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ON MODIFICATIONS OF THE ZAKHAROV EQUATION

FOR SURFACE GRAVITY WAVES

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Michael Stiassnie* & Lev Shemer**

Zakharov integral equation for surface gravity waves is modified
to include higher order (Class II) interactions, for water of
constant (finite or infinite) depth. This new equation is used
to study some aspects of Class I and Class II instabilities of a
Stokes wave.

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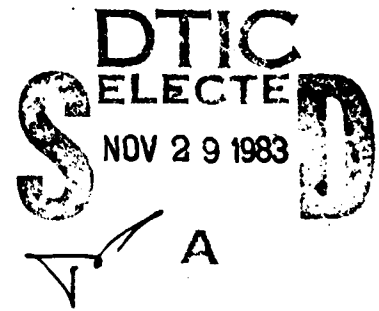
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1. Introduction

Our understanding of nonlinear dynamics of deep water gravity waves has grown substantially in recent years. We feel that the lion's share of this progress should be attributed to the staff of the TRW Fluid Mechanics Department. Most of their findings are summarized in an extensive review article by Yuen & Lake (1982), which served as our main reference. Much of this progress is based on applications of the so-called Zakharov equation which was originally derived by Zakharov (1968) for infinitely deep water. In order to extend the range of application we re-derive Zakharov's equation for finite water depth (in section 2) and show its relations to the cubic Schrödinger equation and to Hasselmann's nonlinear interaction model (in section 3). It is generally accepted now that the Zakharov equation is superior to all other existing approximate models as far as Class I interactions are concerned.

The term 'Class I interactions' refers to nonlinear interaction processes at the lowest possible order; for surface gravity waves this occurs at third order in the nonlinearity parameter ε. Generally speaking, Class I interactions require the coexistence of resonating, or nearly resonating, wave quartets. The time scale of Class I interactions is $\varepsilon^{-2} P$ where P is a typical wave period.

The structure of the surface gravity wave dispersion relation does not enable nonlinear interaction at shorter time scales ($\varepsilon^{-1} P$) which occur in many other physical systems, (e.g., capillary waves).

While Class I interactions are basically four wave interactions, the special case where one of the waves is taken into account twice so that only three waves are considered, has attracted much attention. These cases which lead to what sometimes is called Benjamin-Feir instabilities, display many of the features of the more general quartet interaction. Interactions including a smaller number of waves — as two waves each taken into account twice, or one wave taken into account four times — are also possible, but display a degenerated type of interaction which manifests itself in Stokes-type second order corrections of the frequency (see Longuet-Higgins and

Phillips 1962). Numerical linear stability analysis of the exact finite amplitude Stokes wave, by McLean (1982a,b), as well as experimental evidence by Su et. al. (1982) and Su (1982), reveal the importance of Class II interactions, which are basically quintet interactions. These, much less studied interactions, occur at fourth order in ε and have a typical time scale of $\varepsilon^{-3}p$. Nevertheless, for high enough steepnesses McLean's study shows that Class II instabilities become dominant. Here again, three waves — one of them is taken into account three times — form a nearly resonating quintet and display many interesting features. In the second half of section 2 we extend the derivation to fourth order and derive a modified form of the Zakharov equation which accounts for both Class I, and the higher order, Class II, interactions.

In section 4, we use this equation to study the linear stability of a uniform wave train. The solution of certain long line evolution problems is under way and will be reported at a later stage.

2. The governing equations

The equations governing the irrotational flow of an incompressible, inviscid fluid with a free surface are:

$$\nabla^2\phi = 0 , \qquad -h \leq z \leq n(\underline{x},t), \qquad (2.1)$$

$$\left.\begin{array}{l} n_t + (\nabla\phi)\cdot(\nabla n) - \phi_z = 0 \\[2mm] \phi_t + \frac{1}{2}(\nabla\phi)^2 + gz = 0 \end{array}\right\} \quad z = n(\underline{x},t) \qquad (2.2a,b)$$

$$\phi_z = 0 , \qquad z = -h , \qquad (2.3)$$

where ϕ is the velocity potential, n is the free surface, and g is the gravitational acceleration. The horizontal coordinates are $(x_1,x_2) = \underline{x}$, the vertical coordinate z is pointing upwards, h is the mean water depth, and t is the time.

The free surface boundary conditions, Eq. (2.2), are rewritten in terms of ϕ^s and $w^s = (\frac{\partial\phi}{\partial z})|_{z=n}$, the velocity potential and the vertical velocity component at the free surface, respectively

$$n_t + (\nabla_{\underline{x}}\phi^s)\cdot(\nabla_{\underline{x}}n) - w^s[1 + (\nabla_{\underline{x}}n)^2] = 0 \qquad (2.4a)$$

$$\phi_t^s + gn + \frac{1}{2}(\nabla_{\underline{x}}\phi^s)^2 - \frac{(w^s)^2}{2}[1+(\nabla_{\underline{x}}n)^2] = 0 \qquad (2.4b)$$

The horizontal Fourier transform of these equations yields

$$\hat{n}_t(\underline{k},t) - \frac{1}{2\pi}\iint_{-\infty}^{\infty}(\underline{k}_1\cdot\underline{k}_2)\hat{\phi}^s(\underline{k}_1,t)\,\hat{n}(\underline{k}_2,t)\delta(\underline{k}-\underline{k}_1-\underline{k}_2)d\underline{k}_1d\underline{k}_2 - \hat{w}^s +$$

$$+\frac{1}{(2\pi)^2}\iiint_{-\infty}^{\infty}(\underline{k}_2\cdot\underline{k}_3)\,\hat{w}^s(\underline{k}_1,t)\hat{n}(\underline{k}_2,t)\,\hat{n}(\underline{k}_3,t)\,\delta(\underline{k}-\underline{k}_1-\underline{k}_2-\underline{k}_3)d\underline{k}_1d\underline{k}_2d\underline{k}_3 = 0$$

$$(2.5a)$$

$$\hat{\phi}_t^s(\underline{k},t) + g\hat{\eta}(\underline{k},t) - \frac{1}{4\pi} \iint\limits_{-\infty}^{\infty} (\underline{k}_1 \cdot \underline{k}_2)\, \hat{\phi}^s(\underline{k}_1,t)\hat{\phi}^s(\underline{k}_2,t)\delta(\underline{k}-\underline{k}_1-\underline{k}_2)d\underline{k}_1 d\underline{k}_2 -$$

$$- \frac{1}{4\pi} \iint\limits_{-\infty}^{\infty} \hat{w}^s(\underline{k}_1,t)\, \hat{w}^s(\underline{k}_2,t)\, \delta(\underline{k}-\underline{k}_1-\underline{k}_2)\, d\underline{k}_1\, d\underline{k}_2 +$$

$$+ \frac{1}{16\pi^3} \iiiint\limits_{-\infty}^{\infty} (\underline{k}_3 \cdot \underline{k}_4)\hat{w}^s(\underline{k}_1,t)\,\hat{w}^s(\underline{k}_2,t)\hat{\eta}(\underline{k}_3,t)\hat{\eta}(\underline{k}_4,t)\delta(\underline{k}-\underline{k}_1-\underline{k}_2-\underline{k}_3-\underline{k}_4) \cdot$$

$$d\underline{k}_1\, d\underline{k}_2\, d\underline{k}_3\, d\underline{k}_4 \tag{2.5b}$$

where the two dimensional Fourier transform of a function $f(\underline{x})$ is given by

$$\hat{f}(\underline{k}) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} f(\underline{x})e^{-i\underline{k}\cdot\underline{x}}d\underline{x},$$

and Dirac δ function is defined as

$$\delta(\underline{k}) = \frac{1}{(2\pi)^2} \int\limits_{-\infty}^{\infty} e^{i\underline{k}\cdot\underline{x}}\, d\underline{x}\ .$$

Taking the Fourier transform of Laplace Eq. (2.1), and satisfying the boundary condition at the bottom, Eq. (2.3), gives

$$\hat{\phi}(\underline{k},z,t) = \hat{\phi}(\underline{k},t)ch(|\underline{k}|(z+h)), \tag{2.6}$$

which enables one to write ϕ^s and w^s in terms of $\hat{\phi}(\underline{k},t)$ and $\eta(\underline{x},t)$ as follows

$$\phi^s(\underline{x},t) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \hat{\phi}(\underline{k},t)[ch(|\underline{k}|h)ch(|\underline{k}|\eta(\underline{x},t)) + sh(|\underline{k}|h)sh(|\underline{k}|\eta(\underline{x},t))]\, e^{i\underline{k}\cdot\underline{x}}d\underline{k}$$

$$\tag{2.7a}$$

$$w^s(\underline{x},t) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} |k|\hat{\phi}(\underline{k},t)[ch(|k|h)sh(|\underline{k}|\eta(\underline{x},t)) + sh(|\underline{k}|h)ch(|\underline{k}|\eta(\underline{x},t))]e^{i\underline{k}\cdot\underline{x}}d\underline{k}$$

$$\tag{2.7b}$$

The next step in the derivation is to express $\hat{w}^s$ as a function of $\hat{\eta}$ and $\hat{\phi}^s$. This is the first step which requires an additional physical assumption. Assuming that $|\underline{k}|\eta$ is small, we pursue the following procedure: (i) replace $\text{sh}(|\underline{k}|\eta)$ and $\text{ch}(|\underline{k}|\eta)$, in Eqs (2.7a,b), by their Taylor series expansions up to order $(|k|\eta)^3$; (ii) express η by means of its Fourier tranform $\hat{\eta}$; and finally, (iii) take the Fourier transform of Eqs.(2.7a,b),

$$\hat{\phi}^s(\underline{k},t) = \hat{\phi}(\underline{k},t)\text{ch}(|\underline{k}|h) + \frac{1}{2\pi}\iint_{-\infty}^{\infty}|\underline{k}_1|\text{sh}(|\underline{k}_1|h)\hat{\phi}(\underline{k}_1,t)\hat{\eta}(\underline{k}_2,t)\delta(\underline{k}-\underline{k}_1-\underline{k}_2)\,d\underline{k}_1 d\underline{k}_2 +$$

$$+\frac{1}{(2\pi)^2}\iiint_{-\infty}^{\infty}\frac{|\underline{k}_1|^2}{2}\text{ch}(|\underline{k}_1|h)\hat{\phi}(\underline{k}_1,t)\hat{\eta}(\underline{k}_2,t)\hat{\eta}(\underline{k}_3,t)\delta(\underline{k}-\underline{k}_1-\underline{k}_2-\underline{k}_3)\,d\underline{k}_1 d\underline{k}_2 d\underline{k}_3 +$$

$$+\frac{1}{(2\pi)^3}\iiiint_{-\infty}^{\infty}\frac{|\underline{k}_1|^3}{6}\text{sh}(|\underline{k}_1|h)\hat{\phi}(\underline{k}_1,t)\hat{\eta}(\underline{k}_2,t)\hat{\eta}(\underline{k}_3,t)\hat{\eta}(\underline{k}_4,t)\delta(\underline{k}-\underline{k}_1-\underline{k}_2-\underline{k}_3-\underline{k}_4)$$

$$\cdot\,d\underline{k}_1 d\underline{k}_2 d\underline{k}_3 d\underline{k}_4 \tag{2.8a}$$

$$\hat{w}^s(\underline{k},t) = |\underline{k}|\hat{\phi}(\underline{k},t)\text{sh}(|\underline{k}|h) + \frac{1}{2\pi}\iint_{-\infty}^{\infty}|\underline{k}_1|^2\text{ch}(|\underline{k}_1|h)\hat{\phi}(\underline{k}_1,t)\hat{\eta}(\underline{k}_2,t)\delta(\underline{k}-\underline{k}_1-\underline{k}_2)\,d\underline{k}_1 d\underline{k}_2 +$$

$$+\frac{1}{(2\pi)^2}\iiint_{-\infty}^{\infty}\frac{|\underline{k}_1|^3}{2}\text{sh}(|\underline{k}_1|h)\hat{\phi}(\underline{k}_1,t)\hat{\eta}(\underline{k}_2,t)\hat{\eta}(\underline{k}_3,t)\delta(\underline{k}-\underline{k}_1-\underline{k}_2-\underline{k}_3)\,d\underline{k}_1 d\underline{k}_2 d\underline{k}_3 +$$

$$+\frac{1}{(2\pi)^3}\iiiint_{-\infty}^{\infty}\frac{|\underline{k}_1|^4}{6}\text{ch}(|\underline{k}_1|h)\hat{\phi}(\underline{k}_1,t)\hat{\eta}(\underline{k}_2,t)\hat{\eta}(\underline{k}_3,t)\hat{\eta}(\underline{k}_4,t)\,(\underline{k}-\underline{k}_1-\underline{k}_2-\underline{k}_3-\underline{k}_4)\cdot$$

$$\cdot\,d\underline{k}_1 d\underline{k}_2 d\underline{k}_3 d\underline{k}_4 \tag{2.8b}$$

Inverting Eq. (2.8a) iteratively, in order to obtain $\hat{\phi} = \hat{\phi}(\hat{\phi}^s)$, and substituting the result into Eq.(2.8b), yields

$$\hat{w}^s(\underline{k},t) = |\underline{k}|\,\text{th}(|\underline{k}|h)\hat{\phi}^s(\underline{k},t) -$$

$$- \frac{1}{2\pi}\iint\limits_{-\infty}^{\infty} |\underline{k}_1|\,[|\underline{k}|\,\text{th}(|\underline{k}|h)\,\text{th}(|\underline{k}_1|h) - |\underline{k}_1|]\hat{\phi}^s(\underline{k}_1,t)\hat{\eta}(\underline{k}_2,t)\delta(\underline{k}-\underline{k}_1-\underline{k}_2)d\underline{k}_1 d\underline{k}_2 -$$

$$- \frac{1}{(2\pi)^2}\iiint\limits_{-\infty}^{\infty} S^{(1)}(\underline{k},\underline{k}_1,\underline{k}_2,\underline{k}_3)\hat{\phi}^s(\underline{k}_1,t)\hat{\eta}(\underline{k}_2,t)\hat{\eta}(\underline{k}_3,t)\delta(\underline{k}-\underline{k}_1-\underline{k}_2-\underline{k}_3)d\underline{k}_1 d\underline{k}_2 d\underline{k}_3 -$$

$$- \frac{1}{(2\pi)^2}\iiint\limits_{-\infty}^{\infty} S^{(2)}(\underline{k},\underline{k}_1,\underline{k}_2,\underline{k}_3,\underline{k}_4)\hat{\phi}^s(\underline{k}_1,t)\hat{\eta}(\underline{k}_2,t)\hat{\eta}(\underline{k}_3,t)\hat{\eta}(\underline{k}_4,t)\delta(\underline{k}-\underline{k}_1-\underline{k}_2-\underline{k}_3-\underline{k}_4)\cdot$$

$$d\underline{k}_1 d\underline{k}_2 d\underline{k}_3 d\underline{k}_4 \tag{2.9}$$

the kernels, $S^{(1)}(\underline{k},\underline{k}_1,\underline{k}_2,\underline{k}_3)$, $S^{(2)}(\underline{k},\underline{k}_1,\underline{k}_2,\underline{k}_3,\underline{k}_4)$, as well as other kernels, which appear throughout the derivation, are given in the Appendix. Substituting $\hat{w}^s$, from Eq. (2.9) into Eqs. (2.5a,b), multiplying Eq. (2.5a) by $\left[\dfrac{g}{2\omega(\underline{k})}\right]^{\frac{1}{2}}$, and Eq. (2.5b) by $i\left[\dfrac{\omega(\underline{k})}{2g}\right]^{\frac{1}{2}}$

where $\qquad \omega(\underline{k}) = [g|\underline{k}|\,\text{th}(|\underline{k}|h)]^{\frac{1}{2}}$, $\hfill (2.10)$

adding these equations together, and defining the new complex variable

$$b(\underline{k},t) = \left[\frac{g}{2\omega(\underline{k})}\right]^{\frac{1}{2}}\hat{\eta}(\underline{k},t) + i\left[\frac{\omega(\underline{k})}{2g}\right]^{\frac{1}{2}}\hat{\phi}^s(\underline{k},t)$$

yields the following equation:

$$b_t(\underline{k},t) + i\omega(\underline{k})b(\underline{k},t) +$$

$$+ i\iint\limits_{-\infty}^{\infty} V^{(1)}(\underline{k},\underline{k}_1,\underline{k}_2)b(\underline{k}_1,t)b(\underline{k}_2,t)\delta(\underline{k}-\underline{k}_1-\underline{k}_2)d\underline{k}_1 d\underline{k}_2 +$$

$$+ i\iint\limits_{-\infty}^{\infty} V^{(2)}(\underline{k},\underline{k}_1,\underline{k}_2)b^*(\underline{k}_1,t)b(\underline{k}_2,t)\delta(\underline{k}+\underline{k}_1-\underline{k}_2)d\underline{k}_1 d\underline{k}_2 +$$

$$+ i \iint V^{(3)}(\underline{k},\underline{k}_1,\underline{k}_2)b^*(\underline{k}_1,t)b^*(\underline{k}_2,t)\delta(\underline{k}+\underline{k}_1+\underline{k}_2)d\underline{k}_1 d\underline{k}_2 +$$

$$+ i \iiint W^{(1)}(\underline{k},\underline{k}_1,\underline{k}_2,\underline{k}_3)b(\underline{k}_1,t)b(\underline{k}_2,t)b(\underline{k}_3,t)\delta(\underline{k}-\underline{k}_1-\underline{k}_2-\underline{k}_3)d\underline{k}_1 d\underline{k}_2 d\underline{k}_3 +$$

$$+ i \iiint W^{(2)}(\underline{k},\underline{k}_1,\underline{k}_2,\underline{k}_3)b^*(\underline{k}_1,t)b(\underline{k}_2,t)b(\underline{k}_3,t)\delta(\underline{k}+\underline{k}_1-\underline{k}_2-\underline{k}_3)d\underline{k}_1 d\underline{k}_2 d\underline{k}_3 +$$

$$+ i \iiint W^{(3)}(\underline{k},\underline{k}_1,\underline{k}_2,\underline{k}_3)b^*(\underline{k}_1,t)b^*(\underline{k}_2,t)b(\underline{k}_3,t)\delta(\underline{k}+\underline{k}_1+\underline{k}_2-\underline{k}_3)d\underline{k}_1 d\underline{k}_2 d\underline{k}_3 +$$

$$+ i \iiint W^{(4)}(\underline{k},\underline{k}_1,\underline{k}_2,\underline{k}_3)b^*(\underline{k}_1,t)b^*(\underline{k}_2,t)b^*(\underline{k}_3,t)\delta(\underline{k}+\underline{k}_1+\underline{k}_2+\underline{k}_3)d\underline{k}_1 d\underline{k}_2 d\underline{k}_3 +$$

$$+ i \iiiint X^{(1)}(\underline{k},\underline{k}_1,\underline{k}_2,\underline{k}_3,\underline{k}_4)b(\underline{k}_1,t)b(\underline{k}_2,t)b(\underline{k}_3,t)b(\underline{k}_4,t)\delta(\underline{k}-\underline{k}_1-\underline{k}_2-\underline{k}_3-\underline{k}_4)d\underline{k}_1 d\underline{k}_2 d\underline{k}_3 d\underline{k}_4 +$$

$$+ i \iiiint X^{(2)}(\underline{k},\underline{k}_1,\underline{k}_2,\underline{k}_3,\underline{k}_4)b^*(\underline{k}_1,t)b(\underline{k}_2,t)b(\underline{k}_3,t)b(\underline{k}_4,t)\delta(\underline{k}+\underline{k}_1-\underline{k}_2-\underline{k}_3-\underline{k}_4)d\underline{k}_1 d\underline{k}_2 d\underline{k}_3 d\underline{k}_4 +$$

$$+ i \iiiint X^{(3)}(\underline{k},\underline{k}_1,\underline{k}_2,\underline{k}_3,\underline{k}_4)b^*(\underline{k}_1,t)b^*(\underline{k}_2,t)b(\underline{k}_3,t)b(\underline{k}_4,t)\delta(\underline{k}+\underline{k}_1+\underline{k}_2-\underline{k}_3-\underline{k}_4)d\underline{k}_1 d\underline{k}_2 d\underline{k}_3 d\underline{k}_4 +$$

$$+ i \iiiint X^{(4)}(\underline{k},\underline{k}_1,\underline{k}_2,\underline{k}_3,\underline{k}_4)b^*(\underline{k}_1,t)b^*(\underline{k}_2,t)b^*(\underline{k}_3,t)b(\underline{k}_4,t)\delta(\underline{k}+\underline{k}_1+\underline{k}_2+\underline{k}_3-\underline{k}_4)d\underline{k}_1 d\underline{k}_2 d\underline{k}_3 d\underline{k}_4 +$$

$$+ i \iiiint X^{(5)}(\underline{k},\underline{k}_1,\underline{k}_2,\underline{k}_3,\underline{k}_4)b^*(\underline{k}_1,t)b^*(\underline{k}_2,t)b^*(\underline{k}_3,t)b^*(\underline{k}_4,t)\delta(\underline{k}+\underline{k}_1+\underline{k}_2+\underline{k}_3+\underline{k}_4)d\underline{k}_1 d\underline{k}_2 d\underline{k}_3 d\underline{k}_4 = 0$$

$$(2.11)$$

where * denotes the complex conjugate

The relations between η, ϕ^S and the complex "amplitude spectrum" b are

$$\hat{\eta}(\underline{k},t) = \left|\frac{\omega(\underline{k})}{2g}\right|^{1/2}[b(\underline{k},t) + b^*(-k,t)] \tag{2.12a}$$

$$\hat{\phi}^S(\underline{k},t) = -i\left|\frac{g}{2\omega(\underline{k})}\right|^{1/2}[b(\underline{k},t) - b^*(-\underline{k},t)] \tag{2.12b}$$

We assume that the wave field can be divided into a slowly varying (in time) component B and small rapidly varying components B', B", B"' and that most of the energy in the wave field is contained in B. These assumptions permit one to write

$$b(\underline{k},t) = [\varepsilon\tilde{B}(\underline{k},t_2,t_3) + \varepsilon^2 B'(\underline{k},t,t_2,t_3) + \varepsilon^3 B''(\underline{k},t,t_2,t_3) +$$

$$+ \varepsilon^4 B'''(\underline{k},t,t_2,t_3)]e^{-i\omega(\underline{k})t} \qquad (2.13)$$

where ε is a small parameter representing the magnitude of nonlinearity, and the slow time scales are defined by $t_2 = \varepsilon^2 t$, $t_3 = \varepsilon^3 t$. The omission of the slow time $t_1 = \varepsilon t$ from Eq. (2.13) results from the fact that resonating triads do not exist for surface gravity waves. Substituting b, from Eq. (2.13), into Eq. (2.11) and arranging the terms according to their order in ε yields the following results:

Order ε - is satisfied identically

Order ε^2 - gives an equation for B'

$$i\frac{\partial B'}{\partial t} = \int\int_{-\infty}^{\infty} \{V^{(1)}_{0,1,2}\tilde{B}_1\tilde{B}_2\delta_{0-1-2}e^{i(\omega-\omega_1-\omega_2)t} + V^{(2)}_{0,1,2}\tilde{B}_1^*\tilde{B}_2\delta_{0+1-2}e^{i(\omega+\omega_1-\omega_2)t} +$$

$$+ V^{(3)}_{0,1,2}\tilde{B}_1^*\tilde{B}_2^*\delta_{0+1+2}e^{i(\omega+\omega_1+\omega_2)t}\} \, dk_1 dk_2 \, , \qquad (2.14a)$$

where we have introduced the compact notation in which the arguments $\underline{k}_i$ in V, $\tilde{B}$, δ, ω, and in other functions in the sequel, are replaced by subscripts i, with the subscript zero assigned to $\underline{k}$. Integrating Eq. (2.14a) with respect to t and keeping t_2, t_3 fixed, gives

$$B' = -\int\int_{-\infty}^{\infty} \{V^{(1)}_{0,1,2}\tilde{B}_1\tilde{B}_2\delta_{0-1-2}\frac{e^{i(\omega-\omega_1-\omega_2)t}}{\omega-\omega_1-\omega_2} +$$

$$+ V^{(2)}_{0,1,2}\tilde{B}_1^*\tilde{B}_2\delta_{0+1-2}\frac{e^{i(\omega+\omega_1-\omega_2)t}}{\omega+\omega_1-\omega_2} +$$

$$+ V^{(3)}_{0,1,2}\tilde{B}_1^*\tilde{B}_2^*\delta_{0+1+2}\frac{e^{i(\omega+\omega_1+\omega_2)t}}{\omega+\omega_1+\omega_2}\} \, dk_1 dk_2 \qquad (2.14b)$$

The constant of integration, which corresponds to the initial phase, has been set to zero without loss of generality.

<u>Order ϵ^3</u> - gives the following equation

$$i \frac{\partial \tilde{B}}{\partial t_2} + i \frac{\partial B''}{\partial t} = \iiint_{-\infty}^{\infty} \{ \tilde{T}^{(1)}_{0,1,2,3} \tilde{B}_1 \tilde{B}_2 \tilde{B}_3 \delta_{0-1-2-3} e^{i(\omega-\omega_1-\omega_2-\omega_3)t} +$$
$$+ \tilde{T}^{(2)}_{0,1,2,3} \tilde{B}^*_1 \tilde{B}_2 \tilde{B}_3 \delta_{0+1-2-3} e^{i(\omega+\omega_1-\omega_2-\omega_3)t} +$$
$$+ \tilde{T}^{(3)}_{0,1,2,3} \tilde{B}^*_1 \tilde{B}^*_2 \tilde{B}_3 \delta_{0+1+2-3} e^{i(\omega+\omega_1+\omega_2-\omega_3)t} +$$
$$+ \tilde{T}^{(4)}_{0,1,2,3} \tilde{B}^*_1 \tilde{B}^*_2 \tilde{B}^*_3 \delta_{0+1+2+3} e^{i(\omega+\omega_1+\omega_2+\omega_3)t} \} d\underline{k}_1 d\underline{k}_2$$

(2

The above equation consists of terms of two types; those which depend on the fast time t, and those which do not. This enables us to split Eq. (2.15) into separate equations:

$$i \frac{\partial \tilde{B}}{\partial t_2} = \iiint_{-\infty}^{\infty} T^{(2)}_{0,1,2,3} \tilde{B}^*_1 \tilde{B}_2 \tilde{B}_3 \delta_{0+1-2-3} e^{i(\omega+\omega_1-\omega_2-\omega_3)t} d\underline{k}_1 d\underline{k}_2 d\underline{k}_3 \qquad (2.16)$$

$$i \frac{\partial B''}{\partial t} = \iiint_{-\infty}^{\infty} \{ \tilde{T}^{(1)}_{0,1,2,3} \tilde{B}_1 \tilde{B}_2 \tilde{B}_3 \delta_{0-1-2-3} e^{i(\omega-\omega_1-\omega_2-\omega_3)t} +$$
$$+ (\tilde{T}^{(2)}_{0,1,2,3} - T^{(2)}_{0,1,2,3}) \tilde{B}^*_1 \tilde{B}_2 \tilde{B}_3 \delta_{0+1-2-3} e^{i(\omega+\omega_1-\omega_2-\omega_3)t} +$$
$$+ \tilde{T}^{(3)}_{0,1,2,3} \tilde{B}^*_1 \tilde{B}^*_2 \tilde{B}_3 \delta_{0+1+2-3} e^{i(\omega+\omega_1+\omega_2-\omega_3)t} +$$
$$+ \tilde{T}^{(4)}_{0,1,2,3} \tilde{B}^*_1 \tilde{B}^*_2 \tilde{B}^*_3 \delta_{0+1+2+3} e^{i(\omega+\omega_1+\omega_2+\omega_3)t} \} d\underline{k}_1 d\underline{k}_2 d\underline{k}_3 \qquad (2.17a)$$

Here we made use of the fact that the only exponent of e which may become zero, under the restriction of the δ functions, is the one in the second term in the r.h.s. of Eq. (2.15). This fact is directly related to the definition of a nearly resonating quartet.

$$\delta_{0+1-2-3} = 0 \; ; \quad |\omega_0+\omega_1-\omega_2-\omega_3| \leq 0(\varepsilon^2) \qquad (2.18a,b)$$

Equation (2.16) is the so-called Zakharov equation, with the kernel

$$T_{0,1,2,3}^{(2)} = \begin{cases} \tilde{T}_{0,1,2,3}^{(2)}, & \text{for near resonance quartets} \\ 0 & , \text{ otherwise} \end{cases} \qquad (2.19)$$

used as a mathematical model for Class I nonlinear interactions. Integrating Eq. (2.17a) with respect to t gives the following result for B",

$$B'' = -\iiint\limits_{-\infty}^{\infty} \{ \tilde{T}_{0,1,2,3}^{(1)} \tilde{B}_1\tilde{B}_2\tilde{B}_3\delta_{0-1-2-3} \frac{e^{i(\omega-\omega_1-\omega_2-\omega_3)t}}{\omega-\omega_1-\omega_2-\omega_3} +$$

$$+ (\tilde{T}_{0,1,2,3}^{(2)} - T_{0,1,2,3}^{(2)}) \tilde{B}_1^*\tilde{B}_2\tilde{B}_3\delta_{0+1-2-3} \frac{e^{i(\omega+\omega_1-\omega_2-\omega_3)t}}{\omega+\omega_1-\omega_2-\omega_3} +$$

$$+ \tilde{T}_{0,1,2,3}^{(3)} \tilde{B}_1^*\tilde{B}_2^*\tilde{B}_3\delta_{0+1+2-3} \frac{e^{i(\omega+\omega_1+\omega_2-\omega_3)t}}{\omega+\omega_1+\omega_2-\omega_3} +$$

$$+ \tilde{T}_{0,1,2,3}^{(4)} \tilde{B}_1^*\tilde{B}_2^*\tilde{B}_3^*\delta_{0+1+2+3} \frac{e^{i(\omega+\omega_1+\omega_2+\omega_3)t}}{\omega+\omega_1+\omega_2+\omega_3} \} \, d\underline{k}_1 d\underline{k}_2 d\underline{k}_3$$

$$(2.17b)$$

<u>Order</u> ε^4:

$$i \frac{\partial\tilde{B}}{\partial t_3} + i \frac{\partial B'''}{\partial t} = -i \frac{\partial B'}{\partial t_2} + \iiiint\limits_{-\infty}^{\infty} \{ \tilde{U}_{0,1,2,3,4}^{(1)}\tilde{B}_1\tilde{B}_2\tilde{B}_3\tilde{B}_4\delta_{0-1-2-3-4}e^{i(\omega-\omega_1-\omega_2-\omega_3-\omega_4)t} +$$

$$+ \tilde{U}_{0,1,2,3,4}^{(2)}\tilde{B}_1^*\tilde{B}_2\tilde{B}_3\tilde{B}_4\delta_{0+1-2-3-4}e^{i(\omega+\omega_1-\omega_2-\omega_3-\omega_4)t} +$$

$$+ \tilde{U}_{0,1,2,3,4}^{(3)}\tilde{B}_1^*\tilde{B}_2^*\tilde{B}_3\tilde{B}_4\delta_{0+1+2-3-4}e^{i(\omega+\omega_1+\omega_2-\omega_3-\omega_4)t} +$$

$$+ \tilde{U}_{0,1,2,3,4}^{(4)}\tilde{B}_1^*\tilde{B}_2^*\tilde{B}_3^*\tilde{B}_4\delta_{0+1+2+3-4}e^{i(\omega+\omega_1+\omega_2+\omega_3-\omega_4)t} +$$

$$+ \tilde{U}_{0,1,2,3,4}^{(5)}\tilde{B}_1^*\tilde{B}_2^*\tilde{B}_3^*\tilde{B}_4^*\delta_{0+1+2+3+4}e^{i(\omega+\omega_1+\omega_2+\omega_3+\omega_4)t} \} d\underline{k}_1 d\underline{k}_2 d\underline{k}_3 d\underline{k}_4$$

$$(2.20)$$

In order to split Eq. (2.20) appropriately into two separate equations, one for $\frac{\partial \tilde{B}}{\partial t_3}$ and the other for $\frac{\partial B'''}{\partial t}$ which becomes relevant only in sextet interactions, we make use of the fact that only the second and third integrands in Eq. (2.20) enable resonating quintets. Similarly to Eq. (2.18), the nearly resonating quintets are defined

$$\delta_{0+1-2-3-4} = 0 \quad , \quad |\Omega_0 + \Omega_1 - \Omega_2 - \Omega_3 - \Omega_4| \leq 0(\varepsilon^3) \qquad (2.21a,b)$$

where Ω_j, the "Stokes corrected" frequencies, are given by

$$\Omega_j = \omega_j + \varepsilon^2 \int_{-\infty}^{\infty} e_{j1} T_{j1j1} |\tilde{B}_1|^2 \, d\underline{k}_1 \quad , \quad e_{j1} = \begin{cases} 2 & j \neq 1 \\ 1 & j = 1 \end{cases} \qquad (2.21c)$$

The Stokes corrected frequencies are obtained by solving Eq. (2.16) for degenerated interactions, namely: "quartets" formed by two waves, each taken into account twice. These corrections become necessary at the order of derivation considered here. Defining

$$U^{(2)}_{0,1,2,3,4} = \begin{cases} \tilde{U}^{(2)}_{0,1,2,3,4}, & \text{for nearly resonating quintets} \\ \\ 0 & , \text{ otherwise} \end{cases} \qquad (2.22a)$$

$$U^{(3)}_{0,1,2,3,4} = \begin{cases} \tilde{U}^{(3)}_{0,1,2,3,4}, & \text{for nearly resonating quintets} \\ \\ 0 & , \text{ otherwise} \end{cases} \qquad (2.22b)$$

we obtain

$$i \frac{\partial \tilde{B}}{\partial t_3} = \iiiint_{-\infty}^{\infty} \{ U^{(2)}_{0,1,2,3,4} \tilde{B}^*_1 \tilde{B}_2 \tilde{B}_3 \tilde{B}_4 \delta_{0+1-2-3-4} \, e^{i(\omega + \omega_1 - \omega_2 - \omega_3 - \omega_4)t} +$$

$$+ U^{(3)}_{0,1,2,3,4} \tilde{B}^*_1 \tilde{B}^*_2 \tilde{B}_3 \tilde{B}_4 \delta_{0+1+2-3-4} \, e^{i(\omega + \omega_1 + \omega_2 - \omega_3 - \omega_4)t} \} \, d\underline{k}_1 d\underline{k}_2 d\underline{k}_3 d\underline{k}_4$$

$$(2.23)$$

Finally, the two orders, Eqs. (2.16) and (2.23), are combined into a single equation for $B = \epsilon \tilde{B}$,

$$i \frac{\partial B}{\partial t} = \iiint_{-\infty}^{\infty} T^{(2)}_{0,1,2,3} \, B^*_1 B_2 B_3 \delta_{0+1-2-3} \, e^{i(\omega+\omega_1-\omega_2-\omega_3)t} \, d\underline{k}_1 d\underline{k}_2 d\underline{k}_3 \; +$$

$$+ \iiiint_{-\infty}^{\infty} U^{(2)}_{0,1,2,3,4} \, B^*_1 B_2 B_3 B_4 \, \delta_{0+1-2-3-4} \, e^{i(\omega+\omega_1-\omega_2-\omega_3-\omega_4)t} \, d\underline{k}_1 d\underline{k}_2 d\underline{k}_3 d\underline{k}_4 \; +$$

$$+ \iiiint_{-\infty}^{\infty} U^{(3)}_{0,1,2,3,4} \, B^*_1 B^*_2 B_3 B_4 \delta_{0+1+2-3-4} \, e^{i(\omega+\omega_1+\omega_2-\omega_3-\omega_4)t} \, d\underline{k}_1 d\underline{k}_2 d\underline{k}_3 d\underline{k}_4$$

$$\text{(2.24)}$$

Equation (2.24) is a modification of the Zakharov equation that accounts for higher order interactions.

3. Comments on Zakharov equation for finite water depth

Denoting $\varepsilon\tilde{B}$ by B we rewrite Eq. (2.16)

$$i\frac{\partial B(\underline{k},t)}{\partial t} =$$

$$\iiint_{-\infty}^{\infty} dk_1 dk_2 dk_3\ T^{(2)}(\underline{k},\underline{k}_1,\underline{k}_2,\underline{k}_3)B^*(\underline{k}_1,t)B(\underline{k}_2,t)B(\underline{k}_3,t)\delta(\underline{k}+\underline{k}_1-\underline{k}_2-\underline{k}_3)e^{i(\omega+\omega_1-\omega_2-\omega_3)t} \qquad (3.1)$$

The first order free surface elevation is related to B through Eqs. (2.12a), (2.13), and is given by

$$\eta(\underline{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{\omega(\underline{k})}{2g}\right]^{1/2}\{B(\underline{k},t)e^{i(\underline{k}\cdot\underline{x}-\omega t)} + c.c\}\ d\underline{k} \qquad (3.2)$$

Equation (3.1) is the now well-known Zakharov equation, generalized for water of any constant depth. The fact that Eq. (3.1) is valid for finite depth affects only the expressions for $\omega(\underline{k})$ and $T^{(2)}(\underline{k},\underline{k}_1,\underline{k}_2,\underline{k}_3)$, which become depth dependent.

The purpose of this section is to show the connections between the Zakharov equation and other model equations, as well as to check our depth dependent expression for $T^{(2)}$. Note that for $h \to \infty$ our equation for $T^{(2)}$, Eq. (A.5c), gives the same result as does Appendix A of Crawford, et al. (1981). This result is different from that given in Yuen & Lake (1982) (even after corrections of minor misprints). This apparent discrepancy is related to the special, almost-symmetric (with respect to $\underline{k}_2$ and $\underline{k}_3$) structure of the Zakharov equation. This structure allows some freedom in the choice of $T^{(2)}$, i.e., $T^{(2)}(\underline{k},\underline{k}_1,\underline{k}_2,\underline{k}_3)$ can be replaced by $\alpha T^{(2)}(\underline{k},\underline{k}_1,\underline{k}_2,\underline{k}_3) + (1-\alpha)\ T^{(2)}(\underline{k},\underline{k}_1,\underline{k}_3,\underline{k}_2)$, with arbitrary α, without altering the value of the integral on the r.h.s. of Eq. (3.1). Any $T^{(2)}$, obtained in some legitimate derivation can be made symmetric in $\underline{k}_2,\underline{k}_3$ by choosing $\alpha = 0.5$. This symmetric $T^{(2)}$, denoted by T, is a uniquely defined function of $\underline{k},\underline{k}_1,\underline{k}_2,\underline{k}_3$ and h and will be used in the sequel.

Relation to Hasselmann's energy transfer model

The energy transfer equation for a finite-depth gravity-wave spectrum, originally obtained in Hasselmann (1961), was rederived by Herterich & Hasselmann

(1980). This last paper served as a reference for the verification of the expression for T, Eq. (A.5c). By reasonings similar to those in Longuet-Higgins (1976), but starting from the Zakharov equation (instead of the cubic Schrödinger equation, used by Longuet-Higgins) the following energy transfer equation is obtained

$$\frac{\partial C(\underline{k},t)}{\partial t} = 4\pi \iiint_{-\infty}^{\infty} T(\underline{k},\underline{k}_1,\underline{k}_2,\underline{k}_3)[C(\underline{k}_2)C(\underline{k}_3)(C(\underline{k}) + C(\underline{k}_1)) - C(\underline{k})C(\underline{k}_1)(C(\underline{k}_2) +$$

$$+ C(\underline{k}_3))]\delta(\underline{k} + \underline{k}_1 - \underline{k}_2 - \underline{k}_3)\delta(\omega + \omega_1 - \omega_2 - \omega_3)\, d\underline{k}_1 d\underline{k}_2 d\underline{k}_3 \qquad (3.3)$$

where the wave-action spectrum $C = |B|^2$.

For strict resonance conditions, which are implied by the two δ functions in Eq. (3.3), T is also symmetric in its two first arguments, $\underline{k}$ and $\underline{k}_1$. Herterich & Hasselmann's $F(\underline{k},t)$ is given by $\omega(\underline{k})C(\underline{k},t)/4g\pi^2$ and their interaction coefficient D is given by

$$D(\underline{k}_3,\underline{k}_2,\underline{k}_1) \equiv -\frac{16\pi^2}{3g}(\omega\omega_1\omega_2\omega_3)^{1/2} T(\underline{k},\underline{k}_1,\underline{k}_2,\underline{k}_3) , \qquad (3.4)$$

for resonating quartets.

The above identity, Eq. (3.4), has been verified numerically, and thus serves as a mutual check of the rather lengthy algebra involved in the derivation of both models.

Relation to the nonlinear Schrödinger equation

The derivation here follows the lines of Zakharov (1968), who showed that, in the case of infinitely deep water, the cubic Schrödinger equation is a particular case of the more general Zakharov equation. In the case of finite water depth, the value of $T(\underline{k},\underline{k}_1,\underline{k}_2,\underline{k}_3)$, in the limit when $\underline{k}_1,\underline{k}_2,\underline{k}_3$ tend to $\underline{k}$ and h is fixed is not unique. In order to provide a better grasp of this nonuniqueness, we include here an outline of the derivation of the finite depth nonlinear Schrödinger equation.

Restricting the analysis to narrow spectra around $\underline{k}_0 = (k_0, 0)$, we rewrite all wavenumbers as $\underline{k}_i = \underline{k}_0 + \underline{\psi}_i$, $\underline{\psi}_i = (\psi_i, \lambda_i)$ and $|\underline{\psi}|/k_0 \ll 1$. Introducing a new variable $A(\underline{\psi},t) = B(\underline{k},t)e^{-i[\omega(\underline{k})-\omega(\underline{k}_0)]t}$ into Eq. (3.1) gives

$$i\frac{\partial A(\underline{\psi},t)}{\partial t} - [\omega(\underline{k}) - \omega(\underline{k}_0)]A(\underline{\psi},t) =$$

$$= \iiint_{-\infty}^{\infty} T(\underline{k}_0 + \underline{\psi}, \underline{k}_0 + \underline{\psi}_1, \underline{k}_0 + \underline{\psi}_2, \underline{k}_0 + \underline{\psi}_3)A^*(\underline{\psi}_1)A(\underline{\psi}_2)A(\underline{\psi}_3)\delta(\underline{\psi}+\underline{\psi}_1-\underline{\psi}_2-\underline{\psi}_3)\, d\underline{\psi}_1 d\underline{\psi}_2 d\underline{\psi}_3$$

$$(3.5)$$

Equation (3.2) is then expanded to the lowest order in the spectral width

$$\eta(\underline{x},t) = \frac{1}{2\pi}\left(\frac{\omega(\underline{k}_0)}{2g}\right)^{1/2}\left\{e^{i[k_0 x_1-\omega(k_0)t]}\int_{-\infty}^{\infty} A(\underline{\psi},t)e^{i\underline{\psi}\cdot\underline{x}}d\underline{\psi} + c.c\right\} =$$

$$= Re\left\{a(\underline{x},t)\,e^{i[k_0 x_1-\omega(k_0)t]}\right\} \qquad (3.6)$$

where the complex wave envelope $a(\underline{x},t)$ is the inverse Fourier transform of $A(\underline{\psi},t)$, multiplied by the coefficient $(2\omega(k_0)/g)^{1/2}$. The frequency difference on the l.h.s. of Eq. (3.5) is replaced by its Taylor series expansion up to the second order in the spectral width

$$\omega(\underline{k}) - \omega(\underline{k}_0) = c_g\psi + \frac{c_g}{2k_0}\lambda^2 + \frac{c_g'}{2}\psi^2 + 0(|\underline{\psi}|^2) \qquad (3.7)$$

where $c_g = \partial\omega_0/\partial k_0$, $c_g' = \partial^2\omega_0/\partial k_0^2$ and $\omega_0^2 = gk_0 th(k_0 h)$. Multiplying Eq. (3.5) by $(2\omega_0/g)^{1/2}$ and taking its inverse Fourier transform yields

$$i\left(\frac{\partial a}{\partial t} + c_g\frac{\partial a}{\partial x_1}\right) + \frac{c_g}{2k_0}\frac{\partial^2 a}{\partial x_2^2} + \frac{c_g'}{2}\frac{\partial^2 a}{\partial x_1^2} =$$

$$= \frac{1}{2\pi}\left(\frac{2\omega_0}{g}\right)^{1/2}\iiint_{-\infty}^{\infty} T(\underline{k}_0+\underline{\psi}_2+\underline{\psi}_3-\underline{\psi}_1,\underline{k}_0+\underline{\psi}_1,\underline{k}_0+\underline{\psi}_2,\underline{k}_0+\underline{\psi}_3)A^*(\underline{\psi}_1)A(\underline{\psi}_2)A(\underline{\psi}_3)e^{i(\underline{\psi}_2+\underline{\psi}_3-\underline{\psi}_1)\cdot\underline{x}}d\underline{\psi}_1 d\underline{\psi}_2 d\underline{\psi}_3$$

$$(3.8)$$

One can show that the Taylor series expansion of T, to the lowest order in the spectral width, is given by

$$T(\underline{k}_0, \underline{k}_0, \underline{k}_0, \underline{k}_0) = T_I + T_{II} \equiv T_{III} + T_{IV} \qquad (3.9a)$$

where

$$T_I = \frac{k_0^3}{32\pi^2\sigma^3} [9\sigma^4 - 10\sigma^2 + 9] \; ; \quad \sigma = th(k_0 h) \qquad (3.9b)$$

$$T_{II} = -\frac{k_0^3}{32\pi^2\sigma} \sum_{j=2}^{3} \frac{4c_p^2(\psi_j-\psi_1)^2 + 4c_p c_g(1-\sigma^2)(\psi_j-\psi_1)^2 + g(1-\sigma^2)^2|\underline{\psi}_j-\underline{\psi}_1|th(h|\underline{\psi}_j-\underline{\psi}_1|)}{g|\underline{\psi}_j-\underline{\psi}_1|th(h|\underline{\psi}_j-\underline{\psi}_1|) - c_g^2(\psi_j-\psi_1)^2} \qquad (3.9c)$$

$$T_{III} = \frac{k_0^3}{32\pi^2\sigma} [\frac{g}{\sigma^2} - 12 + 13\sigma^2 - 2\sigma^4] \qquad (3.9d)$$

$$T_{IV} = -\frac{k_0^3}{16\pi^2\sigma} \frac{[2c_p + c_g(1-\sigma^2)^2]^2(\psi_2-\psi_1)^2}{gth(h|\underline{\psi}_2-\underline{\psi}_1|)|\underline{\psi}_2-\underline{\psi}_1| - c_g^2(\psi_2-\psi_1)^2} \; , \quad c_p = \omega/k_c \qquad (3.9e)$$

Substituting Eqs. (3.9d,e) into Eq. (3.8) gives

$$i\left(\frac{\partial a}{\partial t} + c_g\frac{\partial a}{\partial x_1}\right) + \frac{c_g}{2k_0}\frac{\partial^2 a}{\partial x_2^2} + \frac{c_g'}{2}\frac{\partial^2 a}{\partial x_1^2} = \alpha_1|a|^2 a + c_g\,\alpha_2\,I\bar{a} \qquad (3.10)$$

where

$$\alpha_1 = \frac{g^2 k_0^4}{16\omega_0^3}(\frac{g}{\sigma^2} - 12 + 13\sigma^2 - 2\sigma^4) \; ; \quad \alpha_2 = \frac{k_0^2}{2\omega_0}[2\frac{c_p}{c_g} + (1 - \sigma^2)], \qquad (3.11a,b)$$

and

$$I = -\frac{\alpha_2}{4\pi^2}\iint_{-\infty}^{\infty} \frac{(\psi_2-\psi_1)^2 A^*(\underline{\psi}_1)A(\underline{\psi}_2)e^{i(\underline{\psi}_2-\underline{\psi}_1)\cdot\underline{x}}}{|\underline{\psi}_2-\underline{\psi}_1|th(h|\underline{\psi}_2-\underline{\psi}_1|) - \frac{c_g^2}{gh}(\psi_2-\psi_1)^2} d\underline{\psi}_1\,d\underline{\psi}_2 \qquad (3.11c)$$

Note that for any finite depth and for $\underline{\psi}_j-\underline{\psi}_1 \to 0$, $j = 2,3$, the values of T_{II}, T_{IV} and that of the integrand in Eq. (3.11c) depend on angles θ_j, the "directions" of approach to the limit, where

$$\cos\theta_j = \lim_{\underline{\psi}_j \to \underline{\psi}_1} \left\{ \frac{\psi_j - \psi_1}{|\underline{\psi}_j - \underline{\psi}_1|} \right\}$$

This nonuniqueness disappears for infinitely deep water.

Regarding the integral I, Eq. (3.11c), one can show that $I \equiv \left. \frac{\partial \phi_0}{\partial x_1} \right|_{z=0}$ where ϕ_0 is a solution of the following boundary value problem

$$\frac{\partial^2 \phi_0}{\partial x_1^2} + \frac{\partial^2 \phi_0}{\partial x_2^2} + \frac{\partial^2 \phi_0}{\partial z^2} = 0, \quad \text{for } z \le 0 \tag{3.12a}$$

$$\frac{\partial \phi_0}{\partial z} + \frac{c_g^2}{g} \frac{\partial^2 \phi_0}{\partial x_1^2} = \frac{c_g g \alpha_2}{2\omega_0} \frac{\partial |a|^2}{\partial x_1}, \quad \text{at } z = 0 \tag{3.12b}$$

$$\frac{\partial \phi_0}{\partial z} = 0, \quad \text{at } z = -h \tag{3.12c}$$

Thus ϕ_0 appears to be the mean flow potential. The system of Eqs. (3.10), (3.12) was obtained by Iusim & Stiassnie (1982) using a multiple-scale approach. In the particular case where the water depth is shallow compared to the group length $\text{th}(h|\underline{\psi}_j - \underline{\psi}_1|)$ can be replaced by $h|\underline{\psi}_j - \underline{\psi}_1|$ and the set of equations given by Davey & Stewartson (1974) is recovered. For water of infinite depth, Stiassnie (1983) extended the analysis to one order higher in the spectral width, and rederived Dysthe (1979) set of equations.

4. Linear stability of a uniform wave train

This section deals with the mathematical formulation of one of the simplest possible non-trivial nonlinear interaction problems and its linearized (short time) solution. The smallest number of wave trains required to enable significant nonlinear interaction is three for class I as well as for class II interactions. In what follows, we denote these 3 waves by the subscripts a, b, and c. For anything exciting to happen, these 3 waves have to form a nearly resonating "quartet" for a class I interaction, and a nearly resonating "quintet" for a class II interaction, see Eqs. (2.18) and (2.21), respectively.

To form a "quartet", or a "quintet", out of three waves, one can "count" one of the waves, k_a, twice for class I interactions, and three times for class II interactions.

The governing equations for class I interactions are a discretized form of Eq. (3.1)

$$i \frac{dB_a}{dt} = (T_{aaaa}|B_a|^2 + 2T_{abab}|B_b|^2 + 2T_{acac}|B_c|^2)B_a + 2T_{aabc}B_a^*B_bB_c e^{i\Omega_I t}$$

(4.1a)

$$i \frac{dB_b}{dt} = (2T_{baba}|B_a|^2 + T_{bbbb}|B_b|^2 + 2T_{bcbc}|B_c|^2)B_b + T_{bcaa}B_c^*B_a^2 e^{-i\Omega_I t}$$

(4.1b)

$$i \frac{dB_c}{dt} = (2T_{caca}|B_a|^2 + 2T_{cbcb}|B_b|^2 + T_{cccc}|B_c|^2)B_c + T_{cbaa}B_b^*B_a^2 e^{-i\Omega_I t}$$

(4.1c)

where $\Omega_I = 2\omega_a - \omega_b - \omega_c$.

For class II 3-wave problems, which do not satisfy Eq. (2.18), Eq. (2.24) similarly gives

$$i \frac{dB_a}{dt} = (T_{aaaa}|B_a|^2 + 2T_{abab}|B_b|^2 + 2T_{acac}|B_c|^2)B_a + 2U_{aaabc}^{(3)}(B_a^*)^2 B_bB_c e^{i\Omega_{II} t}$$

(4.2a)

$$i \frac{dB_b}{dt} = (2T_{baba}|B_a|^2 + T_{bbbb}|B_b|^2 + 2T_{bcbc}|B_c|^2)B_b + U_{bcaaa}^{(2)}B_c^*B_a^3 e^{-i\Omega_{II}t}$$

(4.2b)

$$i \frac{dB_c}{dt} = (2T_{caca}|B_a|^2 + 2T_{cbcb}|B_b|^2 + T_{cccc}|B_c|^2)B_b + U_{cbaaa}^{(2)}B_b^*B_a^3 e^{-i\Omega_{II}t}$$

(4.2c)

where $\Omega_{II} = 3\omega_a - \omega_b - \omega_c$ and $U_{aaabc}^{(3)}$ is assumed to be symmetric with respect to b and c.

To complete the mathematical formulation of either of the above systems of equations, Eq. (4.1) or Eq. (4.2), one has to specify the following initial conditions:

$$B_a(0) = b_a, \; B_b(0) = b_b, \; \text{and} \; B_c(0) = b_c \quad ;$$

where the relation between B_j and the actual physical amplitude a_j is

$$a_j = \frac{1}{\pi} \left(\frac{\omega_j}{2g}\right)^{1/2} |B_j|$$

One can assume that the initial amplitude of one of the waves, which is called "the carrier" and denoted by the subscript a, is much larger than the amplitudes of the other two waves, denoted by b and c, to be called "the disturbances": $|b_b|$, $|b_c| \ll |b_a|$. Only linear terms in the disturbances B_b, B_c are retained, so that the carrier wave remains unaffected in this short time analysis, and is given by $B_a = b_a e^{-iT_{aaaa}|b_a|^2 t}$; b_a is assumed to be real without loss of generality.

Class I instabilities:

The wave numbers of the carrier and the disturbances are

$$\underline{k}_a = k_0(1, 0)$$

(4.3a)

$$\underline{k}_b = k_0(1+p, q), \; \underline{k}_c = k_0(1-p; -q)$$

(4.3b,c)

so that Eq. (2.18a) is satisfied identically. The linearized version of Eqs. (4.1b,c) is

$$i \frac{dB_b}{dt} = 2T_{baba} \, b_a^2 B_b + T_{bcaa} B_c^* b_a^2 e^{-i\tilde{\Omega}_I t} \qquad (4.4a)$$

$$i \frac{dB_c}{dt} = 2T_{caca} \, b_a^2 B_c + T_{cbaa} B_b^* b_a^2 e^{-i\tilde{\Omega}_I t} \qquad (4.4b)$$

where $\tilde{\Omega}_I = \Omega_I + 2T_{aaaa} b_a^2$.

Assuming a solution of the form

$$B_b = b_b e^{-i(0.5\tilde{\Omega}_I + \delta_I)t} ; \; B_c = b_c e^{-i(0.5\tilde{\Omega}_I - \delta_I)t}$$

one can show that δ_I must be given by

$$\delta_I = (T_{baba} - T_{caca}) \, b_a^2 \pm D_I^{1/2} \qquad (4.5a)$$

where

$$D_I = [0.5\tilde{\Omega}_I - (T_{baba} + T_{caca})b_a^2]^2 - T_{bcaa} T_{cbaa} b_a^4 \qquad (4.5b)$$

Positive values of $D_I(p,q)$ correspond to stability regions in the p,q plane and vice versa. The curves $D_I = 0$ form the stability boundaries, and the point where D_I attains its minimum is called the most unstable mode. The value of $\sigma_I = (-D_I/gk_0)^{1/2}$ for the most unstable mode is called the maximum growth-rate.

Class II instabilities

For this case, the carrier wave numbers $\underline{k}_a$, and $\underline{k}_b$ are still given by Eqs. (4.3a,b) but

$$\underline{k}_c = k_0(2-p, -q) \qquad (4.6)$$

so that Eq. (2.21a) is now satisfied identically.

The linearized, short-time, version of Eqs. (4.2b,c) is

$$i \frac{dB_b}{dt} = 2T_{baba}b_a^2 B_b + U_{bcaaa}^{(2)}B_c^* b_a^3 e^{-i\tilde{\Omega}_{II}t} \qquad (4.7a)$$

$$i \frac{dB_c}{dt} = 2T_{caca}b_a^2 B_c + U_{cbaaa}^{(2)}B_b^* b_a^3 e^{-i\tilde{\Omega}_{II}t} \qquad (4.7b)$$

where $\tilde{\Omega}_{II} = \Omega_{II} + 3T_{aaaa}b_a^2$.

Assuming, again, a solution of the form

$$B_b = b_b e^{-i(0.5\tilde{\Omega}_{II}+\delta_{II})t}, \quad B_c = b_c e^{-i(0.5\tilde{\Omega}_{II}-\delta_{II})t}$$

one finds that

$$\delta_{II} = (T_{baba} - T_{caca})b_a^2 \pm D_{II}^{1/2} \qquad (4.8a)$$

$$D_{II} = [0.5\tilde{\Omega}_{II} - (T_{baba} + T_{caca})b_a^2]^2 - U_{bcaaa}^{(2)}U_{cbaaa}^{(2)}b_a^6 \qquad (4.8b)$$

The stability boundary and maximum growth rate for class II interactions are obtained from Eq. (4.8b).

Results

In Fig. 1, we show the class I and class II instability regions (as shaded zones) for $k_o h = 2$. The solid lines represent the calculated results and the dashed curves are those of McLean (1982b, Figs. 2b and 2c). In Fig. 1a, $k_o a_o = 0.195$ (where a_o is the first order amplitude of the carrier in Stokes' expansion), which is equivalent to $(ka)_m = 0.2$ (the subscript m stands for McLean). As a conversion formula, we used the following expression

$$(ka)_m = k_o a_o + \frac{24ch^6(k_o h) + 3}{64sh^6(k_o h)}(k_o a_o)^3 + 0(k_o a_o)^5 , \qquad (4.9)$$

given by Sbjelbreia and Hendrickson (1961).

> Fig. 1 about here please

In Fig. 1b, $k_o a_o = 0.326$ (corresponding to $(ka)_m = 0.35$).

The locations of the maximum growth rates (p_I, q_I), (p_{II}, q_{II}), for class I and class II instabilities, respectively, are marked by x for our results and by a dot for McLean's results, and their numerical values, as well as those of the maximum growth rates σ_I, σ_{II} are given in the figures. The overall agreement in Fig. 1a (which is for actual amplitude, which is 47% of the theoretical maximum amplitude, Cokelet, 1977) is quite satisfactory. For smaller steepnesses, the agreement becomes even better. On the other hand, for very steep waves (in Fig. 1b, the actual amplitude is 82% of the theoretical maximum amplitude), the agreement is less impressive. Nevertheless, a somewhat better agreement is obtained if we compare McLean's results for $(ka)_m = 0.35$ with the artificially amplified value $a_o k_o = 0.41$, see Fig. 1c.

A similar degree of agreement was obtained for several other water depths, and should give the reader some indication about the validity of the present model.

> Fig. 2 about here please

Figure 2, which is quite typical, is used to demonstrate some general features as well as clarify some of the terminology which is used later.

One can see that a certain similarity exists between class I and class II instability regions. Both can be regarded as consisting of two domains: a wider band at lower values of p and usually a much narrower region at higher values of p. The first region will be referred to as the main region, and the other as the secondary instability region. The qualitative difference between class I and class II instability regions is that for class I the two domains are usually disconnected while in the case of class II they are bound by a line of infinitesimal thickness. The secondary regions sometimes disappear completely, and for class I, the instability region in these cases terminates at some $q > 0$ (compare with Crawford, et al., 1981, for infinite water depth).

Figure 2 shows the three wave-number vectors $\underline{k}_a$, $\underline{k}_b$, $\underline{k}_c$, as well as the location of four points of local maximum growth rates:

A. Class I point (p_I, q_I) with local maximum growth rate σ_I,

B. The secondary, Class I point (p_I^S, q_I^S) with local maximum growth rate σ_I^S,

C. Class II point (p_{II}, q_{II}) with local maximum growth rate σ_{II}, and

D. The secondary Class II point (p_{II}^S, q_{II}^S) with local maximum growth rate σ_{II}^S.

For the particular data of Fig. 2 ($k_o h = 0.35$, $k_o a = 0.04$) $\sigma_{II} > \sigma_I > \sigma_{II}^S > \sigma_I^S$. These inequalities are by no means general as will be shown in the sequel. Nevertheless, for most cases, $\sigma^{II} > \sigma_s^{II}$.

<div style="border:1px solid">Fig. 3 about here please</div>

Figure 3 is a summary of the results for Class I instabilities. Figures 3a and 3b give the values of p_I and q_I, respectively, as functions of the water depth and wave steepness. The depth is expressed by th($k_o h$) (the range covered is $0.357 \le k_o h \le \infty$); and the wave steepness by Cokelet's (1977) ε^2, denoted here by ε_c^2 (the range $0 \le \varepsilon_c^2 < 0.7$ is covered).

The isolines in Figs. 3 and 4 were drawn using interpolation and are based on about forty computed data points, almost equally distributed over the figure domain. Figure 3c is a plot of $\sigma_I^m = \max(\sigma_I, \sigma_I^S)$ isolines. Note that for the region confined by the broken lines $\sigma_I^S > \sigma_I$ (sometimes by a factor of three), whereas the opposite is true in the outside region. For the case where $\sigma_I^S > \sigma_I$, p_I^S is in the range $1.05 \div 1.30$ and $q_I = 0$ which implies that the most unstable mode is two dimensional.

<div style="border:1px solid">Fig. 4 about here please</div>

The results for Class II are given in Fig. 4. Note for this case p_{II} is always 0.5. Figure 4a gives the values of q_{II}, and σ_{II} is shown in Fig. 4b. For the domain above the dashed line in Fig. 4b, $\sigma_{II} > \sigma_I^m$, which indicates that for this region Class II instabilities may become dominant.

The question whether the disturbances related to the highest value of σ will dominate the physical process remains open, and awaits additional evidence. The authors hope that their current study of the long time evolution of Class I and Class II instabilities will throw some light on this and on other relevant aspects of these important processes.

APPENDIX

The kernels in Eq. (2.9)

$$S^{(1)}_{0,1,2,3} = \frac{\omega_o^2}{g} J^{(1)}_{0,1,2,3} - \frac{\omega_1^2}{2g} \{|\underline{k}_1|^2 - \frac{1}{2}(|\underline{k}-\underline{k}_2|^2 + |\underline{k}-\underline{k}_3|^2 + |\underline{k}_1+\underline{k}_2|^2 + |\underline{k}_1+\underline{k}_3|^2)\} \quad \text{(A.1a)}$$

$$S^{(2)}_{0,1,2,3,4} = -\frac{|\underline{k}_1|^4}{6} + \frac{\omega_o^2}{g} J^{(2)}_{0,1,2,3,4} + |\underline{k}-\underline{k}_2|^2 J^{(1)}_{0\ -2,1,3,4} + \frac{|\underline{k}_1+\underline{k}_2|^2}{2}\left(\frac{\omega_1\omega_{1+2}}{g}\right)^2$$

$$\text{(A.1b)}$$

where

$$J^{(1)}_{0,1,2,3} = \frac{|\underline{k}_1|}{4}\{2|\underline{k}_1| - \frac{th(|\underline{k}_1|h)}{g}(\omega^2_{0-2} + \omega^2_{0-3} + \omega^2_{1+2} + \omega^2_{1+3})\} \quad \text{(A.1c)}$$

$$J^{(2)}_{0,1,2,3,4} = \frac{|\underline{k}_1|^3}{6} th(|\underline{k}_1|h) - \frac{\omega_1^2}{2g}|\underline{k}_1+\underline{k}_2|^2 - \frac{\omega^2_{0-2}}{g} J^{(1)}_{0-2,1,3,4} \quad \text{(A.1d)}$$

The interaction coefficients in Eq. (2.11)

Second order:

$$V^{(1)}_{0,1,2} = -2V_{-0,1,2} + V_{1,2,0} \quad \text{(A.2a)}$$

$$V^{(2)}_{0,1,2} = 2(V_{0,1,2} - V_{-0,2,1} - V_{-1,2,0}) \quad \text{(A.2b)}$$

$$V^{(3)}_{0,1,2} = 2V_{0,1,2} + V_{1,2,0} \quad \text{(A.2c)}$$

where

$$V_{0,1,2} = \frac{1}{8\pi}\left(\frac{g\omega_2}{2\omega_o\omega_1}\right)^{1/2}[\underline{k}\cdot\underline{k}_1 + \left(\frac{\omega_o\omega_1}{g}\right)^2] \quad \text{(A.2d)}$$

Third order:

$$W^{(1)}_{0,1,2,3} = W_{1,2,-0,3} - W_{-0,1,2,3} \quad \text{(A.3a)}$$

$$W^{(2)}_{0,1,2,3} = W_{-0,-1,2,3} + W_{2,3,-0,-1} - W_{2,-1,-0,3} - W_{-0,2,-1,3} - W_{-0,3,2,-1} - W_{3,-1,2,-0}$$

$$\text{(A.3b)}$$

$$W^{(3)}_{0,1,2,3} = 2W_{-0,-1,-2,3} - W_{-0,3,-1,-2} + W_{-1,-2,-0,3} - 2W_{-1,3,-0,-2} \quad \text{(A.3c)}$$

$$W^{(4)}_{0,1,2,3} = W_{0,1,2,3} + W_{1,2,0,3} \quad \text{(A.3d)}$$

where

$$W_{0,1,2,3} = \frac{1}{64\pi^2} \left(\frac{\dot{\omega}_2 \omega_3}{\omega_0 \omega_1}\right)^{1/2} |\underline{k}||\underline{k}_1| \left\{2|\underline{k}|\text{th}(|\underline{k}_1|h) + 2|\underline{k}_1|\text{th}(|\underline{k}|h) - \right.$$

$$\left. - \frac{1}{g}\text{th}(|\underline{k}|h)\text{th}(|\underline{k}_1|h) \left[\omega^2_{0+2} + \omega^2_{0+3} + \omega^2_{1+2} + \omega^2_{1+3}\right]\right\} \quad \text{(A.3e)}$$

Fourth order:

$$\chi^{(1)}_{0,1,2,3,4} = \alpha_{0,1,2,3,4} + \beta_{0,1,2,3,4} \quad \text{(A.4a)}$$

$$\chi^{(2)}_{0,1,2,3,4} = \alpha_{0,4,2,3,-1} + \beta_{0,4,2,3,-1} + \alpha_{0,3,2,-1,4} + \beta_{0,3,2,-1,4} +$$

$$+ \alpha_{0,2,-1,3,4} - \beta_{0,2,-1,3,4} - \alpha_{0,-1,2,3,4} - \beta_{0,-1,2,3,4} \quad \text{(A.4b)}$$

$$\chi^{(3)}_{0,1,2,3,4} = \alpha_{0,3,4,-1,2} + \beta_{0,3,4,-1,-2} - \alpha_{0,-1,3,-2,4} - \beta_{0,-1,3,-2,4} -$$

$$- \alpha_{0,-1,4,3,-2} - \beta_{0,-1,4,3,-2} + \alpha_{0,3,-2,-1,4} - \beta_{0,3,-2,-1,4,} +$$

$$+ \alpha_{0,4,-2,3,-1} - \beta_{0,4,-2,3,-1} - \alpha_{0,-1,-2,3,4} + \beta_{0,-1,-2,3,4}$$

$$\text{(A.4c)}$$

$$\chi^{(4)}_{0,1,2,3,4} = -\alpha_{0,-1,4,-2,-3} - \beta_{0,-1,4,-2,3} + \alpha_{0,4,-2,-3,-1} - \beta_{0,4,-2,-3,-1} -$$

$$- \alpha_{0,-1,-2,-3,4} + \beta_{0,-1,-2,-3,4} - \alpha_{0,-1,-2,4,-3} + \beta_{0,-1,-2,4,-3}$$

$$\text{(A.4d)}$$

$$\chi^{(5)}_{0,1,2,3,4} = -\,\alpha_{0,-1,-2,-3,-4} + \beta_{0,-1,-2,-3,-4} \tag{A.4e}$$

where

$$\alpha_{0,1,2,3,4} = -\frac{1}{32\pi^3}\left(\frac{\omega_2\omega_3\omega_4}{2g\omega_0\omega_1}\right)^{1/2}\left\{S^{(2)}_{0,1,2,3,4} - |\underline{k}_1|(\underline{k}_2\cdot\underline{k}_3)\left(\frac{\omega^2_{1+4}}{g} - |\underline{k}_1|\right)\right\} \tag{A.4f}$$

$$\beta_{0,1,2,3,4} = -\frac{1}{32\pi^3}\left(\frac{\omega_0\omega_3\omega_4}{2g\omega_1\omega_2}\right)^{1/2}\left\{\frac{\omega^2_1}{g}S^{(1)}_{0-1,2,3,4} + \frac{(k_2\cdot k_4)}{2}\left[\frac{\omega_1\omega_2}{g}\right]^2 - \right.$$
$$\left. -\frac{|\underline{k}_1||\underline{k}_2|}{2}\left[\frac{\omega^2_{1+4}}{g}\,\text{th}(|\underline{k}_1|h) - |\underline{k}_1|\right]\left[\frac{\omega^2_{2+3}}{g}\,\text{th}(|\underline{k}_2|h) - |\underline{k}_2|\right]\right\} \tag{A.4g}$$

The kernels of Eq. (2.15)

$$\tilde{T}^{(1)}_{0,1,2,3} = W^{(1)}_{0,1,2,3} - \frac{V^{(1)}_{0,1,2+3}V^{(1)}_{2+3,3,2}}{\omega_{2+3} - \omega_2 - \omega_3} - \frac{V^{(1)}_{0,1+3,2}V^{(1)}_{1+3,3,1}}{\omega_{1+3} - \omega_1 - \omega_3} -$$
$$-\frac{V^{(2)}_{0,-1-3,2}V^{(3)}_{-1-3,3,1}}{\omega_{1+3} + \omega_1 + \omega_3} \tag{A.5a}$$

$$\tilde{T}^{(2)}_{0,1,2,3} = W^{(2)}_{0,1,2,3} - \frac{V^{(1)}_{0,2,3-1}V^{(2)}_{3-1,1,3}}{\omega_{3-1} + \omega_1 - \omega_3} - \frac{V^{(1)}_{0,3-1,2}V^{(2)}_{3-1,1,3}}{\omega_{3-1} + \omega_1 - \omega_3} - \frac{V^{(2)}_{0,2-0,2}V^{(2)}_{1-3,3,1}}{\omega_{1-3} + \omega_3 - \omega_1} -$$
$$-\frac{V^{(3)}_{0,-0-1,1}V^{(3)}_{-2-3,3,2}}{\omega_{2+3} + \omega_2 + \omega_3} - \frac{V^{(3)}_{0,1,-0-1}V^{(3)}_{-2-3,3,2}}{\omega_{2+3} + \omega_2 + \omega_3} - \frac{V^{(1)}_{2+3,2,3}V^{(2)}_{0,1,0+1}}{\omega_{2+3} - \omega_2 - \omega_3} \tag{A.5b}$$

$$T_{0,1,2,3} = 0.5(\tilde{T}^{(2)}_{0,1,2,3} + \tilde{T}^{(2)}_{0,1,3,2}) \tag{A.5c}$$

$$\tilde{T}^{(3)}_{0,1,2,3} = W^{(3)}_{0,1,2,3} - \frac{V^{(2)}_{0,1+2,3}V^{(1)}_{1+2,2,1}}{\omega_{1+2} - \omega_1 - \omega_2} - \frac{V^{(2)}_{0,1,-2+3}V^{(2)}_{-2+3,2,3}}{\omega_{2-3} + \omega_2 - \omega_3} - \frac{V^{(3)}_{0,1,2-3}V^{(2)}_{2-3,3,2}}{\omega_{2-3} + \omega_3 - \omega_2} -$$
$$-\frac{V^{(3)}_{0,1-3,2}V^{(2)}_{1-3,3,1}}{\omega_{1-3} + \omega_3 - \omega_1} - \frac{V^{(1)}_{0,3,-1-2}V^{(3)}_{-1-2,1,2}}{\omega_{1+2} + \omega_1 + \omega_2} - \frac{V^{(1)}_{0,-1-2,3}V^{(3)}_{-1-2,2,1}}{\omega_{1+2} + \omega_1 + \omega_2} \tag{A.5d}$$

$$\tilde{T}^{(4)}_{0,1,2,3} = W^{(4)}_{0,1,2,3} - \frac{V^{(3)}_{0,1,2+3}V^{(1)}_{2+3,3,2}}{\omega_{2+3}-\omega_2-\omega_3} - \frac{V^{(3)}_{0,1+3,2}V^{(1)}_{1+3,3,1}}{\omega_{1+3}-\omega_1-\omega_3} - \frac{V^{(2)}_{0,1,-2,-3}V^{(3)}_{-2-3,3,2}}{\omega_{2+3}+\omega_2+\omega_3}$$

$$(A.5e)$$

The kernels of Eq. (2.20)

$$\tilde{U}^{(2)}_{0,1,2,3,4} = \chi^{(2)}_{0,1,2,3,4} + \frac{V^{(1)}_{0,3+4,2-1}V^{(1)}_{3+4,3,4}V^{(2)}_{2-1,1,2}}{(\omega_{3+4}-\omega_3-\omega_4)(\omega_{2-1}+\omega_1-\omega_2)} + \frac{V^{(1)}_{0,4-1,2+3}V^{(2)}_{4-1,1,4}V^{(1)}_{2+3,2,3}}{(\omega_{4-1}+\omega_1-\omega_4)(\omega_{2+3}-\omega_2-\omega_3)} +$$

$$+ \frac{V^{(2)}_{0,1-3,2+4}V^{(2)}_{1-3,3,1}V^{(1)}_{2+4,2,4}}{(\omega_{1-3}+\omega_3-\omega_1)(\omega_{2+4}-\omega_2-\omega_4)} + \frac{V^{(2)}_{0,-3-4,2-1}V^{(3)}_{-3-4,3,4}V^{(2)}_{2-1,1,2}}{(\omega_{3+4}+\omega_3+\omega_4)(\omega_{2-1}+\omega_1-\omega_2)} +$$

$$+ \frac{V^{(3)}_{0,1-3,-2-4}V^{(2)}_{1-3,3,1}V^{(3)}_{-2-4,4,2}}{(\omega_{1-3}+\omega_3-\omega_1)(\omega_{2+4}+\omega_2+\omega_4)} + \frac{V^{(3)}_{0,-3-4,1-2}V^{(3)}_{-3-4,3,4}V^{(2)}_{1-2,2,1}}{(\omega_{3+4}+\omega_3+\omega_4)(\omega_{1-2}+\omega_2-\omega_1)} -$$

$$- \frac{V^{(1)}_{0,2,3+4-1}\tilde{T}^{(2)}_{3+4-1,1,4,3}}{\omega_{3+4-1}+\omega_1-\omega_3-\omega_4} - \frac{V^{(1)}_{0,3+4-1,2}\tilde{T}^{(2)}_{3+4-1,1,4,3}}{\omega_{3+4-1}+\omega_1-\omega_3-\omega_4} - \frac{V^{(2)}_{0,1,2+3+4}\tilde{T}^{(1)}_{2+3+4,3,4,2}}{\omega_{2+3+4}-\omega_2-\omega_3-\omega_4} -$$

$$- \frac{V^{(2)}_{0,1-3-4,2}\tilde{T}^{(3)}_{1-3-4,3,4,1}}{\omega_{1-3-4}+\omega_3+\omega_4-\omega_1} - \frac{V^{(3)}_{0,1,-2-3-4}\tilde{T}^{(4)}_{-2-3-4,3,4,2}}{\omega_{2+3+4}+\omega_2+\omega_3+\omega_4} - \frac{V^{(3)}_{0,-2-3-4,1}\tilde{T}^{(4)}_{-2-3-4,3,4,2}}{\omega_{2+3+4}+\omega_2+\omega_3+\omega_4} -$$

$$- \frac{V^{(2)}_{4-1,1,4}W^{(1)}_{0,4-1,2,3}}{\omega_{4-1}+\omega_1-\omega_4} - \frac{V^{(2)}_{4-1,1,4}W^{(1)}_{0,2,4-1,3}}{\omega_{4-1}+\omega_1-\omega_4} - \frac{V^{(2)}_{3-1,1,3}W^{(1)}_{0,4,2,3-1}}{\omega_{3-1}+\omega_1-\omega_3} -$$

$$- \frac{V^{(2)}_{1-4,4,1}W^{(2)}_{0,1-4,2,3}}{\omega_{1-4}+\omega_4-\omega_1} - \frac{V^{(1)}_{2+4,4,2}W^{(2)}_{0,1,2+4,3}}{\omega_{2+4}-\omega_2-\omega_4} - \frac{V^{(1)}_{3+4,4,3}W^{(2)}_{0,1,2,3+4}}{\omega_{3+4}-\omega_3-\omega_4} -$$

$$- \frac{V^{(3)}_{-2-4,4,2}W^{(3)}_{0,-2-4,1,3}}{\omega_{2+4}+\omega_2+\omega_4} - \frac{V^{(3)}_{-2-4,4,2}W^{(3)}_{0,1,-2-4,3}}{\omega_{2+4}+\omega_2+\omega_4}$$

$$(A.6a)$$

$$\tilde{U}^{(3)}_{0,1,2,3,4} = \chi^{(3)}_{0,1,2,3,4} + \frac{V^{(1)}_{0,3+4,-1-2}V^{(1)}_{3+4,3,4}V^{(3)}_{-1-2,1,2}}{(\omega_{1+2}+\omega_1+\omega_2)(\omega_{3+4}-\omega_3-\omega_4)} + \frac{V^{(1)}_{0,4-1,3-2}V^{(2)}_{4-1,1,4}V^{(2)}_{3-2,2,3}}{(\omega_{4-1}+\omega_1-\omega_4)(\omega_{3-2}+\omega_2-\omega_3)} +$$

$$+ \frac{V^{(1)}_{0,-1-2,3+4}V^{(3)}_{-1-2,1,2}V^{(1)}_{3+4,3,4}}{(\omega_{1+2}+\omega_1+\omega_2)(\omega_{3+4}-\omega_3-\omega_4)} + \frac{V^{(2)}_{0,1+2,3+4}V^{(1)}_{1+2,1,2}V^{(1)}_{3+4,3,4}}{(\omega_{1+2}-\omega_1-\omega_2)(\omega_{3+4}-\omega_3-\omega_4)} +$$

$$+ \frac{V^{(2)}_{0,1-3,4-2}V^{(2)}_{1-3,3,1}V^{(2)}_{4-2,2,4}}{(\omega_{1-3}+\omega_3-\omega_1)(\omega_{4-2}+\omega_2-\omega_4)} + \frac{V^{(2)}_{0,-3-4,-1-2}V^{(3)}_{-3-4,3,4}V^{(3)}_{-1-2,1,2}}{(\omega_{3+4}+\omega_3+\omega_4)(\omega_{1+2}+\omega_1+\omega_2)} +$$

$$+ \frac{V^{(3)}_{0,1+2,-3-4}V^{(1)}_{1+2,1,2}V^{(3)}_{-3-4,3,4}}{(\omega_{1+2}-\omega_1-\omega_2)(\omega_{3+4}+\omega_3+\omega_4)} + \frac{V^{(3)}_{0,1-3,2-4}V^{(2)}_{1-3,3,1}V^{(2)}_{2-4,4,2}}{(\omega_{1-3}+\omega_3-\omega_1)(\omega_{2-4}+\omega_4-\omega_2)} +$$

$$+ \frac{V^{(3)}_{0,-3-4,1+2}V^{(3)}_{-3-4,3,4}V^{(1)}_{1+2,1,2}}{(\omega_{1+2}-\omega_1-\omega_2)(\omega_{3+4}+\omega_3+\omega_4)} - \frac{V^{(1)}_{0,3,4-1-2}\tilde{T}^{(3)}_{4-1-2,1,2,4}}{\omega_{4-1-2}+\omega_1+\omega_2-\omega_4} - \frac{V^{(1)}_{0,4-1-2,3}\tilde{T}^{(3)}_{4-1-2,1,2,4}}{\omega_{4-1-2}+\omega_1+\omega_2-\omega_4} -$$

$$- \frac{V^{(2)}_{0,1,4+3-2}\tilde{T}^{(2)}_{4+3-2,2,4,3}}{\omega_{4+3+2}+\omega_2-\omega_3-\omega_4} - \frac{V^{(2)}_{0,1+2-3,4}\tilde{T}^{(2)}_{1+2-3,3,1,2}}{\omega_{1+2-3}+\omega_3-\omega_1-\omega_2} - \frac{V^{(3)}_{0,1,2-3-4}\tilde{T}^{(3)}_{2-3-4,3,4,2}}{\omega_{2-3-4}+\omega_3+\omega_4-\omega_2} -$$

$$- \frac{V^{(3)}_{0,1-3-4,2}\tilde{T}^{(3)}_{1-3-4,3,4,1}}{\omega_{1-3-4}+\omega_3+\omega_4-\omega_1} - \frac{V^{(3)}_{-1-2,1,2}W^{(1)}_{0,-1-2,4,3}}{\omega_{1+2}+\omega_1+\omega_2} - \frac{V^{(3)}_{-1-2,1,2}W^{(1)}_{0,4,-1-2,3}}{\omega_{1+2}+\omega_1+\omega_2} -$$

$$- \frac{V^{(3)}_{-1-2,1,2}W^{(1)}_{0,3,4,-1-2}}{\omega_{1+2}+\omega_1+\omega_2} - \frac{V^{(1)}_{1+2,1,2}W^{(2)}_{0,1+2,4,3}}{\omega_{1+2}-\omega_1-\omega_2} - \frac{V^{(2)}_{4-2,2,4}W^{(2)}_{0,1,4-2,3}}{\omega_{4-2}+\omega_2-\omega_4} -$$

$$- \frac{V^{(2)}_{4-2,2,4}W^{(2)}_{0,1,3,4-2}}{\omega_{4-2}+\omega_2-\omega_4} - \frac{V^{(2)}_{1-4,4,1}W^{(3)}_{0,1-4,2,3}}{\omega_{1-4}+\omega_4-\omega_1} - \frac{V^{(2)}_{2-4,4,2}W^{(3)}_{0,1,2-4,3}}{\omega_{2-4}+\omega_4-\omega_2} -$$

$$- \frac{V^{(1)}_{3+4,4,3}W^{(3)}_{0,1,2,3+4}}{\omega_{3+4}-\omega_3-\omega_4} - \frac{V^{(3)}_{-4-3,4,3}W^{(4)}_{0,-4-3,2,1}}{\omega_{4+3}+\omega_4+\omega_3} - \frac{V^{(3)}_{-4-3,4,3}W^{(4)}_{0,1,-4-3,2}}{\omega_{4+3}+\omega_3+\omega_4} -$$

$$- \frac{V^{(3)}_{-3-4,4,3}W^{(4)}_{0,1,2,-3-4}}{\omega_{3+4}+\omega_3+\omega_4} \qquad (A.6b)$$

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Figure Captions

Fig. 1: Bands of instability for $k_o h = 2$. The instability boundaries are given by the solid lines and the points of maximum growth rate are labeled by X. McLean's results are marked by the dashed lines and by ●. (a) $k_o a_o = 0.195$, $(ka)_m = 0.2$; (b) $k_o a_o = 0.326$, $(ka)_m = 0.35$; (c) $k_o a_o = 0.41$, $(ka)_m = 0.35$.

Fig. 2: Bands of instability for $k_o h = 0.35$, $k_o a_o = 0.04$ and notation.

Fig. 3: Summary of results for Class I instabilities. (a) Isolines of p_I (for $\varepsilon_c^2 = 0$, $p_I = 0$); (b) Isolines of q_I (for either $\varepsilon_c^2 = 0$ or $th(k_o h) = 1$, $q_I = 0$); (c) Isolines of $10\sigma_I^m$, the maximum growth rate (for $\varepsilon_c^2 = 0$, $\sigma_I^m = 0$).

Fig. 4: Summary of results for Class II instabilities. (a) Isolines of q_{II}; (b) Isolines of $10\sigma_{II}$.

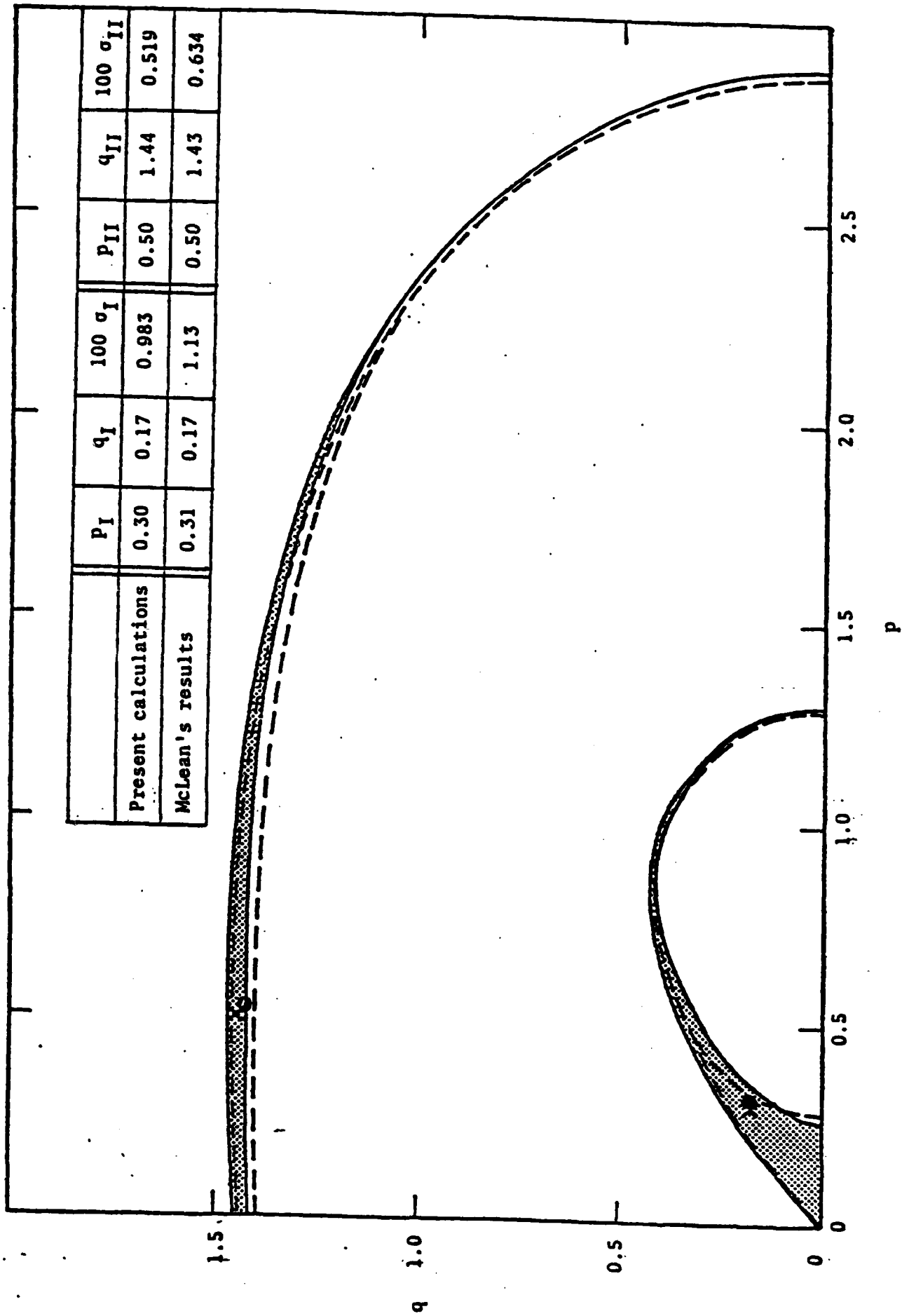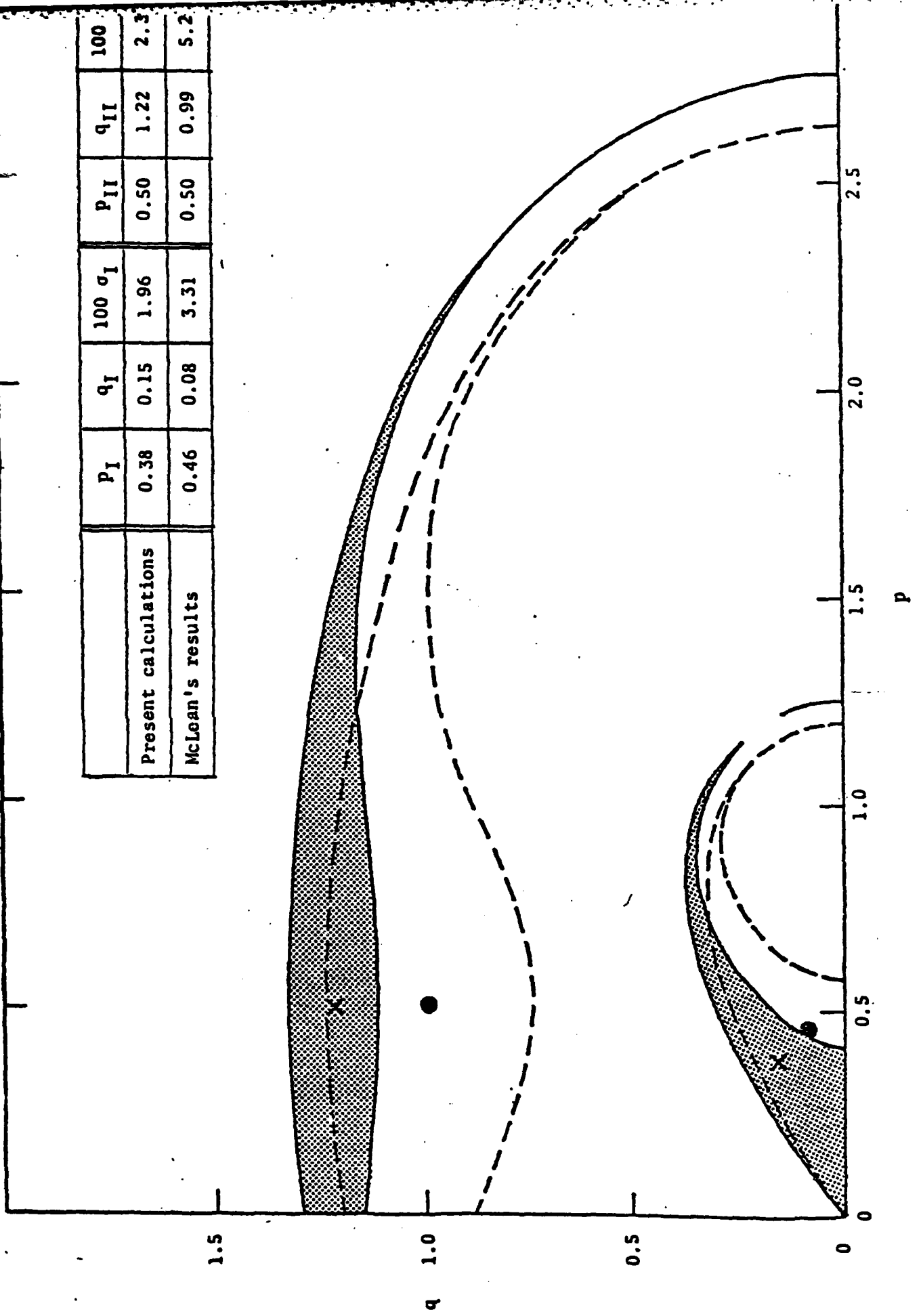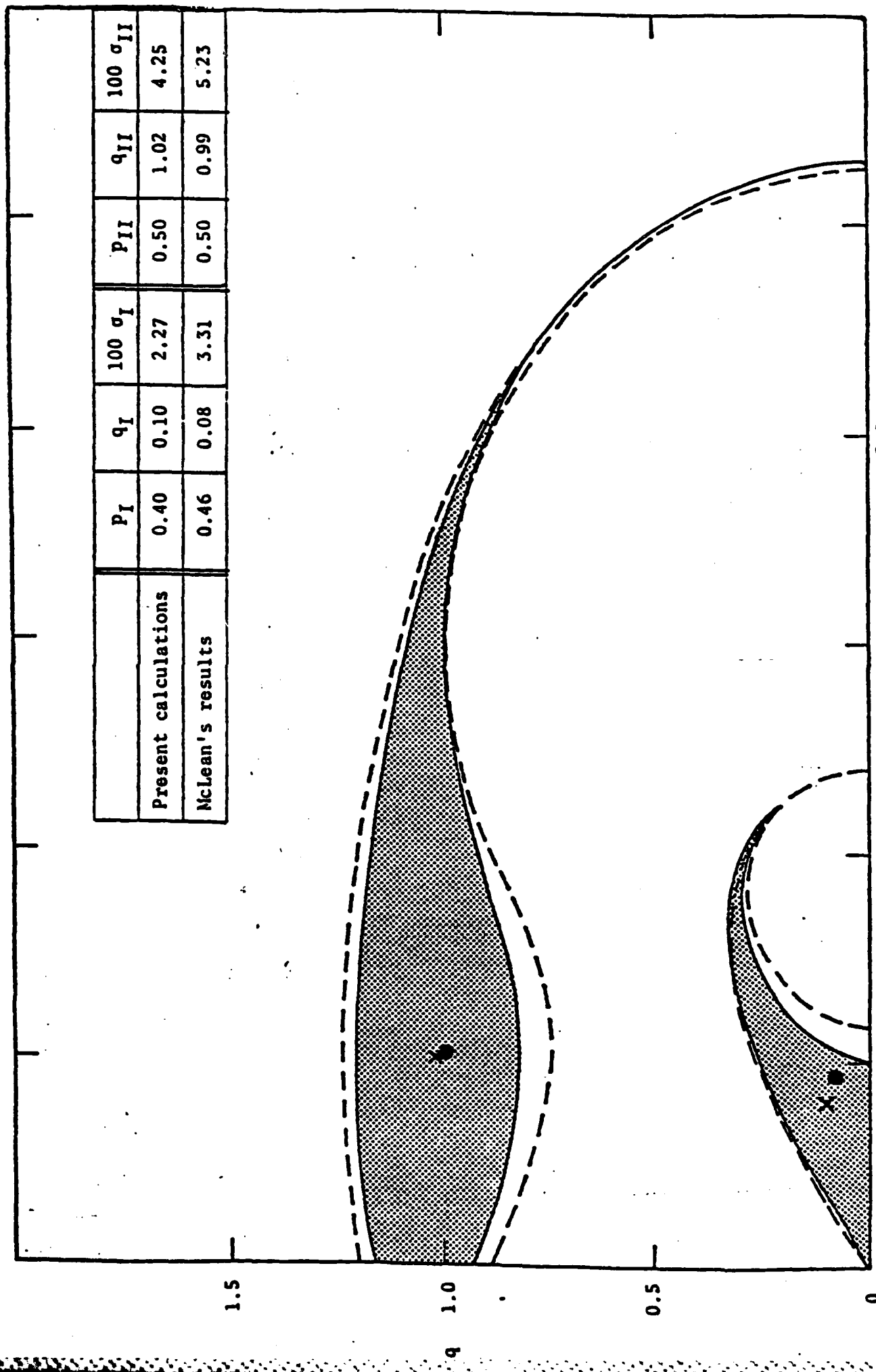|	p_I	q_I	$100\ \sigma_I$	p_{II}	q_{II}	$100\ \sigma_{II}$
Present calculations	0.30	0.17	0.983	0.50	1.44	0.519
McLean's results	0.31	0.17	1.13	0.50	1.43	0.634

Figure 1a

	P_I	q_I	$100\ \sigma_I$	P_{II}	q_{II}	100
Present calculations	0.38	0.15	1.96	0.50	1.22	2.3
McLean's results	0.46	0.08	3.31	0.50	0.99	5.2

Figure 1b

	p_I	q_I	$100\,\sigma_I$	p_{II}	q_{II}	$100\,\sigma_{II}$
Present calculations	0.40	0.10	2.27	0.50	1.02	4.25
McLean's results	0.46	0.08	3.31	0.50	0.99	5.23

Fig. 2

Fig. 3a

Fig. 3b

Axis labels: ε_c^2 (vertical), th(k_oh) (horizontal)

Contour labels: 0.275, 0.250, 0.225, 0.200, 0.175

Fig. 3c

th(k_0h)

Fc

1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4

0.9
0.8
0.7
0.6
0.5
0.4
0.35
0

0.6
0.4
0.2

Fig. 4b

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