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# PROCEEDINGS OF THE TWENTY-SEVENTH CONFERENCE ON THE DESIGN OF EXPERIMENTS IN ARMY RESEARCH DEVELOPMENT AND TESTING



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# THE GEORGE WASHINGTON UNIVERSITY School of Engineering and Applied Science Institute for Reliability and Risk Analysis

Abstract of Serial GWU/IRRA/TR-81/4 21 December 1981

THE U.S. ARMY (BRL'S) KINETIC ENERGY PENETRATOR PROBLEM: ESTIMATING THE PROBABILITY OF RESPONSE FOR A GIVEN STIMULUS

> Thomas A. Mazzuchi Nozer D. Singpurwalla

The crew compartment of an army vehicle is protected by an armor plate. It is desired to test the strength of this armor plate in order to assess its appropriateness for use in the vehicle.

A specimen of the plate is taken and projectiles are fired at different points on the plate at different striking velocities. If a projectile penetrates the armor it is said to have defeated the armor. Our goal is to determine the relationship between the striking velocity and the probability of penetration. Due to the expensive nature of all items involved, this goal must be achieved with a minimum amount of testing. A Bayesian approach for solving this problem is presented here and illustrated using some real data.

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### THE GEORGE WASHINGTON UNIVERSITY School of Engineering and Applied Science institute for Reliability and Risk Analysis

THE U.S. ARMY (BRL'S) KINETIC ENERGY PENETRATOR PROBLEM: ESTIMATING THE PROBABILITY OF RESPONSE FOR A GIVEN STIMULUS

> Thomas A. Mazzuchi Nozer D. Singpurwalla

### 1. STATEMENT OF THE PROBLEM

The following statement of the problem is based on our several discussions with Dr. Robert L. Launer of the Army Research Office, Research Triangle Park, North Carolina, and Dr. J. Richard Moore of the Ballistic Research Laboratory (BRL), Aberdeen Proving Ground, Maryland.

The crew compartment of an army vehicle is protected by a certain kind of material which we will refer to as an "armor plate." It is desired to test the strength of this armor plate so that we may be able to assess its appropriateness for use on the vehicle.

In order to do this, a  $10' \times 10'$  specimen of the armor plate is taken, and a projectile is fired from a gun which is aimed at different points on the plate. In Figure 1.1 below, we indicate a possible firing pattern according to which the gun is aimed.

Typically, the distance between the muzzle of the gun and the target is about 200 meters, and the velocity of the projectile, measured

ŀ	•	-10'-		+
T	0	0	0	
10'	0	0	0	
	0	0	0	

Figure 1.1--Illustration of a firing pattern of a gun.

between two conveniently located points between the gun and the target is about 5000 feet per second.

The projectile is known as the "penetrator," and the outcome of each firing is described by a binary variable which takes the value 1 if the penetrator defeats the target, and the value 0 if the penetrator fails to defeat the target. The penetrator induces a stress on the armor; the stress is a function of two quantities, the "striking velocity" and the "angle of fire." The striking velocity, also known as the "stimulus," is the velocity with which the penetrator strikes the armor, whereas the angle of fire  $\theta$  (indicated in Figure 1.2 below) is the amount by which the armor plate is tilted.



Figure 1.2--Illustration of the angle of fire.

Both the armor specimen and the penetrator are very expensive and thus the testing has to be kept to a bare minimum. One strategy that has been adopted is to fix the angle of fire, say at  $\theta^*$ , and then to fire the penetrator at different striking velocities. After each firing, a record is made of whether the penetrator defeated the target or not. It is assumed that the striking velocity can be measured without any error.

### 2. GOALS, OBJECTIVES, AND SOME COMMENTS ON CURRENT APPROACHES

1 Miles Constraint

Given that our goal is to be able to assess the appropriateness of the armor plate for use on a vehicle, our objective should be to estimate the relationship between the striking velocity (the stimulus) and the probability of penetration (a response of 1). This is illustrated in Figure 2.1, wherein it is assumed that the probability of penetration is a nondecreasing function of the stimulus.

The situation described above is identical to the one encountered in "bioassay experiments," and "low dose radiation experiments," in which the relationship mentioned before is known as the <u>quantal response curve</u>. The dose level of a drug is the stimulus, and interest generally centers around  $V_{.5}$ , the stimulur at which the probability of response is .5. Since it is possible to subject more than one animal to a particular dose level, the number of tests at each value of the stimulus can be more than one. Furthermore, tests are often conducted at several dose levels, and thus the large sample theory which typically justifies inference from bioassay experiments is adequately substantiated.



Figure 2.1--Probability of penetration vs. stimulus.

Despite these conspicuous differences between bioassay experimentation and the problem described here, the methodology and techniques of the former have been directly adopted for use in the latter. In so doing, a serious compromise has been made--the estimation of  $V_{.5}$ , rather than the entire quantal response curve, has been made the dominant issue of the kinetic energy penetration problem. Specifically, the BRL's commonly used "Langley Method" [Rothman, Alexander, and Zimmerman (1965, pp. 55-58)] and the "Up and Down Method" [op. cit., pp. 101-103] focus exclusive attention on the estimation of  $V_{.5}$ .

The typical approach used in bioassay for estimating  $V_{.5}$  is to assume that the probability of response p is an <u>arbitrary</u> nondecreasing function of the stimulus V, specified via the relationship

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$$p = F((v-\mu)/\sigma)$$
,

where F is a distribution function determined by a symmetrical density function with location parameter  $\mu$  and scale parameter  $\sigma$ . Often F is taken to be the normal distribution function

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} ds$$
,

or the logistic distribution function  $F(x) = (1 - e^{-x})^{-1}$ .

The data from a bioassay experiment consists of  $n_i$ , the number of subjects receiving stimulus  $V_i$ ,  $i=1,\ldots,K$ , and  $X_{ij}$ ,  $j=1,\ldots,n_i$ , where

X<sub>ij</sub> = 1 , if the j subject responds under stimulus V<sub>i</sub>, and = 0 , otherwise.

Given the data  $(n_i, X_{ij})$ ,  $i=1, \ldots, K$ ,  $j=1, \ldots, n_i$ , the parameters  $\mu$  and  $\sigma$  are estimated using the method of maximum likelihood,

under the assumption that the test results can be judged independent. Once  $\mu$  and  $\sigma$  are estimated, the estimation of  $V_{.5}$  follows from the fact that F, the <u>tolerance distribution</u>, has been specified. Nonparametric and robust estimators of  $V_{.5}$ , such as the Spearman-Karber estimator, the L-estimator, the M-estimator, and the Tukey Biweight estimator, have also been obtained, all under the assumption that the density function giving F is symmetric. These estimators have been discussed by Miller and Halpern (1979). Furthermore, it has been empirically shown that for the estimation of  $V_{.5}$  it does not matter what specific form is chosen for F; many of the commonly used nonparametric estimators yield identical estimates of  $V_{.5}$ , as long as symmetry is assumed.

A drawback of the assumption of symmetry is that the estimate of the probability of response when the stimulus is zero is nonzero. Whereas this may not be too disturbing in bioassay with its emphasis on  $V_{.5}$ , in the problem considered here and the low dose radiation experimentation, such an estimate would be clearly unacceptable. A zero value of the stimulus should correspond to a zero value for the probability of response.

In view of the above difficulty, the paucity of data at each level of the stimulus, and our inability to specify a functional form of F which has some practical merit, we are motivated to advocate a Bayesian approach for the solution of this problem. Our approach is described in Section 3.

### 3. AN OUTLINE OF A BAYESIAN APPROACH

A Bayesian approach to the bioassay problem was first proposed by Kraft and Van Eeden in 1964, and was more fully developed by Ramsey in 1972. We consider here the theme proposed by Ramsey; extensions of this theme are considered by Shaked and Singpurwalla (1982).

Let  $0 \equiv V_0 < V_1 < \ldots < V_M < V_{M+1} \leq \infty$ , be M distinct levels of the stimulus at which the target (armor plate) is tested; M is chosen in advance. The outcome of a test at  $V_i$  is described by a binary (0,1) variable  $X_i$ , where  $X_i = 1$  if the penetrator with a striking velocity  $V_i$  defeats the target. Let  $p_i = P\{X_i=1\}$ ,  $i=1,\ldots,M$ , and without loss of generality, we assume that

$$D \equiv p_0 < p_1 < p_2 < \dots < p_M < p_{M+1} \equiv 1;$$
 (3.1)

it is always possible to choose  $V_1$  and  $V_M$  which satisfy the above inequality.

Given  $X = (X_1, \ldots, X_M)$ , one goal is to estimate the unknown  $p_i$ 's,  $i=1,\ldots,M$ , subject to the inequalities (3.1). Another goal is to estimate  $p_j$ , for some  $j \neq i$ , such that if  $V_i < V_j < V_{i+1}$ , the estimates satisfy  $p_i < p_j < p_{i+1}$ ,  $i=1,\ldots,M$ ; this pertains to estimating the probability of response at a stimulus where no target was tested. Yet a third goal would be to estimate the largest stimulus, say  $V_{\alpha}$ , for which  $p_{\alpha} \leq \alpha$ , where  $0 < \alpha < 1$  is specified.

Ramsey's approach for achieving the above goals is to assign a Dirichlet as a prior distribution for the successive differences  $P_1, P_2^{-P_1}, \dots, P_M^{-P_{M-1}}$ , and then to use the <u>modal</u> value of the

resulting joint posterior distribution as a Bayes point estimate of  $(p_1, \ldots, p_M)$ . The modal value is computed with the inequalities (3.1) being satisfied. The modal value of the posterior distribution, if unique, is also known as the generalized maximum likelihood estimator [see DeGroot (1970, p. 236)], and is used as a Bayes estimator when we do not wish to specify a particular loss function. Having estimated the  $p_i$ 's, the estimation of  $p_j$  and  $V_{\alpha}$  is undertaken via an interpolation procedure.

Specifically, if  $\alpha_i > 0$ , i=1,...,M, and  $\beta > 0$  are conducts such that  $\sum_{i=1}^{M+1} \alpha_i = 1$ , then the prior density function  $\pi$  is on the form

$$\pi \propto \begin{cases} M+1 & \alpha_{i-1} \\ \Pi & (p_{i} - p_{i-1}) \\ i=1 \end{cases}^{\alpha_{i}} \end{cases}^{\beta} .$$
 (3.2)

It is important to note that when averaging according to  $\pi$  integration must be done with respect to  $\prod_{i=1}^{M} dp_i / \prod_{i=1}^{M+1} (p_i - p_{i-1})$ .

Since M has been prechosen, the stopping rule is clearly delineated, and so the likelihood for the response probabilities at the observed stresses is

$$\begin{array}{c} M & X_{i} & 1 - X_{i} \\ \Pi & P_{i} & (1 - P_{i}) \end{array}$$

$$(3.3)$$

The joint density function of the posterior distribution of  $P_1, \ldots, P_M$  is proportional to the product of the prior density function (3.2) and the likelihood function (3.3). Thus

$$f(p_{1}, \dots, p_{M} \mid X_{1}, \dots, X_{M})$$

$$\underset{i=1}{\overset{M+1}{\underset{i=1}{X_{i}}}} (1 - p_{i})^{1-X_{i}} \left[ \frac{\Gamma(\beta)}{\underset{i=1}{\overset{M+1}{\underset{i=1}{\Pi}}} \right] \left\{ \overset{M+1}{\underset{i=1}{\Pi}} (p_{i} - p_{i-1})^{\alpha_{i}} \right\}^{\beta} . \quad (3.4)$$

Ramsey has not been able to obtain the posterior marginal distributions of  $p_i$ , i=1,...,M, nor has he commented on any aspects of these distributions. He uses a nonlinear programming algorithm to obtain  $(\hat{p}_1, \ldots, \hat{p}_M)$  , the modal value of (3.4), subject to the constraint that  $\hat{p}_1 \leq \hat{p}_2 \leq \ldots \leq \hat{p}_M$ ; this is his Bayes estimator of  $(p_1, \ldots, p_M)$ . In contrast to this Mazzuchi (1982) has been able to obtain all the moments of the marginal posterior distribution of the  $p_i$ , i=1,...,M. This work of Mazzuchi's represents an extension of Ramsey's results, and is one that takes us a step closer to a fully Bayesian analysis. The moments can be used to approximate the marginal posterior distributions of the p<sub>i</sub>'s using the techniques given in Elderton and Johnson (1969). The approximated posterior distributions give us a measure of uncertainty associated with our using the first moment of the marginal posterior distribution of  $p_i$ ,  $i=1,\ldots,M$ , as our Bayes estimate of  $\boldsymbol{p}_{\mathbf{i}}$  . The first moment of the marginal posterior distribution is used as a Bayes estimator when we are willing to assume the square error as a loss function. The formulae for the moments and their use for approximating the marginal posterior distributions are given in Appendix A.

The computational effort required to compute the moments mentioned above increases with M. Thus there is a trade-off between the convenience of using an optimization algorithm to obtain the modal value

of (3.4), versus the laborious computational effort involved in obtaining several moments of each of the M posterior marginal distributions. The optimization algorithm cited above is based on the "Sequential Unconstrained Minimization Technique" (SUMT) of Fiacco and McCormick (1968). A computer code which adopts SUMT for the problem considered here is described by Mazzuchi and Soyer (1982). This code can also be used for the computation of the moments of the marginal posterior distributions of the  $p_i$ ,  $i=1,\ldots,M$ .

### 3.1 Specification of the Prior Parameters

In order to implement the Bayesian procedure, we need to specify the prior parameters  $\alpha_i$ , i=1,...,M, and  $\beta$ , given in (3.2). In order to do this, we observe (see Ramsey) that  $u_i = p_i - p_{i-1}$ , i=1,...,M, has a beta distribution on the unit interval (denoted as  $u_i \sim \text{Beta}(\beta \alpha_i, \beta(1 - \alpha_i); 0, 1))$ ,

$$f(u_{i}; \beta \alpha_{i}, \beta(1-\alpha_{i})) = \frac{\Gamma(\beta)}{\Gamma(\beta \alpha_{i}) \Gamma(\beta(1-\alpha_{i}))} u_{i}^{\beta \alpha_{i}} (1-u_{i})^{\beta(1-\alpha_{i})}, \quad 0 \leq u_{i} \leq 1$$

with

$$E(u_i) = \alpha_i , \text{ and } (3.5)$$

$$Var(u_i) = \frac{\alpha_i (1 - \alpha_i)}{(\beta + 1)}$$
 (3.6)

If  $P_i^*$  denotes our <u>best prior guess</u> about  $P_i$ , consistent with the fact that the  $P_i^*$ 's increase in i, then the  $\alpha_i$ 's can be obtained via (3.5) as

$$\alpha_{1} = P_{1}^{*}$$
  
 $\alpha_{i} = P_{i}^{*} - P_{i-1}^{*}, \quad i=2,...,M,$ 

and

 $\alpha_{M+1} = 1 - P_M^* .$ 

In order to choose the parameter  $\beta$ , we need to have some idea about the uncertainty associated with our choice of  $p_1^*$ . This in practice can be done in one of the following two ways:

(i) Suppose that in addition to  $p_1^*$ , our best guess about the variance of  $p_1$  is  $Var(p_1)$ . Then, substituting  $\alpha_1 = p_1^*$  in (3.6), we have

$$Var(u_1) = Var(p_1) = \frac{\alpha_1(1 - \alpha_1)}{(\beta + 1)}$$
,

so that

$$\beta = \begin{cases} \frac{p_1^*(1 - p_1^*)}{Var(p_1)} - 1, & \text{if } \beta > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Note that  $\beta = 0$  corresponds to the case of isotonic regression.

(ii) Often in practice [cf. McDonald (1979)], associated with the best guess value  $p_1^*$ , a user is able to specify two numbers  $a_1^* > 0$  and  $b_1^* < 1$ , such that for some  $\gamma_1$ (specified by the user),  $0 < \gamma_1 < 1$ ,

 $P(a_1^* < p_1 < b_1^*) = 1 - \gamma_1$ .

Since  $p_1 \sim Beta(\beta \alpha_i, \beta(1-\alpha_i); 0, 1)$ , given  $p_1^*$ , we set  $\alpha_1 = p_1^*$ , and find that value of  $\beta$  such that

$$\int_{a_{1}}^{b_{1}} \frac{\Gamma(\beta)}{\Gamma(\beta\alpha_{1})} \frac{\Gamma(\beta(1-\alpha_{1}))}{\Gamma(\beta(1-\alpha_{1}))} p_{1}^{\beta\alpha_{1}-1} (1-p_{1})^{\beta(1-\alpha_{1})-1} dp_{1} = 1-\gamma_{1}$$
(3.7)

Suppose, further, that for any one or more of the indices i, i=2,...,M, a user is also able to specify two numbers  $a_i^* > p_{i-1}^*$ , and  $b_i^* < 1$ , such that for some  $\gamma_i$  (specified by the user),  $0 < \gamma_i < 1$ ,  $P(a_i^* < (p_i^* \mid p_{i-1}^*, \dots, p_1^*) < b_i^*) = 1 - \gamma_i$ . Then, using the fact (see Ramsey) that  $(p_i \mid p_{i-1}^*) \sim \text{Beta}(\beta \alpha_i, \beta(1 - \alpha_1 - \dots - \alpha_i); p_{i-1}^*, 1)$  $= f(p_i \mid p_{i-1}^*; \beta, \alpha_i)$ , say,

we can find the smallest value of  $\beta$ ,  $\beta^*$ , which satisfies (3.7) and (3.8), where

$$\int_{i}^{b^{*}} f(p_{i} | p_{i-1}^{*}; \beta, \alpha_{i}) dp_{i} = 1 - \gamma_{i}, \qquad (3.8)$$

$$a_{i}^{*}$$

with 
$$\alpha_{i} = p_{i}^{*} - p_{i-1}^{*}$$
,  $i=2,...,M$ .

A computer code which determines the smallest value of  $\beta$  described above is available; the details of this program are given by Mazzuchi and Soyer (1982). Our reason for choosing the smallest value of  $\beta$ stems from the fact that large values of  $\beta$  give a very strong prior, with the result that even a large amount of failure data will not change our prior distribution.

# 3.2 Interpolation Procedure and the Estimation of Quantiles

Let the M-dimensional point

 $(p_1^+, \dots, p_M^+) = \begin{cases} (\hat{p}_1^-, \dots, \hat{p}_M^-) &, \text{ if the mode of the joint posterior is} \\ (\tilde{p}_1^-, \dots, \tilde{p}_M^-) &, \text{ if the first moments of the marginal posterior are} \end{cases}$ 

used as the Bayes estimator of  $(p_1, \ldots, p_M)$ .

Suppose that we wish to estimate  $p_j$ , for some  $j \neq i$ ,  $i=1,\ldots,M$ , where  $V_i < V_j < V_{i+1}$ . Let  $p_j^*$  be our best prior guess of  $p_j$ , the probability of response at a nonexperimental impulse  $V_j$ . Then, following Ramsey, we pick  $p_j^+$  in such a manner that

$$\frac{p_{i+1}^{*} - p_{j}^{*}}{p_{i+1}^{*} - p_{j}^{*}} = \frac{p_{j}^{*} - p_{i}^{*}}{p_{j}^{*} - p_{i}^{*}}.$$
(3.9)

For the estimation of  $V_{\alpha}$ , the  $\alpha$ th quantile  $(0 < \alpha < 1)$ , we first see if there is an observation stimulus, say  $V_i$ , for which  $p_i^+ = \alpha$ . If so, then  $V_i$  is our Bayes estimate of  $V_{\alpha}$ . If not, we determine the pair of observational impulses, say  $V_i$  and  $V_{i+1}$ , for which  $p_i^+ < \alpha < p_{i+1}^+$ . Since the probability of response curve is assumed to be increasing, the straight line segment joining the points 0,  $p_1^+$ , ...,  $p_i^+$ ,  $p_{i+1}^+$ , ...,  $p_M^+$ , 1, will be an increasing function of i. We shall find that value of the impulse, say  $V_{\alpha}^+$ ,  $V_i < V_i^+ < V_{i+1}$ , for which  $p_{\alpha}^+ = \alpha$ .

### 4. APPLICATION TO SOME BRL DATA

In Appendix B we present eight sets of data labelled 1, 2, 3, 4, 6, 7, 8, and 9, pertaining to 60 kinetic energy penetration tests. These data were given to us by Dr. Moore of BRL and have been carefully sanitized to maintain confidentiality. Data sets labelled 5 and 10. also given to us by Dr. Moore, have been eliminated from consideration because the striking velocity for these data is much too different from those of the other sets. All the 10 sets of data were obtained sequentially over time, in the sense that data set 1 was the first one to be obtained, followed by data set 2 (obtained after some lapse of time), and so on, until we reach data set 9, which is the last considered here. To the best of our knowledge, all eight data sets are assumed to have been collected under identical conditions. That is, there is no indication that, except for differences in striking velocity, the material and the methods of testing used for data set 1 are different from those used in data set 2, and so on. This, plus the sequential nature of the data, enables us to use the posterior obtained from one data set as the prior for the next set, and so on, until we obtain the posterior using data set 9, which then gives our final estimate of the response curve.

Data set 1 consists of 13 observations taken at striking velocities ranging from 128.60 (in some unspecified units) to 166.16. The result of each test is indicated by a binary variable  $X_i$ . The best prior guess values  $p_i^*$ , necessary to choose the prior parameters  $\alpha_i$ , were not specified by BRL. However, what appears to be reasonable is to assume that the probability of response at a striking velocity of 100 is close to zero, and that at a striking velocity of 200 it is almost 1. Thus we make an arbitrary choice for  $p_i^*$ , say  $p_{i0}^*$ , by letting  $p_{i0}^* = 1 - \exp[-.07(V_i - 100)]$ . Data on striking velocities outside the range of 100 to 200 were excluded. Despite this arbitrary choice of  $p_{i0}^*$ , we shall see how even a scant amount of data significantly changes the posterior response curve, provided that the smoothing parameter  $\beta$ is not too large. Three values of  $\beta$  were also chosen arbitrarily; these are 1, 10, and 25. Recall that small values of  $\beta$  tend to emphasize the data, whereas large values of  $\beta$  tend to emphasize the prior distribution. In Appendix B we show our analysis for the case of  $\beta = 10$ .

Since, in reality, the data are generated sequentially over time, our first step would be to revise the best prior guess values  $p_{10}^*$ , i=1,...,61, based on data set 1 alone. The posterior (modal) values corresponding to the striking velocities of data set 1,  $p_{11}^+$ , will be the revised values of  $p_{10}^*$ , for i=1,...,13; these are given in column 5 of the table in Appendix B. The revised values of  $p_{10}^*$ , for i=4,...,61, are obtained via the interpolation formula (3.9), using  $p_{11}^+$ , i=1,...,13, and  $p_{10}^*$ , i=14,...,61. Let the revised values of  $p_{10}^*$ , i=14,...,61, be denoted by  $p_{11}^*$ ; these too are shown in column 5 of the table in Appendix B.

Upon receiving data set 2, we revise the values  $p_{11}^{\star}$ , i=14,...,19, by the posterior modal values corresponding to the six striking velocities of data set 2. We denote these revised values by  $p_{12}^{+}$ , i=14,...,19; these are given in column 5 of the table in Appendix B. The revised values of  $p_{11}^{\star}$ , i=20,...,61, are obtained by interpolation, using

 $p_{i1}^+$ ,  $i=1,\ldots,13$ ,  $p_{i2}^+$ ,  $i=14,\ldots,19$ , and  $p_{i1}^*$ ,  $i=20,\ldots,61$ ; we denote these revised values by  $p_{i2}^*$ ,  $i=20,\ldots,61$ , and show them in column 6.

We continue the above scheme of systematically revising the  $\mathbf{p}_{i}$  's , either via the posterior modal values or by interpolation, until we incorporate the effect of all eight sets of data. Data set 9, the last one considered here, consists of eight observations taken at starting velocities ranging from  $V_{54} = 144.83$  to  $V_{61} = 198.94$ . The posterior modal values corresponding to the striking velocities of data set 9,  $p_{i8}^+$ , i=54,...,61, are given in column 12; the interpolated values  $p_{i7}^*$  required to obtain the  $p_{i8}^+$ 's are given in column 11. Since the  $p_{i7}^*$ 's incorporate the results of the previous seven sets of data, we claim that the final posterior modal values  $p_{18}^+$ , i=54,...,61, are based on the results of all the testing. Had we ignored the sequential nature of the data and computed the posterior modal values by using Bayes Theorem on the best prior guess values  $p_{i0}^*$ , i=1,...,61, then the posterior modal values corresponding to  $V_{14}$  through  $V_{61}$  would be different from the  $p_{i}^+$  values, i=14,...,61, given in the table. This difference is due to the interpolation scheme that is used to constantly revise the best prior guess values, when we consider the data sets sequentially.

A plot of  $p_{i8}^+$  versus  $V_i$ , i=54,...,61, represents our final estimate of the quantal response curve. Estimates of the probabilities of response at striking velocities different from  $V_i$ , i=54,...,61, can be obtained using the interpolation formula (3.9). When we use the interpolation formula to obtain an estimate of  $p_i$ , for some

j=1,...,53 , we need to specify a value  $p_j^*$ , the best prior guess value of  $p_j$ . Suppose that the index j appears in data set k, for some k < 9; then for  $p_j^*$  we will use  $p_{jk}^+$ . In so doing, we will have incorporated the effect of the last data set, data set 9, in our obtaining the estimate of  $p_j$ , and thus achieve a certain amount of smoothness. Note that the effect of the data sets between k and 9 is already present in our estimates  $p_{18}^+$ , i=54,...,61, and these are used in our interpolation scheme. For example, suppose that we wish to estimate the probability of response at a striking velocity of 158.52. This striking velocity occurs in data set 2, and lies between the striking velocities 148.97 and 159.15 of data set 9. The index j corresponding to the value 158.82 is 17. To use (3.9), we identify  $p_{i+1}^*$  and  $p_{i+1}^+$  as being .70499 and .53014, respectively,  $p_i^*$  and  $p_i^+$  as .62881 and .42386 (see data set 9), and  $p_j^*$  as .64436 (see data set 2), and compute  $p_i^+$  as our estimate of  $p_j$ .

In Figures 4.1, 4.2, and 4.3, we show plots of our Bayes estimate of the probability of response at the eight striking velocities of data set 9, for  $\beta = 1$ , 10, and 25, respectively. Also shown are the 90% probability of coverage intervals for each estimate. These intervals are obtained using the moments of the posterior distributions of  $p_i$ , i=54,...,61, and then using the techniques of Elderton and Johnson (1969) to approximate the posterior distributions--see Appendix A. On each of these figures we also show a graph of our best guess values  $p_{10}^*$ , i=1,...,61; these enable us to see how the data have changed our prior estimates. We observe that the 90% probability of coverage intervals tend to be small in the middle of the range of the striking velocities.

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In Figure 4.4, we superimpose the plots of Figures 4.1, 4.2, and 4.3, in order to give a perspective of the effect of  $\beta$  in our computations. It appears that our Bayes estimates for the three cases of  $\beta = 1$ , 10, and 25 tend to converge toward each other; this is to be expected, since we have 61 observations with which we revise our prior probabilities.



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### ACKNOWLEDGMENTS

The use of the "Sequential Unconstrained Minimization Technique" (SUMT) was central to our undertaking and accomplishing the work reported here; without it, we would not have been able to implement our Bayesian ideas. We gratefully acknowledge the help of Professors A. V. Fiacco and G. P. McCormick in connection with our use of SUMT. We also acknowledge the several helpful conversations with Drs. J. Richard Moore and Robert Launer regarding several aspects of this problem, and are particularly grateful to the latter for encouraging us to write this up. Mr. Refik Soyer's assistance with developing the computer programs cited here is also recognized. Mr. W. McDonald of the Naval Surface Weapons Center, made several comments which led us to develop the material of Section 3.1.

### APPENDIX A

#### Moments of the Marginal Posterior Distributions

The moments of the posterior distribution of  $p_i$ , i=1,...,M, have been obtained by Mazzuchi (1982); a formula for obtaining these is given below. A computer code which facilitates the computation of the moments is described by Mazzuchi and Soyer (1982).

Let  $\bar{X}_i = 1 - X_i$ , i=1,...,M, B(a,b) =  $\Gamma(a)\Gamma(b) / \Gamma(a+b)$ , and

$$\kappa = \sum_{\substack{r_1=0 \\ r_1=0}}^{\overline{X}_i} \dots \sum_{\substack{r_M=0 \\ m=0}}^{\overline{X}_M} (-1)^{\sum_{i=1}^M r_i} \prod_{\substack{i=1 \\ i=1}}^M B\left(\sum_{j=1}^i X_j + \beta\alpha_j + r_j, \beta\alpha_{i+1}\right).$$

Then, for *l*=1,2,...,

$$E(p_{s}^{\ell}) = \frac{1}{K} \sum_{r_{1}=0}^{\overline{X}_{1}} \dots \sum_{r_{M}=0}^{\overline{X}_{M}} (-1)^{\sum_{i=1}^{M} r_{i}} \prod_{i=1}^{M} B\left(\sum_{j=1}^{i} X_{j}^{\star} + \beta \alpha_{j} + r_{j}, \beta \alpha_{i+1}\right),$$

where

$$X_{j}^{*} = \begin{cases} X_{j} + \ell, & j = s \\ \\ X_{j}, & otherwise. \end{cases}$$

These moments can be used to approximate the posterior distribution of  $p_i$ ,  $f(p_i)$ ,  $i=1,\ldots,M$ . In order to do this, we consider a system of frequency curves described by Elderton and Johnson (1969) which are based on the transforms of a standard normal variate Z. The system of curves which is appropriate to our problem is that referred to as the "bounded system of curves," denoted by Elderton and Johnson (1969, p. 123) as  $S_B$ , and described by

$$Z = \gamma + \delta Ln[(p_i - \epsilon) / (\epsilon + \lambda - p_i)], \quad \epsilon < p_i < \epsilon + \lambda,$$

where  $\gamma$ ,  $\delta$ ,  $\lambda$ , and  $\epsilon$  are parameters whose values are determined by the first four moments of  $f(p_i)$  about its mean.

Hill, Hill, and Holder (1976) give a computer code which determines  $\gamma$ ,  $\delta$ ,  $\lambda$ , and  $\varepsilon$  from the first four moments of  $f(p_i)$  about its mean. Since it was assumed that  $p_{i-1} < p_i < p_{i+1}$ , we estimate  $\lambda$  and  $\varepsilon$  from the Bayesian estimates of the  $p_i$ ;  $\gamma$  and  $\delta$  are obtained from the computer code. Having obtained these parameters, the distribution  $f(p_i)$  is obtained from Elderton and Johnson (1969, p. 130) as

$$f(p_{i}) = \frac{N}{\lambda\sqrt{2\pi}} \left[ \left( \frac{p_{i} - \varepsilon}{\lambda} \right) \left( 1 - \frac{p_{i} - \varepsilon}{\lambda} \right) \right]^{-1} \exp \left[ -\frac{1}{2} \left( \gamma + \delta \ln \left( \frac{p_{i} - \varepsilon}{\varepsilon + \lambda - p_{i}} \right) \right)^{2} \right],$$
  
$$\varepsilon < p_{i} < \varepsilon + \lambda$$

where N in our case is 1.

In order to obtain the approximate  $(1-\gamma)$ % probability of coverage intervals for each  $p_i$ , which contain its Bayes estimate  $p_i^+$  (mode or mean), we use the fact that since

$$z = \gamma + \delta \ln[(p_i - \varepsilon)/(\varepsilon + \lambda - p_i)], \quad \varepsilon < p_i < \varepsilon + \lambda,$$
$$p_i = \lambda \exp[(\frac{\gamma - z}{\delta}) + 1]^{-1} + \varepsilon.$$

Thus, to find two numbers, a and b, such that

$$P\{p_{i}^{+} = a \leq p_{i} \leq p_{i}^{+} = b \} = 1 - \delta,$$

we use

$$P\left\{-\delta \ln\left(\frac{\lambda}{p_{1}^{+}-a-\varepsilon}-1\right)+\gamma\leq z\leq -\delta \ln\left(\frac{\lambda}{p_{1}^{+}+b-\varepsilon}-1\right)+\gamma\right\} = 1-\delta,$$

and solve for a and b by setting

$$-\delta \ln \left( \frac{\lambda}{p_i^+ - a - \varepsilon} - 1 \right) + \gamma = z_{1 - (\delta/2)}$$

and

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$$-\delta \ln \left( \frac{\lambda}{p_i^+ + b - \epsilon} - 1 \right) + \gamma = z_{\delta/2},$$

where  $z_{\delta/2}$  is the  $(1-(\delta/2))$ th percentile of a standard normal distribution. Taking c = max(a,b), we form our interval

$$\Pr\{p_{i}^{+} - c < p_{i} < p_{i}^{+} + c\} \ge 1 - \delta.$$

These intervals may not be symmetric about the mean or modal estimate. This case arises when the boundaries of the probability of coverage interval exceed the boundary of the variable. In such cases the variable boundary is used as the boundary of the probability of coverage interval. The probability of any symmetric interval about the mean or modal estimate may be obtained by proceeding in the reverse or the above and evaluating the interval for the standard normal variate.

## APPENDIX B

In the table below we give values of the striking velocity  $V_i$ , the response  $X_i$ , and the best prior guess values  $p_{i0}^*$ ,  $i=1,\ldots,61$ , for the eight sets of data described in Section 4. We also show, for  $\beta = 10$ , the revised values of  $p_{10}^*$ ,  $p_{1j}^+$ , or  $p_{1j}^*$  based on data set j, j=1,2,3,4,6,7,8,9.

TABLE B.1

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