30*	

V=30

Interceptors 800 — α = 0

79*	62	71	79*
83	83 (78,89)	72 (51,94)	83
101	101 (102,100)	96 (92,100)	101
79*	62	71	

V=79

Figure 11. MATRIX GAMES FOR 200 MISSILES, 2000 SHELTERS, ATTACKER KILL PROBABILITY = .7, DEFENDER KILL PROBABILITY = .7, UNIFORM DEFENSE WITHIN AREAS

Interceptors: 200, 400, 800

Warheads: 1,000, 2,000, 4,000, 8,000

1,000 Warheads — 200 Interceptors ($\alpha = -.5$)

174	163	152	174
172	162 (169,154)	153 (166,139)	172
170*	162 (176,148)	153 (183,126)	170*
170*	162	152	

V=170

1,000 Warheads — 400 Interceptors ($\alpha = -1.5$)

191*	185	165	191*
192	186 (190,181)	166 (178,154)	192
191*	186 (195,178)	165 (192,138)	191*
191*	185	165	

V=191

1,000 Warheads — 800 Interceptors ($\alpha = -3.5$)

200	197	180	200
199*	197 (198,196)	179 (188,169)	199*
199*	197 (199,195)	179 (196,163)	199*
199*	197	179	

V=199

2,000 Warheads — 200 Interceptors ($\alpha = 0$)

143	126	105	143	
127	120 (124,116)	111 (119,103)	127	.29
112	115* (125,106)	119 (132,102)	119*	.71
112	115*	105		
	.63	.37		

V=117

2,000 Warheads — 400 Interceptors ($\alpha = -1$)

183	172	129	183
170	158 (167,149)	134 (149,119)	170
159*	146 (166,125)	137 (167,107)	159*
159*	146	129	

V-159

2,000 Warheads — 800 Interceptors ($\alpha = -3$)

198	195	159	198
193	186 (194,179)	155 (175,135)	193
186	179 (193,166)	151 (189,114)	186*
186*	179	151	

V=186

4,000 Warheads — 200 Interceptors ($\alpha = 1$)

8	24	43*	43*
60	55 (41,69)	52 (26,78)	60
89	84 (76,91	82 (70,95)	89
8	24	43*	

V=43

4,000 Warheads — 400 Interceptors ($\alpha = 0$)

102	74	77	102	.08
94*	97 (100,93)	85 (75,95)	97*	.92
97	101 (103,98)	100 (100,100)	101	
94*	74	77		
.75	.25			

V=95

4,000 Warheads — 800 Interceptors ($\alpha = -2$)

167	150	99	167
150	141 (160,121)	117 (134,100)	150
135*	128 (150,107)	125 (150,100)	135*
135*	128	99	

V=135

8,000 Warheads — 200 Interceptors ($\alpha = 3$)

0*	0*	0*	0*
2	8 (1,14)	12 (1,24)	12
78	69 (56,83)	62 (35,90)	78
0*	0*	0*	

V=0

8,000 Warheads — 400 Interceptors ($\alpha = 2$)

0	1	17*	17*
31	28 (8,49)	32 (1,63)	32
93	89 (81,97)	72 (45,99)	93
0	1	17*	

V=17

8,000 Warheads — 800 Interceptors ($\alpha = 0$)

53	40	58*	58*
69	70 (56,84)	58 (24,91)	70
100	98 (96,100)	87 (75,100)	100
53	40	58*	

V=58

Figure 12. MATRIX GAMES FOR 200 MISSILES, 2000 SHELTERS, ATTACKER KILL PROBABILITY = .95, DEFENDER KILL PROBABILITY = .7, UNIFORM DEFENSE WITHIN AREAS

Interceptors

Warheads

1,000

200

α = -.5

197	183	171	197	
196*	185 (190,179)	171 (181,160)	196*	
196*	184 (195,174)	171 (194,148)	196*	
196*	183	171		

V=196

400

α = -1.5

200*	198	179	200*	
200*	199 (199,198)	178 (187,169)	200*	
200*	199 (200,198)	179 (196,162)	200*	
200*	198	178		

V=200

800

α = -3.5

200*	200*	192	200*	
200*	200* (200,200)	192 (195,189)	200*	
200*	200* (200,200)	192 (198,185)	200*	
200*	200*	192		

V=200

2,000

α = 0

194	168	143	194	
167	156 (162,149)	143 (157,130)	167	.17
141	143* (157,130)	146* (173,119)	146*	.83
141	143*	143*		
.10		.90		

V=146

α = -1

200	197	158	200	
197	186 (197,176)	158 (174,143)	197	
195*	175 (196,154)	158 (190,125)	195*	
195*	175	158		

V=195

α = -3

200*	200	183	200*	
200*	199 (200,198)	175 (191,160)	200*	
200*	198 (200,195)	167 (196,137)	200*	
200*	198	167		

V=200

4,000

α = 1

57	78	93*	93*	.29
95	92 (85,99)	86 (71,101)	95	.71
103	99 (96,103)	97 (93,102)	103	
57	78	86*		
	.31	.69		

V=88

α = 0

186	138	117	186	.15
141	143 (167,118)	124 (141,106)	143	.05
117	126* (144,107)	131 (158,104)	131*	.80
117	126*	117		
.13	.13	.74		

V=128

α = -2

199	194	125	199	
194	180 (198,162)	142 (172,111)	194	
190*	162 (194,130)	147 (190,104)	190*	
190*	162	125		

V=190

8,000

α = 3

4	7	9*	9*	
20	28 (16,40)	35 (11,60)	35	
98	87 (76,98)	77 (55,99)	98	
4	7	9*		

V=9

α = 2

17	28	64*	64*	.15
69	61 (34,88)	60 (24,96)	69	.85
100	99 (98,100)	83 (66,100)	100	
17	28	60*		
.08		.92		

V=61

α = 0

173	107	99	173	.13
111	125* (148,102)	103 (105,100)	125*	.39
102	108 (115,101)	128 (157,100)	128	.48
102	107*	99		
.16	.47	.37		

V=114

Figure 13. MATRIX GAMES FOR 200 MISSILES, 2000 SHELTERS, ATTACKER KILL PROBABILITY = .7, DEFENDER KILL PROBABILITY = .95, UNIFORM DEFENSE WITHIN AREAS

Interceptors: 200, 400, 800

Warheads 1,000

Interceptors 200, $\alpha = -.5$

196	179	161	196	
196	178 (186,170)	161 (176,146)	196	
195*	178 (193,164)	162 (192,132)	195*	
195*	178	161		V=195

Interceptors 400, $\alpha = -1.5$

200*	198	169	200*	
200*	198 (199,197)	170 (182,158)	200*	
200*	198 (200,197)	170 (194,147)	200*	
200*	198	169		V=200

Interceptors 800, $\alpha = -3.5$

200*	200*	189	200*	
200*	200* (200,200)	189 (193,185)	200*	
200*	200* (200,200)	189 (198,179)	200*	
200*	200*	189		V=200

Warheads 2,000

Interceptors 200, $\alpha = 0$

191	156	123	191	.14	
155	143 (150,135)	128 (143,114)	155		
122	129* (143,114)	134 (164,105)	134*	.86	
122	129*	123			
	.30	.70			V=133

Interceptors 400, $\alpha = -1$

200	196	140	200	
195	181 (196,167)	145 (164,125)	195	
193*	165 (195,136)	148 (186,110)	193*	
193*	165	140		V=193

Interceptors 800, $\alpha = -3$

200	200	177	200	
200	199 (200,197)	167 (186,149)	200	
199	196 (200,193)	157 (196,118)	199*	
199*	196	157		V=199

Warheads 4,000

Interceptors 200, $\alpha = 1$

9	41	75*	75*	.17	
78	72 (57,86)	65 (36,94)	78	.83	
98	91 (84,99)	85 (70,100)	98		
9	41	65*			
	.25	.75			V=66

Interceptors 400, $\alpha = 0$

180	117	100	180	.17	
120	130 (156,103)	112 (123,100)	130	.11	
102	115* (129,100)	121 (143,100)	121*	.72	
102	115*	100			
.16	.21	.63			V=117

Interceptors 800, $\alpha = -2$

199	193	104	199	
192	172 (196,148)	132 (163,101)	192	
185*	154 (192,116)	144 (188,100)	185*	
185*	154	104		V=185

Warheads 8,000

Interceptors 200, $\alpha = 3$

0*	0*	0*	0*	
3	13 (2,24)	23 (1,46)	23	
96	82 (67,97)	69 (39,99)	96	
0*	0*	0*		V=0

Interceptors 400, $\alpha = 2$

0	4	51*	51*	.16	
58	50 (14,85)	48 (2,94)	58	.84	
100	98 (97,100)	76 (52,100)	100		
0	4	48*			
.05		.95			V=49

Interceptors 800, $\alpha = 0$

165	90	95	165	.11	
97*	115 (130,99)	93 (86,100)	115*	.44	
100	101 (102,100)	121 (141,100)	121	.45	
97*	90	93			
.19	.55	.26			V=106

Figure 14. MATRIX GAMES FOR 200 MISSILES, 2000 SHELTERS, ATTACKER KILL PROBABILITY - .95, DEFENDER KILL PROBABILITY - .95, UNIFORM DEFENSE WITHIN AREAS

Table 3. EXPECTED SURVIVING MISSILES FOR 200 MISSILES AND 1000 AND 2000 SHELTERS, UNIFORM DEFENSE WITHIN AREAS

		1000 Shelters			2000 Shelters		
		Interceptors					
Kill Probabilities	Warheads	200	400	800	200	400	800
a=.7, d=.7	1000	132	167	190	179	194	199
	2000	67	110	146	132	167	190
	4000	6	30	78	66	111	144
	8000	0	0	7	6	30	79
a=.95, d=.7	1000	116	156	189	170	191	199
	2000	42	96	133	117	159	186
	4000	0	16	58	43	95	135
	8000	0	0	2	0	17	58
a=.7, d=.95	1000	145	194	200	196	200	200
	2000	88	128	190	146	195	200
	4000	8	60	114	88	128	190
	8000	0	1	33	9	61	114
a=.95, d=.95	1000	133	193	200	195	200	200
	2000	67	117	187	133	193	199
	4000	0	49	105	66	117	185
	8000	0	0	17	0	49	106

I. RESULTS FOR 200 MISSILES AND 2000 SHELTERS WITH PREFERENTIAL DEFENSE WITHIN AREAS

Computations have not been performed for 2000 shelters and preferential defense within areas. However, an analogous argument to that given in the preceding section applies here.

If warheads and shelters are doubled while interceptors and missiles stay the same, the average number of unopposed missiles per shelter, α, is the same. Within each area, assigning preferentially a certain number of interceptors to defend a certain number of missiles yields engagements within areas which are identical for 2000 shelters to engagements within areas for 1000 shelters. Therefore, for 2000 shelters, the results presented in Figures 7, 8, 9 and 10 and in the right-hand side of Table 2 should hold for equal values of α.

J. SUMMARY OF RESULTS

When the defender is limited to uniform defense within areas and the situation is either parity with moderate defender kill probability ($\alpha = 0$ and d=.7) or small attacker advantage with high defender kill probability ($0 \leq \alpha \leq 2$ and d=.95), mixed strategies are optimal. If the situation is either attacker advantage with moderate defender kill probability ($\alpha > 0$ and d=.7) or significant attacker advantage with high defender kill probability ($\alpha > 2$ and d=.95), uniform attack across areas and non-uniform defense across areas are optimal. If the situation is defender advantage ($\alpha < 0$), non-uniform attack across areas and uniform defense across areas are optimal.

When the defender can employ preferential defense within areas the expected number of surviving missiles increases significantly in the cases of parity and attacker advantage ($\alpha \geq 0$). All three defense allocations yield essentially the same payoff; thus non-uniform interceptor allocations across areas are not necessary and the more-easily-verifiable uniform allocations are satisfactory.

Obtaining information on the other side's allocation can yield a much-improved payoff. This has particular practical significance in the situation where the defender is limited to uniform defense within areas, the attacker is superior, and the defender's best allocation is non-uniform across areas.

Increasing shelters from 1000 to 2000 yields the same results for the same values of α since engagements at the missiles within the areas are the same. This is true for both uniform defense within areas and preferential defense within areas.

The figures permit identification of defensive options which are robust against increases in attacking warheads. This is particularly important for uniform defense within areas.

Two limitations should be noted with respect to the scope of the analysis:

(1) The attacker and defender may choose to allocate warheads and interceptors within areas non-uniformly. Optimal assignment within one area when warheads and interceptors are preallocated to particular missiles is treated in References [3] and [4]. The methods of the present paper could be applied to the multi-area problem to (a) optimize attack and defense across and within areas or (b) optimize attack across and within areas and defense within areas (the latter serving to investigate uniform interceptor deployment across areas.)

(2) When the attacker advantage is very substantial (very high α's) there may be integer assignment problems resulting in inefficient use of interceptors. Variations of the Monte Carlo model could be used to explore this issue.

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APPENDIX

EXPECTED NUMBER OF SURVIVORS CONSIDERING ALL POSSIBLE MATCHES OF OFFENSE AND DEFENSE

APPENDIX

In this appendix we prove the claim that the expected number of surviving missiles can be calculated using only the results of perfectly matched and perfectly mismatched allocations.

The general set-up of the problem is this. There are 2N areas which receive warheads and endos as follows. The attacker allocates P percent of his warheads evenly among N randomly chosen areas; the remaining 100-P percent of the warheads are evenly allocated to the remaining N areas. The defender performs a similar allocation using Q and 100-Q percent of his endos. The attacker and defender make area selections independently from one another. Given all such random choices of areas, what is the expected number of surviving missiles?

In our analysis, the total number of missiles, warheads, and endos is fixed, as are P and Q. We assume these numbers are such that each area receives an integral number of warheads and endos, and that an equal number of missiles is assigned to each area. The allocation method provides that N areas receive the same "high" number of warheads, and N areas receive the same "low" number of warheads. These are the N areas receiving either P or 100-P percent of the warheads. Similarly, N areas receive "high" and "low" numbers of endos. In either case, the high and low numbers may be equal if P or Q equals 50 percent. In this event, the random choice of areas has no effect on the expected value, which is computed directly by the model. Henceforth, we may assume P, Q do not equal 50 percent.

There are four characteristically different situations occurring within the areas. These are determined by the high or low number of warheads or endos assigned to a given area. For example, the case (high, high) occurs if an area receives the high number of warheads and the high number of endos. In an analogous manner we define the last three cases: (high, low), (low, high), (low, low). Let the expected number of surviving missiles for each area case be α, β, γ, δ, respectively.

For a given allocation, let X be the number of areas of the (high, high) case which results. Note that $0 \leq X \leq N$. Then the other three cases occur with the multiplicities shown in Table A-1.

For such an allocation, the total number of expected survivors over all 2N areas is

$$
\begin{aligned}
& X\alpha + (N-X)\beta + (N-X)\gamma + X\delta \\
&= X(\alpha+\delta) + (N-X)(\beta+\gamma) \\
&= XA + (N-X)B, \text{ where } A = \alpha+\delta,\ B = \beta+\gamma.
\end{aligned}
$$

Table A-1: CASE TYPE, MULTIPLICITY OF OCCURRENCE, AND AREA EXPECTED SURVIVORS

Case Type (Attacker number, Endo number)	Multiplicity of Occurrence	Area Expected Survivors
(high, high)	X	α
(high, low)	N-X	β
(low, high)	N-X	γ
(low, low)	X	δ

Note that the perfectly matched allocation corresponds to X = N, with expected number of survivors NA. The perfectly mismatched allocation corresponds to X = 0, with NB expected survivors.

In the following, the combinatorial notation $\binom{M}{P} = \frac{M!}{P!(M-P)!}$ equals the number of distinct ways P identical objects can be placed in M distinct boxes, no more than one object to a box ($P \leq M$).

To compute the expected number of survivors over all attacker and defender allocations, we lose no generality by fixing an arbitrary defender allocation, and letting the attacker allocations vary completely. There are $\binom{2N}{N}$ ways the attacker can allocate N high and N low warhead levels to 2N areas. Of these, there are

$$\binom{N}{X}\binom{N}{N-X}$$

allocations that yield X areas of the (high, high) case. Thus, the expected number of survivors, over all possible allocations, is

$$\frac{1}{\binom{2N}{N}} \sum_{X=0}^{N} \binom{N}{X}\binom{N}{N-X}(XA + (N-X)B).$$

The main result shows this equals

$$\frac{N(A+B)}{2} .$$

First, the following lemma.

<u>Lemma</u>:

$$\sum_{X=0}^{N} \binom{N}{X}\binom{N}{N-X} = \binom{2N}{N}.$$

<u>Proof</u>: Let there be 2N boxes labeled 1, ..., N and N+1, ..., 2N. There are $\binom{2N}{N}$ ways of assigning N identical balls to these 2N boxes. Counting a different way, there are $\binom{N}{X}\binom{N}{N-X}$ ways of putting X balls in the first N boxes and N-X in the second N boxes. Summing X from 0 to N yields the left hand side.

The lemma can also be proved by an induction argument where we induct on J in the following formula:

$$\binom{2N}{N} = \sum_{\ell=0}^{J} \binom{J}{\ell}\binom{2N-J}{N-J+\ell}.$$

Returning to the main assertion, let

$$d = \frac{1}{\binom{2N}{N}} \sum_{X=0}^{N} \binom{N}{X}\binom{N}{N-X}(XA + (N-X)B).$$

Then

$$2d = \frac{1}{\binom{2N}{N}} \left\{ \cdot\sum_{X=0}^{N} \binom{N}{X}\binom{N}{N-X}(XA + (N-X)B) \right.$$

$$\left. + \sum_{X=0}^{N} \binom{N}{N-X}\binom{N}{X}((N-X)A + XB)) \right\}$$

$$= \frac{1}{\binom{2N}{N}} \sum_{X=0}^{N} \binom{N}{X}\binom{N}{N-X}(NA + NB)$$

$$= \frac{1}{\binom{2N}{N}} \cdot \binom{2N}{N} \cdot (NA + NB).$$

Thus,

$$d = \frac{N(A+B)}{2}.$$

In summary, the expected number of surviving missiles over all possible attacker and defender allocations is the average of the perfectly matched and the perfectly mismatched allocations.