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ASYMPTOTIC PROPERTIES OF EXTENDED YULE-WALKER ESTIMATES OF THE AR PARAMETERS OF AN ARMA TIME SERIES

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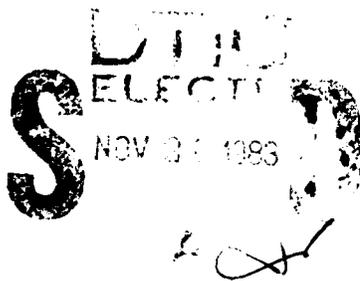Prepared for
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20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

The extended Yule-Walker equations are used to estimate the autoregressive parameters of an autoregressive moving-average time series. The asymptotic statistical properties of these estimates are derived. It is shown that they are asymptotically unbiased and normal; the covariance matrix of the limit distribution is calculated. The special case of estimating the autoregressive parameters of a noise corrupted autoregressive series is also treated.

SUMMARY

This report shows that when the extended Yule-Walker equations are used to estimate the autoregressive parameters of an autoregressive moving-average time series, the parameter estimates are asymptotically unbiased and normal. The covariance matrix is evaluated for the general autoregressive moving-average case and for the special case of autoregressive plus noise series. The evaluation of the asymptotic statistics of the corresponding spectral estimate remains to be completed.

I. INTRODUCTION

In many applications, such as those found in radar, sonar, speech analysis, and econometrics, one may assume that the observed time series is generated (or adequately modeled) by an autoregressive moving-average (ARMA) model. In these cases the first estimation problem is that of estimating the model parameters from the observed time series. In some applications, only the autoregressive (AR) parameters will be of interest. The problem considered in this report is the evaluation of asymptotic statistical properties for estimates of the AR parameters of an ARMA series from a finite set of observations. We also consider the related special case of estimating the AR parameters of an AR series corrupted by additive noise. In both cases the extended Yule-Walker (Y-W) equations (see Gersch[1]) are used for parameter estimation.

Similar problems have been considered by Gersch[1], Walker[2], and Pagano[3]. Gersch uses the extended Y-W equations to form asymptotically unbiased estimates of the AR parameters of an ARMA series. He indicates (but does not prove) that the limit distribution of the estimates is normal and evaluates the asymptotic covariance matrix. Walker was the first to consider the problem of estimating the AR parameters of an AR series corrupted by additive noise. He evaluated the asymptotic efficiency and variance of the parameter estimates for a first order AR series. Pagano proves that the correct model for an AR series plus noise is an ARMA model and through the use of nonlinear regression methods develops strongly consistent efficient parameter estimates.

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In this report, under the assumption that the AR or ARMA series is Gaussian distributed, we derive the asymptotic statistical properties for the AR parameter estimates based on well known asymptotic statistical properties of standard covariance estimates (Parzen[4]). For the estimation of the AR parameters of an AR plus noise series, Walker's results are extended proving asymptotic normality and obtaining the asymptotic covariance for arbitrary model order. For the estimation of the AR parameters of an ARMA series, Gersch's results are extended by providing a more precise derivation of the asymptotic covariance matrix and by proving that the limit distribution is normal.

The organization of the report is as follows: In Section II we define the extended Y-W equation estimates and establish certain definitions required in Section III. In Section III we prove that the limit distribution is normal and evaluate the asymptotic covariance matrix for the extended Y-W equation estimates. A separate covariance matrix is derived for the AR series plus noise case.

II. PRELIMINARIES

Let $\{Y_n\}_{n=-\infty}^{\infty}$ be a discrete parameter time series generated (or modeled) by a zero-mean autoregressive moving average process of known order, ARMA(p,q). We write

$$Y_n - a_1 Y_{n-1} - \ldots - a_p Y_{n-p} = \eta_n - b_1 \eta_{n-1} - \ldots - b_q \eta_{n-q} \tag{1}$$

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where the sequence $\{\eta_n\}$ is assumed to be stationary, independent, identically distributed (i.i.d.) with zero mean and variance σ_η^2. The set $\{a_j\}_{j=1}^{p}$ is referred to as the autoregressive (AR) parameters and the set $\{b_j\}_{j=1}^{q}$ is referred to as the moving average (MA) parameters.

Define the polynomials in z, z complex, by

$$A^p(z) = 1 - \sum_{j=1}^{p} a_j z^j \tag{2a}$$

$$B^q(z) = 1 - \sum_{j=1}^{q} b_j z^j \; . \tag{2b}$$

We assume that all zeroes of $A^p(z)$ and $B^q(z)$ lie outside the unit circle on the complex z-plane and that the polynomials have no common zeroes. The process Y is guaranteed to be stationary by this assumption. For the stationary process the spectral density is given by:

$$\phi(\lambda) = \frac{\sigma_\eta^2}{2\pi} \frac{|B^q(e^{i\lambda})|^2}{|A^p(e^{i\lambda})|^2} \; . \tag{3}$$

We evaluate the autocovariance sequence for the Y process by multiplying Equation 1 through by Y_{n-k} and taking expectations term by term, we obtain

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$$E [Y_{n-k} Y_n] - a_1 E [Y_{n-k} Y_{n-1}] - \ldots - a_p E [Y_{n-k} Y_{n-p}]$$

$$= E [Y_{n-k} \eta_n] - b_1 E [Y_{n-k} \eta_{n-1}] - \ldots - b_q E [Y_{n-k} \eta_{n-q}]. \qquad (4)$$

Define the covariance sequence for the Y process to be $\{r_k\}$, where $r_k = E [Y_n Y_{n-k}]$, since Y_{n-k} does not depend on inputs η_{n-j} for $n-j>n-k$, it follows that $E [Y_{n-k} \eta_{n-j}] = 0$ for $k>j$, thus we can rewrite Equation 4 as

$$r_k - a_1 r_{k-1} - \ldots - a_p r_{k-p} = 0 \quad k = q+1, q+2, \ldots, q+p \quad . \qquad (5)$$

This set of equations is referred to as the extended Yule-Walker equations. Let $\underline{\Gamma}_q$ be a (pxp) covariance matrix with elements $\Gamma_{kj} = r_{q+(k-j)}$.

$$\underline{\Gamma}_q \triangleq \begin{bmatrix} r_q & r_{q-1} & \cdots & r_{q-p+1} \\ r_{q+1} & r_q & \cdots & r_{q-p+2} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ r_{q+p-1} & r_{q+p-2} & \cdots & r_q \end{bmatrix} \qquad (6)$$

and define the (px1) vectors by

$$\underline{a}^T = [a_1, a_2, \ldots a_p]$$

$$\underline{R}^T_q = [r_{q+1}, r_{q+2}, \ldots r_{q+p}]$$

then Equation 5 can be written in matrix form as

$$\underline{\Gamma}_q \underline{a} = \underline{R}_q. \qquad (7)$$

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Gersch proved that the non-symmetric Toeplitz matrix $\underline{\Gamma}_q$, q finite, is nonsingular; thus a solution for Equation 7 always exists.

Given a finite set of observations $\{Y_n\}_{n=1}^{N}$, $N \geq p+q$, we estimate the covariance sequence $\{r_k\}$ by

$$
\hat{r}_k = \begin{cases} (\frac{1}{N}) \sum_{n=1}^{N-|k|} Y_n\, Y_{n+|k|} & |k| \leq N-1 \\[2mm] 0 & |k| > N-1. \end{cases} \tag{8}
$$

When the covariances r_k in the matrix $\underline{\Gamma}_q$ and vector $\underline{R}_q$ are replaced by their corresponding estimates using Equation 8, the resulting matrix and vector are denoted by $\hat{\underline{\Gamma}}_q$ and $\hat{\underline{R}}_q$. The extended Y-W equations, Equation 7, can be written in terms of the estimated quantities as

$$
\hat{\underline{\Gamma}}_q\, \hat{\underline{a}} = \hat{\underline{R}}_q. \tag{9}
$$

These equations are used for estimating the AR parameters of an ARMA series.

In subsequent developments of the asymptotic statistical properties of the estimates $\hat{\underline{a}}$ formed using Equations 9 and 8, we use the following vectors and matrices

$$
\underline{R}^T \overset{\Delta}{=} [r_0,\ r_1,\ \ldots,\ r_{p+q}]
$$

$$
\underline{U} \overset{\Delta}{=} \{e^{i(k+j)\lambda} + e^{i(k-j)\lambda}\} \quad k=0,\ 1,\ \ldots,\ p+q;\ j=0,\ 1,\ \ldots,\ p+q
$$

$$\underline{\phi} \overset{\Delta}{=} [0, 0, \ldots, 0]$$

where the dimension of $\underline{\phi}$ will be clear by the context in which it is used.

III. ASYMPTOTIC STATISTICAL PROPERTIES

A. FUNDAMENTAL PROPERTIES AND ASSUMPTIONS

Let $Y = \{Y_n\}_{n=-\infty}^{\infty}$ be a discrete time stochastic process defined on $(\Omega, \mathcal{F}, P)$ satisfying:

ASSUMPTION 1: $Y = \{Y_n\}_{n=-\infty}^{\infty}$ is a real-valued stationary Gaussian process with zero mean and absolutely continuous spectral distribution function with spectral density $\phi(\lambda)$ which is strictly positive and bounded.

Under Assumption 1, it follows by Kromer[5], Theorem 2.1, that there is a real sequence $\{c_j\}_{j=1}^{\infty}$ and a positive real number σ such that the process Y admits an infinite MA representation with spectral density

$$\phi(\lambda) = \frac{\sigma^2}{2\pi} \left| 1 + \sum_{j=1}^{\infty} c_j \, e^{ij\lambda} \right|^2 . \tag{10}$$

ASSUMPTION 2: The infinite sequence $\{c_j\}$ satisfies

$$\sum_{j=1}^{\infty} |c_j| < \infty .$$

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We proceed to establish the asymptotic distribution of the covariance estimates (Equation 8)

THEOREM 1:   Under Assumptions 1 and 2, the elements of the covariance error vector

$$N^{1/2}(\hat{\underline{R}} - \underline{R})$$

are asymptotically jointly multivariate normal with mean zero and covaria structure given by

$$\lim_{N \to \infty} \text{cov} \{N^{1/2}(\hat{\underline{R}} - \underline{R}), N^{1/2}(\hat{\underline{R}} - \underline{R})^T\} = 2\pi \int_{-\pi}^{\pi} \underline{U} \phi^2(\lambda) \, d\lambda. \tag{11}$$

This result was proven by Kromer in Theorem 3.2. Define for later use the p+q+1 x p+q+1 covariance matrix $\underline{Y}$ by

$$\underline{Y} \triangleq 2\pi \int_{-\pi}^{\pi} \underline{U} \phi^2(\lambda) \, d\lambda . \tag{12}$$

The following lemma establishes a convergence in probability result that is fundamental to the proof of the main asymptotic results.

LEMMA 1:   Under Assumptions 1 and 2

$$\hat{\underline{\Gamma}}_q^{-1} \xrightarrow{P} \underline{\Gamma}_q^{-1} \quad \text{as } N \longrightarrow \infty$$

for all finite q.

PROOF: Parzen[4] states, under assumptions satisfied by our Assumptions 1 and 2, that the estimate $\hat{r}_k$ of Equation 8 converges almost surely to r_k as $N \rightarrow \infty$. It follows directly that

$$\hat{\underline{r}}_q \xrightarrow{P} \underline{r}_q \quad \text{as } N \rightarrow \infty$$

for all finite q. Gersch proves that the matrix $\underline{r}_q$ is nonsingular and thus guarantees the existence of $\underline{r}_q^{-1}$. Since the matrix inverse is a continuous function, then on a neighborhood containing $\underline{r}_q$ it follows (Rao[6]) that

$$\hat{\underline{r}}_q^{-1} \xrightarrow{P} \underline{r}_q^{-1} \quad \text{as } N \rightarrow \infty.$$

The next lemma establishes a relationship between a vector of covariance estimates, $\hat{\underline{R}}$, a matrix of the AR parameters $\underline{D}$, and the extended Y-W equations. It is through this relationship, in conjunction with Theorem 1, that we will establish the asymptotic distributional properties of the AR parameter estimates of Equation 9.

LEMMA 2: For any ARMA(p,q) process there is a pxp+q+1 matrix $\underline{D}$ such that

$$\underline{D}(\hat{\underline{R}} - \underline{R}) = \hat{\underline{R}}_q - \hat{\underline{r}}_q \underline{a}$$

where the matrix $\underline{D}$ is comprised of the AR parameters $\{a_j\}_{j=1}^{p}$, ones and zeroes.

This result follows directly from the definition of the matrix $\underline{D}$ and the extended Y-W equations, see Equations 7 and 9. The matrix $\underline{D}$ is defined in the appendix.

LEMMA 3: For an ARMA(p,q) process satisfying Assumptions 1 and 2, the elements of the vector

$$N^{1/2}(\hat{\underline{R}}_q - \hat{\underline{\Gamma}}_q \underline{a})$$

are asymptotically jointly multivariate normal with mean zero and covariance matrix structure given by

$$\lim_{N \to \infty} \text{cov } \{N^{1/2}(\hat{\underline{R}}_q - \hat{\underline{\Gamma}}_q \underline{a}), N^{1/2}(\hat{\underline{R}}_q - \hat{\underline{\Gamma}}_q \underline{a})^T\} = \underline{D} \ \underline{\Psi} \ \underline{D}^T. \tag{13}$$

This result follows directly from Lemma 2 and Theorem 1 with the matrix $\underline{\Psi}$ defined by Equation 12.

B. ASYMPTOTIC PROPERTIES OF THE AR PARAMETER ESTIMATES

We now establish the asymptotic distributional properties for the vector of AR parameter estimate errors $(\hat{\underline{a}} - \underline{a})$. In Lemma 3 we established asymptotic normality and evaluated the asymptotic covariance matrix for the vector $N^{1/2}(\hat{\underline{R}}_q - \hat{\underline{\Gamma}}_q \underline{a})$; in Lemma 1 we proved that the inverse estimated covariance matrix $\hat{\underline{\Gamma}}_q^{-1}$ converges in probability to $\underline{\Gamma}_q^{-1}$ as $N \to \infty$. Using these two results, we prove the following asymptotic distribution result.

THEOREM 2: For AR parameter estimates of Equation 9, under Assumptions 1 and 2, the elements of the error vector $N^{1/2}(\hat{\underline{a}} - \underline{a})$ are asymptotically jointly normal with mean zero and covariance matrix structure given by

$$\lim_{N \to \infty} \text{cov} \{N^{1/2}(\hat{\underline{a}} - \underline{a}), N^{1/2}(\hat{\underline{a}} - \underline{a})^T\} = \underline{\Gamma}_q^{-1} \underline{D} \, \underline{\Psi} \, \underline{D}^T (\underline{\Gamma}_q^{-1})^T. \tag{14}$$

<u>PROOF</u>: By Lemma 3 it suffices to show

$$N^{1/2}(\hat{\underline{a}} - \underline{a}) - N^{1/2}\underline{\Gamma}_q^{-1}(\hat{\underline{R}}_q - \hat{\underline{\Gamma}}_q\underline{a}) \xrightarrow{P} \underline{\phi} \quad \text{as } N \to \infty.$$

Define the px1 vector $\underline{Z}$ by

$$\underline{Z} = \begin{bmatrix} Z_{1,N} \\ Z_{2,N} \\ \vdots \\ Z_{p,N} \end{bmatrix} \stackrel{\Delta}{=} N^{1/2} \, \hat{\underline{\Gamma}}_q^{-1} \, (\hat{\underline{R}}_q - \hat{\underline{\Gamma}}_q\underline{a}) - N^{1/2} \, \underline{\Gamma}_q^{-1} \, (\hat{\underline{R}}_q - \hat{\underline{\Gamma}}_q\underline{a}).$$

Lemma 3 implies that

$$\lim_{N \to \infty} N^{1/2}(\hat{\underline{R}}_q - \hat{\underline{\Gamma}}_q\underline{a}) < \infty$$

and by Lemma 1 it follows that

$$\underline{Z} = N^{1/2}(\hat{\underline{\Gamma}}_q^{-1} - \underline{\Gamma}_q^{-1})(\hat{\underline{R}}_q - \hat{\underline{\Gamma}}_q\underline{a}) \xrightarrow{P} \underline{\phi} \quad \text{as } N \to \infty. \tag{15}$$

On $(\Omega, \mathcal{F}, P)$ define the set $\Lambda_{\epsilon,N}$ for $\epsilon > 0$ and $N > p$ by

$$\Lambda_{\epsilon,N} = \{\omega \in \Omega: |Z_{j,N}| < \epsilon, \; j = 1, 2, \ldots, p\}.$$

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By Equation 15, for every $\alpha \in [0,1]$ there exists a $N^*_{\epsilon,\alpha}$ such that

$$P(\Lambda_{\epsilon,N}) \geq 1 - \alpha \quad \text{for } N \geq N^*_{\epsilon,\alpha}.$$

Since $|Z_{j,N}| < \epsilon$ for $j = 1, 2, \ldots, p$ then the vector $\hat{\underline{\Gamma}}_q^{-1}(\hat{\underline{R}}_q - \hat{\underline{\Gamma}}_q \underline{a})$ exists for all $\omega \in \Lambda_\epsilon$. Hence, by Equation 9, we write

$$\hat{\underline{\Gamma}}_q^{-1}(\hat{\underline{R}}_q - \hat{\underline{\Gamma}}_q \underline{a}) = \hat{\underline{a}} - \underline{a}$$

for all $\omega \in \Lambda_{\epsilon,N}$ and we write $\underline{Z}$ as

$$\underline{Z} = N^{1/2}(\hat{\underline{a}} - \underline{a}) - N^{1/2} \hat{\underline{\Gamma}}_q^{-1} (\hat{\underline{R}}_q - \hat{\underline{\Gamma}}_q \underline{a})$$

for all $\omega \in \Lambda_{\epsilon,N}$. Since the selection of ϵ and α is arbitrary we conclude that

$$N^{1/2}(\hat{\underline{a}} - \underline{a}) - N^{1/2} \underline{\Gamma}_q^{-1} (\hat{\underline{R}}_q - \hat{\underline{\Gamma}}_q \underline{a}) \xrightarrow{P} \underline{\Phi} \text{ as } N \to \infty.$$

We define the matrix $\underline{\theta}$ to be the asymptotic covariance matrix established by Theorem 2.

$$\underline{\theta} = \underline{\Gamma}_q^{-1} \underline{D} \underline{\Psi} \underline{D}^T (\underline{\Gamma}_q^{-1})^T . \tag{16}$$

We now examine the detailed form of the asymptotic covariance matrix $\underline{\theta}$ as a function of the assumed form for $\phi(\lambda)$. First, we use the ARMA(p,q) form (Equation 3) and, secondly, we use the form for the special ARMA case of AR(p) plus noise.

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C. GENERAL ARMA(p,q) CASE

We first examine the matrix $\underline{D} \underline{\Psi} \underline{D}^T$ which by Equation 12 can be written as

$$\underline{D} \underline{\Psi} \underline{D}^T = 2\pi \int_{-\pi}^{\pi} \underline{D} \underline{U} \underline{D}^T \phi^2(\lambda) \, d\lambda. \tag{17}$$

Let $\xi_{k,j}$ be an element of $\underline{D} \underline{U} \underline{D}^T$, let d_{kj} be an element of $\underline{D}$ and let u_{kj} be an element of U. We can write

$$\xi_{k,j} = \sum_{\ell=0}^{p+q} \sum_{m=0}^{p+q} d_{k\ell} u_{\ell m} d_{jm} \qquad k = 1, 2, \ldots, p; \; j = 1, 2, \ldots, p$$

but by definition

$$u_{\ell m} = \exp\left[i(\ell-m)\lambda\right] + \exp\left[i(\ell+m)\lambda\right]$$

thus

$$\xi_{k,j} = \sum_{\ell=0}^{p+q} \sum_{m=0}^{p+q} d_{k\ell} d_{jm} \left\{\exp\left[i(\ell-m)\lambda\right] + \exp\left[i(\ell+m)\lambda\right]\right\}$$

$$= \sum_{\ell=0}^{p+q} d_{k\ell} \exp(i\ell\lambda) \left[\sum_{m=0}^{p+q} d_{jm} \exp(-im\lambda) + \sum_{m=0}^{p+q} d_{jm} \exp(im\lambda)\right]. \tag{18}$$

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Using the structure of the matrix $\underline{D}$ and Equation 2a we write

$$\sum_{\ell=0}^{p+q} d_{k\ell} \exp{(i\ell\lambda)} = \exp{[i(q+k)\lambda]}\ A^p(e^{-i\lambda})$$

$$- 2i \sum_{t=q+k+1}^{p} a_t \sin{(t-q-k)\lambda} \tag{19}$$

the second term of Equation 19 vanishes for $k \geq p-q$. We also write

$$\sum_{m=0}^{p+q} d_{jm} \exp{(-im\lambda)} + \sum_{m=0}^{p+q} d_{jm} \exp{(im\lambda)}$$

$$= \exp{[-i(q+j)\lambda]}\ A^p(e^{i\lambda}) + \exp{[i(q+j)\lambda]}A^p(e^{-i\lambda}). \tag{20}$$

By Equations 18, 19 and 20, after further manipulation, we obtain

$$\xi_{k,j} = A^p(e^{-i\lambda})A^p(e^{i\lambda})\exp{[i(k-j)\lambda]} + A^p(e^{-i\lambda})A^p(e^{-i\lambda})\exp{[i(k+j+2q)\lambda]}$$

$$-4i \sum_{t=q+k+1}^{p} \sum_{s=0}^{p} a_t a_s \sin{[(t-k-q)\lambda]} \cos{[(s-j-q)\lambda]} \tag{21}$$

the third term of Equation 21 vanishes for $k \geq p-q$.

Let $\rho_{k,j}$ be an element of $\underline{D}\ \underline{\Psi}\ \underline{D}^T$, then

$$\rho_{k,j} = 2\pi \int_{-\pi}^{\pi} \xi_{k,j}\ \phi^2(\lambda)\ d\lambda \qquad k = 1, 2, \ldots, p;\ j = 1, 2, \ldots, p$$

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Using Equation 21 we obtain

$$\rho_{k,j} = 2\pi \int_{-\pi}^{\pi} A^P(e^{-i\lambda})A^P(e^{i\lambda})\exp[i(k-j)\lambda]\ \phi^2(\lambda)\ d\lambda$$

$$+ 2\pi \int_{-\pi}^{\pi} A^P(e^{-i\lambda})A^P(e^{-i\lambda})\exp[i(k+j+2q)\lambda]\ \phi^2(\lambda)\ d\lambda \tag{22}$$

$$- 8\pi i \sum_{t=q+k+1}^{p} \sum_{s=0}^{p} a_t a_s \int_{-\pi}^{\pi} \sin[(t-k-q)\lambda]\ \cos[(s-j-q)\lambda]\ \phi^2(\lambda)\ d\lambda$$

the third term of Equation 22 vanishes for $k \geq p-q$. To simplify further evaluation of $\underline{\theta}$ we state the following lemma whose proof can be found in the appendix.

LEMMA 4: For any ARMA(p,q) series satisfying Assumption 1 we have

$$2\pi \int_{-\pi}^{\pi} A^P(e^{-i\lambda})A^P(e^{-i\lambda})\ \exp[i(k+j+2q)\lambda]\ \phi^2(\lambda)\ d\lambda = 0$$

for $k = 1, 2, \ldots , p; j = 1, 2, \ldots , p$.

By Lemma 4 it follows that the contribution of the second term of Equation 22 is zero over the range of k and j; since the integrand of the third term is an odd function in λ, its contribution is also zero over the range of k and j, therefore, we have

$$\rho_{k,j} = 2\pi \int_{-\pi}^{\pi} A^P(e^{-i\lambda})A^P(e^{i\lambda})\ \exp[i(k-j)\lambda]\ \phi^2(\lambda)\ d\lambda. \tag{23}$$

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Using the ARMA spectral density representation for $\phi(\lambda)$, as given by Equation 3, we obtain

$$\rho_{k,j} = \sigma_\eta^2 \int_{-\pi}^{\pi} B^q(e^{i\lambda})B^q(e^{-i\lambda}) \exp[i(k-j)\lambda] \; \phi(\lambda) \; d\lambda. \tag{24}$$

Let $\phi_{MA}(\lambda)$ represent the spectral density for a moving-average process, that is with $B^q(e^{i\lambda})$ defined by Equation 2b, we write

$$\phi_{MA}(\lambda) = \left(\frac{\sigma_\eta^2}{2\pi}\right) B^q(e^{i\lambda})B^q(e^{-i\lambda}) \tag{25}$$

and $\rho_{k,j}$ becomes

$$\rho_{k,j} = 2\pi \int_{-\pi}^{\pi} \phi_{MA}(\lambda) \; \phi(\lambda) \; \exp[i(k-j)\lambda] \; d\lambda. \tag{26}$$

Let $\{c_\ell\}$ represent the covariance sequence of the moving-average process, then by Box and Jenkins[7] we can write

$$c_\ell = \begin{cases} \sigma_\eta^2 \sum_{j=0}^{q-|\ell|} b_j b_{j+|\ell|} & |\ell| \leq q \\ \\ 0 & |\ell| > q \end{cases} \tag{27}$$

where $b_0 = -1$; then using the Fourier transform pairs

$$\phi_{MA}(\lambda) = \frac{1}{2\pi} \sum_{\ell=-\infty}^{\infty} c_\ell \exp(-i\ell\lambda)$$

$$r_k = \int_{-\pi}^{\pi} \phi(\lambda) \exp(ik\lambda) \, d\lambda$$

in Equation 26 we obtain

$$\rho_{k,j} = c_0 r_{k-j} + \sum_{\ell=1}^{q} c_\ell r_{k-j-\ell} + \sum_{\ell=1}^{q} c_\ell r_{k-j+\ell}.$$

Since $\rho_{k,j}$ is an element of $\underline{D} \, \underline{\Psi} \, \underline{D}^T$, then with Γ_ℓ as defined by Equation 6 with $q = \ell$ we have

$$\underline{D} \, \underline{\Psi} \, \underline{D}^T = c_0 \underline{\Gamma}_0 + \sum_{\ell=1}^{q} c_\ell \, [\underline{\Gamma}_\ell + \underline{\Gamma}_\ell^T]. \tag{28}$$

Using Equation 28 in Equation 16 we have a final form for the asymptotic covariance matrix $\underline{\theta}$

$$\underline{\theta} = c_0 \underline{\Gamma}_q^{-1} \underline{\Gamma}_0 (\underline{\Gamma}_q^{-1})^T + \sum_{\ell=1}^{q} c_\ell \underline{\Gamma}_q^{-1} [\underline{\Gamma}_\ell + \underline{\Gamma}_\ell^T](\underline{\Gamma}_q^{-1})^T. \tag{29}$$

This form for the asymptotic covariance matrix is analogous to that obtained by Gersch and Sakai[8]. We note that $\underline{\theta}$ contains two main terms; the first term is similar to the covariance result obtained for estimating the AR parameters

of an AR process. In fact, if we let q=0, this term reduces to the classic result obtained by Mann and Wald[9]. When q is non-zero the second term provides the contribution to the covariance due to the presence of the moving-average process. We also note that both $\underline{\Gamma}_0$ and $\sum_{\ell=1}^{q} [\underline{\Gamma}_\ell + \underline{\Gamma}_\ell^T]$ are real symmetric matrices.

D.   SPECIAL CASE: AUTOREGRESSIVE PROCESS PLUS NOISE

In the previous section we developed the asymptotic distribution and form of the asymptotic covariance matrix for estimates of the AR parameters from observations of an ARMA series. We now consider a special ARMA series. When the observed series consists of an AR(p) process plus i.i.d. noise, Walker[2] and Pagano[3] showed that an ARMA(p,p) model can be used to represent the observed series. Thus, let q=p and the results of the previous section can be applied to this case.

Let $\{Z_n\}$ be a discrete parameter time series generated by the sum of a zero mean autoregressive process of known order and an i.i.d. noise sequence. We write

$$Z_n = Y_n + \epsilon_n \qquad (30)$$

where $\{Y_n\}$ is an AR(p) process generated by Equation 1 with q=0 and $\{\epsilon_n\}$ is assumed to be an i.i.d. noise sequence with mean zero and variance σ_ϵ^2. Given

a finite set of observations $\{Z_n\}_{n=1}^N$, $N > p$, we estimate the AR parameters using the extended Y-W equations. Since the process Z can be modeled as an ARMA(p,p) series, the result of Theorem 2 applies to this case and only the detailed form of the asymptotic covariance matrix $\underline{\theta}$ as given by Equation 16 need be evaluated. As in the previous case we first examine the structure of the matrix $\underline{D} \; \underline{\Psi} \; \underline{D}^T$, whose elements, ρ_{kj}, are given by Equation 23 and which is repeated below

$$\rho_{k,j} = 2\pi \int_{-\pi}^{\pi} A^P(e^{i\lambda}) A^P(e^{-i\lambda}) \exp[i(k-j)\lambda] \; \phi^2(\lambda) \; d\lambda. \tag{31}$$

For the AR plus noise process, Z, we can express the spectral density as

$$\phi(\lambda) = \frac{\sigma_\epsilon^2}{2\pi} + \frac{\sigma_\eta^2}{2\pi \; A^P(e^{i\lambda}) \; A^P(e^{-i\lambda})} \; . \tag{32}$$

Substituting this expression into Equation 31 we obtain

$$\rho_{k,j} = \frac{\sigma_\eta^2}{2\pi} \int_{-\pi}^{\pi} \frac{\sigma_\eta^2}{A^P(e^{i\lambda})A^P(e^{-i\lambda})} \; \exp[i(k-j)\lambda] \; d\lambda$$

$$+ \frac{2\sigma_\epsilon^2 \sigma_\eta^2}{2\pi} \int_{-\pi}^{\pi} \exp[i(k-j)\lambda] \; d\lambda \tag{33}$$

$$+ \frac{\sigma_\epsilon^4}{2\pi} \int_{-\pi}^{\pi} A^P(e^{i\lambda})A^P(e^{-i\lambda}) \; \exp[i(k-j)\lambda] \; d\lambda.$$

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Performing the integrations we obtain

$$\rho_{k,j} = \sigma_\eta^2 r(k-j) + \sigma_\epsilon^2 \sigma_\eta^2 \delta(k-j) + \sigma_\epsilon^4 \sum_{m=0}^{p-|k-j|} a_m a_{m+|k-j|}$$

$$k = 1, 2, \ldots, p; \ j = 1, 2, \ldots, p.$$

Since $\rho_{k,j}$ is an element of $D \underline{\Psi} D^T$ we can write

$$D \underline{\Psi} D^T = \sigma_\eta^2 \underline{\Gamma}_0 + \sigma_\epsilon^2 \sigma_\eta^2 \underline{I} + \sigma_\epsilon^4 \underline{Q} , \tag{34}$$

where the pxp matrix $\underline{Q}$ is defined by

$$\underline{Q} \triangleq \begin{bmatrix} \sum_{m=0}^{p} a_m^2 & \sum_{m=0}^{p-1} a_m a_{m+1} & \cdots & \sum_{m=0}^{1} a_m a_{m+(p-1)} \\ \sum_{m=0}^{p-1} a_m a_{m+1} & & & \\ \vdots & & & \vdots \\ \sum_{m=0}^{1} a_m a_{m+(p-1)} & & & \sum_{m=0}^{p} a_m^2 \end{bmatrix} .$$

Using Equation 34 in Equation 16 for the AR plus noise process we obtain a final form for the asymptotic covariance matrix $\underline{\theta}$

$$\underline{\theta} = \sigma_\eta^2 \underline{\Gamma}_p^{-1} \underline{\Gamma}_0 (\underline{\Gamma}_p^{-1})^T + \sigma_\epsilon^2 \sigma_\eta^2 \underline{\Gamma}_p^{-1} (\underline{\Gamma}_p^{-1})^T + \sigma_\epsilon^4 \underline{\Gamma}_p^{-1} \underline{Q} (\underline{\Gamma}_p^{-1})^T . \tag{35}$$

The first term of Equation 35 represents the contribution to the covariance due to estimating the AR parameters of a non-noise corrupted process using the extended Y-W equations. If there were no additive noise present, this would be the only term in Equation 35. The second and third terms represent the contributions due to the presence of the additive noise.

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The structure of the (pxp+q+1) matrix $\underline{D}$ is most easily explained by defining two submatrices $\underline{D}_1$ and $\underline{D}_2$. Define $\underline{D}$ in terms of $\underline{D}_1$ and $\underline{D}_2$ by:

$$
\underline{D} \;=\;
\left.
\begin{bmatrix}
 & \underline{D}_1 & \\[2em]
\hline
 & \underline{D}_2 & \\[3em]
\end{bmatrix}
\right.
\begin{array}{l}
j=1 \\[2em]
j=p-q-1 \\
j=p-q \\[3em]
j=p
\end{array}
$$

$$\underbrace{}_{p+q+1}$$

The structure of $\underline{D}_1$ is dependent on the relation between p and q and must be defined separately for two cases:

For $q \geq [p/2]$ the jth row of $\underline{D}_1$ is given by

$$
-a_{q+j},\; -(a_{q+j-1} + a_{q+j+1}),\; -(a_{q+j-2} + a_{q+j+2}),\; \ldots ,
$$

$$
-(a_{q+j-(p-q-j)} + a_{q+j+(p-q-j)}),\; -(a_{q-(p-q-2j+1)}),\ldots,\; -a_1,\; 1,\; 0,\; \underbrace{\ldots,\; 0.}_{p-j} \tag{A1}
$$
$$(*)$$

For $q < [p/2]$ the jth row of $\underline{D}_1$ ($j = 1, 2, \ldots , [p/2]-q$) is given by

$$
-a_{q+j},\; -(a_{q+j-1} + a_{q+j+1}),\; -(a_{q+j-2} + a_{q+j+2}),\; \ldots ,
$$

$$
-(a_1 + a_{2q+2j-1}),\; -(a_{2q+2j} - 1),\; -a_{2q+2j+1},\; \ldots,\; -a_p,\; 0,\; \underbrace{\ldots,\; 0}_{2q+j}
$$

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the remaining rows are given by Equation A1.

The submatrix $\underline{D}_2$ is defined by:

$$
\begin{bmatrix}
-a_p & -a_{p-1} & \cdots & -a_1 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\
0 & -a_p & -a_{p-1} & \cdots & -a_1 & 1 & 0 & \cdots & & 0 \\
\cdot & & & & & & & & & \cdot \\
\cdot & & & & & & & & & \cdot \\
\cdot & & & & & & & & & \cdot \\
0 & \cdot & \cdot & \cdot & 0 & -a_p & -a_{p-1} & \cdots & -a_1 & 1
\end{bmatrix}
$$

Proof of Lemma 4: Define $\beta_{k,j}$ by

$$
\beta_{k,j} \overset{\Delta}{=} 2\pi \int_{-\pi}^{\pi} A^p(e^{i\lambda}) A^p(e^{-i\lambda}) \exp\,[i(k + j + 2q)\lambda]\, \phi^2(\lambda)d\lambda. \tag{A2}
$$

Using $\phi(\lambda)$, as given by Equation 3, in Equation A2 we get

$$
\beta_{k,j} = \frac{\sigma_n^4}{2\pi} \int_{-\pi}^{\pi} \frac{[B^q(e^{i\lambda})\, B^q(e^{-i\lambda})]^2}{[A^p(e^{i\lambda})]^2} \exp\,[i(k + j + 2q)\lambda]d\lambda. \tag{A3}
$$

Since the zeroes of $A^p(Z)$ lie outside of the unit circle on the complex Z-plane, then $1/A^p(Z)$ is analytic inside and on the unit circle, thus we can write

$$
1/[A^p(e^{i\lambda})]^2 = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} c_{m_1} c_{m_2} \exp\,[i(m_1 + m_2)\lambda] \tag{A4}
$$

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and we also have

$$[B^q(e^{i\lambda})\ B^q(e^{-i\lambda})]^2 = \sum_{\ell_1=0}^{q} \cdots \sum_{\ell_4=0}^{q} b_{\ell_1} b_{\ell_2} b_{\ell_3} b_{\ell_4}$$

$$\exp\{i(\ell_1 - \ell_2 + \ell_3 - \ell_4)\lambda\}\ . \tag{A5}$$

Using Equations A4 and A5, we rewrite Equation A3

$$\beta_{k,j} = \frac{\sigma_\eta^4}{2\pi}\int_{-\pi}^{\pi} \sum_{\ell_1=0}^{q} \cdots \sum_{\ell_4=0}^{q} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} b_{\ell_1} b_{\ell_2} b_{\ell_3} b_{\ell_4} c_{m_1} c_{m_2}$$

$$\exp[i(m_1 + m_2 + d)\lambda]d\lambda$$

where

$d = k + j + 2q + \ell_1 - \ell_2 + \ell_3 - \ell_4$. Since $d > 0$, it follows that $\beta_{k,j} = 0$ for $k = 1, 2, \ldots, p$; $j = 1, 2, \ldots, p$.

END

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