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Technical Memorandum

SIGNAL-TO-NOISE RATIO REQUIREMENTS FOR GREATEST-OF DEVICE FOLLOWED BY INTEGRATOR

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ABSTRACT

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Derivations and programs are presented for evaluation of the required input signal-to-noise ratio for specified output performance of a Greatest-Of-device followed by an integrator, as a function of: N, the number of channels in the Greatest-Of; M, the number of independent samples in the integrator; P_F, the false alarm probability; and P_D, the detection probability. Several numerical examples are presented, for the special case of $P_D = 0.5$.

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ADMINISTRATIVE INFORMATION

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INTRODUCTION

In an effort to minimize equipment, searching, or display requirements, a number, N, of channels of data are occasionally represented, instead, by their largest member. Hopefully in this fashion, detection of signal presence in a particular channel will still be accomplished reliably because the largest member will exceed a threshold. However, for $N > 1$, the threshold must be raised in order to maintain an acceptable false alarm probability; this effect makes signal detection less likely. Also, for low signal-to-noise ratio in one channel at the input to the Greatest-Of device, virtually a random selection takes place in the Greatest-Of device; whether the integration that follows can make up for this non-linear action is not clear.

Here we will derive the equations that characterize performance of the Greatest-Of device followed by an integrator, in terms of detection and false alarm probabilities. We also will present BASIC programs for the HP9830A calculator so that the reader can investigate numerical examples of his own interest. Some related past work is cited in reference 1.

DERIVATIONS

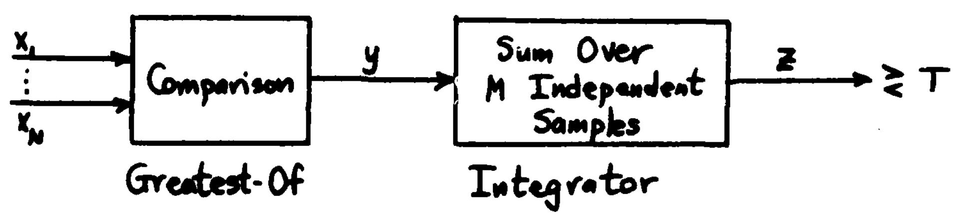

Figure 1. Greatest-Of Device Followed by Integrator

The system of interest is depicted in figure 1. Input channels $x_1, \ldots x_N$ are statistically independent and contain either (a) no signal in any channel, or (b) signal in one channel only. The probability densities of x under these two hypotheses are denoted by $p_0(x)$ and

$p_1(x)$ respectively. The integrator in figure 1 sums over M independent samples and subjects the output to a comparison with a threshold T.

The Greatest-Of device output is

$$y = \max(x_1, \ldots, x_N). \tag{1}$$

Then the cumulative distribution of y for no signal present is

$$\begin{aligned} Q_0(Y) &= \text{Prob}(y < Y \mid \text{no signal}) \\ &= \text{Prob}(x_1, \ldots, x_N < Y \mid \text{no signal}) = P_0^N(Y), \end{aligned} \tag{2}$$

where $P_0(\ )$ is the cumulative distribution corresponding to density $p_0(\ )$. Therefore, the density of y for no signal present is

$$q_0(y) = \frac{d}{dy} Q_0(y) = N\, P_0^{N-1}(y)\, p_0(y). \tag{3}$$

In a similar fashion, the distribution of y for a signal present is

$$\begin{aligned} Q_1(Y) &= \text{Prob}(y < Y \mid \text{signal}) \\ &= \text{Prob}(x_1, \ldots, x_N < Y \mid \text{signal}) = P_0^{N-1}(Y) P_1(Y) \end{aligned} \tag{4}$$

since only one channel contains a signal. The density of y is

$$q_1(y) = \frac{d}{dy} Q_1(y) = (N-1) P_0^{N-2}(y) p_0(y) P_1(y) + P_0^{N-1}(y) p_1(y). \tag{5}$$

In order to evaluate the statistics of the integrator output z, we assume that $M \gg 1$, thereby making z approximately Gaussian. Since

$$z = \sum_{k=1}^{M} y_k , \tag{6}$$

we have

$$\mu_z = E\{z\} = M \mu_y \quad , \quad \sigma_z^2 = M \sigma_y^2 . \tag{7}$$

Therefore the densities of z, for no signal and signal respectively, are

$$r_0(z) = \frac{1}{\sigma_{z0}}\phi\left(\frac{z-\mu_{z0}}{\sigma_{z0}}\right), \quad r_1(z) = \frac{1}{\sigma_{z1}}\phi\left(\frac{z-\mu_{z1}}{\sigma_{z1}}\right), \tag{8}$$

where

$$\phi(x) \equiv (2\pi)^{-1/2}\exp(-x^2/2), \tag{9}$$

and where we define for future use

$$\Phi(x) \equiv \int_{-\infty}^{x} dt\, \phi(t). \tag{10}$$

The probability of detection of the system in figure 1 is given by

$$P_D = \int_T^{\infty} dz\, r_1(z) = \Phi\left(\frac{\mu_{z1}-T}{\sigma_{z1}}\right) \tag{11}$$

using (8) and (10). The false alarm probability is

$$P_F = \int_T^{\infty} dz\, r_0(z) = \Phi\left(\frac{\mu_{z0}-T}{\sigma_{z0}}\right). \tag{12}$$

We now specialize to the case where we desire

$$P_D = 0.5. \tag{13}$$

In this case, (11) dictates that the threshold be set at

$$T = \mu_{z1} \text{ for } P_D = 0.5. \tag{14}$$

Then the false alarm probability (12) becomes

$$P_F = \Phi\left(\frac{\mu_{z0}-\mu_{z1}}{\sigma_{z0}}\right) \text{ for } P_D = 0.5. \tag{15}$$

For a specified performance (i.e., specified P_F and P_D), (15) can be solved for

$$\frac{\mu_{z1}-\mu_{z0}}{\sigma_{z0}} = -\Phi^{-1}(P_F) \quad \text{for } P_D = 0.5. \tag{16}$$

The left side of (16) is the (linear) deflection criterion at the output of figure 1.

In order to complete the analysis, we must now relate μ_{z0}, μ_{z1}, and σ_{z0} to the statistics of y given in (3) and (5). We have

$$\mu_{y0} = \int dy\, y\, q_0(y) = N \int dy\, y\, p_0(y) P_0^{N-1}(y), \tag{17}$$

and

$$\mu_{y1} = \int dy\, y\, q_1(y) = \int dy\, y \left[(N-1) P_0^{N-2}(y) p_0(y) P_1(y) + P_0^{N-1}(y) p_1(y)\right]. \tag{18}$$

At this point, we make the assumption that the density functions p_0 and p_1 of the inputs are Gaussian, as they might be, for example, if the variables $x_1, \ldots, x_N$ themselves were sums of a number of independent samples. That is,

$$p_0(x) = \frac{1}{\sigma}\phi\left(\frac{x-m_0}{\sigma}\right), \quad p_1(x) = \frac{1}{\sigma}\phi\left(\frac{x-m_1}{\sigma}\right), \tag{19}$$

which are depicted in figure 2. The standard deviations

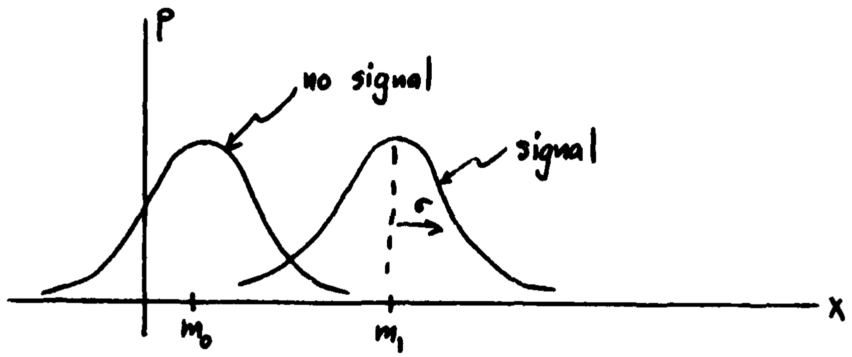

Figure 2. Density Functions of Input

on both hypotheses have been assumed equal; this could be easily generalized if desired.

Substitution of (19) in (17) yields (after some labor)

$$\mu_{y_0} = m_0 + \sigma a_N \tag{20}$$

where

$$a_N = N \int dx\, x\, \phi(x)\, \Phi^{N-1}(x). \tag{21}$$

The integral in (21) is a function only of N and need be numerically evaluated only once*; as special cases we have

$$a_1 = 0,\ a_2 = \frac{1}{\sqrt{\pi}},\ a_3 = \frac{1.5}{\sqrt{\pi}}. \tag{22}$$

Substitution of (19) in (18) yields

$$\mu_{y_1} = m_0 + \sigma c_N(r), \tag{23}$$

* A BASIC program for the HP9830A calculator for the evaluation of a_N is presented in appendix A.

where

$$C_N(r) = (N-1)\int dx\, x\, \phi(x)\, \Phi^{N-2}(x)\, \Phi(x-r) + \int dx\, x\, \Phi^{N-1}(x)\, \phi(x-r), \tag{24}$$

and where we have defined

$$r = \frac{m_1 - m_0}{\sigma}, \tag{25}$$

which is a (linear) deflection criterion at the system input. As special cases of (24), we have

$$C_1(r) = r, \quad C_2(r) = \pi^{-1/2} e^{-r^2/4} + r\,\Phi\left(\frac{r}{\sqrt{2}}\right), \quad C_N(0) = a_N. \tag{26}$$

Combining (7), (20), and (23), there follows

$$\mu_{z1} - \mu_{z0} = M\sigma\left[C_N(r) - a_N\right]. \tag{27}$$

Finally, since from (7)

$$\sigma_{z0}^2 = M\,\sigma_{y0}^2, \tag{28}$$

we must evaluate

$$\sigma_{y0}^2 = E\{y^2 \mid \text{no signal}\} - \mu_{y0}^2. \tag{29}$$

But using (3), (19), and (10), we find

$$E\{y^2 \mid \text{no signal}\} = \int dy\, y^2\, q_0(y)$$

$$= m_0^2 + 2m_0\sigma a_N + \sigma^2 b_N \tag{30}$$

(after some reductions) where

$$b_N = N\int dx\ x^2 \phi(x)\, \Phi^{N-1}(x). \tag{31}$$

This integral is a function only of N and need be numerically evaluated once once*; as special cases, we have

$$b_1 = 1, \quad b_2 = 1. \tag{32}$$

Combining (29), (30), and (20), there follows

$$\sigma_{yo}^2 = \sigma^2\left(b_N - a_N^2\right). \tag{33}$$

Then finally, the left side of (16) can be expressed (by use of (27), (28), and (33)) as

$$\frac{\mu_{21} - \mu_{20}}{\sigma_{20}} = \sqrt{M}\ \frac{c_N(r) - a_N}{\sqrt{b_N - a_N^2}}. \tag{34}$$

Employing this relation in (16) and solving, we obtain the form

$$c_N(r) - a_N = -\Phi^{-1}(P_F)\sqrt{\frac{b_N - a_N^2}{M}} \quad \text{for } P_D = 0.5. \tag{35}$$

* A program for the evaluation of b_N is presented in appendix A.

This equation gives an explicit expression for the minimum deflection r (defined in (25)) required for specified performance $P_F, P_D = .5$, as a function of N and M. It requires the numerical evaluation of a_N, b_N, and $c_N(r)$ as given in (21), (31), and (24) respectively. The left side of (35) can be re-arranged into the form

$$c_N(r) - a_N = \int dx\, x\, \Phi^{N-2}(x)\left[(N-1)\,\phi(x)\,\Phi(x-r) + \Phi(x)\,\phi(x-r) - N\,\phi(x)\,\Phi(x)\right], \tag{36}$$

and has been adopted for use here.

A short table for the constants a_N and b_N is given below.

N	a_N	b_N	$\sqrt{b_N - a_N^2}$
1	0	1	1
2	0.564190	1	.825645
3	0.846284	1.275664	.747975
4	1.029375	1.551329	.701224
5	1.162964	1.800020	.668980
6	1.267206	2.021739	.644924
7	1.352178	2.220304	.626033
8	1.423600	2.399535	.610653

Table 1. Constants a_N and b_N in (21) and (31).

EXAMPLES

For the example of*

$$M = 15000, \qquad P_F = 10^{-3} \text{ and } 10^{-6}, \tag{37}$$

a tabulation of the right side of (35) follows. The numerical

*The inverse Φ-function of (10) is tabulated in reference 10, pages 976-7.

N	$-\Phi^{-1}(P_F)\sqrt{\frac{b_N - a_N^2}{M}}$	
	$P_F = 10^{-3}$	$P_F = 10^{-6}$
1	.02523	.03881
2	.02083	.03205
3	.01887	.02903
4	.01769	.02722
5	.01688	.02596
6	.01627	.02503
7	.01580	.02430
8	.01541	.02370

Table 2. Right-Hand Side of (35)

problem is now to determine r from (36) such that (35) is satisfied. A program for the evaluation of the integral in (36) is presented in appendix B. The required values of r for the example of (37) is presented in Table 3.

N	Required r	
	$P_F = 10^{-3}$	$P_F = 10^{-6}$
1	.0252	.0388
2	.0412	.0630
3	.0553	.0841
4	.0683	.1033
5	.0805	.1210
6	.0921	.1377
7	.1032	.1534
8	.1137	.1683

Table 3. Required Values of r for M = 15000, P_D = .5

A plot of the required SNR is given in figure 3, where the conversion to dB is via

$$20 \log r = 20 \log \frac{m_1 - m_0}{\sigma}, \qquad (38)$$

since the inputs to the Greatest-Of device are voltages. It is observed, for example, that 13 dB additional signal-to-noise ratio is required at the Greatest-Of input for 8 channels rather than 1. The dependence on N in figure 3 is well approximated by 14.3 log N. There is a 3.4 to 3.75 dB difference in the two curves in figure 3.

To ascertain how the curve in figure 3 depends on M, a plot for M = 1000 and 15000 is presented in figure 4 for $P_F = 10^{-3}$. The required values of r are given in table 4.

N	Required r for $P_F = 10^{-3}$
1	.0977
2	.1546
3	.2017
4	.2426
5	.2788
6	.3116
7	.3414
8	.3688

Table 4. Required Values of r for $M = 1000$, $P_D = 0.5$

The plot in figure 4 for M = 1000 is well approximated by the dependence 13 log N. The difference in the two curves is 10 log [illegible] = 11.8 dB at N = 1, and decreases to 10.2 dB at N =8.

A comparison of the required signal-to-noise ratio at N=8 versus that at N=1 is presented in table 5 and figure 5 as a function of M from* M=10 to 15000. Thus, less than 5 dB difference exists at M = 10.

M	Right-hand side of (35) for N=1	Right-hand side of (35) for N=8	Required r for N=8
10	.9772	.5967	1.704
30	.5642	.3445	1.240
100	.3090	.1887	.8513
300	.1784	.1090	.5831
1000	.09772	.05967	.3688
4000	.04886	.02984	.2064
15000	.02523	.01541	.1137

Table 5. Required Signal-to-Noise Ratio for $P_D = .5, P_F = 10^{-3}, N=8$

* The Gaussian approximation for z in figure 1 is being stretched at the left end of figure 5.

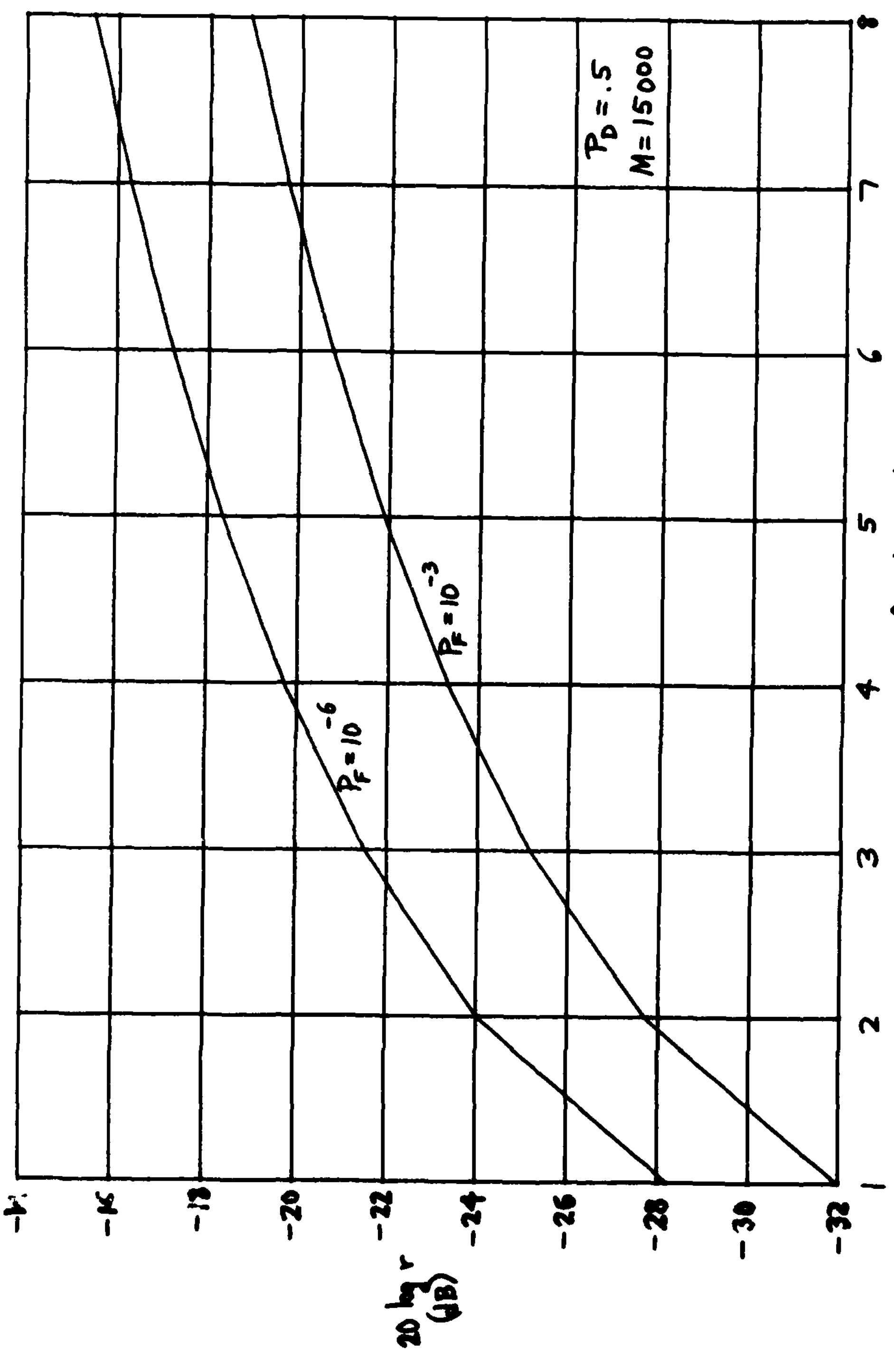

Figure 3. Required Input Signal-to-Noise Ratio; Fixed M

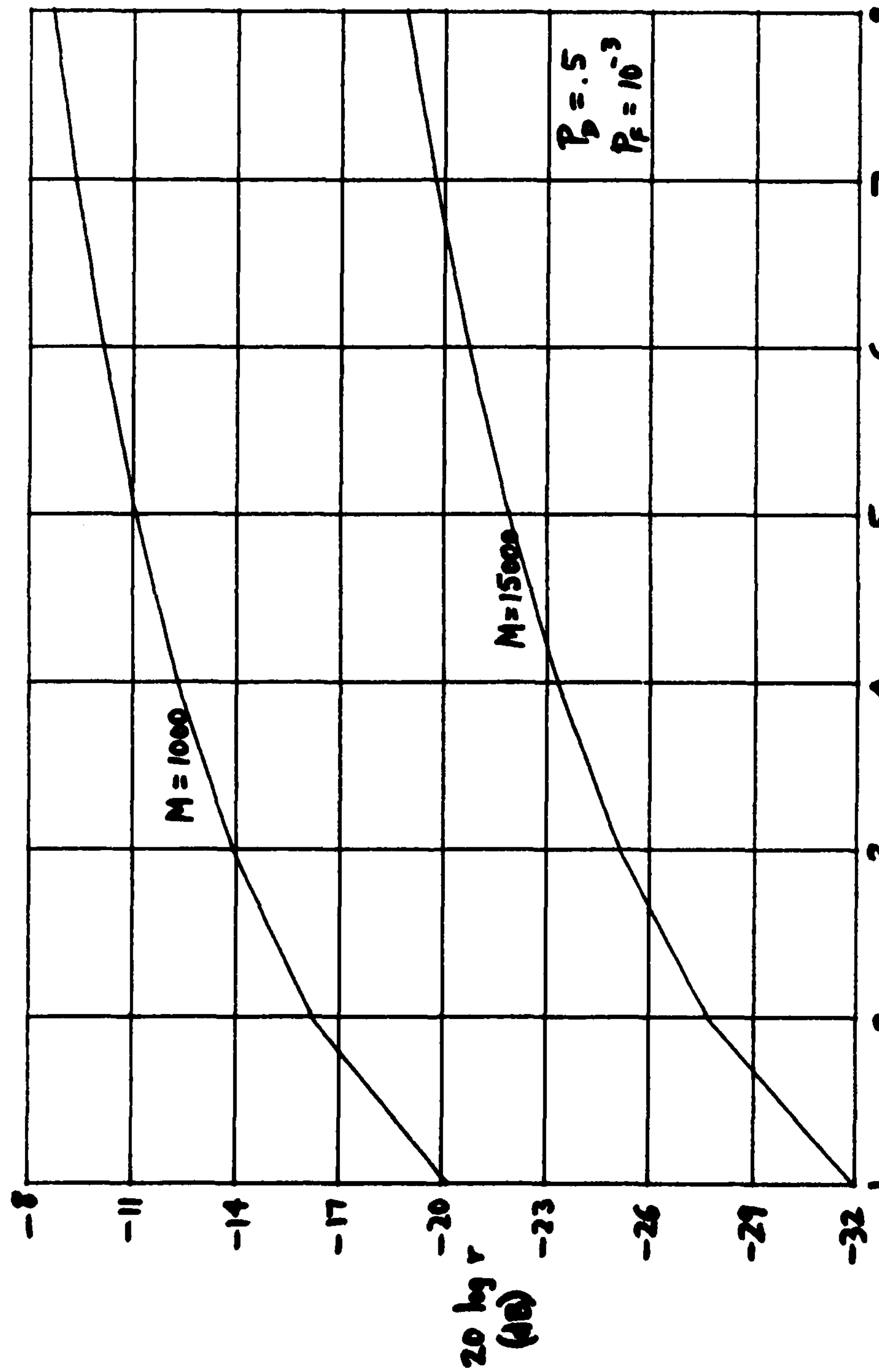

Figure 4. Required Input Signal-to-Noise Ratio; Fixed P_F

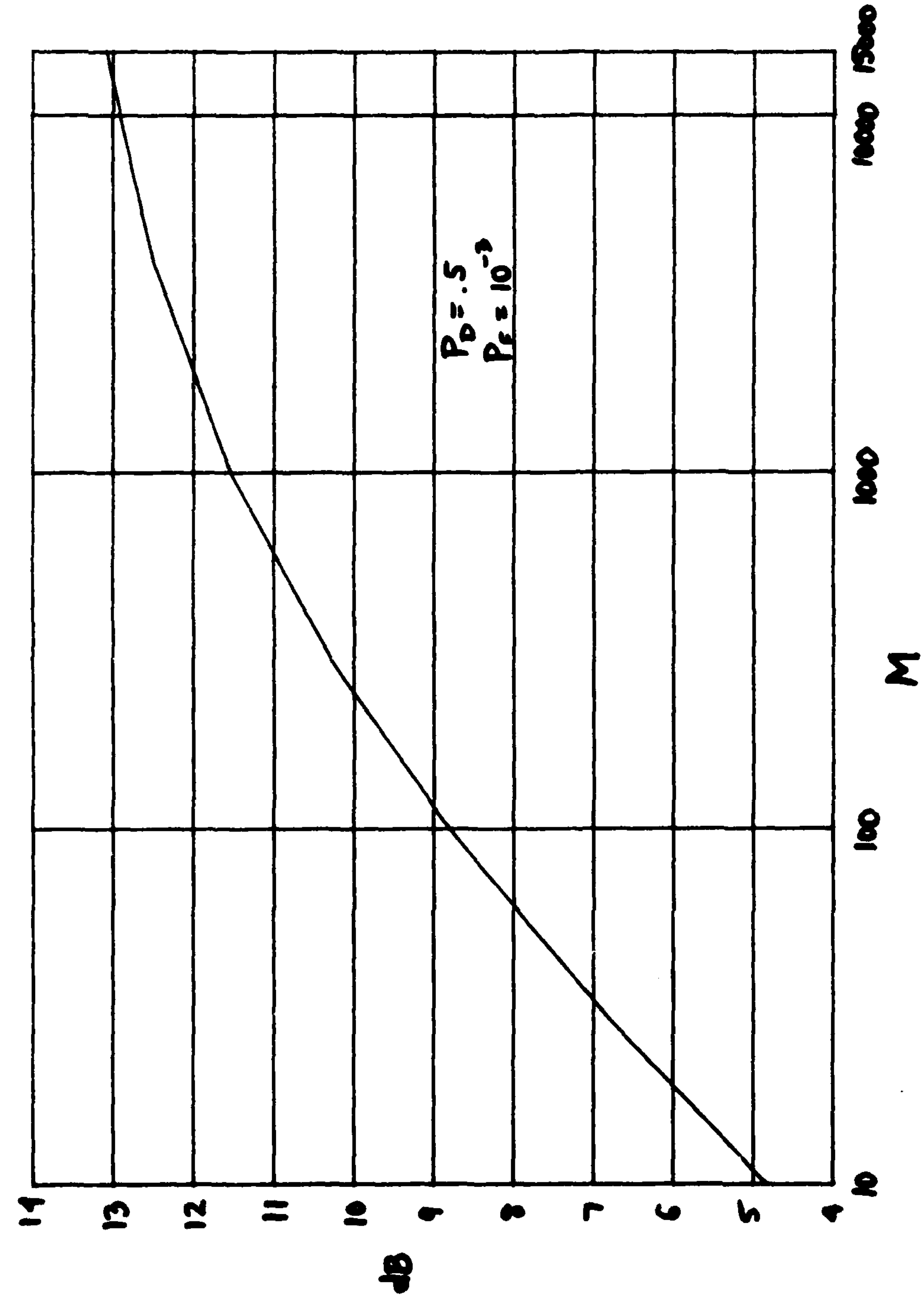

Figure 5. Difference in Signal-to-Noise Ratios Required at N=8 vs. N=1

REFERENCES

1. "Loss of S/N Ratio in Peak Detecting N Channels of Data," ASW Technical Note, Hughes Aircraft Co., Fullerton, Calif., 12 Feb. 1968.

2. Handbook of Mathematical Functions, U.S. Dept. Comm., Nat. Bur. Std., App. Math. Ser. 55, June 1964.

APPENDIX A

Program for (21) and (31)

The program below computes a_N in (21). In order to compute c_N in (31), line 230 is changed by replacing the first X on the right side by X↑2 . Line 10 is for inputting N; lines 20-220 is Simpson's rule for integration with automatic halving; line 230 is the integrand of (21); and lines 240-340 are for Φ of (10).

```
10 N=8
20 C1=N/SQR(2*PI)
30 S1=-8
40 S2=8
50 REM    INTEGRAL OF S(X) OVER (S1,S2)
60 PRINT
70 PRINT  N ="N
80 S3=(FNS(S1)+FNS(S2))*0.5
90 S4=0
100 S5=2
110 S6=(S2-S1)*0.5
120 S7=2/3
130 FOR S8=1 TO S5-1 STEP 2
140 S4=S4+FNS(S1+S6*S8)
150 NEXT S8
160 WRITE (15,170)(S3+2*S4)*S6*S7,S5
170 FORMAT E20.11,F10.0
180 S3=S3+S4
190 S4=0
200 S5=S5*2
210 S6=S6*0.5
220 GOTO 130
230 DEF FNS(X)=C1*X*EXP(-0.5*X↑2)*FNP(X)↑(N-1)
240 DEF FNP(P0)
250 P=ABS(P0)
260 IF P<7 THEN 290
270 P=0
280 GOTO 320
290 P=1/(1+0.2316419*P)
300 P=P*(0.31938153-P*(0.3565[illegible]82-P*(1.781477937-P*
                         (1.[illegible]1255978-P*1.330274431))))
310 P=P*EXP(-0.5*P0↑2)/SQR(2*[illegible])
320 IF P0<0 THEN 340
330 P=1-P
340 RETURN P
350 END
```

APPENDIX B.

Program for (36)

The program below computes $C_N(r)$ in (36). Lines 10-20 are for inputting r and N; lines 30-230 is Simpson's rule; lines 240-330 is the integrand of (36); and lines 340-440 are for Φ of (10). See lines 240-340 in appendix A for Φ.

```
10 R=0.1137
20 N=8
30 C1=1/SQR(2*PI)
40 S1=-8
50 S2=8
60 REM    INTEGRAL OF S(X) OVER (S1,S2)
70 PRINT
80 PRINT "N ="N,"R ="R
90 S3=(FNS(S1)+FNS(S2))*0.5
100 S4=0
110 S5=2
120 S6=(S2-S1)*0.5
130 S7=2/3
140 FOR S8=1 TO S5-1 STEP 2
150 S4=S4+FNS(S1+S6*S8)
160 NEXT S8
170 WRITE (15,180)(S3+2*S4)*S6*S7,S5
180 FORMAT E20.11,F10.0
190 S3=S3+S4
200 S4=0
210 S5=S5*2
220 S6=S6*0.5
230 GOTO 140
240 DEF FNS(X)
250 T1=FNP(X)
260 T2=EXP(-0.5*X↑2)
270 T3=X-R
280 IF N=2 THEN 310
290 T4=T1↑(N-2)
300 GOTO 320
310 T4=1
320 S=C1*X*T4*((N-1)*T2*FNP(T3)+T1*EXP(-0.5*T3↑2)-N*T2*T1)
330 RETURN S
```