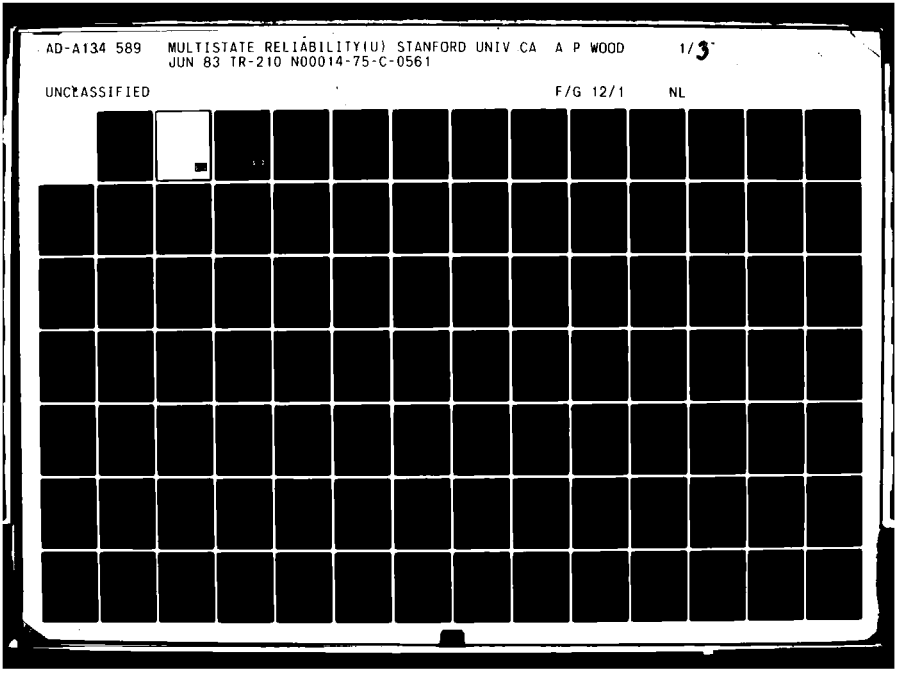
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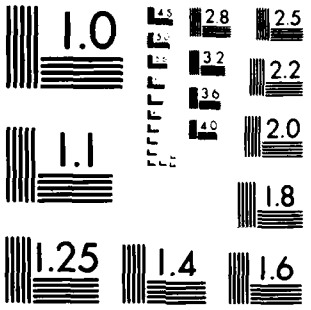

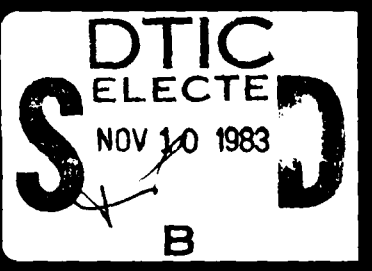
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MULTISTATE RELIABILITY

by

ALAN P. WOOD

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DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

PART I

MULTISTATE COHERENT STRUCTURE FUNCTION THEORY

In military systems, power plants, and many other areas of modern society, the consequences of failure can be catastrophic. The study of reliability has evolved from the desire to prevent failure, or at least mitigate the consequences thereof. Reliability is usually defined as the probability that a component or system is able to perform its specified function. A reliability analysis is performed to determine that probability. This analysis consists of mathematically modelling the system, assessing failure probabilities, and generating a numerical result. This thesis deals with aspects of the theory underlying the mathematical model.

Most reliability calculations are performed assuming that components and systems are either functioning or failed. This dichotomy is often a reasonable assumption, but the assumption is sometimes made simply because there are no applicable results dealing with more complicated state spaces. There are many situations in which the ability to consider multiple states would be useful in a reliability context. A component or system may have a useful partially operating mode. For example, if one of two turbines in a power plant is undergoing repair, the plant may be able to generate 50% of its rated electric capacity which is significantly better than being completely shut down. It may also be useful to differentiate among different modes of failure. A valve may fail to open, fail closed, or fail ruptured, and these failure modes may have very different effects on system operation. The enlarged

state space can be used for actual quantities rather than just qualitative measures.For example, states 0, 1, ..., 100 might be water temperature in degrees centigrade.

This thesis considers reliability in a general multistate setting. The number of states used in describing the status of each component and the operation of the system is allowed to vary, and the dependence of the system state on the component states can be very general. The first part of the thesis deals with the extension of binary coherent structure function theory to the multistate case. Some results in this area have previously been obtained. The existing results are generalized, and several new results are derived. The first part of the thesis could be applied to maintenance policies by letting the component states represent the total available number of a certain binary component and letting the system state represent the total number of operating machines. However, the second part of the thesis is more useful for determining optimal maintenance policies for multistate components. Most previous maintenance models deal with discrete time replacement policies. Components which may be repaired or replaced at any time are considered in this thesis.

1. INTRODUCTION AND HISTORY

Coherent structure theory is an attempt to treat reliability theory in an axiomatic way. Since reliability is an engineering discipline, the axioms and definitions are intended to reasonably portray the operation of systems and components in the "real world". The first systems to be considered from a reliability viewpoint consisted of binary components, components for which one state represents operational status and the other state indicates failure. The structure function, which represents the state of the system given the states of the components, was usually assumed to be binary and, in addition, was usually assumed to be coherent. Binary coherence means that each component of the system is relevant, i.e., important in determining the value of the system structure function, and that the system state cannot decrease when the state of one of its components increases. It seems reasonable to expect most real systems to operate in this fashion.

Because many components may partially function or have several modes of operation, there have been attempts to generalize the theory to include components or systems with several states. However, there are many gaps in the theory of multistate coherent structures which need to be filled, and the theory is not as general as is desirable for engineering purposes. In particular the concepts of relevance and coherence for components with several states need to be broadened to represent useful ideas in the operation of multistate systems. The first part of this thesis is an attempt to remedy the situation. After a review of the binary case and recent generalizations in this chapter, definitions

are given in Chapter 2 which are meant to reconcile coherent structure function theory with the "real world" operation of multistate components. The rest of Chapter 2 and all of Chapter 3 show that most of the results in the binary case have analogues in the multistate case. Block diagrams and fault trees and their extension to multistate components are the subject of Chapter 4. Chapter 5 pertains to components which have a continuum of states rather than multiple discrete states.

1.1. Binary Coherence and Notation

Reliability is a relatively new field and is primarily the outgrowth of the concern about military hardware reliability following World War II. The reliability literature of the 1940's and 1950's is primarily devoted to quality control, renewal theory, and properties of various life distributions. The first paper to treat reliability from a functional point-of-view seems to be Birnbaum, Esary, and Saunders [1961]. That paper defined a binary coherent system and its dual and developed many of their properties including minimum paths and cuts. The first reliability bounds for a coherent system with independent components were developed by Esary and Proschan [1963]. A paper by Birnbaum and Esary [1965] introduced the concept of modules and developed some of their basic properties. Bodin [1970] then used modules to obtain better bounds on system reliability. The idea of associated random variables was introduced by Esary, Proschan, and Walkup [1967], and Esary and Proschan [1970] used that concept to obtain reliability bounds for systems with dependent components. The concept of reliability importance is due to Birnbaum [1969]. A complete

treatment of binary coherent structure functions and their properties may be found in Barlow and Proschan [1975a]. The notation of that book is used throughout this section.

In binary coherent structure function theory, the components and system can be in one of two states represented by the numbers 0 (failed) and 1 (functioning). Let X_i be the state of the ith component in a system composed of n components.

Notation: $X_i \equiv 1_{\{\text{component } i \text{ is function}\}}$

$$= \begin{cases} 1 & \text{if component } i \text{ is functioning} \\ 0 & \text{if component } i \text{ is failed} \end{cases}$$

$1_{\{\cdot\}}$ is called the indicator function

$\underline{X} \equiv (X_1, X_2, \ldots, X_n)$

$\phi(\underline{X})$ = the state of the system $\equiv 1_{\{\text{system functioning}\}}$

$\phi : \{0,1\}^n \to \{0,1\}$

$\phi(j_i, \underline{X}) \equiv \phi(X_1, X_2, \ldots, X_{i-1}, j, X_{i+1}, \ldots, X_n)$

Definition: Component i is relevant if there exists $\underline{X}$ such that $\phi(1_i, \underline{X}) = 1$ and $\phi(0_i, \underline{X}) = 0$.

It will be assumed that all components are relevant since irrelevant components have no bearing on the system state.

Definition: A system represented by $\phi(\underline{X})$ is coherent if

(1) $\phi(\underline{0}) = 0$, $\phi(\underline{1}) = 1$, and

(2) $\phi(\underline{X})$ is increasing in $\underline{X}$.

Note: Increasing is used to mean $\underline{X}_1 > \underline{X}_2 \Rightarrow \phi(\underline{X}_1) \geq \phi(\underline{X}_2)$ while strictly increasing will mean $\underline{X}_1 > \underline{X}_2 \Rightarrow \phi(\underline{X}_1) > \phi(\underline{X}_2)$. The same applies to decreasing and strictly decreasing. $\underline{X} > \underline{Y}$ means $X_i \geq Y_i \ \forall\, i$ and $X_i > Y_i$ for some i.

Definition: A path vector is a vector $\underline{X}$ such that $\phi(\underline{X}) = 1$. The corresponding path set is $\{i : X_i = 1\}$. A path vector $\underline{X}$ such that $\underline{Y} < \underline{X} \Rightarrow \phi(\underline{Y}) = 0$ is called a minimal path vector and the associated path set is called a minimal path set. The jth minimal path set is denoted P_j.

Definition: A cut vector is a vector $\underline{X}$ such that $\phi(\underline{X}) =$ The corresponding cut set is $\{i : X_i = 0\}$. A cut vector $\underline{X}$ s[illegible]hat $\underline{Y} > \underline{X} \Rightarrow \phi(\underline{Y}) = 1$ is called a minimal cut vector and the a[illegible]ated cut set is called a minimal cut set. The jth minimum cut set is denoted K_j.

Minimal path sets and minimal cut sets are sometimes called min paths and min cuts, respectively.

Definition: The dual of ϕ is denoted by ϕ^D and is defined by $\phi^D(\underline{X}) \equiv 1 - \phi(\underline{1}-\underline{X})$.

It is easy to show that $(\phi^D)^D = \phi$. Minimal path sets for ϕ are minimal cut sets for ϕ^D and vice versa. The following are classic examples of coherent systems.

Example 1.1: A series system is one in which every component must function in order for the system to function.

$$\phi(\underline{X}) = \prod_{i=1}^{n} X_i = \min_i (X_i) \quad .$$

Each component represents a minimum cut set, and the only path vector is $\underline{1}$. □

Example 1.2: A parallel system is one in which the system will function if any component functions.

$$\phi(\underline{X}) = \coprod_{i=1}^{n} X_i \equiv 1 - \prod_{i=1}^{n} (1-X_i) = \max_i (X_i)$$

Each component is a minimum path set, and the only cut vector is $\underline{0}$. The dual of a parallel system is a series system and vice versa. □

Example 1.3: A k-out-of-n system is an n-component system that will function if k or more of its components function.

$$\phi(\underline{X}) = 1_{\{\sum_{i=1}^{n} X_i \geq k\}} \quad .$$

The dual of a k-out-of-n system is an (n-k+1)-out-of-n system. A series system is an n-out-of-n system and a parallel system is a 1-out-of-n system. □

Let (C,ϕ) denote a set of components C and a coherent structure function ϕ.

Definition: (A,χ) is a module of (C,ϕ) if $A \subseteq C$ and $\phi(\underline{X}) = \phi(\chi(\underline{X}^A),\underline{X}^{A^C})$ where ϕ is also a coherent structure function.

The notation $\underline{X}^A$ means the vector with elements X_i, $i \in A$, and A^C means the set complementary to A.

Definition: A modular decomposition of a coherent system (C,ϕ) is a set of disjoint modules $\{(A_1,\chi_1), \ldots, (A_r,\chi_r)\}$ together with an organizing structure ψ such that

(1) $C = \bigcup_{i=1}^{r} A_i$ and $A_i \cap A_j = \{\phi\}$ $\forall$ $i \neq j$, and

(2) $\phi(\underline{X}) = \psi(\chi_1(X^{A_1}), \chi_2(X^{A_2}), \ldots, \chi_r(X^{A_r}))$.

Example 1.4: A trivial modular decomposition is $\phi(\underline{X}) = \phi(\chi_1(X^{A_1}),\ldots,\chi_r(X^{A_r}))$ where $\chi(X_i) = X_i$ $\forall i$. As a more useful example, consider a system composed of two elements in series followed by two elements in parallel as in Figure 1.1.

$$\phi(\underline{X}) = \psi(\chi_1(X_1,X_2), \chi_2(X_3,X_4))$$

where

$$\chi_1(X_1,X_2) = X_1X_2$$

$$\chi_2(X_3,X_4) = X_3 + X_4 - X_3X_4$$

$$\phi(\chi_1,\chi_2) = \chi_1\chi_2 = X_1X_2(X_3 + X_4 - X_3X_4) \quad . \square$$

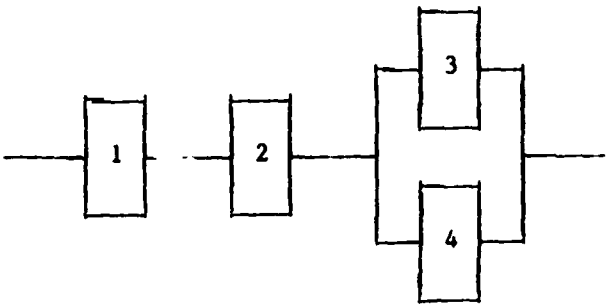

Figure 1.1 System Diagram

The preceeding discussion and definitions have dealt with the deterministic aspects of coherent systems. The following discussion relates to probabilistic evaluation of system operation with the component and system states considered as random variables. Reliability is defined as the probability of successful operation.

Notation: $P_i \equiv$ component reliability $= P(X_i = 1) = EX_i$.

$h \equiv$ system reliability $= P(\phi(\underline{X}) = 1) = E\phi(\underline{X})$.

$h = h(\underline{P})$ where $\underline{P} = (P_1, \ldots, P_n)$ if it is assumed that all components are independent.

$h(P) = h(\underline{P})$ when $P = P_1 = P_2 = \cdots = P_n$.

Components are often subject to the same loads and a common environment so that component failures may be highly correlated rather than independent. The following definition is a type of correlation useful in reliability theory.

Definition: Random variables $T_1, \ldots, T_n$ are associated if $COV[F(\underline{T}), G(\underline{T})] \geq 0$ for all pairs of increasing binary functions F and G (when the covariance exists).

Many reliability bounds have been developed for systems composed of associated components. These bounds can usually be improved through modular decomposition.

An implicit assumption in the preceeding discussion is that time is not considered or is fixed. In practice each component will have a random life length T_i governed by a life distribution $F_i(t)$.

Notation: $\bar{F}_i(t) \equiv 1-F_i(t) = P(X_i(t) = 1) = P(T_i > t)$

$$\bar{F}(s|t) \equiv \bar{F}(t+s)/\bar{F}(t) \quad \text{if} \quad \bar{F}(t) > 0$$

$$= \text{conditional reliability for a component of age } t$$

$$h(\underline{F}) \equiv P(\phi(\underline{X}(t) = 1)) \quad \text{where} \quad \underline{F} = (F_1(t), \ldots, F_n(t))$$

$$r(t) = \text{failure rate} \equiv \lim_{s \to 0} \frac{F(t+s)-F(t)}{\bar{F}(t)}$$

$$= f(t)/\bar{F}(t) \quad \text{when} \quad f(t) \quad \text{exists and} \quad \bar{F}(t) > 0$$

$$\bar{F}(t) = \exp\left(-\int_0^t r(s)ds\right)$$

Components are sometimes classified according to their life distributions. Several categories of life distributions are defined below. Useful reliability bounds have been developed for each category.

Definition: A distribution $F(t)$ or a random variable (component) with that life distribution is said to be

-Increasing Failure Rate (IFR) if $\bar{F}(s|t)$ is decreasing in $t \; \forall \; s \geq 0$.

-Decreasing Failure Rate (DFR) if $\bar{F}(s|t)$ is increasing in $t \ \forall \ s \geq 0$.

-Increasing Failure Rate Average (IFRA) if $-(1/t)\log[\bar{F}(t)]$ is increasing in t.

-Decreasing Failure Rate Average (DFRA) if $-(1/t)\log[\bar{F}(t)]$ is decreasing in t.

-New Better than Used (NBU) if $\bar{F}(s+t) \leq \bar{F}(s)\bar{F}(t) \ \forall \ s \geq 0, \ t \geq 0$.

-New Worse than Used (NWU) if $\bar{F}(s+t) \geq \bar{F}(s)\bar{F}(t) \ \forall \ s \geq 0, \ t \geq 0$.

-New Better than Used in Expectation (NBUE) if $\int_t^\infty \bar{F}(x)dx \leq \mu\bar{F}(t)$ where $\mu = ET < \infty$.

-New Worse than Used in Expectation (NWUE) if $\int_t^\infty \bar{F}(x)dx \geq \mu\bar{F}(t)$.

If $F(t)$ has a density $f(t)$, then IFR(DFR) means that the failure rate is increasing (decreasing) in t, and IFRA(DFRA) means that $\int_0^t r(s)ds$ is increasing (decreasing) in t. It can be shown that IFR $\Rightarrow$ IFRA $\Rightarrow$ NBU and that DFR $\Rightarrow$ DFRA $\Rightarrow$ NWU. It can also be shown that IFRA and NBU distributions are closed under the formation of coherent systems, that IFR, IFRA, and NBU distributions are closed under convolutions, and that DFR, DFRA, and NWU distributions are closed under mixtures.

1.2 Previous Generalizations

The first attempts to treat multistate reliability from a functional viewpoint were entitled cannibalization. Cannibalization is the use of parts from several failed units to form operational equipment. If one aircraft has a damaged tire and one has an inoperative

radio, a simple transfer allows the fleet to have an operational aircraft. The first article on this subject was Hirsch, Meisner, and Boll [1968]. This paper treated binary components and a multistate structure function with restrictions similar to coherence. Each component is allowed to be used in several locations. A cannibalization operation is the transfer of some or all components to different positions within the system, providing an increase in the system state. Conditions are derived under which the state of the system can be determined from the number of each type of component available. Although different terminology is used, ideas such as cut sets and k-out-of-n structures are described. Hochberg [1973] extended these results from systems with binary components to systems with multistate components. Simon [1972] obtains bounds on $P(\phi(\underline{X}(t)) \geq j)$ in cannibalized systems. This is accomplished by placing restrictions on the cannibalization operations.

The first extension of coherent structure function theory to continuous components is contained in Postelnicu [1970]. In this paper the component and system states are any values in the unit interval. The structure function used is closely related to coherent structure functions. Two reliability bounds were obtained in the paper. A second paper dealing with a continuous state space was Ross [1979]. He defines an IFRA process and an NBU process. These definitions are used to generalize the IFRA and NBU closure theorems (closure under the formation of coherent systems).

The first paper intended primarily to generalize coherent structure function theory was Barlow and Wu [1978]. In this paper a system is modelled as though it were binary, and the minimum path sets and minimum

cut sets are determined. The components and structure function are then allowed to be any integer among $(0,1,\ldots,M)$. The system state is defined as the state of the worst component in the best min path which is the same as the state of the best component in the worst min cut, i.e.

$$\phi(\underline{X}) = \max_j \min_{i \in P_j} (X_i) = \min_j \max_{i \in K_j} (X_i)$$

where P_j is the jth min path and K_j is the jth min cut. This is very restrictive since the relationship between the system and components is not allowed to vary as the system level varies. Reliability and stochastic system performance are considered, and a variant of the IFRA closure theorem is proved.

Another article containing a restrictive generalization of coherence is El-Neweihi, Proschan, and Sethuraman [1978]. The system and components may be any integer in $(0,1,\ldots,M)$, and ϕ is coherent if

(1) $\phi(\underline{X})$ is increasing in $\underline{X}$,

(2) there exists $\underline{X}$ such that $\phi(j_i,\underline{X}) = j$ while $\phi(\ell_i,\underline{X}) \neq j$ $\forall$ component i and system level j, and

(3) $\phi(\underline{j}) = j$ where $\underline{j} = (j_1,j_2,\ldots,j_n)$.

Minimum path sets, minimum cut sets, and utility functions are discussed in the paper. It is shown that $EU(\phi(\underline{X}))$ is stochastically increasing in $\underline{X}$ where $U(\cdot)$ is a utility function. Some reliability bounds are given, and the NBU closure theorem is generalized using a different definition for NBU process than the one in Ross [1979].

A less restrictive generalization of coherence may be found in a discussion of ternary (3-state) systems by Butler [1979]. Component i is defined to be relevant if there exists $\underline{X}$ such that $\phi(2_i,\underline{X}) \neq \phi(0_i,\underline{X})$. Component i is fully relevant if there exists $\underline{X}$ such that $\phi(2_i,\underline{X}) \neq \phi(1_i,\underline{X})$ and there exists $\underline{Y}$ such that $\phi(1_i,\underline{Y}) \neq \phi(0_i,\underline{Y})$. Fully relevant means that every state of every component is relevant. It is assumed that all components are relevant. ϕ is coherent if

(1) $\phi(\underline{0}) = 0$, $\phi(\underline{2}) = 2$, and

(2) $\phi(\underline{X})$ is increasing in $\underline{X}$.

Using this definition of coherence, Butler extends the ideas of reliability importance and structural importance, including new importance measures introduced in the paper, to multistate systems. In a second paper, Butler [1982] extended several bounds on system reliability to the multistate case. Another paper using this definition of coherence in a ternary system is Hatoyama [1979], although it is not explicitly stated that $\phi(\underline{0}) = 0$ and $\phi(\underline{1}) = 1$. Duals, path sets, and cut sets are defined in this paper, and some bounds on system reliability are derived. Most of the paper deals with a specialized type of system composed of modules of series or parallel elements with a series organizing structure.

A discussion of several possible generalizations of coherent structure functions may be found in Griffith [1980]. A function $\phi : \{0,1,\ldots,M\}^n \to \{0,1,\ldots,M\}$ is called a multistate monotone system if

(1) $\phi(\underline{X})$ is increasing in $\underline{X}$, and

(2) $\min_i (X_i) \leq \phi(\underline{X}) \leq \max_i (X_i)$.

If $\phi(\underline{X})$ is a MMS, it is called

(A) Strongly coherent if there exists $\underline{X}$ such that $\phi(j_i, \underline{X}) = j$ and $\phi(\ell_i, \underline{X}) \neq j \; \forall \; \ell \neq j$, $\forall$ component i, and $\forall$ state j.

(B) Coherent if there exists $\underline{X}$ such that $\phi((j-1)_i, \underline{X}) < \phi(j_i, \underline{X})$ $\forall$ component i and $\forall$ state $j \geq 1$.

(C) Weakly coherent if there is $\underline{X}$ such that $\phi(0_i, \underline{X}) < \phi(M_i, \underline{X})$ $\forall$ component i.

Note that conditions (1), (2), and (A) correspond to coherence in El-Neweihi, Proschan, and Sethuraman [1978]. If condition (2) is weakened to $\phi(\underline{0}) = 0$ and $\phi(\underline{2}) = 2$, then conditions (1), (2), and (C) correspond to coherence in Butler [1979]. The difference between relevant and fully relevant in that article is the same as the difference between conditions (B) and (C). Griffith defines the dual of a MMS, ϕ^D, and shows that it possesses the same type of coherence as ϕ. He also discusses modules and utility functions in the multistate setting. Reliability importance is defined, and it is shown that system utility can be expressed as the product of the reliability importance of a component and the probability vector of the component.

The analysis of specialized multistate systems may be found in Fardis and Cornell [1981]. This paper considers systems for which there are modules of completely interchangable components in series or parallel. Truth tables for these types of systems are analyzed.

2. THE MULTISTATE MODEL

In binary coherent structure function theory and in previous extensions to the multistate case, each component and the system structure function were all assumed to have the same number of states, labelled 0, 1, ..., M. When a component or the structure function had only $J < M + 1$ natural distinct states, more states were added by making the last states $J + 1, J + 2, \ldots, M$ identical to state J. This could significantly enlarge the state space which is computationally inefficient. In addition, some theoretical results are unnecessarily weakened as discussed later. This enlargement of the state space is herein eliminated by allowing every component and the system structure function to have a different number of states. This means that every state of every component will be relevant for determination of the system state since component states that are not relevant are eliminated.

In Section 2.2.1 it is shown that previous definitions of coherence lead to a situation in which every system can be modelled as a coherent system, rendering the concept meaningless. This problem arises because the states of the structure function do not have to be monotonically ordered by increasing utility. A new definition of coherence incorporating a utility function is proposed, and various types of coherence are discussed. This definition requires that every state of every component be relevant. Previously, it was only possible to require that every component be relevant (called weak coherence) because of the aforementioned enlargement of the state space. Using coherence rather than weak coherence strengthens some of the results in Chapters 2 and 3. The concepts of minimum path sets, minimum cut sets, series, parallel, and

k-out-of-n have been redefined in terms of the value of the structure function. Section 2.2 presents definitions for terms such as minimum cut set at level j and series system at level k.

The dual of a structure function is defined in Section 2.2.4, and it is shown that the dual possesses the same type of coherence as the original structure function. Rather than dealing separately with each component in a coherent system, it is often easier to consider subsets of components called modules together with an organizing structure for the modules. The main result of Section 2.2.5 is the relationships that exist between the type of coherence possessed by the overall system and the various types of coherence associated with the modules and their organizing structure. The general model discussed herein has an advantage in this modular decomposition since it turns out that coherence of the modules and their organizing structure permits a stronger conclusion than is possible with weak coherence assumptions.

2.1. Description of the Model

Consider a system composed of n components, and let $X_i \in \{0,1,\ldots,N_i\}$ be the state of the ith component. Let $\phi(\underline{X}) \in \{0,1,\ldots,M\}$ be the state of the system.

$$\phi : \{0,1,\ldots,N_1\} \times \{0,1,\ldots,N_2\} \times \cdots \times \{0,1,\ldots,N_n\} \to \{0,1,\ldots,M\} .$$

The total number of vectors $\underline{X}$ is $(N_1+1)(N_2+1)\cdots(N_n+1)$. In probabilistic evaluations, the component states and the system state become random variables.

Notation: $p_{ij} = P(X_i = j)$ $\qquad q_{ij} = P(X_i \geq j) = \sum_{k=j}^{N_i} p_{ik}$

$\underline{P}_i = (p_{i0},\ldots,p_{iN_i}) \qquad \underline{q}_i = (q_{i0},\ldots,q_{iN_i})$

$$\underline{P} = \begin{bmatrix} \underline{P}_1 \\ \cdot \\ \cdot \\ \underline{P}_n \end{bmatrix} \qquad \underline{q} = \begin{bmatrix} \underline{q}_1 \\ \cdot \\ \cdot \\ \underline{q}_n \end{bmatrix}$$

Note that $q_{i0} = 1 \ \forall\ i$. $\underline{P}$ is not really a matrix since its ith row has length $N_i + 1$, and thus its row length may vary. However, it will be called a probability matrix for lack of a better term. Reliability can no longer be defined as the probability that the system functions since the system may operate at one of several levels.

Definition: Reliability at level k is $h^k \equiv P(\phi(\underline{X}) \geq k)$. When the components are independent, this probability depends only on $\underline{P}$ and is denoted $h^k(\underline{P})$. When the system level is clear or when the discussion applies to any level k, the superscript will sometimes be dropped, and reliability at any level k will be denoted h or $h(\underline{P})$.

When time is a variable in reliability calculations, it is assumed that each component i begins in state N_i at $t = 0$ and has a distribution $F_i^j(t)$ which represents the time until the component state drops to or below a level j.

Notation: $F_{ji}(t) \equiv 1-F_i(t) = P(X_i(t) > j) = P(X_i(t) \geq j+1)$

$$\underline{F}_i \equiv (F_i^0(t), \ldots, F_i^{N_i}(t)) \qquad \underline{F} \equiv \begin{bmatrix} \underline{F}_1 \\ \vdots \\ \underline{F}_n \end{bmatrix}$$

$$h^k(\underline{F}) \equiv P(\phi(\underline{X}(t)) \geq k) \quad .$$

Again $\underline{F}$ is a matrix possibly with rows of unequal length. The reliability at level k may be denoted $h^k(\underline{F}(t))$ when it is desirable to explicitly show the time dependence, and the superscript may be dropped when it is unnecessary.

2.2 Definitions

2.2.1 Coherence

The definition of binary coherent structure function includes two main premises:

(1) All components must be relevant, and

(2) An increase in the state of a component cannot cause a decrease in the state of the system.

These premises seem very reasonable, so reasonable, in fact, that it is difficult to think of a situation in which they would not hold. The first premise - all components relevant - will always hold since if it did not, the irrelevant components could be disregarded and a new structure formed with only the relevant components. However, the second premise may fail to hold because of problems in defining success and failure. If two incoming water lines feed a pipe in which the desired

water temperature is 50^{o} F., then success for each incoming water line might be defined as being at 50^{o} F. (with some tolerance). If the water in one line is 60^{o} F. and the water in the other line is 40^{o} F., then the system might be considered a success even though both incoming water lines are failed. Furthermore, if the first water line is repaired, i.e., the water temperature is lowered from 60^{o} F. to 50^{o} F., then the system will fail. In fault trees there is a concept called an "exclusive or gate" which means that one of two inputs but not both is necessary for success. In structure function notation, this means that $\phi(0,0) = \phi(1,1) = 0$ and $\phi(1,0) = 1$, which does not fit the definition of coherence. These examples of non-coherent systems are rather contrived, however, and an engineer would never expect to find a non-coherent system in practice.

It seems a relatively easy matter to extend binary coherence to multistate systems. The following definition is based on the second premise of coherence discussed in the previous paragraph.

Definition: $\phi(\underline{X})$ is a Monotone Structure Function (MSF) if:

(1) $\phi(\underline{0}) = 0$, $\phi(\underline{N}) = M$ where $\underline{N} = (N_1, N_2, \ldots, N_n)$, and

(2) $\phi(\underline{X})$ is increasing in $\underline{X}$.

The concept of relevance may be used in a multistate setting to mean either that every component is relevant or that every state of every component is relevant. Following Griffith [1980], the following definition is presented.

Definition: A component i is said to be

(A) Relevant if there exists $\underline{X}$ such that $\phi(N_i,\underline{X}) > \phi(0_i,\underline{X})$.

(B) Fully Relevant if $\forall\, j = 1, 2, \ldots, N_i$, there exists $\underline{X}$ such that $\phi(j_i,\underline{X}) > \phi((j-1)_i,\underline{X})$.

Clearly, a component which is fully relevant is also revelant but not vice versa.

A reasonable definition of coherence would seem to be a MSF with all components either relevant or fully relevant. There is a fly in the ointment, however, as shown in the following proposition.

Proposition: All multistate systems may be modelled by a structure function which has properties (1), (2), and (B).

Proof: The proof is constructive. Assume that each of n components has a natural state space $\{0,1,\ldots,N_i\}$. Arbitrarily assign

$$\phi(\underline{0}) = 0,\ \phi(1_1,\underline{0}) = 1,\ \phi(1_2,\underline{0}) = 2,\ \ldots,\ \phi(1_n,\underline{0}) = n,$$

$$\phi(1_1,1_2,\underline{0}) = n+1,\ \ldots,\ \phi(\underline{1}) = 2^n-1,\ \phi(2_1,\underline{0}) = 2^n,\ \ldots,$$

$$\phi(\underline{N}) = \prod_{i=1}^{n}(N_i+1) - 1 \equiv M$$

By construction, $\phi(\underline{0}) = 0$, $\phi(\underline{N}) = M$, $\phi(\underline{X})$ is increasing in $\underline{X}$, and $\phi(j_i,\underline{X}) > \phi((j-1)_i,\underline{X})\ \forall\, j,i,\underline{X}$. □

Thus, using (1), (2), and (A) or (B) as a definition of coherence means that every system is coherent. Some people might view this as desirable, but it would be nice if there was a feature which separated coherent and non-coherent systems.

Implicit in the use of binary coherence is that state 1 is desirable (success) while state 0 is undesirable (failure). This notion has not yet been incorporated into multistate coherence. In the proof of the previous proposition, $\phi(2_1,\underline{0}) = 2^n > \phi(\underline{1}) = 2^n-1$, but it is very possible that $U(\phi(2_1,\underline{0})) < U(\phi(\underline{1}))$ where $U(\cdot)$ is a utility function. Normally, an engineer would specify system states in order of increasing utility rather than constructing a structure function as in the preceding proposition. The conditions (1), (2), and (B) would then be tested in the framework of the specified structure function. Thus, although a system can always be assigned a structure function which makes the system coherent, a more natural structure function with states monotonically ordered by increasing utility may be non-coherent. This leads to the following definition of coherence.

Definition: Let $\phi(\underline{X})$ be a MSF and let $U(\phi(\underline{X}))$ be the corresponding utility function that assigns utility a_j to state j. $\phi(\underline{X})$ is coherent if

(1) every component is fully relevant, and

(2) $a_j > a_{j-1}$ $\forall$ $j = 1, 2, \ldots, M$.

The first part of the definition can always be satisfied by eliminating irrelevant states. The second part of the definition means that the operational value of the system increases as the system state increases. Obviously, the utility function must reflect the true value of each system state since otherwise setting $U(\phi(\underline{X})) = \phi(\underline{X})$ would return the problem to its original status. The definition also implies that irrelevant system states will be eliminated. If $a_j = a_{j-1}$ for ϕ, define ϕ' and a' by

$$\phi'(\underline{X}) = \begin{cases} \phi(\underline{X}) & \text{whenever} \quad \phi(\underline{X}) \leq j - 1 \\ \phi(\underline{X}) - 1 & \text{whenever} \quad \phi(\underline{X}) \geq j \end{cases}$$

$$a_i' = \begin{cases} a_i & \text{whenever} \quad i \leq j - 1 \\ a_{i-1} & \text{whenever} \quad i \geq j \end{cases}$$

Thus, every state of the system, as well as every state of every component, will be relevant.

If the components and system are required to have the same state space, say $\{0,1,\ldots,M\}$, it may not be possible for each component to be fully relevant. This is the situation in the papers reviewed in Section 1.2. To allow the situation described herein to be compared with previous literature, the following two definitions are presented (using terminology from Griffith [1980]).

Definition: Let $\phi(\underline{X})$ be a MSF, and let $U(\phi(\underline{X}))$ be the corresponding utility function that assigns utility a_j to state j.

$\phi(\underline{X})$ is weakly coherent if

(1) every component is relevant, and

(2) $a_j \geq a_{j-1}$ $\forall$ $j = 1, \ldots, M$.

$\phi(\underline{X})$ is strongly coherent if

(1) $\forall$ component i and state j, there exists $\underline{X}$ such that $\phi(j_i,\underline{X}) = j$ while $\phi(\ell_i,\underline{X}) \neq j$ for $\ell \neq j$, and

(2) $a_j > a_{j-1}$ $\forall$ $j = 1, \ldots, M$.

It is also assumed in Griffith [1980] and El-Neweihi, Proschan, and Sethuraman [1978] that $\phi(j) = j$, but this restriction is not necessary for any of the results in those papers or the results herein.

It is clear that strong coherence $\Rightarrow$ coherence $\Rightarrow$ weak coherence. Examples 2.1 and 2.2 are presented to show that the reverse implications are not true.

Example 2.1: Consider two ternary components each with state space $\{0,1,2\}$, and let $a_j = j \ \forall\ j$. Let

$$\phi(0,0) = 0,$$

$$\phi(1,0) = \phi(0,1) = \phi(1,1) = \phi(0,2) = \phi(2,0) = 1, \text{ and}$$

$$\phi(1,2) = \phi(2,1) = \phi(2,2) = 2.$$

Since $\phi(1,0) > \phi(0,0)$ and $\phi(2,1) > \phi(1,1)$, component 1 is fully relevant. Since the system state is symmetric with respect to the components, component 2 is also fully relevant, and the system is coherent. To show that ϕ is not strongly coherent, consider $\phi(1,X_2)$.

$$X_2 = 0 \Rightarrow \phi(1,X_2) = \phi(2,X_2) = 1$$

$$X_2 = 1 \Rightarrow \phi(0,X_2) = \phi(1,X_2) = 1$$

$$X_2 = 2 \Rightarrow \phi(1,X_2) = \phi(2,X_2) = 2 \quad .$$

Thus, there is no X_2 such that $\phi(1,X_2) = 1$ and $\phi(j_1,X_2) \neq 1$ for $j = 0, 2$, so part (1) in the definition of strong coherence is not satisfied. □

Example 2.2: Again consider two ternary components with $a_j = j$.

Let $\phi(0,0) = \phi(1,0) = 0,$

$\phi(0,1) = \phi(1,1) = 1, \text{ and}$

$\phi(0,2) = \phi(2,0) = \phi(2,1) = \phi(1,2) = \phi(2,2) = 2.$

Since $\phi(2,0) > \phi(0,0)$ and $\phi(0,2) > \phi(0,0)$, ϕ is weakly coherent.

However, $\phi(0,X_2) = \phi(1,X_2) \vee X_2$, so X_1 is not fully relevant, and the system is not coherent. Note that we can combine states 0 and 1 of component 1 to get a coherent structure ϕ' which is equivalent to ϕ as follows. Let $X_i' = \max(X_i - 1, 0)$, and define ϕ' by

$$\phi'(0',0) = 0 \quad , \quad \phi'(0',1) = 1, \text{ and}$$
$$\phi'(0',2) = \phi'(1',0) = \phi'(1',1) = \phi'(1',2) = 2 \quad .$$

This is a special case of Theorem 2.1. □

Theorem 2.1: A relevant component can be made fully relevant.

Proof: Assume $\phi(N_1,\underline{X}) > \phi(0_1,\underline{X})$ for some $\underline{X}$ and $\phi(j_1,\underline{X}) = \phi((j-1)_1, \underline{X}) \vee \underline{X}$. Define ϕ' and X' by

$$\phi'(k_1',X) = \begin{cases} \phi(k_1,\underline{X}) & \text{for } k' \leq j-1 \\ \phi((k+1)_1,\underline{X}) & \text{for } k' \geq j \end{cases}$$

$$X_1' = \begin{cases} X_1 \vee X_1' \leq j-1 \\ X_1 - 1 \vee X_1' \geq j \quad . \end{cases}$$

The new structure function ϕ' is the same as ϕ except states j and $j-1$ of component 1 have been combined. Repeat the process if any components are still not fully relevant. □

Theorem 2.1 shows that weak coherence is a useful concept only when it is desirable to have the same state space for all components.

For simplicity, throughout the remainder of this thesis, it is assumed that all components are relevant if all components are required to have the same number of states and fully relevant otherwise. The

results contained herein will often hold assuming just that ϕ is a MSF which need not be coherent.

2.2.2. Series, Parallel, and k-out-of-n

The definitions of series, parallel, and k-out-of-n systems are straightforward generalizations of their binary counterparts.

Definition: The system represented by a MSF ϕ is

(1) Series if $\phi(\underline{X}) = \min_i (X_i)$

(2) Parallel if $\phi(\underline{X}) = \max_i (X_i)$, and

(3) k-out-of-n if $\phi(\underline{X}) = \max \{j : \sum_{i=1}^{n} 1_{\{X_i \geq j\}} \geq k\}$.

Clearly, a series system is an n-out-of-n system, and a parallel system is a 1-out-of-n system. Also, $M = \min_i (N_i)$ for a series system, and $M = \max_i (N_i)$ for a parallel system. These types of systems are useful since the position of the component within a system is irrelevant; only the numerical value of the component state has an impact. When this occurs, ϕ is said to have interchangeable components.

Example 2.3: Let ϕ be a ternary MSF composed of 3 interchangeable ternary components.

(A) ϕ is series if: $\phi(0,0,0) = \phi(1,0,0) = (1,1,0) = \phi(2,0,0)$
$= \phi(2,1,0) = \phi(2,2,0) = 0$
$\phi(1,1,1) = \phi(2,1,1) = \phi(2,2,1) = 1$
$\phi(2,2,2) = 2$.

(B) ϕ is parallel if: $\phi(0,0,0) = 0$

$$\phi(1,0,0) = \phi(1,1,0) = \phi(1,1,1) = 1$$

$$\phi(2,0,0) = \phi(2,1,0) = \phi(2,1,1) = \phi(2,2,0)$$

$$= \phi(2,2,1) = \phi(2,2,2) = 2$$

(C) ϕ is 2-out-of-3 if: $\phi(0,0,0) = \phi(1,0,0) = \phi(2,0,0) = 0$

$$\phi(1,1,0) = \phi(1,1,1) = \phi(2,1,0)$$

$$= \phi(2,1,1) = 1$$

$$\phi(2,2,0) = \phi(2,2,1) = \phi(2,2,2) = 2 \quad . \square$$

Theorem 2.2: Let ϕ be a MSF.

(i) $\phi(\underline{X} \vee \underline{Y}) \geq \phi(\underline{X}) \vee \phi(\underline{Y})$ where

$\underline{X} \vee \underline{Y} = (\max(X_1,Y_1), \ldots, \max(X_n,Y_n))$, and

(ii) $\phi(\underline{X} \wedge \underline{Y}) \leq \phi(\underline{X}) \wedge \phi(\underline{Y})$ where

$\underline{X} \wedge \underline{Y} = (\min(X_1,Y_1), \ldots, \min(X_n,Y_n))$.

If ϕ is a coherent MSF with $M = N_1 = \cdots = N_n$, then equality in (i) $\Longleftrightarrow$ ϕ is a parallel structure, and equality in (ii) $\Longleftrightarrow$ ϕ is a series structure.

Proof: (i) By definition $\underline{X} \vee \underline{Y} \geq \underline{X}$ and $\underline{X} \vee \underline{Y} \geq \underline{Y}$. Since $\phi(\underline{X})$ is increasing in $\underline{X}$, $\phi(\underline{X} \vee \underline{Y}) \geq \phi(\underline{X})$ and $\phi(\underline{X} \vee \underline{Y}) \geq \phi(\underline{Y})$. Thus, $\phi(\underline{X} \vee \underline{Y}) \geq \phi(\underline{X}) \vee \phi(\underline{Y})$.

Assume that $\phi(\underline{X}) = \max_i (X_i)$ (parallel). Then $\phi(\underline{X} \vee \underline{Y}) = \max_i (X_i \vee Y_i)$ $= [\max_i (X_i)] \vee [\max_i (Y_i)] = \phi(\underline{X}) \vee \phi(\underline{Y})$. Now assusme that ϕ is coherent and that $\phi(\underline{X} \vee \underline{Y}) = \phi(\underline{X}) \vee \phi(\underline{Y})$. For each i, there exists $\underline{X}$ such that $\phi((j-1)_i,\underline{X}) < \phi(j_i,\underline{X})$.

$$\phi(j_i,\underline{X}) = \phi((j_i,\underline{0}) \vee (0_i,\underline{X})) = \phi(j_i,\underline{0}) \vee \phi(0_i,\underline{X})$$
$$= \phi(j_i,\underline{0}) \vee [\phi(0_i,(X_k)_k,\underline{0}) \vee \phi(0_i,0_k,\underline{X})]$$
$$= \cdots = \phi(j_i,\underline{0}) \vee [\max_{k\neq i} \phi((X_k)_k,\underline{0})] \quad .$$

Similarly, $\phi((j-1)_i,\underline{X}) = \phi((j-1)_i,\underline{0}) \vee [\max_{k\neq i} \phi((X_k)_k,\underline{0}]$.

Since $\phi((j-1)_i,\underline{X}) < \phi(j_i,\underline{X})$, the above equations imply that $\phi((j-1)_i,\underline{0}) < \phi(j_i,\underline{0}) \vee j$ or $0 = \phi(\underline{0}) < \phi(1_i,\underline{0}) < \cdots < \phi(N_i,\underline{0}) \leq M$. If $M = N_1 = \cdots = N_n$, this means that $\phi(j_i,\underline{0}) = j \vee i$ and j or $\phi(\underline{X}) = \max_i (X_i)$.

The proof of (ii) is similar. □

Example 2.4: To show that $M = N_1 = \cdots = N_n$ is a necessary part of the proof, let X_1 be binary and X_2 be ternary. Let $\phi(0,0) = 0$, $\phi(0,1) = 1$, and $\phi(0,2) = \phi(1,0) = \phi(1,1) = \phi(1,2) = 2$. It is easy to show that ϕ is a coherent MSF and $\phi(\underline{X} \vee \underline{Y}) = \phi(\underline{X}) \vee \phi(\underline{Y})$, but $\phi(\underline{X}) \neq \max_i (X_i)$. To show that weak coherence does not suffice to prove the theorem, make component 1 ternary with $\phi(2,0) = \phi(2,1) = \phi(2,2) = 2$. Then ϕ is weakly coherent, but $\phi(1,0) = 2 \neq \max_i (X_i)$.□

These concepts may be applied to a single system state rather than the entire system.

Definition: The system represented by a MSF ϕ is

(1) Series at level j if $\phi(\underline{X}) = j \iff \min_i (X_i) = j$.

(2) Parallel at level j if $\phi(\underline{X}) = j \iff \max_i (X_i) = j$.

(3) k-out-of-n at level j if $\phi(\underline{X}) \iff \max_{\ell} \{\ell: \sum_{i=1}^{n} 1_{\{X_i \geq \ell\}} \geq k\} = j$.

A series system is a series system at level $j \neq j$. Note that if ϕ is k-out-of-n at level j, it must be at least k-out-of-n for all system levels larger than j. To see this, let ϕ be k-out-of-n at level j and (k-1)-out-of-n at level j + 1. Then $\phi((j+1)_1,\ldots,(j+1)_{k-1},\underline{0}) < j$ by the first criterion, but $\phi((j+1)_1,\ldots,(j+1)_{k-1},\underline{0}) \geq j + 1$ by the second criterion which is clearly impossible.

Example 2.5: Let ϕ be a MSF composed of 3 interchangeable components with $M = N_1 = N_2 = N_3 = 3$. We construct ϕ to be parallel at level 1, 2-out-of-3 at level 2, and series at level 3 as follows.

$$\phi(0,0,0) = 0 \qquad \phi(3,3,3) = 3$$

$$\phi(\underline{X}) = 1 \quad \text{for} \quad \underline{X} \in \{(1,0,0),(1,1,0),(1,1,1),$$
$$(2,0,0),(2,1,0),(2,1,1),$$
$$(3,0,0),(3,1,0),(3,1,1)\}$$

$$\phi(\underline{X}) = 2 \quad \text{for} \quad \underline{X} \in \{(2,2,0),(2,2,1),(2,2,2),$$
$$(3,2,0),(3,2,1),(3,2,2),$$
$$(3,3,0),(3,3,1),(3,3,2)\} \quad . \square$$

2.2.3. Min Paths and Min Cuts

In the binary case, a minimum path set is a list of components. If every component in a minimum path set functions, then so does the system. In the multistate case, however, path sets have to be associated with the appropriate system state, and each component must have a specified minimum state. Thus, in the multistate case, a minimum

path set is a list of minimum requirements for the state of each component.

Definition: Let $\phi(\underline{X})$ be a MSF. $\underline{X}$ is a path vector at level m if $\phi(\underline{X}) \geq m$. It is a path vector at maximal level m if $\phi(\underline{X}) = m$. If, in addition, $\phi(\underline{Y}) < m$ whenever $\underline{Y} < \underline{X}$, then $\underline{X}$ is called a minimum path vector at maximal level m. Let $\underline{m} = (m_1,\ldots,m_n)$ be a vector such that if $\underline{X} \geq \underline{m}$, then $\phi(\underline{X}) \geq m$ and if $\underline{Y} < \underline{m}$, then $\phi(\underline{Y}) < m$. The vector $\underline{m}$ will be called a min path (minimum path set at system level m). The jth min path at system level m will be denoted $\underline{m}^j = (m_1^j,\ldots,m_n^j)$. There are s_m min paths for each system level (s when the system level is clear). Also,

$$\rho_j(\underline{X}) = 1_{\{\underline{X} \geq \underline{m}\}} = \prod_{i=1}^{n} 1_{\{X_i \geq m_i^j\}}$$

is the jth minimum path structure function at level m.

Definition: Let $\phi(\underline{X})$ be a MSF. $\underline{X}$ is a cut vector at level m if $\phi(\underline{X}) < m$. It is a cut vector at minimal level m if $\phi(\underline{X}) = m-1$. If, in addition, $\phi(\underline{Y}) \geq m$ whenever $\underline{Y} > \underline{X}$, then $\underline{X}$ is called a minimum cut vector at minimal level m. Let $\bar{\underline{m}} = (\bar{m}_1,\ldots,\bar{m}_n)$ be a vector such that if $\underline{X} \leq \bar{\underline{m}}$, then $\phi(\underline{X}) < m$ and if $\underline{Y} > \bar{\underline{m}}$, then $\phi(\underline{Y}) \geq m$. The vector $\bar{\underline{m}}$ will be called a min cut (minimum cut set at system level m). The jth min cut at system level m will be denoted $\bar{\underline{m}}^j = (\bar{m}_1^j,\ldots,\bar{m}_n^j)$. There are t_m min cuts for each system level m (t when the system level is clear). Also,

$$\kappa_j(\underline{X}) = 1_{\{\underline{X} \not\geq \underline{\bar{m}}^j\}} = 1 - \prod_{i=1}^{n} 1_{\{X_i \leq \bar{m}_i^j\}}$$

is the jth <u>minimum cut structure function at level m</u>.

<u>Example 2.6</u>: Let $\phi(X_1,X_2,X_3)$ be as in Example 2.5.

Min paths at level 1 = $\{(1,0,0),(0,1,0),(0,0,1)\}$

Min cuts at level 1 = $\{(0,0,0)\}$

Min paths at level 2 = $\{(2,2,0),(2,0,2),(0,2,2)\}$

Min cuts at level 2 = $\{(3,1,1),(1,3,1),(1,1,3)\}$

Min paths at level 3 = $\{(3,3,3)\}$

Min cuts at level 3 = $\{(3,3,2),(3,2,3),(2,3,3)\}$.

Looking at the min paths and min cuts, it is easy to see that ϕ is parallel at level 1, 2-out-of-3 at level 2, and series at level 3. □

<u>Theorem 2.3</u>:

(i) $\quad \phi(\underline{X}) \geq m \iff \coprod_{j=1}^{s} \rho_j(\underline{X}) = 1 \iff \coprod_{j=1}^{t} \kappa_j(\underline{X}) = 1.$

(ii) $\quad h^m = P(\coprod_{j=1}^{s} \rho_j(\underline{X}) = 1) = P(\coprod_{j=1}^{t} \kappa_j(\underline{X}) = 1)$

<u>Proof</u>:

(i) $\coprod_{j=1}^{s} \rho_j(\underline{X}) = 1$ means that $\rho_j(\underline{X}) = 1$ for some j.

Thus, $\underline{X} \geq \underline{m}^j$ for some j so $\phi(\underline{X}) \geq m$ by definition. Now assume $\phi(\underline{X}) \geq m$. $\underline{X}$ is then a path vector at level m, and a min path can be constructed such that $\underline{X} \geq \underline{m}^j$. This implies that $\coprod_{j=1}^{s} \rho_j(\underline{X}) = 1$

since $\{\rho_j, j = 1,2,\ldots,s_m\}$ contains all min paths at system level m. The proof is similar for min cuts.

(ii) Part (ii) follows by taking expected values in (i). □

For independent components, it is true that $P(\rho_j(\underline{X}) = 1) = \prod_{i=1}^{s} P(X_i \geq m_i^j)$. However, it is not necessarily true that $P(\coprod_{j=1}^{s} \rho_j(\underline{X}) = 1) = \coprod_{j=1}^{s} P(\rho_j(\underline{X}) = 1)$. This situation arises because independence among components does not imply independence among path sets since two path sets can have a component in common. The same remarks apply to cut sets. This problem is illustrated in Example 2.7.

Example 2.7: Consider a ternary system composed of two ternary components with two path sets at level 2.

$$(2_1^1 = 2,\ 2_2^1 = 1) \quad , \quad (2_1^2 = 1,\ 2_2^2 = 2)$$

$$\underline{P}_1 = (1/4,1/4,1/2) \quad , \quad \underline{P}_2 = (1/3,1/3,1/3)$$

$$P(\coprod_{j=1}^{2} \rho_j(\underline{X}) = 1) = P(X_1 = 2 \text{ and } X_2 \geq 1 \text{ or } X_1 \geq 1 \text{ and } X_2 = 2)$$

$$= P(X_2 \geq 1)\ p_{12} + P(X_2 = 2)\ p_{11} + 0 \cdot p_{10} = 5/12$$

$$\coprod_{j=1}^{2} P(\rho_j(\underline{X}) = 1) = 1-[1-P(X_1 = 2, X_2 \geq 1)] \cdot [1-P(X_1 \geq 1,\ X_2 = 2)]$$

$$= 1/2. \ \square$$

2.2.4. Duals

The dual of a binary coherent structure function is useful in reliability modelling since the minimum path sets of ϕ are the minimum cut sets of ϕ^D and vice versa. Thus, solving for the reliability of

the dual system is equivalent to solving for the unreliability of the original system. A similar relationship exists in the multistate case. This is mathematical justification for computing system reliability using block diagrams and fault trees which model either system success or failure.

Definition: Let ϕ be a MSF. Its dual, ϕ^D, is defined by $\phi^D(\underline{X}) \equiv M - \phi(\underline{N}-\underline{X})$.

Theorem 2.4: $(\phi^D)^D = \phi$.

Proof: $(\phi^D)^D(\underline{X}) = M-\phi^D(\underline{N}-\underline{X}) = M-[M-\phi(\underline{N}-(\underline{N}-\underline{X}))] = \phi(\underline{X})$. □

Theorem 2.5: (Griffith [1980]): The dual of a MSF is a MSF and possesses the same type of coherence as the original structure function.

Proof:
$$\phi^D(\underline{0}) = M-\phi(\underline{N}-\underline{0}) = M-M = 0.$$
$$\phi^D(\underline{N}) = M-\phi(\underline{N}-\underline{N}) = M.$$

If $\underline{X} \geq \underline{Y}$, $\phi^D(\underline{X}) = M-\phi(\underline{N}-\underline{X}) \geq M-\phi(\underline{N}-\underline{Y}) = \phi^D(\underline{Y})$. This proves that if ϕ is a MSF, then ϕ^D is a MSF. Now assume ϕ is coherent. Let $\underline{Y}$ be such that $\phi((N_i-j)_i,\underline{Y}) < \phi((N_i-j+1)_i,\underline{Y})$, and let $\underline{X} = \underline{N}-\underline{Y}$.

$$\phi^D((j-1)_i,\underline{X}) = M - \phi(\underline{N}-((j-1)_i,\underline{X})) = M - \phi((N_i-j+1)_i,\underline{Y})$$
$$< M - \phi((N_i-j)_i,\underline{Y}) = M-\phi(\underline{N}-(j_i,\underline{X})) = \phi^D(j_i,\underline{X}).$$

The previous inequality shows that ϕ is coherent. If ϕ is weak coherent, let $\underline{Y}$ be such that $\phi(0_i,\underline{Y}) < \phi(N_i,\underline{Y})$, and use the same argument. If ϕ is strong coherent, let $\underline{Y}$ be such that $\phi((N_i-j)_i,\underline{Y}) = N_i-j$ and $\phi((N_i-\ell)_i,\underline{Y}) \neq N_i-j \ \forall \ \ell \neq j$, and use the same argument. □

Theorem 2.6: Let $\underline{X}$ be a path (cut) vector at level m for a MSF ϕ. Then $\underline{N}-\underline{X}$ is a cut (path) vector for ϕ^D at level $M-m+1$. Furthermore, if $\underline{m}$ is a min path (cut) at maximum (minimum) level m for ϕ, then $\underline{N}-\underline{m}$ is a min cut (path) at minimum (maximum) level $M-m+1$ for ϕ^D.

Proof: If $\underline{X}$ is a path vector at level m for ϕ, $\phi(\underline{X}) \geq m$. Thus, $\phi^D(\underline{N}-\underline{X}) = M-\phi(\underline{X}) \leq M-m < M-m+1$, and $\underline{N}-\underline{X}$ is a cut vector for ϕ^D at level $M-m+1$. Now let $\underline{m}$ be a min path for ϕ. Then $\phi(\underline{m}) = m$ and $\forall\ \underline{Y} < \underline{m}$, $\phi(\underline{Y}) \leq m-1$. Thus,

$$\phi^D(\underline{N}-\underline{m}) = M-\phi(\underline{m}) = M-m < M-m+1, \text{ and}$$
$$\phi^D(\underline{N}-\underline{Y}) = M-\phi(\underline{Y}) \geq M-m+1 \ \forall\ \underline{Y} < \underline{m} \text{ or } \underline{N}-\underline{Y} > \underline{N}-\underline{m}.$$

Thus, $\underline{N}-\underline{m}$ is a min cut for ϕ^D. The results in parentheses hold by considering ϕ^D as the original MSF and remembering that $(\phi^D)^D = \phi$. □

Corollary 2.7: If ϕ is a k-out-of-n system at level m, ϕ^D is a $(M-k+1)$-out-of-n system at level $M-m+1$.

Proof: Since a k-out-of-n system is uniquely determined by its min paths or min cuts, this is immediate from Theorem 2.6. □

Example 2.7: Let ϕ have 3 components with 4 states each as in Example 2.5 - parallel at level 1, 2-out-of-3 at level 2, and series at level 3. Then $\phi^D(\underline{X}) = \phi(\underline{X})$, i.e., the system is self-dual. It is easy to see that Theorem 2.5 and Corollary 2.7 are satisfied. □

Example 2.8: Let ϕ be a series system composed of 3 ternary components.

$$\phi^D(0,0,0) = 0$$

$$\phi^D(1,1,1) = \phi^D(0,1,1) = \phi^D(0,0,1) = 1$$

$$\phi^D(2,2,2) = \phi^D(1,2,2) = \phi^D(1,1,2) = \phi^D(0,2,2)$$
$$= \phi^D(0,1,2) = \phi^D(0,0,2) = 2.$$

Thus, ϕ^D is a parallel MSF. □

2.2.5 Modules

A module is essentially an assembly of components which can itself be treated as a component. Modules are useful for breaking up a large system into several smaller ones which may be more readily analyzed. They can also be used to determine bounds on system reliability which are as good as or better than bounds obtained by considering the original system. The concept of a module is easily generalized to the multistate case. Let (C,ϕ) denote a set of components C and a MSF ϕ.

Definition: (A,χ) is called a module of (C,ϕ) if $A \subseteq C$ and $\phi(\underline{X}) = \psi(\chi(X^A), X^{A^C})$ where ψ is a MSF. Note that the number of system states for χ must be equal to the number of states for the first component of ψ.

Definition: A modular decomposition of the system (C,ϕ) is a set of disjoint modules $\{(A_1,\chi_1),\ldots,(A_r,\chi_r)\}$ together with an organizing structure ψ such that

(1) $C = \bigcup_{i=1}^{r} A_i$ and $A_i \cap A_j = \{\phi\}$ $\forall$ i and j, and

$$(2)\quad \phi(\underline{X}) = \psi(\chi_1(\underline{X}^{A_1}), \ldots, \chi_r(\underline{X}^{A_r})) \quad .$$

Parts (i), (ii), and (iii) of Theorem 2.8 are due to Griffith [1980].

Theorem 2.8: Let (A,χ) be a module of (C,ϕ), i.e.,

$$\phi(\underline{X}) = \psi(\chi(\underline{X}^A), \underline{X}^{A^C}).$$

(i) If χ and ψ are both coherent, ϕ is coherent.

(ii) If χ and ψ are both strong coherent, ϕ is strong coherent.

(iii) If χ and ψ are both weak coherent, ϕ is not necessarily weak coherent.

(iv) If ψ is coherent and χ is weak coherent, ϕ is weak coherent.

Proof: Consider a component i. If $i \notin A$, the theorem is obvious, so assume $i \in A$.

(i) Since χ is coherent, there exists $\underline{X}$ such that $\chi((j-1)_i, \underline{X}^A) < \chi(j_i, \underline{X}^A)$. Since ψ is coherent, there exists $\underline{X}^{A^C}$ such that

$$\phi((j-1)_i, \underline{X}) = \psi(\chi((j-1)_i, \underline{X}^A), \underline{X}^{A^C})$$

$$< \psi(\chi(j_i, \underline{X}^A), \underline{X}^{A^C}) = \phi(j_i, \underline{X}) \quad .$$

(ii) Since χ is strong coherent, there exists $\underline{X}^A$ such that $\chi(j_i, \underline{X}^A) = j$ and $\chi(\ell_i, \underline{X}^A) \neq j \vee \ell \neq j$. Since ψ is coherent, there exists $\underline{X}^{A^C}$ such that

$$\phi(j_i,\underline{X}) = \psi(\chi(j_i,\underline{X}^A),\underline{X}^{A^C}) = \psi(j_i,\underline{X}^{A^C}) = j, \text{ and}$$

$$\phi(\ell_i,\underline{X}) = \psi(\chi(\ell_i,\underline{X}^A),\underline{X}^{A^C}) \neq j \;\forall\; \ell \neq j \quad .$$

(iii) The problem here is that there exists $\underline{X}^A$ such that $\chi(N_i,\underline{X}^A) > \chi(0_i\underline{X}^A)$, and there exists $\underline{X}^{A^C}$ such that $\psi((M_\chi)_1,\underline{X}^{A^C}) > \psi(0_1,\underline{X}^{A^C})$, but $\chi(N_i,\underline{X}^A)$ is not necessarily equal to M_χ. This problem exists even if χ is coherent or strong coherent. As an example, let $\chi(X_1,X_2) = \psi(X_1,X_2)$ be ternary systems with ternary components.

$$\psi(0,0) = \psi(1,0) = 0$$

$$\psi(0,1) = \psi(1,1) = \psi(0,2) = \psi(1,2) = 1$$

$$\psi(2,0) = \psi(2,1) = \psi(2,2) = 2 \quad .$$

Note that states 0 and 1 of component 1 are indistinguishable. Let $\phi(X_1,X_2,X_3) = \psi(\chi(X_1,X_2),X_3)$. Enumeration shows that $\phi(X_1,0_2,X_3) = \phi(X_1,2_2,X_3) \;\forall\; X_1$ and X_3. Thus, component 2 is irrelevant, so ϕ is not weak coherent.

(iv) Since χ is weak coherent, there exists $\underline{X}^A$ such that $\chi(0,\underline{X}^A) < \chi(N_i\,\underline{X}^A)$. Since ψ is coherent, there exists $\underline{X}^{A^C}$ such that

$$\phi(0_i,\underline{X}) = \psi(\chi(0_i,\underline{X}^A),\,\underline{X}^{A^C}) < \psi(\chi(N_i,\underline{X}^A),\underline{X}^{A^C})$$
$$= \phi(N_i,\underline{X}) \quad . \;\square$$

Theorem 2.8 clearly applies to a modular decomposition by replacing χ with $\chi_1, \ldots, \chi_r$.

Example 2.9: Consider $\phi(X_1,X_2,X_3,X_4)$ where everything is ternary, X_1 and X_2 are in parallel, X_3 and X_4 are in parallel, and $X_1 - X_2$ are in series with $X_3 - X_4$. This system is shown in Figure 2.1.

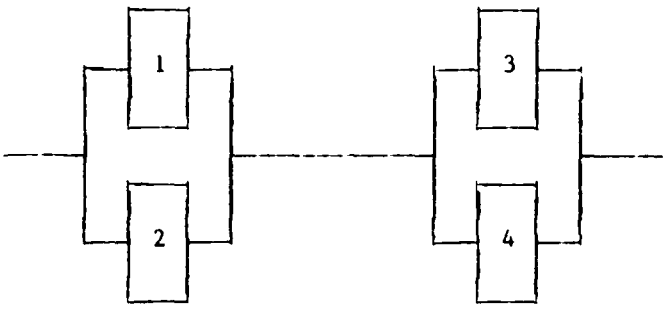

Figure 2.1 Block Diagram

It is easy to see that

$$\phi(X_1,X_2,X_3,X_4) = \min (X_1 \vee X_2, X_3 \vee X_4)$$
$$= \min (\chi_1(X_1,X_2), \chi_2(X_3,X_4)) = \psi(\chi_1(X_1,X_2), \chi(X_3,X_4))$$

where χ is a parallel structure function and ψ is a series structure function.

As a sample calculation,

$$\phi(2,0,1,1) = \psi(\chi(2,0), \chi(1,1)) = \psi(2,1) = 1. \quad \square$$

The next theorem shows that the dual of a module is a module in the dual.

Theorem 2.9: If ϕ is a MSF and (A,χ) is a module of (C,ϕ), then $\phi^D(\underline{X}) = \phi^D(\chi^D(\underline{X}^A),\underline{X}^{A^C})$.

Proof:

$$\phi^D(\chi^D(\underline{X}^A),\underline{X}^{A^C}) = \phi^D(M_\chi - \chi(\underline{N}^A - \underline{X}^A),\underline{X}^{A^C})$$

$$= M - \phi(\chi(\underline{N}^A - \underline{X}^A),\underline{N}^{A^C} - \underline{X}^{A^C})$$

$$= M - \phi(\underline{N} - \underline{X}) = \phi^D(\underline{X}) \quad .\square$$

3. STOCHASTIC RESULTS

Although the definition of coherence is important in engineering considerations, most of the stochastic results for the binary case can be extended to the multistate case assuming only that the structure function is nondecreasing in each component. Many of the multistate analogues to binary results exist in the literature although theorem hypotheses are generally stronger than necessary. These theorems have been extended to the general multistate model described in Chapter 2 with hypotheses weakened when appropriate. The proofs of these theorems usually require only minor modifications to the proofs given for more restrictive situations. Some results which had not previously been extended to the multistate case also appear in this chapter.

Results pertaining to system utility are contained in Section 3.1. These are important because reliability is thought of as the probability that the system is operational. Since the system can be partially operational in the multistate case, expected system utility is a better measure of system performance than is reliability. In Section 3.2 the relative importance of each component to the system is considered. Many of the reliability importance measures in this section had not been previously extended to the multistate case. It is known that certain classes of life distributions are closed under convolutions, mixtures, and the formation of coherent systems. By extending the idea of life distribution classes to life distribution processes, these closure theorems are shown to apply to the multistate case in Section 3.3. The results for closure under the formation of coherent systems previously

existed in the literature, but the results for closure under convolutions and mixtures are new. Reliability bounds on both system and component reliability are considered in Section 3.4. Butler [1978] extended most of the bounds on system reliability to the multistate case using the probability that the system is at a certain level or higher as a measure of reliability. In Section 3.4.2 these bounds are shown to apply to expected system utility. In addition, bounds on the expected utility of non-coherent systems are obtained. Bounds on component reliability are usually based on properties of the life distribution class to which the component belongs. Using life distribution processes, these bounds are applied to multistate components in Section 3.4.3.

3.1. System Utility

In Section 2.1, the reliability of the system at level k was defined as $h^k = P(\phi(\underline{X}) \geq k)$. Note that $h^k \neq E\phi(\underline{X})$ as it was in the binary case. In the multistate setting, $E\phi(\underline{X}) = \sum_{k=1}^{M} P(\phi(\underline{X}) \geq k) = \sum_{k=1}^{M} h^k$. However, neither h^k nor $E\phi(\underline{X})$ is necessarily the best measure of system performance. It is possible that system states j and $j+1$ have nearly identical utility while there is a large difference in the utility of states k and $k+1$. The measure of system performance that will be used in this thesis is expected system utility.

Let $U(\cdot)$ be a utility function which assigns value a_k to system state k, $k = 1, 2, \ldots, M$ where $a_0 \equiv 0$ without loss of generality.

$$U(\phi(\underline{X})) = \sum_{k=1}^{M} a_k 1\{\phi(\underline{X})=k\} = \sum_{k=1}^{M} b_k 1\{\phi(\underline{X})\geq k\}$$

where $b_k = a_k - a_{k-1}$.

$$EU(\phi(\underline{X})) = \sum_{k=1}^{M} a_k P(\phi(\underline{X}) = k) = \sum_{k=1}^{M} b_k h^k \quad .$$

When $a_k = k \; \forall \; k$, $EU(\phi(\underline{X})) = E\phi(\underline{X})$. When $a_j = 0 \; \forall \; j \leq k-1$ and $a_j = 1 \; \forall \; j \geq k$, $EU(\phi(\underline{X})) = h^k$. Thus, expected utility includes expected value and reliability as special cases. If $\phi(\underline{X})$ is weak coherent, $0 = a_0 \leq a_1 \leq \cdots \leq a_M$ and $b_k \geq 0 \; \forall \; k$. If $\phi(\underline{X})$ is coherent, $0 = a_0 < a_1 < \cdots < a_M$ and $b_k > 0 \; \forall \; k$.

Theorem 3.1:

$$EU(\phi(X)) = \sum_{j=0}^{N_i} p_{ij} EU(\phi(j_i, \underline{X})) \quad .$$

Proof:

$$\begin{aligned} EU(\phi(\underline{X})) &= \sum_{k=1}^{M} b_k \sum_{j=0}^{N_i} p_{ij} P(\phi(\underline{X}) \geq k \mid X_i = j) \\ &= \sum_{j=0}^{N_i} p_{ij} \sum_{k=1}^{M} b_k P(\phi(j_i, \underline{X}) \geq k) \\ &= \sum_{j=0}^{N_i} p_{ij} \; EU(\phi(j_i, \underline{X})) \quad . \; \square \end{aligned}$$

Using Theorem 3.1, the following pivotal decomposition can be derived where the first equality holds only for independent components.

$$\begin{aligned} EU(\phi(\underline{X})) &= \sum_{j_1=0}^{N_1} \cdots \sum_{j_n=0}^{N_n} p_{1j_1} \cdots p_{nj_n} \sum_{j=0}^{M} b_k \, 1_{\{\phi(j_1, \ldots, j_n) \geq k\}} \\ &= \sum_{\underline{x}} P(\underline{X} = \underline{x}) \sum_{j=0}^{M} b_k \, 1_{\{\phi(\underline{x}) \geq k\}} \quad . \end{aligned}$$

Theorem 3.2: If ϕ is a coherent MSF, then

(i) $EU(\phi(\underline{X} \vee \underline{Y})) \geq [EU(\phi(\underline{X}))] \vee [EU(\phi(\underline{Y}))]$ where equality holds if and only if ϕ is a parallel structure function.

(ii) $EU(\phi(\underline{X} \wedge \underline{Y})) \leq [EU(\phi(\underline{X}))] \wedge [EU(\phi(\underline{Y}))]$ where equality holds if and only if ϕ is a series structure function.

Proof: (i) From the pivotal decomposition,

$$P(\phi(\underline{X} \vee \underline{Y}) \geq k) - P(\phi(\underline{X}) \geq k) \vee P(\phi(\underline{Y}) \geq k)$$
$$= \sum_{\underline{x}} \sum_{\underline{y}} \left[1_{\{\phi(\underline{x} \vee \underline{y}) \geq k\}} - 1_{\{\phi(\underline{x}) \vee \phi(\underline{y}) \geq k\}}\right] P(\underline{X} = \underline{x}, \underline{Y} = \underline{y}) \quad .$$

From Theorem 2.2, the quantity in brackets is always nonnegative and is 0 if and only if ϕ is a parallel structure function. The result follows by multiplying by $b_k > 0$ and summing over k.

(ii) The proof is similar. □

The next result shows that expected system utility is increasing in q_{ij} (recall $q_{ij} = P(X_i \geq j)$. If ϕ is coherent, then the expected system utility is strictly increasing in q_{ij}.

Theorem 3.3: Let $\underline{X}(\underline{X}')$ have distribution $\underline{q}(\underline{q}')$.

(i) If $\underline{q} \leq \underline{q}'$ and ϕ is weak coherent, $EU(\phi(\underline{X})) \leq EU(\phi(\underline{X}'))$.

(ii) If $\underline{q} < \underline{q}'$ and ϕ is coherent, $EU(\phi(\underline{X})) < EU(\phi(\underline{X}'))$.

Note: $\underline{q} \leq \underline{q}'$ is the same as saying $\underline{X}$ is stochastically smaller than $\underline{X}'$ or $\underline{X} \leq_{st} \underline{X}'$.

Proof: (i) Since $\underline{q} \leq \underline{q}'$ and $\phi(\underline{X})$ is increasing in $\underline{X}$,

$$P(\phi(\underline{X}) \geq k) \leq P(\phi(\underline{X}') \geq k) \; \forall \; k.$$

The result follows by multiplying by $b_k \geq 0$ and summing over k.

(ii) Since $\underline{q} < \underline{q}'$, $P(\phi(\underline{X}) \geq k) \leq P(\phi(\underline{X}') \geq k)$.

Inequality must hold for some k since ϕ is coherent, and the result follows by multiplying by $b_k > 0$ and summing over k. To see that coherence implies inequality for some k, assume $\underline{q}$ and $\underline{q}'$ differ only in the ith row.

$$P(\phi(\underline{X}) \geq k) = \sum_{j=0}^{N_i} p_{ij} P(\phi(j_i, \underline{X}) \geq k)$$

$$= \sum_{j=0}^{N_i} q_{ij} [P(\phi(j_i, \underline{X}) \geq k) - P(\phi((j-1)_i, \underline{X}) \geq k)]$$

where $$P(\phi(-1_i, \underline{X}) \geq k) \equiv 0$$

$$= \sum_{j=0}^{N_i} q_{ij} [P(\phi(j_i, \underline{X}') \geq k) - P(\phi((j-1)_i, \underline{X}') \geq k)]$$

$$< \sum_{j=0}^{N_i} q'_{ij} [P(\phi(j_i, \underline{X}') \geq k) - P(\phi((j-1)_i, \underline{X}') \geq k)]$$

by coherence

$$= P(\phi(\underline{X}') \geq k) \quad . \square$$

Theorem 3.3 is a special case of the economic theory of stochastic dominance and utility functions. See, for example, Hadar and Russell [1969].

3.2. Reliability Importance

When deciding whether or how to improve a system, it is useful to know where a given improvement would do the most good. Obviously a budget constraint is necessary, but a good heuristic approach to

improving system reliability is to start with the component that yields the greatest increase in system reliability for a given improvement in component reliability. Several measures of component importance have been suggested. A measure for which component probability distributions are not necessary is called structural importance while reliability importance is the term given to a measure which involves those distributions. The first definitions of structural and reliability importance were proposed by Birnbaum [1969]. These measures and their attributes have been extended to the multistate case by Butler [1979] and Griffith [1980] in different ways. The theorems and proofs in those papers have been slightly modified to fit the more general multistate case presented in this thesis. Other importance measures are due to Barlow and Proschan [1975b] and Fussell [1976]. These measures and their attributes are herein extended to the multistate case. For a complete discussion of binary reliability importance, see Lambert [1975]. Throughout this section, it is assumed that the components are independent so that $h^k = h^k(\underline{P})$ and upgrading one component will not affect other components.

Definition: The Birnbaum Reliability Importance of component i in a binary system, denoted $I_h(i,\underline{P})$, is defined by

$$I_h(i,\underline{P}) \equiv \frac{\partial h(P)}{\partial P_i} = h(1_i,\underline{P}) - h(0_i,\underline{P})$$

Definition: (Butler [1979]). The r,s reliability importance of component i at level m is defined by

$$I_h^{r,s}(i,\underline{P}) = h^m(r_i,\underline{P}) - h^m(s_i,\underline{P}) = P(\phi(\underline{X}) \geq m | X_i = r)$$

where $0 \leq s < r \leq N_i$.

Theorem 3.4: (Butler [1979]). Let $\delta_0 + \delta_1 + \cdots + \delta_{N_k} = 0$, and let $0 \leq p_{kj} + \delta_j \leq 1 \; \forall \; k,j$. Define $\hat{\underline{P}}$ by

$$\hat{p}_{ij} = \begin{cases} p_{ij} & \text{if } i \neq k \\ p_{kj} + \delta_j & \text{if } i = k \end{cases} .$$

Then,

$$h(\hat{\underline{P}}) = h(\underline{P}) + \delta_{N_k} I_h^{N_k,0}(k,\underline{P}) + \cdots + \delta_1 I_h^{1,0}(k,\underline{P}) .$$

Proof:

$$h(\underline{P}) + \delta_{N_k} I_h^{N_k,0}(k,\underline{P}) + \cdots + \delta_1 I_h^{1,0}(k,\underline{P})$$

$$= \sum_{j=0}^{N_k} p_{kj} h(j_k,\underline{P}) + \delta_{N_k}[h(N_k,\underline{P}) - h(0_k,\underline{P})] + \cdots + \delta_1[h(1_k,\underline{P}) - h(0_k,\underline{P})]$$

$$= \sum_{j=1}^{N_k} (p_{kj} + \delta_j) h(j_k,\underline{P}) + p_{k0} h(0_k,\underline{P}) - h(0_k,\underline{P}) \sum_{j=1}^{N_k} \delta_j$$

$$= \sum_{j=0}^{N_k} (p_{kj} + \delta_j) h(j_k,\underline{P}) - h(0_k,\underline{P}) \sum_{j=0}^{N_K} \delta_j$$

$$= \sum_{j=0}^{N_k} \hat{p}_{kj} h(j_k,\underline{P}) = \sum_{j=0}^{N_k} \hat{p}_{kj} h(j_k,\hat{\underline{P}}) = h(\hat{\underline{P}})$$

where the last line follows since $\hat{\underline{P}}$ and $\underline{P}$ differ only in the kth row. □

Theorem 3.5: Let $\phi(\underline{X}) = \psi(\chi(\underline{X}^A), \underline{X}^{A^C})$, let $i \in A$, and let χ be a binary MSF. Then

$$I_{h_\phi}^{r,s}(i) = I_{h_\psi}^{1,0}(1) I_{h_\chi}^{r,s}(i) \quad .$$

Proof:

$$h_\phi(r_i, \underline{P}) = h_\psi(1_i, \underline{X}^{A^C}) h_\chi(r_i, \underline{X}^A) + h_\psi(0_i, \underline{X}^{A^C})[1 - h_\chi(r_i, \underline{X}^A)]$$

$$h_\phi(s_i, \underline{P}) = h_\psi(1_i, \underline{X}^{A^C}) h_\chi(s_i, \underline{X}^A) + h_\psi(0_i, \underline{X}^{A^C})[1 - h_\chi(s_i, \underline{X}^A)]$$

$$I_{h_\phi}^{r,s}(i) = h_\phi(r_i, \underline{P}) - h_\phi(s_i, \underline{P})$$

$$= [h_\psi(1_i, \underline{X}^{A^C}) - h_\psi(0_i, \underline{X}^{A^C})][h_\chi(r_i, \underline{X}^A) - h_\chi(s_i, \underline{X}^A)]$$

$$= I_{h_\psi}^{1,0}(1)\, I_{h_\chi}^{r,s}(i) \quad . \; \square$$

Definition: $\underline{X}$ is a critical vector for binary component i if $\phi(1_i, \underline{X}) - \phi(0_i, \underline{X}) = 1$. Let $n_\phi(i)$ be the number of critical vectors for component i. The Birnbaum Structural Importance of component i is

$$I_\phi(i) \equiv \frac{1}{2^n} n_\phi(i) = \frac{1}{2^n} \sum_{\underline{X}} [\phi(1_i, \underline{X}) - \phi(0_i, \underline{X})] \quad .$$

Definition: (Butler [1979]). $\underline{X}$ is an r,s critical vector at level m for component i if $\phi(r_i,\underline{X}) \geq m$ and $\phi(s_i,\underline{X}) < m$, or

$$1_{\{\phi(r_i,\underline{X})\geq m\}} - 1_{\{\phi(s_i,\underline{X})\geq m\}} = 1 \quad .$$

Let $n_\phi^{r,s}(i)$ be the number of r,s critical vectors at level m for component i. The r,s structural importance at level m for component i is

$$I_\phi^{r,s}(i) \equiv [(N_1+1)(N_2+1)\cdots(N_n+1)]^{-1}\, n_\phi^{r,s}(i) \quad .$$

Theorem 3.6: (Butler [1979]).

(i) $I_h^{N_i,0}(i) = I_h^{N_i,N_i-1}(i) + \cdots + I_h^{1,0}(i)$

(ii) $I_\phi^{N_i,0}(i) = I_\phi^{N_i,N_i-1}(i) + \cdots + I_\phi^{1,0}(i)$

(iii) $I_h^{r,s}(i,[1/N]) = I_\phi^{r,s}(i)$ where $[1/N]$ means

$$p_{ij} = 1/(N_i+1) \;\forall\; i,j \quad .$$

Proof:

$$\begin{aligned}(i)\quad I_h^{N_i,0}(i) &= h(N_i,\underline{P}) - h(0_i,\underline{P}) \\ &= [h(N_i,\underline{P}) - h((N_i-1)_i,\underline{P})] + \cdots + [h(1_i,\underline{P}) - h(0_i,\underline{P})] \\ &= I_h^{N_i,N_i-1}(i) + \cdots + I_h^{1,0}(i)\end{aligned}$$

$$\text{(ii)}\quad n_\phi^{N_i,0}(i) = \sum_{\underline{X}} \left[1_{\{\phi(N_i,\underline{X}) \geq m\}} - 1_{\{\phi(0_i,\underline{X}) \geq m\}}\right]$$

$$= \sum_{\underline{X}} \{[1_{\{\phi(N_i,\underline{X}) \geq m\}} - 1_{\{\phi((N_i-1)_i,\underline{X}) \geq m\}}]$$

$$+ \cdots + [1_{\{\phi(1_i,\underline{X}) \geq m\}} - 1_{\{\phi(0_i,\underline{X}) \geq m\}}]\}$$

$$= n_\phi^{N_i,N_i-1}(i) + \cdots + n_\phi^{1,0}(i)$$

Multiplying by $[(N_1+1)\cdots(N_n+1)]^{-1}$ yields the result.

$$\text{(iii)}\quad h(j_i,[1/N]) = \sum_{\underline{x}\in s, x_i=j} P(\phi(j_i,\underline{x}) \geq m)P(\underline{X} = \underline{x})$$

$$= [(N_1+1)\cdots(N_{i-1}+1)(N_{i+1}+1)\cdots(N_n+1)]^{-1}$$

$$\sum_{\underline{x}\in s, x_i=j} 1_{\{\phi(j_i,\underline{x}) \geq m\}}$$

$$= [(N_1+1)\cdots(N_n+1)]^{-1} \sum_{\underline{x}\in s} 1_{\{\phi(j_i,\underline{x}) \geq m\}}$$

$$I_h^{r,s}(i,[1/N]) = [(N_1+1)\cdots(N_n+1)]^{-1} \sum_{\underline{x}\in s} [1_{\{\phi(r_i,\underline{x}\} \geq m\}}$$

$$- 1_{\{\phi(s_i,\underline{x}) \geq m\}}]$$

$$= [(N_1+1)\cdots(N_n+1)]^{-1} n_\phi^{r,s}(i) = I_\phi^{r,s}(i) \quad . \square$$

A different extension of the Birnbaum reliability importance is due to Griffith [1980]. In the binary case, $h(\underline{P}) = h(0_i,\underline{P}) + p_{i1}I_h(i,\underline{P})$. Thus, knowing $I_h(i,\underline{P})$, it is easy to determine the increase in system reliability caused by upgrading a component. There is a vector in the multistate case which is analogous to $I_h(i,\underline{P})$.

Notation:

$$I_{\ell j}(i) \equiv P(\phi(\ell_i, \underline{X}) \geq j) - P(\phi((\ell-1)_i, \underline{X}) \geq j)$$

$$I_{\ell}(i) \equiv \sum_{j=1}^{M} b_j I_{\ell j}(i)$$

$$\underline{I}(i) \equiv (I_1(i), \cdots, I_{N_i}(i)) \quad .$$

Theorem 3.7: (Griffith [1980]). $EU(\phi(\underline{X})) = \sum_{j=1}^{M} b_j P(\phi(0_i, \underline{X}) \geq j) + \underline{I}(i) \cdot \underline{q}_i^t$ where $\underline{q}_i = (q_{i1}, q_{i2}, \cdots, q_{iN_i})$, ($t$ denotes transpose).

Proof:

$$P(\phi(\underline{X}) \geq j) = \sum_{\ell=0}^{N_i} P(\phi(\ell_i, \underline{X}) \geq j) p_{i\ell}$$

$$= \sum_{\ell=0}^{N_i} P(\phi(\ell_i, \underline{X}) \geq j)(q_{i\ell} - q_{i,\ell+1})$$

where $q_{i0} = 1$ and $q_{i,N_{i+1}} \equiv 0$

$$= P(\phi(0_i, \underline{X}) \geq j) + \sum_{\ell=1}^{N_i} [P(\phi(\ell_i, \underline{X}) \geq j) - P(\phi((\ell-1)_i, X) \geq j)] q_{i\ell}$$

$$EU(\phi(\underline{X})) = \sum_{j=1}^{M} b_j \, P(\phi(\underline{X}) \geq j)$$

$$= \sum_{j=1}^{M} b_j \, P(\phi(0_i,\underline{X}) \geq j) + \sum_{j=1}^{M} \sum_{\ell=1}^{N_i} b_j [P(\phi(\ell_i,\underline{X}) \geq j) - P(\phi((\ell-1)_i,\underline{X}) \geq j)] q_{i\ell}$$

$$= \sum_{j=1}^{M} b_j \, P(\phi(0_i,\underline{X}) \geq j) + \sum_{\ell=1}^{N_i} I_\ell(i) q_{i\ell}$$

$$= \sum_{j=1}^{M} b_j \, P(\phi(0_i,\underline{X}) \geq j) + \underline{I}(i) \cdot \underline{q}_i^t \quad . \; \square$$

Corollary 3.8: If component i is stochastically improved from distribution $\underline{q}_i$ to $\underline{q}_i' \geq \underline{q}_i$, the change in expected utility is

$$\Delta EU(\phi(\underline{X})) = \underline{I}(i) \cdot \underline{\Delta}^t \quad \text{where} \quad \Delta_j = q_{ij}' - q_{ij} \; \forall \; i,j \geq 1 \quad .$$

Theorem 3.9: (Griffith [1980]). If $a_j = j$, so that $EU(\phi(\underline{X})) = E\phi(\underline{X})$, then

$$\underline{I}(i) = (E\phi(1_i,\underline{X}) - E\phi(0_i,\underline{X}), \ldots, E\phi(N_i,\underline{X}) - E\phi((N_i-1)_i,\underline{X})) \quad .$$

Several other importance measures are due to Barlow and Proschan [1975b]. Their multistate extensions are defined herein.

Definition: The B-P (binary) reliability importance of component i is

$$I_{BP}(i) \equiv \int_0^\infty [h(1_i,\underline{F}(t)) - h(0_i,\underline{F}(t))] dF_i(t)$$

= probability that component i causes system failure.

A time dependent version of this measure is

$$I_{BP}(i,t) \equiv [1-h(\underline{F}(t))]^{-1} \int_0^t [h(1_i,F(t))-h(0_i,\underline{F}(t))]dF_i(t)$$

$$= \frac{\int_0^t [h(1_i,\underline{F}(t)) - h(0_i,\underline{F}(t))]dF_i(t)}{\sum_{i=1}^{n} \int_0^t [h(1_i,\underline{F}(t))-h(0_i,\underline{F}(t))]dF_i(t)}$$

= P (component i causes system failure | system is failed by time t) .

The second equality above follows from the fact that if the system is failed by time t, one of its components must have been the source of that failure.

Definition: The B-P multistate reliability importance at level m of component i is

$$I_{BP}^m(i) \equiv \sum_{k=0}^{N_i-1} \int_0^\infty [h^m((k+1)_i,\underline{F}(t))-h^m(k_i,\underline{F}(t))]dF_i^k(t) \quad .$$

Recall that $F_i^k(t) = P(X_i(t) \leq k)$.

Each term in the above sum is the probability that a transition of component i from state k+1 to k caused the system state to drop below m. $I_{BP}^m(i)$ is the probability that some transition of component i caused the system state to drop below m. Letting $\underline{P} = \underline{F}(t)$, we can relate the two multistate reliability importance measures via

$$I_{BP}^m(i) = \sum_{k=0}^{N_i-1} \int_0^\infty I_h^{k+1,k}(i,\underline{F}(t))dF_i^k(t) \quad .$$

The time dependent multistate version of this measure is

$$I^m_{BP}(i,t) \equiv [1-h^m(\underline{F}(t))]^{-1} \sum_{k=0}^{N_i-1} \int_0^t [h^m((k+1)_i\underline{F}(t))-h^m(k_i,\underline{F}(t))]dF_i^k(t)$$

$$= \frac{\sum_{k=0}^{N_i-1} \int_0^t [h^m((k+1)_i, \underline{F}(t))-h^m(k_i,\underline{F}(t))]dF_i^k(t)}{\sum_{i=1}^{n} \sum_{k=0}^{N_i-1} \int_0^t [h^m((k+1)_i,\underline{F}(t))-h^m(k_i,\underline{F}(t))]dF_i^k(t)}$$

$$= P \text{ (component } i \text{ caused the system state to be } < m \mid \text{system state at time } t \text{ is } < m) .$$

Note that $I^m_{BP}(i,\infty) = I^m_{BP}(i)$.

Theorem 3.10: If $\phi(\underline{X}) = \psi(\chi(\underline{X}^A),\underline{X}^{A^C})$, $i \in A$, and χ is a binary MSF, then

(i) $I^m_{BP}(i) = \sum_{k=0}^{N_i-1} \int_0^\infty I^{1,0}_{h_\psi}(1)\ I^{k+1,k}_{h_\chi}(i)dF_i^k(t)$

(ii) $I^m_{BP}(M) = \sum_{i\in A} I^m_{BP}(i)$ where M is the module χ of ϕ .

Proof:

(i) Letting $\underline{P} = \underline{F}(t)$, this is an immediate corollary to Theorem 3.5.

(ii) Since failure of the module at time t must be caused by a transition of one of its components at time t,

$$dF_M(t) = \sum_{i\in A} \sum_{k=0}^{N_i-1} [h_\chi((k+1)_i, \underline{X}^A) - h_\chi(k_i, \underline{X}^A)]dF_i^k(t)$$

where M means module

$$= \sum_{i\in A} \sum_{k=0}^{N_i-1} I_{h_\chi}^{k+1,k}(i)dF_i^k(t)$$

$$I_{BP}^m(M) = \int_0^\infty I_{h_\psi}^{1,0}(M)dF_M(t)$$

$$= \int_0^\infty I_{h_\psi}^{1,0}(M) \sum_{i\in A} \sum_{k=0}^{N_i-1} I_{h_\chi}^{k+1,k}(i)dF_i^k(t)$$

$$= \sum_{i\in A} I_{BP}^m(i) \quad \text{using part (i)} \quad . \quad \square$$

The same idea can be applied to determine the reliability importance of a cut set instead of a single component.

Definition: The B-P (binary) cut set importance of cut set K_j is

$$I_{BP}(K_j) \equiv \sum_{i\in K_j} \int_0^\infty h(1_i, \underline{0}^{K_j-\{i\}}, \underline{F}(t)) \prod_{\ell\in K_j-\{i\}} F_\ell(t)dF_i(t)$$

$$= P \quad \text{(failure of cut set } K_j \text{ causes system failure).}$$

Definition: The B-P multistate cut set importance of cut set $\bar{\underline{m}}^j$ (level m) is

$$I_{BP}^m(\bar{\underline{m}}^j+1) \equiv \sum_{i=1}^n \int_0^\infty h^m((\bar{\underline{m}}^j)_i, \underline{F}^{\bar{m}}(t))[\prod_{\ell\neq i} F_\ell^{\bar{m}}(t)]dF_i^{\bar{m}}(t)$$

where $F_\ell^{\bar{m}}(t) = P(X_\ell(t) \leq \bar{m}_\ell^j)$ and

$$h^m((\bar{m}_i^j+1)_i, \underline{F}^{\bar{m}}(t)) = P(\phi(\underline{X}(t)) \geq m | X_i(t) = \bar{m}_i^j+1, X_\ell(t) \leq \bar{m}_\ell^j \ \forall\ \ell \neq i)$$

Note that component ℓ may be unimportant in a particular cut set in which case

$$\bar{m}_\ell^j = N_\ell \quad \text{and} \quad \bar{F}_\ell^m(t) = 1 \; \forall \; t \quad .$$

Definition: The B-P (binary) structural importance of component i is

$$I_s(i) \equiv \int_0^\infty [h(1_i, \underline{F}(t)) - h(0_i, \underline{F}(t))]dt$$

$$= \int_0^\infty [h(1_i, \underline{P}) - h(0_i, \underline{P})]dP$$

where $F_j(t) = F_k(t) = P \; \forall \; j,k$.

The B-P structural importance of component i is derived by letting all components have the same life distribution and averaging over that distribution. This importance measure is more difficult to extend to the multistate case since the life of a component is not well defined. One possible extension is contained in the following definition.

Definition: Let all components and the system have state space $\{0,1,\ldots,M\}$. The B-P multistate structural importance at level m of component i is

$$I_s^m(i) \equiv \sum_{k=0}^{N_i-1} \sum_{\ell=0}^{M} \int_0^\infty [h^m(k+1)_i, \underline{F}^\ell(t)) - h^m(k_i, \underline{F}^\ell(t)]dF^\ell(t)$$

where $F^\ell(t) = P(X_j(t) \leq \ell) = P(X_k(t) \leq \ell) \; \forall \; j,k$.

An importance measure which is used in fault tree analysis because it is easy to calculate is described in Fussell [1976].

Definition: Let $1-h_i(\underline{F}) \equiv P(\phi_k^i(\underline{X}(t)) = 0) = P$ (all components in a cut set containing i are failed at time t)

$$\phi_k^i(\underline{X}(t)) \equiv \prod_{j=1}^{N_k^i} \coprod_{\ell \epsilon K_j} X_\ell(t)$$

where N_k^i is the number of cut sets containing component i.

The V-F (binary) reliability importance of component i is

$$I_{VF}(i) \equiv [1-h_i(\underline{F})]/[1-h(\underline{F})]$$

$$= P \text{ (component } i \text{ is contributing to system failure } | \text{ system failure by time } t) .$$

Definition: Let

$$1-h_i^{\ell,m}(\underline{F}) = P(\phi_i^{\ell,m}(t) = 1)$$

$$= P(X_k(t) \leq \bar{m}_\ell^j \ \forall\ k \text{ in a cut set at system level } m \text{ for which } \bar{m}_i^j = \ell)$$

$$\phi_i^{\ell,m}(t) \equiv \coprod_{j=1}^{N_i^{\ell,m}} \prod_{k=1}^{n} 1_{\{X_k(t) \leq \bar{m}_k^j\}}$$

where $N_i^{\ell,m} \equiv$ the number of cut sets at system level m for which $\bar{m}_i^j = \ell$.

The V-F multistate reliability importance at system level m of component i at level ℓ is

$$I_{VF}^{\ell}(i) = [1-h_i^{\ell,m}(\underline{F})]/[1-h^m(\underline{F})] \quad .$$

Definition: The V-F (binary) reliability importance of cut set K_j is

$$I_{VF}(K_j) = \prod_{i \in K_j} F_i(t)/[1-h(\underline{F})]$$

$$= P \text{ (cut set } K_j \text{ is contributing to system failure at time } t \mid \text{system failure by time } t)$$

Definition: The V-F multistate reliability importance of cut set $\underline{\bar{m}}^j$ is

$$I_{VF}^m(\underline{\bar{m}}^j) = \prod_{i=1}^{n} F_i^{\bar{m}_i^j}(t)/[1-h^m(\underline{F})] \quad .$$

The V-F importance measures are all concerned with the probability that a component is contributing to system failure. Another consideration is the probability that a component is critical to system failure, i.e., that repairing that component will allow the system to function.

Definition: The criticality importance of component i is

$$I_c(i) \equiv [h(1_i,\underline{F}) - h(0_i,\underline{F})]F_i(t)/[1-h(\underline{F})]$$

$$= P \text{ (component } i \text{ is failed at time } t \text{ and repairing it will restore the system} \mid \text{system failure by time } t)$$

Definition: The r,s criticality importance at system level m of component i is

$$I_c^{r,s}(i) \equiv [h^m(r_i,\underline{F})-h^m(s_i,\underline{F})]F_i^s(t)/[1-h^m(\underline{F})]$$

$$= P(X_i(t) \leq s, \text{ and } X_i(t) \geq r \text{ would make}$$

$$\phi(\underline{X}(t)) \geq m \mid \phi(\underline{X}(t) < m) \quad .$$

3.3. Closure Theorems

Several categories of life distributions were defined in Section 1.1. An obvious way to extend these categories to the multistate case is to replace the component life distribution $F_i(t)$ by $F_i^k(t)$ $= P(X_i(t) \leq k)$. The following definition is slightly more general.

Definition: Let $\{X_i(t), t \geq 0\}$ be a real valued stochastic process, and let $T_i^a = \inf \{t : X_i(t) \leq a\}$ be the hitting time of a. $X_i(t)$ is said to be an IFRA (IFR, DFRA, DFR, NBU, NWU, NBUE, NWUE) stochastic process if T_i^a is an IFRA (IFR, etc.) random variable $\forall$ a. The process $X_i(t)$ will usually represent the state of the ith component. This definition allows $X_i(t)$ to be a continuous random variable.

There are several reliability operations for which closure theorems have been developed in the binary case. One type of reliability operation which has been previously discussed is the formation of a coherent system from several components. A second is convolution which yields the distribution for a sum of random variables. This is used to get the distribution for the sum of the life lengths of a component and

several spares. Another reliability operation is mixture of distributions which is useful when a component may have one of several life distributions. This could occur if several manufacturers produce the same type of component, and it is not known which manufacturer produced the operating component. A closure theorem for one of these reliability operations states that if the input distributions all belong to a certain category, then the output distribution also belongs to that category. The following table shows which categories are preserved under each of the three aforementioned reliability operations.

	Reliability Operation			
Life Distribution Class	Formation of Coherent Systems	Convolution of Distributions	Arbitrary Mixtures of Distributions	Mixtures of Distributions that do not cross
IFR	NP	P	NP	NP
IFRA	P	P	NP	NP
DFR	NP	NP	P	P
DFRA	NP	NP	P	P
NBU	P	P	NP	NP
NBUE	NP	P	NP	NP
NWU	NP	NP	NP	P
NWUE	NP	NP	?	P

P = preserved NP = not preserved ? = unknown

TABLE 3.1 Closure Theorems

The proofs of the results in Table 3.1 may be found in Barlow and Proschan [1975a]. The binary counter-examples to preservation of life distribution classes presented therein obviously apply to the multistate case. It will be shown that all of the closure theorems which hold in the binary case have analogues in the multistate case. Not surprisingly, the main technique used to prove the multistate closure

theorems is the reduction of the multistate case to the binary case. The following theorem is due to Ross [1979].

Theorem 3.11 (IFRA Closure): Let $\{X_i(t), t \geq 0\}$ be independent IFRA stochastic processes, and let ϕ be a MSF. Then $\{\phi(\underline{X}(t)), t \geq 0\}$ is an IFRA stochastic process.

In this theorem, ϕ does not need to be a multistate function. It could be continuous, negative, etc., - just as long as it is increasing in its arguments. This is true for all the theorems in this section. The next theorem is due to El-Neweihi, Proschan, and Sethuraman [1978].

Theorem 3.12 (NBU Closure): Let $\{X_i(t), t \geq 0\}$ be independent NBU stochastic processes, and let ϕ be a MSF. Then $\{\phi(\underline{X}(t)), t \geq 0\}$ is a NBU stochastic process.

Definition: Let T_1 and T_2 be random variables with distributions F_1 and F_2. $T_1 + T_2$ has distribution $F(t) = \int_0^t F_1(t-x)dF_2(x)$ which is called the convolution of F_1 and F_2 and is denoted $F_1 * F_2$. In reliability T_1 and T_2 are usually the life lengths of components.

Theorem 3.13: Let $\{X_1(t), t \geq 0\}$ and $\{X_2(t), t \geq 0\}$ be independent IFR (IFRA, NBU, NBUE) stochastic processes corresponding to two components. Let k and ℓ be the respective states for which components 1 and 2 are considered failed. Then $T \equiv T_1^k + T_2^\ell$ is an IFR (IFRA, NBU, NBUE) random variable.

Proof: Since $X_1(t)$ and $X_2(t)$ are IFR stochastic processes, T_1^k and T_2^ℓ are IFR random variables. The result is then immediate from the binary case (see Theorem 4.2 of Barlow and Proschan [1975a]). □

Theorem 3.13 could be considered as the multistate extension of the closure theorem for convolutions, but it is really the same theorem with state $0 \equiv \{0,1,\ldots,k\}$ and state $1 \equiv \{k+1,\ldots,M\}$ for component 1 and a similar equivalence for component 2. Theorem 3.15 contains more of the flavor of an operating component and spares. First, the following lemma is needed.

Lemma 3.14: Let $\{X_i(t), t \geq 0\}$ be a stochastic process corresponding to the state of the ith component for which the only allowable state transitions are from a state to the next lower state. Denote the sojourn time in the kth state by $T_{k,k-1}$. If the $T_{k,k-1}$ are independent IFR (IFRA, NBU, NBUE) random variables $\forall\ k = 1, \ldots, N_i$, then $X_i(t)$ is an IFR (IFRA, NBU, NBUE) stochastic process.

Proof: $T_1^k \equiv \inf\{t : X_i(t) \leq k\} = T_{N_i,N_i-1} + \cdots + T_{k+1,k}$ is an IFR random variables since it is a sum of IFR random variables. Thus, $X_i(t)$ is on IFR stochastic process. □

Now consider a process $X(t)$ as the maximum of several processes $X_1, \ldots, X_n$ which represent components. Only one component, the one which is currently in the largest state, is in operation. Thus, transitions between the states of components not currently in service cannot occur, and the only allowable transition is to the next lower state of the operating component.

Theorem 3.15: If transitions $T_{k,k-1}$ are independent IFR (IFRA, NBU, NBUE) random variables, then the $X(t)$ process described above is an IFR (IFRA, NBU, NBUE) stochastic process.

Proof: $T^k \equiv \inf \{t : X(t) \leq k\} = T_1^k + \cdots + T_n^k$.
By Lemma 3.14, T_i^k are all independent IFR random variables. Since T^k is the sum of independent IFR random variables, it is an IFR random variable. □

Definition: Let $\{X_\alpha\}$ be a set of random variables whose index α is a random variable with distribution G. The mixture of X_α is a random variable with distribution

$$F(x) = \int_{-\infty}^{\infty} F_\alpha(x)dG(\alpha) \quad .$$

The F_α will usually represent component life distributions, and α will usually have a finite range so that

$$F(x) = \sum_{i=1}^{n} F_i(x)P(\alpha = i)$$

Theorem 3.16: Let $\{X_\alpha(t), t \geq 0\}$ be independent DFR (DFRA) stochastic processes indexed by a random variable α with distribution G. Let $X(t)$ be the mixture of $X_\alpha(t)$. Then $X(t)$ is a DFR (DFRA) stochastic process.

Proof: Let $T_\alpha^a \equiv \inf \{t : X_\alpha(t) \leq a\}$, and let F_α^a be the distribution of T_α^a. The distribution of $T^a \equiv \inf \{t : X(t) \leq a\}$ is $F^a(t) = \int_{-\infty}^{\infty} F_\alpha^a(t)dG(\alpha)$. Since each T_α^a is a DFR random variable, T^a is a DFR random variable from the binary DFR closure theorem for mixtures (see Theorem 4.3 of Barlow and Proschan [1975a]). □

Theorem 3.17: Let $\{X_\alpha(t), t \geq 0\}$ be independent NWU (NWUE) stochastic processes such that there is no crossing of F^a_α and $F^a_{\alpha'}$ on $(0,\infty)$ $\forall$ a, α, α'. Let α be a random variable with distribution G, and let $X(t)$ be the mixture of $X_\alpha(t)$. Then $X(t)$ is a NWU (NWUE) stochastic process.

Proof: Same as the preceding theorem using the binary NWU closure theorem for mixtures of distributions that do not cross (see Theorem 5.7 of Barlow and Proschan [1975a]). □

3.4. Bounds

It may often be difficult or time-consuming to compute exact system reliability, especially for large systems. Thus, it can be useful to have upper and lower reliability bounds. If the lower bound is sufficiently large, no further calculations will be necessary to satisfy reliability requirements. If the upper bound is too low, this may be an indication of problems in the system design. In the multistate case, system reliability is the probability that the system structure function meets or exceeds a certain level. Several upper and lower bounds on system reliability are given in Section 3.4.1. These include the use of modular decomposition which helps make the computations feasible for large systems and often leads to improved bounds. Bounds on expected system utility are discussed in Section 3.4.2. These are simple extensions of the bounds in Section 3.4.1 and are useful since expected system utility is a better measure of performance than reliability for multistate systems. Bounds exploiting

the properties of various life distributions are contained in Section 3.4.3. Some of these bounds are formulated specifically for multistate components, and some are for multistate systems.

3.4.1. System Reliability Bounds

The bounds in this section pertain to $h^k = P(\phi(\underline{X}) \geq k)$. The first paper to consider multistate reliability bounds was Postelnicu [1970]. The results from this paper have been slightly modified and are contained in Theorem 3.18.

Theorem 3.18: Let $\mu_i(\underline{X}) = \phi(X_i, \underline{0})$, and let $\nu_i(\underline{X}) = \phi(X_i, \underline{1})$. Let $\bar{F}^{\mu_i}(k) = P(\mu_i(\underline{X}) \geq k)$, and let $\bar{F}^{\nu_i}(k) = P(\nu_i(\underline{X}) \geq k)$. If ϕ is a MSF with independent components, then

(i) $P(\mu_i(\underline{X}) \geq k) \leq h^k(\underline{P}) \leq P(\nu_i(\underline{X}) \geq k) \; \forall \; i$ and k

(ii) $\bar{F}_1^{\mu_1} * \bar{F}^{\mu_2} * \cdots * \bar{F}^{\mu_n}(nk) \leq h^k(\underline{P}) \leq \bar{F}^{\nu_1} * \cdots * \bar{F}^{\nu_n}(nk)$

(iii) $\coprod_{i=1}^{n} P(\mu_i(\underline{X}) \geq k) \leq h^k(\underline{P}) \leq \prod_{i=1}^{n} P(\nu_i(\underline{X}) \geq k)$.

Proof:

(i) $\mu_i(\underline{X}) = \phi(X_i, \underline{0}) \leq \phi(\underline{X}) \leq \phi(X_i, \underline{1}) = \nu_i(\underline{X}) \; \forall \; i.$

The result follows immediately.

(ii) Summing on i and dividing by n in the equation from part (i) yields

$$\frac{\mu_1(\underline{X}) + \cdots + \mu_n(\underline{X})}{n} \leq \phi(\underline{X}) \leq \frac{\nu_1(\underline{X}) + \cdots + \nu_n(\underline{X})}{n} \quad .$$

The result follows from properties of convolutions.

(iii) Since $\max[\mu_1(\underline{X}),\ldots,\mu_n(\underline{X})] \leq \phi(\underline{X}) \leq \min[\nu_1(\underline{X}),\ldots,\nu_n(\underline{X})]$,

$P(\max[\mu_1(\underline{X}),\ldots,\mu_n(\underline{X})] \geq k] \leq h^k(\underline{P}) \leq P(\min[\nu_1(\underline{X}),\ldots,\nu_n(\underline{X})] \geq k)$.

The result follows from the independence of the X_i. □

Bounds on multistate system reliability were also obtained by Barlow and Wu [1978], El-Neweihi, Proschan, and Sethuraman [1978], and Hatoyama [1978]. These bounds and many others are contained in Butler [1982]. The rest of the theorems in this section, except Theorem 3.19, come from Butler's paper, and any proofs which are omitted may be found therein. The following definition may also be found in Section 1.1 and describes components which have similar rather than independent behavior. This might be useful to describe components in a common environment.

Definition: Random variables $X_1, \ldots, X_n$ are associated if

$$COV(\Gamma(\underline{X}), \Delta(\underline{X})) \geq 0 \quad \forall \text{ pairs of increasing binary functions } \Gamma \text{ and } \Delta \text{ (assuming the covariance exists).}$$

Associated random variables have the following properties. Proofs may be found in Esary, Proschan, and Walkup [1967] or Barlow and Proschan [1975a].

(1) Subsets of associated random variables are associated.

(2) Independent random variables are associated.

(3) Increasing functions of associated random variables are associated.

(4) If $X_1, \ldots, X_n$ are associated random variables, then $\forall\ X_1, \ldots, X_n$

$$P(X_1 > x_1, \ldots, X_n > x_n) \geq \prod_{i=1}^{n} P(X_i > x_i), \text{ and}$$

$$P(X_1 \leq x_1, \ldots, X_n \leq x_n) \geq \prod_{i=1}^{n} P(X_i \leq x_i).$$

Theorem 3.19 (Series - Parallel Bounds): Let ϕ be a MSF for which $\min_i (X_i) \leq \phi(\underline{X}) \leq \max_i (X_i)$, and let X_i, $i = 1, \ldots, n$ be associated. Then

$$\prod_{i=1}^{n} P(X_i \geq m) \leq h^m \leq \coprod_{i=1}^{n} P(X_i \geq m) \quad .$$

Proof: Since the X_i are associated, $1_{\{X_i \geq m\}}$ are associated by property (3). Since $\min_i (X_i) \leq \phi(\underline{X}) \leq \max_i (X_i)$,

$$\prod_{i=1}^{n} 1_{\{X_i \geq m\}} \leq 1_{\{\phi(\underline{X}) \geq m\}} \leq \coprod_{i=1}^{n} 1_{\{X_i \geq m\}} \quad .$$

Taking expectations and using property (4) yields

$$\prod_{i=1}^{n} P(X_i \geq m) \leq P(X_1 \geq m, \ldots, X_n \geq m)$$

$$= E\left[\prod_{i=1}^{n} 1_{\{X_i \geq m\}}\right] \leq h^m \leq E\left[\coprod_{i=1}^{n} 1_{\{X_i \geq m\}}\right]$$

$$= 1 - P(X_1 < m, \ldots, X_n < m)$$

$$\leq 1 - \prod_{i=1}^{n} P(X_i < m) = \coprod_{i=1}^{n} P(X_i \geq m). \square$$

Theorem 3.20 (Path-Cut Bounds): Let ϕ be a MSF with associated components $X_1, \ldots, X_n$. Then

$$\prod_{j=1}^{t} P(\kappa_j(\underline{X}) = 1) \leq h^m \leq \coprod_{j=1}^{s} P(\rho_j(\underline{X}) = 1) \quad .$$

Proof: $\kappa_1(\underline{X}), \ldots, \kappa_t(\underline{X})$ and $\rho_1(\underline{X}), \cdots, \rho_s(\underline{X})$ are increasing functions of $\underline{X}$ and thus associated by property (3). Using Theorem 2.3, the fact that κ_i and ρ_i are binary, and property (4) of association yields

$$h^m = P\left(\prod_{j=1}^{t} \kappa_j(\underline{X}) = 1\right) = P(\kappa_1(\underline{X}) > 0, \ldots, \kappa_t(\underline{X}) > 0)$$

$$\geq \prod_{j=1}^{t} P(\kappa_j(\underline{X}) = 1) \quad .$$

$$h^m = P\left(\coprod_{i=1}^{n} \rho_i(\underline{X}) = 1\right) = 1 - P(\rho_1(\underline{X}) \leq 0, \ldots, \rho_s(\underline{X}) \leq 0)$$

$$\leq 1 - \prod_{i=1}^{s} P(\rho_i(\underline{X}) \leq 0) = \coprod_{i=1}^{s} P(\rho_i(\underline{X}) = 1) \quad . \square$$

Corollary 3.21 (Path-cut bounds for independent components): If the X_i in the preceding theorem are independent, then

$$\prod_{j=1}^{t} \coprod_{i=1}^{n} P(X_i > \bar{m}_i^j) \leq h^m(\underline{P}) \leq \coprod_{j=1}^{s} \prod_{i=1}^{n} P(X_i \geq m_i^j) \quad .$$

Proof: From independence

$$P(\rho_j(\underline{X}) = 1) = \prod_{i=1}^{n} P(X_i \geq m_i^j), \quad \text{and}$$

$$P(\kappa_j(\underline{X}) = 1) = 1 - \prod_{i=1}^{n} P(X_i \leq \bar{m}_i^j) = \coprod_{i=1}^{n} P(X_i > \bar{m}_i^j) \quad . \square$$

Note that equality does not necessarily hold in Corollary 3.21 as shown by Example 2.7.

Theorem 3.22 (Max-min bounds): Let ϕ be a MSF. Then

$$\max_j P(\rho_j(\underline{X}) = 1) \leq h^m \leq \min_j P(\kappa_j(\underline{X}) = 1)$$

Proof: From Theorem 2.3, $\rho_j(\underline{X}) \leq 1\{\phi(\underline{X}) \geq m\} \leq \kappa_i(\underline{X})$ $\forall$ i,j. The result follows by taking expectations, maximizing over j, and minimizing over i. $\square$

Corollary 3.23 (Max-min bounds for associated components): If ϕ is a MSF and $X_1, \ldots, X_n$ are associated, then

$$\max_j \prod_{i=1}^{n} P(X_i \geq m_i^j) \leq h^m \leq \min_j \coprod_{i=1}^{n} P(X_i > \bar{m}_i^j) \quad .$$

Lemma 3.24 (Bonferroni Inequalities): The following equations are true for arbitrary events $E_1, \ldots, E_n$.

(i) $P(E_1 \cup E_2 \cup \cdots \cup E_n) \leq \sum_{j=1}^{n} P(E_j)$

(ii) $P(E_1 \cup E_2 \cup \cdots \cup E_n) \geq \sum_{j=1}^{n} P(E_j) - \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} P(E_j \cap E_k)$.

Theorem 3.25 (Bonferroni Bounds): Let ϕ be a MSF.

(i) $h^m \leq \sum_{j=1}^{s} P(\rho(\underline{X}) = 1)$.

(ii) $h^m \geq \sum_{j=1}^{s} P(\rho_j(\underline{X}) = 1)$

$$- \sum_{j=1}^{s-1} \sum_{k=j+1}^{s} P(\rho_j(\underline{X}) = \rho_k(\underline{X}) = 1).$$

(iii) $h^m \geq 1 - \sum_{j=1}^{t} P(\kappa_j(\underline{X}) = 0)$.

(iv) $h^m \leq 1 - \sum_{j=1}^{t} P(\kappa_j(\underline{X}) = 0)$

$$+ \sum_{j=1}^{t-1} \sum_{k=j+1}^{t} P(\kappa_j(\underline{X}) = \kappa_k(\underline{X}) = 0).$$

Proof: Apply the Bonferroni inequalities to

$$h^m = P(\{\rho_1(\underline{X}) = 1\} \cup \cdots \cup \{\rho_s(\underline{X}) = 1\}) = 1 - P(\phi(\underline{X}) < m)$$

$$= 1 - P(\{\kappa_1(\underline{X}) = 0\} \cup \cdots \cup \{\kappa_t(\underline{X}) = 0\}) \quad . \square$$

Note that the idea in Theorem 3.25 could be extended by adding more terms in the Bonferroni inequalities, e.g.,

$$P(E_1 \cup \cdots \cup E_n) \leq \sum_{j=1}^{n} P(E_j) - \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} P(E_j \cap E_k)$$

$$+ \sum_{j=1}^{n-2} \sum_{k=j+1}^{n-1} \sum_{\ell=k+1}^{n} P(E_j \cup E_k \cup E_\ell) \quad .$$

Corollary 3.26 (Bonferroni bounds for independent components): If ϕ is a MSF and $X_1, \ldots, X_n$ are independent, then

$$\text{(i)} \quad h^m(\underline{P}) \leq \sum_{j=1}^{s} \prod_{i=1}^{n} P(X_i \geq m_i^j) \quad .$$

$$\text{(ii)} \quad h^m(\underline{P}) \geq \sum_{j=1}^{s} \prod_{i=1}^{n} P(X_i \geq m_i^j) - \sum_{j=1}^{s-1} \sum_{k=j+1}^{s} \prod_{i=1}^{n} P(X_i \geq \max (m_i^j, m_i^k)).$$

$$\text{(iii)} \quad h^m(\underline{P}) \geq 1 - \sum_{j=1}^{t} \prod_{i=1}^{n} P(X_i \leq \bar{m}_i^j).$$

$$\text{(iv)} \quad h^m(\underline{P}) \leq 1 - \sum_{j=1}^{t} \prod_{i=1}^{n} P(X_i \leq \bar{m}_i^j) + \sum_{j=1}^{t-1} \sum_{k=j+1}^{t} \prod_{i=1}^{n} P(X_i \leq \min (\bar{m}_i^j, \bar{m}_i^k)).$$

Example 3.1: Let $\phi(X_1, X_2, X_3)$ be parallel at level 1, 2-out-of-3 at level 2, and series at level 3. Let $\underline{P}_1 = (.1, .1, .4, .4)$, $\underline{P}_2 = (.2, .2, .2, .4)$, and $\underline{P}_3 = (.1, .2, .2, .5)$. Calculations of exact system reliability are shown below.

$$P(\phi(\underline{X}) \geq 1) = \coprod_{i=1}^{3} P(X_i \geq 1) = .998$$

$$P(\phi(\underline{X}) \geq 2) = P(X_1 \geq 2, X_2 \geq 2, \text{ or } X_1 \geq 2, X_3 \geq 2, \text{ or } X_2 \geq 2, X_3 \geq 2) = .788$$

$$P(\phi(\underline{X}) = 3) = \prod_{i=1}^{3} p_{i3} = .08 \quad .$$

Table 3.2 contains several upper and lower bounds for this system. Notice that several of the bounds are exact at level 1 and level 3. This is reasonable since series and parallel systems are very simple systems. The best bounds at level 2 are the path-cut lower bound (.761) and Bonferroni upper bound (.812). Bonferroni bounds may be outside the unit interval and may not be monotonic in the system level. □

Upper Bounds at	Theorem 3.19C	Series-Parallel	Path-Cut	Max-Min	Bonferroni (i)and(ii)	Bonferroni (iii)and(iv)	Actual Value
Level 1	1	.998	.998	.998	2.6	.998	.998
Level 2	1	.976	.867	.88	1.46	.812	.788
Level 3	.08	.82	.08	.4	.08	.26	.08
Lower Bounds at							
Level 1	.998	.576	.998	.9	.35	.998	.998
Level 2	0	.336	.761	.56	.452	.74	.788
Level 3	0	.08	.08	.08	.08	-.7	.08

Table 3.2 Bounds

For large systems, exact reliability calculations or even computation of the bounds in this section may be difficult. By exploiting modular decomposition, this computational effort can be greatly decreased. A second benefit of modular decompositions is that the resulting reliability bounds are at least as good as the bounds obtained by considering the whole system. The obvious way to precede is to establish upper and lower bounds on the reliability of a module at each level and then use those bounds as if they were the actual probabilities in the organizing structure.

Notation: Let ϕ be a MSF composed of independent components with modular decomposition $\phi(\underline{X}) = \psi(\chi_1(A_1), \ldots, X_r(A_r))$.

$$D_\phi^\ell(\underline{q}) \equiv \prod_{j=1}^{t} \coprod_{i=1}^{n} P(X_i > \bar{\ell}_i^j) \qquad E_\phi^\ell(\underline{q}) \equiv \coprod_{j=1}^{s} \prod_{i=1}^{n} P(X_i \geq \ell_i^j))$$

$$L_\phi^m(\underline{q}) \equiv \max_{m \leq \ell \leq M} (D_\phi^\ell(\underline{q})) \qquad U_\phi^m(\underline{q}) \equiv \min_{0 \leq \ell \leq m} (E_\phi^\ell(\underline{q}))$$

$$L_\phi(\underline{q}) \equiv (L_\phi^0(\underline{q}), \ldots, L_\phi^M(\underline{q})) \qquad U_\phi(\underline{q}) \equiv (U_\phi^0(\underline{q}), \ldots, U_\phi^M(\underline{q}))$$

$$H_\phi(\underline{q}) \equiv (h_\phi^0(\underline{P}), \ldots, h_\phi^M(\underline{P}))$$

$$H_{\underline{\chi}}(\underline{q}) \equiv \begin{bmatrix} H_{\chi_1}(\underline{q}) \\ \vdots \\ H_{\chi_r}(\underline{q}) \end{bmatrix} \quad L_{\underline{\chi}}(\underline{q}) \equiv \begin{bmatrix} L_{\chi_1}(\underline{q}) \\ \vdots \\ L_{\chi_r}(\underline{q}) \end{bmatrix} \quad U_{\underline{\chi}}(\underline{q}) \equiv \begin{bmatrix} U_{\chi_1}(\underline{q}) \\ \vdots \\ U_{\chi_r}(\underline{q}) \end{bmatrix}$$

It can be shown that $D_\phi^m(\underline{q}) \neq L_\phi^m(\underline{q})$ and that $E_\phi^m(\underline{q}) \neq U_\phi^m(\underline{q})$ in general. $D_\phi^\ell(\underline{q})$ and $E_\phi^\ell(\underline{q})$ are the path-cut lower and upper bounds at level ℓ for independent components. Note that $H_{\underline{\chi}}$, $L_{\underline{\chi}}$, and $U_{\underline{\chi}}$ are matrices with r rows and $M+1$ columns.

Theorem 3.27: Let ϕ be a MSF composed of independent components with modular decomposition $\phi(\underline{X}) = \psi(\chi_1(A_1), \ldots, \chi_r(A_r))$.

$$\text{(i)} \quad H_\phi(\underline{q}) = H_\psi(H_{\underline{\chi}}(\underline{q})) \geq \begin{Bmatrix} L_\psi(H_{\underline{\chi}}(\underline{q})) \\ H_\psi(L_{\underline{\chi}}(\underline{q})) \end{Bmatrix} \geq L_\psi(L_{\underline{\chi}}(\underline{q})) \geq L_\phi(\underline{q})$$

$$\text{(ii)} \quad H_\phi(\underline{q}) = H_\psi(H_{\underline{\chi}}(\underline{q})) \leq \begin{Bmatrix} U_\psi(H_{\underline{\chi}}(\underline{q})) \\ H_\psi(U_{\underline{\chi}}(\underline{q})) \end{Bmatrix} \leq U_\psi(U_{\underline{\chi}}(\underline{q})) \leq U_\phi(\underline{q})$$

Note that the equations in Theorem 3.27 show that the exact system reliability calculated by using modular decomposition is the same as that obtained by considering the whole system. This must be true from the definition of a module. Also note that the path-cut bounds obtained by modular decomposition are better than the original path-cut bounds, $L_\phi(\underline{q})$ and $U_\phi(\underline{q})$. Butler [1982] has an example of the calculations involved for each of the quantities in Theorem 3.27.

3.4.2. System Utility Bounds

The bounds in Section 3.4.1 were bounds on the probability that the system meets or exceeds a certain level. A useful facet of multistate systems is the allowance for partial operation. For example the system states might be the percent of rated power generated at a coal plant. In this case there is no minimum level which the system must meet or exceed, but there are often requirements on the long run average capacity. Letting states $0, 1, \ldots, 100$ be the percent of rated power generated, a lower bound on $\sum_{k=0}^{100} kP(\phi(\underline{X})=k)$ is desired. More generally, bounds on $\sum_{k=0}^{M} a_k P(\phi(\underline{X}) = k)$, where a_k is the utility of state k, would be useful.

Theorem 3.28: Let ϕ be a coherent MSF. Let a_k be the utility of state k with $a_0 = 0$, and let $b_k = a_k - a_{k-1}$ for $k = 1, \ldots, M$. Let $L_k \leq P(\phi(\underline{X}) \geq k) \leq U_k \ \forall\ k$. Then,

$$\sum_{k=1}^{M} b_k L_k \leq EU(\phi(\underline{X})) \leq \sum_{k=1}^{M} b_k U_k \quad .$$

Proof: Multiplying $L_k \leq P(\phi(\underline{X}) \geq k) \leq U_k$ by $b_k \geq 0$, summing over k, and using $EU(\phi(\underline{X})) = \sum_{k=1}^{M} b_k P(\phi(\underline{X}) \geq k)$ yields the result. □

The best way to use Theorem 3.28 is to set $L_k = \max(L_k^1, \ldots, L_k^n)$ and $U_k = \min(U_k^1, \ldots, U_k^n)$ where L_k^i and U_k^i are the lower and upper bounds obtained in Section 3.4.1.

Example 3.2: Consider the system from Example 3.1 with $a_k = k$ (or $b_k = 1$).

$$EU(\phi(\underline{X})) = .998 + .788 + .08 = 1.866 \quad .$$

The best upper and lower bounds in Table 3.2 are:

$$U_1 = L_1 = .998$$

$$U_2 = .812, \; L_2 = .761$$

$$U_3 = L_3 = .08$$

$$\sum_{k=1}^{3} b_k L_k = 1.829 \quad \text{and} \quad \sum_{k=1}^{3} b_k U_k = 1.88 \quad .$$

Theorem 3.28 becomes $1.829 \leq 1.866 \leq 1.88$. Both upper and lower bounds are with 2% of the actual value. □

It is possible for a multistate system to be non-coherent. For example, if the state of the system is temperature, there may be a range of temperature which is acceptable while both high and low temperatures are unacceptable. In this case system reliability might be defined as $P(\phi(\underline{X}) \geq k) - P(\phi(\underline{X}) \geq \ell)$ where $\ell - 1$ is the upper temperature limit and k is the lower temperature limit. This could be accomplished by setting $a_j = 0$ for $j < k$ or $j \geq \ell$ and $a_j = 1$ for $j = k, k+1, \ldots, \ell - 1$. Theorem 3.29 provides for bounds on non-coherent systems.

Theorem 3.29: Let ϕ be a MSF, and let $L_k \leq P(\phi(\underline{X}) \geq k) \leq U_k \ \forall\, k$. Let $K^+ = \{k : b_k \geq 0\}$, and $K^- = \{k : b_k \leq 0\}$ where $b_k = a_k - a_{k-1}$. Then

$$\sum_{k\in K^+} b_k L_k + \sum_{k\in K^-} b_k U_k \leq EU(\phi(\underline{X})) \leq \sum_{k\in K^+} b_k U_k + \sum_{k\in K^-} b_k L_k \ .$$

Proof: For $k \in K^+$, $b_k L_k \leq b_k P(\phi(\underline{X}) \geq k) \leq b_k U_k$.

For $k \in K^-$, $b_k L_k \geq b_k P(\phi(\underline{X}) \geq k) \geq b_k U_k$.

Summing on k yields the result. □

Theorem 3.28 is obviously a special case of Theorem 3.29 with K^- being the null set for a coherent system.

3.4.3. Bounds from Life Distribution Processes

In this section properties of life distribution processes are used to determine reliability bounds. In the first part of this section, it is assumed that the underlying distribution is known to be IFR, IFRA, DFR, or DFRA and that one parameter of the distribution, e.g., a moment or a quantile, is known. This kind of analysis is useful in determining bounds on the reliability of binary components. If a component is known to have an IFR distribution due to wear-out or a DFR distribution due to burn-in, it may be possible to estimate the mean component lifetime and then use these bounds. This analysis can also be applied to the multistate case, but it may not be as useful because of a lack of data for a parameter of the distribution $F_i^k(t)$. The upper and lower bounds on component reliability can be used to determine upper and lower bounds on system reliability. In the second part of this

section, bounds for mean system lifetime are obtained. If it was known that h^k had an appropriate distribution with a known parameter, then the bounds from the first part of this section could be directly applied to system reliability. This is unlikely, however.

Theorems 3.30, 3.31, 3.32, and 3.33 appear in Barlow and Proschan [1975a] and are intended only as a representative sample of the many bounds contained in reliability literature.

Theorem 3.30: Let F be IFRA (DFRA) with pth quantile ξ_p, i.e., $F(\xi_p = p)$.

$$\bar{F}(t) \begin{cases} \geq (\leq)\ e^{-\alpha t} & \text{for} \quad 0 \leq t \leq \xi_p \\ \leq (\geq)\ e^{-\alpha t} & \text{for} \quad t \geq \xi_p \end{cases}$$

where $\alpha = -(1/\xi_p)\ \ln(1-p)$.

Theorem 3.31: Let F be DFR with mean μ.

$$\bar{F}(t) \leq \begin{cases} e^{-t/\mu} & \text{for} \quad t \leq \mu \\ \mu e^{-1}/t & \text{for} \quad t \geq \mu \end{cases} .$$

Theorem 3.32: Let F be IFRA with mean μ.

$$\bar{F}(t) \leq \begin{cases} 1 & \text{for} \quad t \leq \mu \\ e^{-wt} & \text{for} \quad t \geq \mu \end{cases}$$

where $w > 0$ satisfies $1 - w\mu = e^{-wt}$.

Theorem 3.33: Let F be a continuous IFR distribution with rth moment μ_r, $r \geq 1$.

$$\bar{F}(t) \geq \begin{cases} \exp(-t/\lambda_r^{1/r}) & \text{for } t < \mu_r^{1/r} \\ 0 & \text{for } t \geq \mu_r^{1/r} \end{cases}$$

where $\lambda_r = \mu_r/\Gamma(r+1)$.

If $\{X_i(t), t \geq 0\}$ is an IFR (IFRA, etc.) stochastic process, then the distribution $F_i^k(t)$ is IFR (IFRA, etc.) for any k. If we know the appropriate parameter of the distribution $F_i^k(t)$ for any level k, then the appropriate bound may be applied. The bounds may also be useful if $F_i^k(t)$ is IFR (IFRA, etc.) for a particular k even though $X_i(t)$ is not an IFR (IFRA, etc.) stochastic process.

In the binary case it is easy to obtain a bound on system reliability from the bounds on component reliability. For example if component 1 has an IFR distribution with known mean and component 2 has an IFRA distribution with a known quantile, and the components are independent, applying Theorems 3.30 and 3.33 yields

$$h(F_1, F_2) \geq h(e^{-t/\lambda_1}, e^{-\alpha t}) \quad \text{for } 0 \leq t < \min(\xi_p, \mu_1) \ .$$

To generalize this to multistate systems, a lower bound would have to be specified for several states of each component. Also, the bounds on $F_i^k(t)$ and $F_i^j(t)$ would probably be correlated so that the calculation of h^k might not be straightforward. A way to avoid this

problem is to use the component bounds together with the bounds on system reliability from Section 3.4.1 as illustrated in Example 3.3.

Example 3.3: *Consider* $\phi(X_1,X_2)$ *where* X_1 *and* X_2 *are independent* ternary components. Let ϕ be parallel at level 1 and series at level 2.

$$\phi(0,0) = 0 \qquad\qquad \phi(2,2) = 2$$

$$\phi(0,1) = \phi(1,0) = \phi(1,1) = \phi(2,0) = \phi(0,2) = \phi(1,2) = \phi(2,1) = 1 \quad .$$

At level 1, the min paths are $(0,1)$ and $(1,0)$, and the only min cut is $(0,0)$.

At level 2, the only min path is $(2,2)$, and the min cuts are $(1,2)$ and $(2,1)$.

Applying the max-min bounds for associated components (Corollary 3.23) yields

$$\max_{j=1,2} \left(\prod_{i=1}^{2} P(X_i \geq 1_i^j) \right) \leq h^1 \leq \coprod_{i=1}^{2} P(X_i > \bar{1}_i^1) \qquad \text{or}$$

$$\max (P(X_1 \geq 1), P(X_2 \geq 1)) \leq h^1 \leq P(X_1>0) + P(X_2>0) - P(X_1>0)P(X_2>0)$$

$$\prod_{i=1}^{2} P(X_i \geq 2_i^j) \leq h^2 \leq \min_{j=1,2} \left(\coprod_{i=1}^{2} P(X_i > \bar{2}_i^j) \right) \qquad \text{or}$$

$$P(X_1=2)P(X_2=2) \leq h^2 \leq \min (P(X_1=2), P(X_2=2)) \quad .$$

Now assume that $X_1(t)$ and $X_2(t)$ are IFR stochastic processes and that F_i^k *has mean* μ_i^k. From Theorem 3.33,

$$\bar{F}_i^k(t) \geq e^{-t/\mu_i^k} \quad \text{for} \quad t < \mu_i^k \quad .$$

Using this result and the max-min bounds yields

$$\max\left(e^{-t/\mu_1^0}, e^{-t/\mu_2^0}\right) \leq h^1(\underline{F}(t)) \quad \text{for} \quad t < \min(\mu_1^0, \mu_2^0)$$

$$e^{-t/\mu_1^1} e^{-t/\mu_2^1} \leq h^2(\underline{F}(t)) \quad \text{for} \quad t < \min(\mu_1^1, \mu_2^1) \quad .$$

Since $F_i^k(t)$ is IFR, it is also IFRA, and Theorem 3.32 applies to give

$$\bar{F}_i^k(t) \leq e^{-w_i^k t} \quad \text{for} \quad t > \mu_i^k \quad \text{where} \quad w_i^k > 0 \quad \text{satisfies}$$

$$1 - w_i^k \mu_i^k = e^{-w_i^k t} \quad .$$

Using these inequalities in the max-min upper bounds yields

$$h^1(\underline{F}(t)) \leq e^{-w_1^0 t} + e^{-w_2^0 t} - e^{-w_1^0 t} e^{-w_2^0 t} \quad \text{for} \quad t > \max(\mu_1^0, \mu_2^0)$$

$$h^2(\underline{F}(t)) \leq \min\left(e^{-w_1^1 t}, e^{-w_2^1 t}\right) \quad \text{for} \quad t > \max(\mu_1^1, \mu_2^1) \quad . \square$$

The technique in Example 3.3 can be used to combine the bounds in this section and Section 3.4.1 to derive bounds on system reliability. The results in Section 3.4.2 could then be applied to get bounds on expected system utility. There are also some bounds which apply

directly to mean system lifetime in the binary case. In the multistate case these bounds apply to the mean time that the system spends above a certain level. The following definition will be useful in this development.

Definition: Let F and G be continuous distributions, let G be strictly increasing on its support, and let $F(0) = G(0) = 0$. F is star-shaped with respect to G, written $F \underset{*}{<} G$, if $(1/x)\, G^{-1}(F(x))$ is increasing in $x \geq 0$. Some properties of this ordering are:

(1) $F \underset{*}{<} G$ is scale invariant but not translation invariant.

(2) Let $G(x) = 1 - e^{-\lambda x}$. Then $F \underset{*}{<} G$ is equivalent to F being IFRA.

(3) If $F \underset{*}{<} G$, then $\bar{F}(x)$ crosses $\bar{G}(\theta x)$ at most once as x increases from 0 to ∞ for each $\theta > 0$. If a crossing occurs, it occurs from above. If F and G have the same mean, the crossing must occur.

Theorems 3.34, 3.35, and 3.36 apply to binary systems of binary components and may be found in Barlow and Proschan [1975a].

Theorem 3.34: Let $X_i(Y_i)$ have distribution $F_i(G_i)$ with mean μ_i, and let $F_i \underset{*}{<} G_i$, $i = 1, 2, \ldots, n$. Let $X_1, \ldots, X_n$ be associated, let $Y_1, \ldots, Y_n$ be associated, and let X_i be independent of Y_j $\forall$ i,j. Then the mean life of a series system using components with lifetimes $X_1, \ldots, X_n$ is greater than the mean life of a series system using components with lifetimes $Y_1, \ldots, Y_n$ while the opposite is true for a parallel system. More precisely:

$$E(\min(X_1,\ldots,X_n)) \geq E(\min(Y_1,\ldots,Y_n)) \text{ and}$$

$$E(\max(X_1,\ldots,X_n)) \leq E(\max(Y_1,\ldots,Y_n)) .$$

Theorem 3.35 is derived using Theorem 3.34 and property (2) of star-shaped ordering.

Theorem 3.35: Let μ_s be the mean life of a series system, and let μ_p be the mean life of a parallel system composed of n associated components. Let the ith component have an IFRA (DFRA) marginal distribution with mean μ_i.

$$\text{(i)} \quad \mu_s \geq (\leq) \int_0^\infty \prod_{i=1}^n e^{-t/\mu_i}\, dt = \left(\sum_{i=1}^n 1/\mu_i\right)^{-1}$$

$$\text{(ii)} \quad \mu_p \leq (\geq) \int_0^\infty \coprod_{i=1}^n e^{-t/\mu_i}\, dt \quad .$$

Theorem 3.36: Let ϕ be a binary coherent system with min cut sets $K_1, \ldots, K_t$ and min path sets $P_1, \ldots, P_s$. Let component lifetimes be associated, and let the ith component have an IFRA marginal distribution with mean μ_i. Let T be the system lifetime.

$$\max_j \left[\sum_{i\in P_j} 1/\mu_i\right]^{-1} \leq ET \leq \min_j \int_0^\infty \coprod_{i\in K_j} e^{-t/\mu_i}\, dt \quad .$$

To apply these results to the multistate case, let X_i^k be the time that component i spends at or above state k, and let the system be series (parallel) at level k. Assuming that the hypotheses of Theorem 3.34 hold for X_i^k and Y_i^k, the mean time spent at or above

system level k by a series system at level k using component lifetimes $X_1^k, \ldots, X_n^k$ is greater than that obtained by using $Y_1^k, \ldots, Y_m^k$. The opposite is true for a parallel system at level k.

Theorem 3.35 (Multistate Version): Let $\mu_s^k (\mu_p^k)$ be the mean life of a series (parallel) system at level k, and let the ith component have an IFRA (DFRA) marginal distribution at level k with mean μ_i^k.

$$\text{(i)} \quad \mu_s^k \geq (\leq) \int_0^\infty \prod_{i=1}^n e^{-t/\mu_i^k} dt = \left(\sum_{i=1}^n 1/\mu_i^k \right)^{-1}$$

$$\text{(ii)} \quad \mu_p^k \leq (\geq) \int_0^\infty \coprod_{i=1}^n e^{-t/\mu_i^k} dt \quad .$$

Theorem 3.36 (Multistate version): Let ϕ be a MSF with minimum cut structure functions $\kappa_1, \ldots, \kappa_t$ at level m and minimum path structure functions $\rho_1, \ldots, \rho_s$ at level m. Let the time that component i spends at or above level j be IFRA with mean μ_i^j, and let all such times be associated. Let T^m be the time that the system spends at or above level m. Then

$$\max_{1 \leq j \leq s} \left[\sum_{i=1}^n 1/\lambda_i^j \right]^{-1} \leq ET^m \leq \min_{1 \leq j \leq t} \int_0^\infty \coprod_{i=1}^n e^{-t/\bar{\lambda}_i^j} dt$$

where $\lambda_i^j \equiv \mu_i^{m^j}$ and $\bar{\lambda}_i^j \equiv \mu_i^{\bar{m}^j + 1}$.

By convention $\mu_i^0 = \infty$, so $1/\mu_i^0 = 0$.

Proof: $E(\text{time } \rho_i(\underline{X}(t)) = 1) \leq ET^m \leq E(\text{time } \kappa_j(\underline{X}(t)) = 1)$ $\forall$ i,j so

$$\max_j E(\text{time } \rho_j(\underline{X}(t)) = 1) \leq ET^m \leq \min_j E(\text{time } \kappa_j(\underline{X}(t)) = 1) .$$

The result follows from the multistate version of Theorem 3.35 since

$$E(\text{time } \rho_j(\underline{X}(t)) = 1) = E(\text{time } X_i \geq m_i^j \ \forall\ i) \geq \Big(\sum_{i=1}^{n} 1/\lambda_i^j\Big)^{-1}, \text{ and}$$

$$E(\text{time } \kappa_j(\underline{X}(t) = 1) = E(\text{time } X_i > \bar{m}_i^j \text{ for any } i)$$

$$= E(\text{time } X_i \geq \bar{m}_i^j + 1 \text{ for any } i) \leq \int_0^\infty \prod_{i=1}^{n} e^{-t/\bar{\lambda}_i^j} dt \ . \ \square$$

Theorem 3.36 obviously applies when each component is an independent IFRA stochastic process. In that case a bound on expected total system utility can be derived assuming that the system runs without repair for $0 \leq t \leq \infty$. Multiplying the inequalities in Theorem 3.36 by b_m and summing on m yields

$$\sum_{m=1}^{M} b_m \max_j \Big[\sum_{i=1}^{n} 1/\lambda_i^j\Big]^{-1} \leq \sum_{m=1}^{M} b_m ET^m \leq \sum_{m=1}^{M} b_m \min_j \int_0^\infty \prod_{i=1}^{n} e^{-t/\bar{\lambda}_i^j} dt$$

Note that the middle term is total expected system utility.

A different idea for applying Theorem 3.36 is contained in Theorem 3.38.

Lemma 3.37: If $X_1, \ldots, X_n$ are associated, then so are $\sum_{i\in I} C_i X_i$ and $\sum_{j\in J} d_j X_j$ where I and J are any subsets of the indices 1, ..., n, and C_i and d_j are nonnegative constants.

Proof:

$$\mathrm{COV}\Big[\sum_{i\in I} C_i X_i,\ \sum_{j\in J} d_j X_j\Big] = \sum_{i\in I}\sum_{j\in J} C_i d_j\ \mathrm{COV}[X_i, X_j] \geq 0$$

since the X_i's are associated. □

Theorem 3.38: Let transitions between the states of a component be independent IFRA random variables T_{ij} for $0 \leq j < i \leq M$. Then $\forall$ k, there exists an IFRA random variable $\bar{T}^k$ such that $P(T^k > t) \geq P(\bar{T}^k > t)$ $\forall$ t.

Proof: Let W^k, k = 1, ..., R, be the distinct paths which can be taken to reach or drop below state k when the state is always larger than k before the last transition.

$$W^k = T_{M,k_1} + T_{k_1,k_2} + \cdots + T_{k_\ell,k_{\ell+1}} \quad \text{where } k_\ell > k,\ k_{\ell+1} \leq k$$

$$T^k = \min_{k=1,\ldots,R} (W^k) \ .$$

Since sums of independent IFRA random variables are IFRA, the W^k are IFRA. From Lemma 3.37, the W^k are associated. Thus,

$$P(T^k > T) = P(\min_k W^k > t) \geq \prod_{k=1}^{R} P(W^k > t)$$

$$= \prod_{k=1}^{R} P(\bar{W}^k > t) \equiv P(\bar{T}^k > t) \ .$$

where $\bar{W}^1, \ldots, \bar{W}^n$ are independent random variables with the same distribution as $W^1, \ldots, W^n$. The distribution of $\bar{T}^k$ is the same as

the distribution of a series system of independent IFRA random variables $(\bar{W}^1, \ldots, \bar{W}^n)$ and is thus IFRA by the multistate IFRA closure theorem. □

The proof of Theorem 3.38 is constructive in that the distribution of $\bar{T}^k$ may be constructed from

$$P(\bar{T}^k > t) = \prod_{k=1}^{R} P(W^k > t) \quad .$$

From Theorem 3.38, $\mu_i^j \geq \bar{\mu}_i^j$, so we can use $\bar{\mu}_i^j$ in place of μ_i^j to obtain the lower bound in Theorem 3.36. A special case of Theorem 3.38 occurs when a component can be represented by a Markov process with transition rates λ_{ij}, $0 \leq j < i \leq M$, and an absorbing state 0. This is important since exponential failure rates are often assumed, especially when the data is scarce.

4. BLOCK DIAGRAMS AND FAULT TREES

The main deterministic tools used by reliability engineers are block diagrams and fault trees. It is assumed that the probability distribution for the state of each component at a certain point in time is known and that the problem is to find the probability distribution for the state of the system. Block diagrams and fault trees are used to find that distribution. There are many computer codes with block diagrams or fault trees as input and system reliability as output in the binary case, and some work has been done on algorithms in the ternary case. In this chapter it is shown that multistate components may be analyzed using existing binary algorithms. Define new binary variables X_{ij} which are 1 if component i is in state j or higher and 0 otherwise. Block diagrams and fault trees may be formulated in terms of these binary variables to calculate the probability that the system is at a certain level or higher. From these calculations, the expected system utility can be determined. Unfortunately, the binary variables are highly correlated since $X_{ij} = 1$ implies $X_{ik} = 1$ for $k < j$, and this correlation decreases computational efficiency. This chapter presents the new formulation, problems associated with it, and a special case in which computational efficiency may be greatly increased.

Notation:

$$X_{ij} \equiv 1_{\{X_i \geq j\}} \qquad X_i = \sum_{j=1}^{N_i} X_{ij}$$

$$\phi^k(\underline{X}) \equiv 1_{\{\phi(\underline{X}) \geq k\}} \qquad \phi(\underline{X}) = \sum_{k=1}^{M} \phi^k(\underline{X})$$

$$\phi^k(\underline{X}) \equiv \phi^k(X_{11}, X_{12}, \ldots, X_{1,N_1}, X_{21}, \ldots, X_{2,N_2}, \ldots, X_{n,N_n}) \quad .$$

Each $\phi^k(\underline{X})$ is a binary structure function that depends only on binary components. Note that $E\phi^k(\underline{X}) = h^k$ and that the minimum cuts for ϕ^k are the minimum cuts for ϕ at level k.

A disadvantage of this formulation is that there are many more components, $(N_1+1)\cdots(N_n+1)$ as opposed to the original n. Many of the binary variables may also be irrelevant. If ϕ is series at level k, then the only relevant components for ϕ^k are $X_{1k}, X_{2k}, \ldots, X_{nk}$. We could, in theory, eliminate these irrelevant components to get M binary coherent structure functions of n components each, but this may be more work than solving the original problem. Another disadvantage of this formulation is that independence among the components is lost. Even if the original multistate components are independent, the binary components X_{ij} will be associated in general since $X_{ij} = 1$ implies $X_{ik} = 1 \; \forall \; k < j$. The binary structure functions are also associated since $\phi^j(\underline{X}) = 1$ implies $\phi^k(\underline{X}) = 1 \; \forall \; k \leq j$. An advantage of this formulation is that results which require only that $\phi(\underline{X})$ be increasing in $\underline{X}$ follow immediately from the results in the binary case. For example, some reliability bounds for associated components follow from the corresponding binary results. However, the major advantage of this formulation is that computer codes for the binary case can be applied to the multistate case with adjustments in the model rather than the code.

A block diagram is a system schematic composed of series and parallel (minimum and maximum) operators. Each block in a block diagram represents a binary random variable. The output of n blocks in series is $\min(X_i) = \prod_{i=1}^{n} X_i$ while the output of n blocks in parallel is $\max(X_i) = \coprod_{i=1}^{n} X_i$.

Example 4.1: Consider the block diagram in Figure 4.1. The system will function if component 1 functions, component 2 functions, and one of components 3, 4, and 5 functions.

$$\phi(\underline{X}) = X_1 X_2 \max(X_3, X_4, X_5) \quad .$$

Computer codes generally perform these calculations in stages, e.g.,

$$X_6 = X_1 X_2, \; X_7 = \coprod_{i=3}^{5} X_i, \text{ and } \phi(\underline{X}) = X_6 X_7 \quad . \square$$

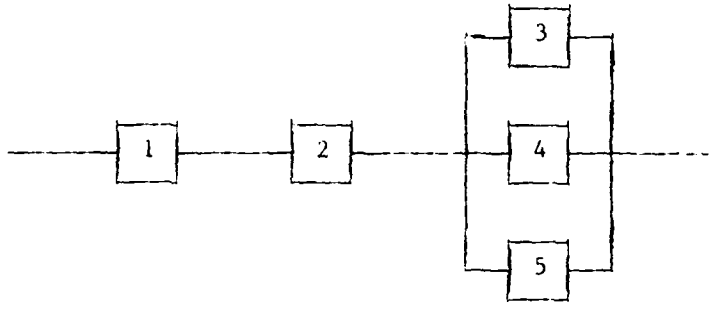

Figure 4.1 Block Diagram

If the components in a block diagram are independent, the calculation of system reliability is simple - just replace X_i by $P_i = P(X_i = 1)$ and be sure to use $\coprod_{i=1}^{n} P_i$ rather than $\max(X_i)$. In Example 4.1, $h(\underline{P}) = P_1 P_2 \coprod_{i=3}^{5} P_i$. If the components are dependent, then conditional probability expansions are used, e.g.,

$$h(\underline{P}) = P_1 E(\phi(\underline{X}) | X_1 = 1) + (1-P_1) E(\phi(\underline{X}) | X_1 = 0) \quad .$$

A block diagram is not limited to systems which have only series and

parallel combinations of components since a system can be represented in terms of its min paths or min cuts (Theorem 2.3). However, if a component appears in more than one min cut or min path, conditional probabilities must be used to correctly compute system reliability. The output of a block diagram is usually the probability of system success, but the output can be the probability of system failure if the dual structure function is used.

Example 4.2: Consider the system shown in Figure 4.2. If component 1 functions, then the system functions. If not, then one of components 2 and 3 must function, and 2-out-of-3 of components 4, 5, and 6 must function in order for the system to function. Since components 4, 5, and 6 appear in more than one place in the block diagram, a conditional probability expansion must be used to calculate system reliability. Conditioning on X_4,

$$h(\underline{P}) = \coprod \{P_1, [P_2+P_3-P_2P_3] \cdot [(P_5+P_6-P_5P_6)P_4 + P_5P_6(1-P_4)]\} \quad .$$

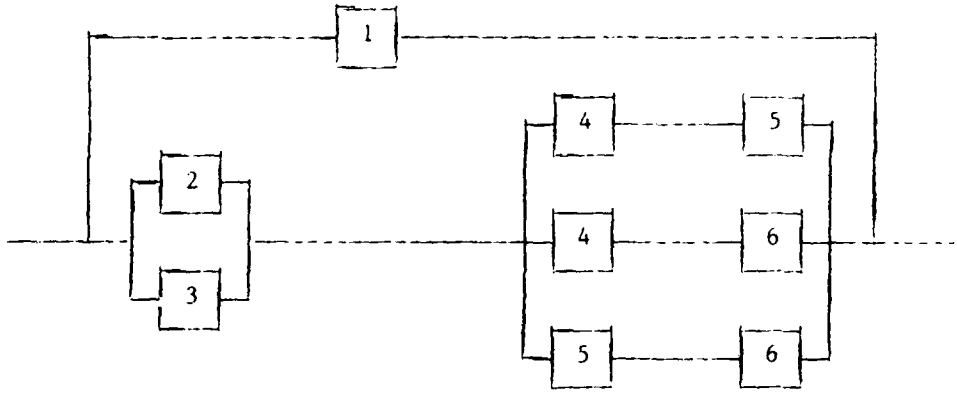

Figure 4.2 Success Block Diagram

The dual of this system is shown in Figure 4.3 with the boxes now representing component failures rather than component success. Note that the dual of a 2-out-of-3 system is a 2-out-of-3 system which can be represented in the two different ways shown in Figure 4.2 and Figure 4.3. □

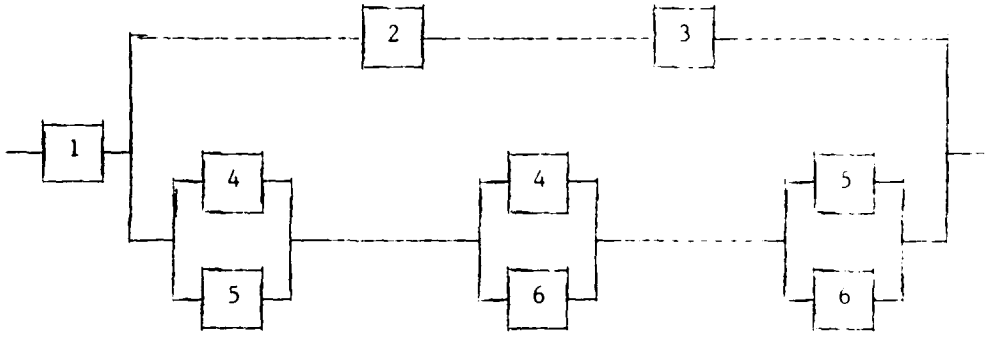

Figure 4.3 Failure Block Diagram

To perform reliability calculations for multistate systems, design block diagrams corresponding to ϕ^k for each system level k, and calculate $h^k = E\phi^k(\underline{X})$. Then $EU(\phi(\underline{X})) = \sum_{k=1}^{M} b_k E\phi^k(\underline{X})$ is the expected system utility. If the components are independent and if at most one level of each component appears at most once in the block diagram, then the reliability calculation may be performed as though the binary components were independent. Otherwise conditional probability expansions are necessary.

Example 4.3: Consider a twin engine jet which can land normally if one engine is at full power and the other engine is at half power. It can land on a foamed runway if one engine is at full power or if both

engines are at half power. It will crash if one engine is at half power and the other engine is inoperable. Let the component state space be {0 = failed, 1 = half power, 2 = full power}, and let the system state space be {0 = crash, 1 = land on foamed runway, 2 = land normally}.

$$\phi(2,2) = \phi(2,1) = \phi(1,2) = 2$$

$$\phi(0,2) = \phi(2,0) = \phi(1,1) = 1$$

$$\phi(0,1) = \phi(1,0) = \phi(0,0) = 0$$

$$\phi^1(\underline{X}) = \max (X_{12}, X_{22}, X_{11}X_{21})$$

$$\phi^2(\underline{X}) = \max (X_{11}X_{22}, X_{12}X_{21})$$

ϕ^1 and ϕ^2 are shown in Figure 4.4. Note that both the component number and the minimum required component state appear inside the blocks in the block diagram. □

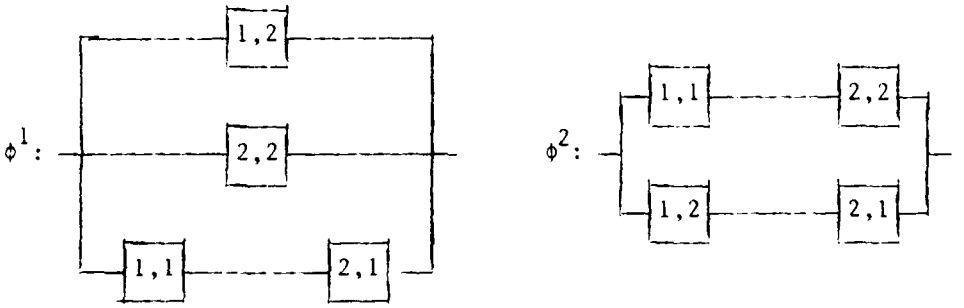

Figure 4.4 Multistate Block Diagrams

There are some cases in which the calculations are much easier. Consider a MSF which is k-out-of-n at each system level m where k is allowed to depend on m. Then, for each level k, $\phi^k(\underline{X})$ $= \phi^k(X_{1k}, X_{2k}, \ldots, X_{nk})$. Component states other than k are unimportant

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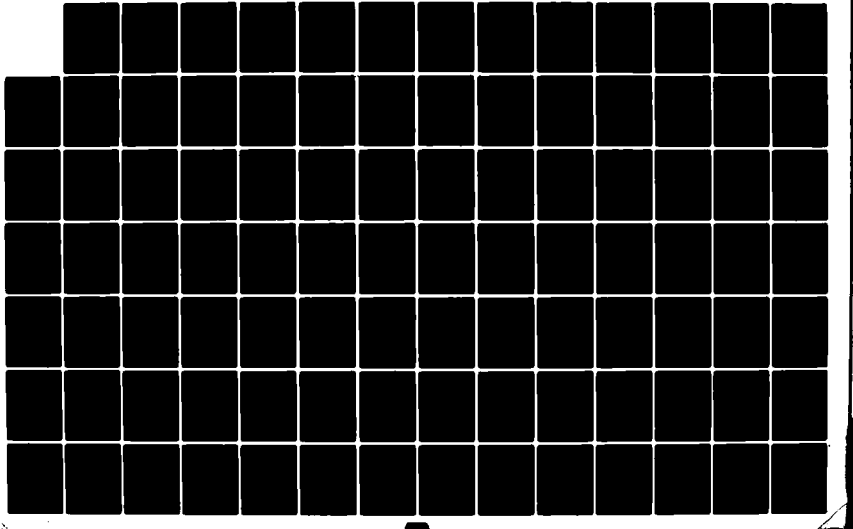

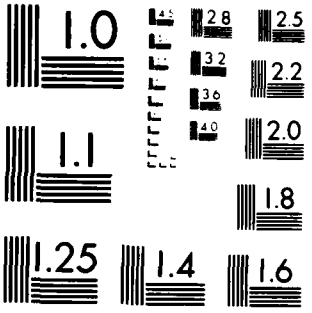

in determining the value of ϕ^k. This means that if the original multistate components were independent, then the binary variables X_{ik} will be independent in each ϕ^k. No conditional probability expansions will be necessary since the X_{ik}'s are independent for fixed k, and for every i, only one X_{ik} appears in the block diagram for ϕ^k. This situation occurs in Example 4.4.

Example 4.4: Let $\phi(X_1,X_2,X_3)$ be parallel at level 1, 2-out-of-3 at level 2, and series at level 3, and let X_1, X_2, and X_3 be independent.

$$\phi^1(\underline{X}) = \coprod_{i=1}^{3} X_{i1} \qquad \phi^2(\underline{X}) = 1_{\{\sum_{i=1}^{3} X_{i2} \geq 2\}}$$

$$\phi^3(\underline{X}) = \prod_{i=1}^{3} X_{i3}$$

The block diagram for each ϕ^k is shown in Figure 4.5. Reliability calculations may be performed using independence for ϕ^1 and ϕ^3, but conditional probabilities are necessary to correctly calculate $P(\phi^2(\underline{X}) = 1)$. □

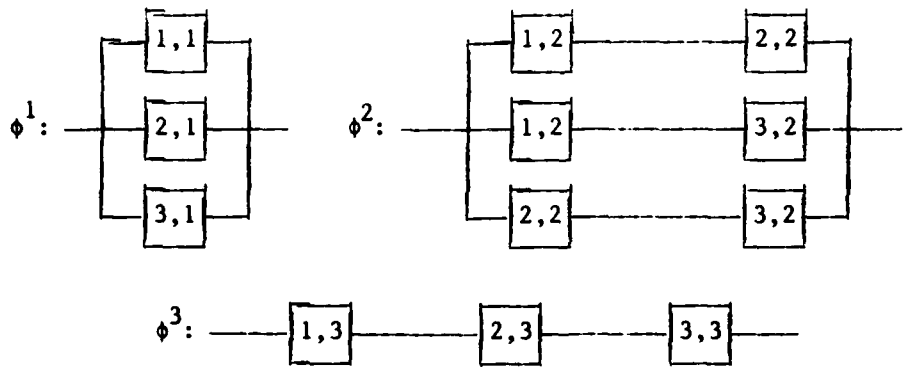

Figure 4.5 Multistate Block Diagrams

An even more specialized case occurs when all of the ϕ^k have identical block diagrams. In this case a system analysis would require only a single block diagram with different input probabilities for different system levels. This would occur for a system which is series, parallel, or k-out-of-n at all levels. It also occurs in Example 4.5.

Example 4.5: A power plant can generate 0%, 25%, 50%, 75%, or 100% (corresponding to states 0, 1, 2, 3, and 4) of rated electric capacity depending on the condition of the turbine and the amount of steam flow reaching the turbine. Three turbines are available, and the one which can maximize power output is always used. Components 1, 2, and 3 are the turbines and component 4 represents the rate of flow at the turbine. The block diagram for all levels of this system is shown in Figure 4.6. □

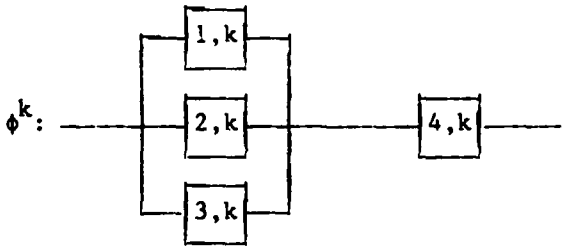

Figure 4.6 Multilevel Block Diagram

Fault trees and event trees are system models consisting of a top event and a structure delineating the ways in which the top event can occur. The term fault tree is used when the top event is system failure while the term event tree is used for system success. The tree

structure consists of "and gates" and "or gates" which respectively perform the same function in the fault tree as the series and parallel operators in the block diagram. The inputs to these gates are generally component successes or failures.

Example 4.6: The fault tree and event tree for the system in Example 4.2 are shown in Figure 4.7. The squares with numbers inside represent an occurrence of component failure in the fault tree and component success in the event tree. The dual of a fault tree is an event tree and vice versa. It is very easy to draw the dual since the only changes are that "and gates" become "or gates" and "or gates" become "and gates". □

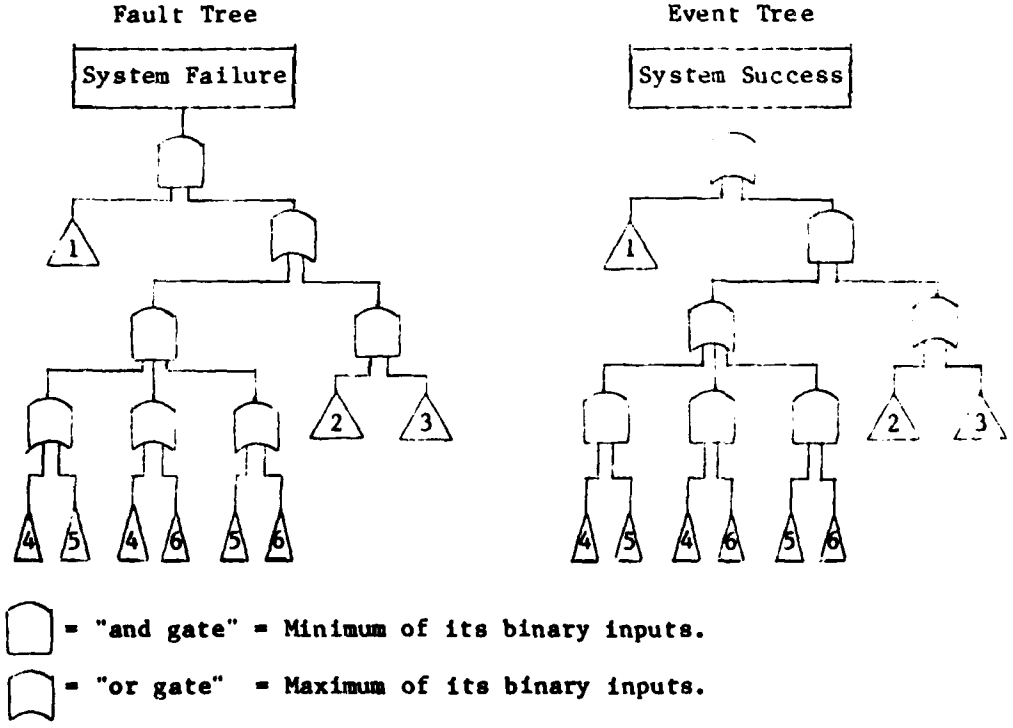

Figure 4.7 Fault Tree and Event Tree

Computer algorithms based on fault trees generally procede by finding the min cut sets of a system. This is a minimal group of elements whose failure to function causes the system to fail to function. In a fault tree the min cut sets for the system are the minimal sets of basic events which cause the top event to occur, i.e., the min path sets in the dual structure. Min cut sets are useful to reliability engineers since they provide a qualitative measure of the most important components in the system. The relationship

$$\phi(\underline{X}) = \prod_{j=1}^{t} \coprod_{i \epsilon K_j} X_i$$

is then used to determine the value of ϕ. If the components are independent and each component appears in at most one min cut set, then

$$E\phi(\underline{X}) = \prod_{j=1}^{t} \coprod_{i \epsilon K_j} P_i \quad .$$

Conditional probabilities must be used if a component appears in more than one min cut set just as in block diagrams. Calculations of system reliability using fault tree computer codes generally take longer than the same calculation with a computer code based on block diagrams, but the output from the fault tree evaluation includes the min cut sets while the block diagram output does not.

The extension of fault tree analysis to the multistate case is very similar to the extension of block diagrams. A fault tree or event tree is constructed for each system level. Basic events are X_{ik} for event

trees and $1-X_{ik}$ for fault trees. A binary fault tree code is used to determine $E\phi^k(\underline{X})$ for each system level k, and expected system utility is calculated from $EU(\phi(\underline{X})) = \sum_{k=1}^{M} b_k E\phi^k(\underline{X})$.

Example 4.7: The event trees for the structure function contained in Example 4.4 are shown in Figure 4.8. □

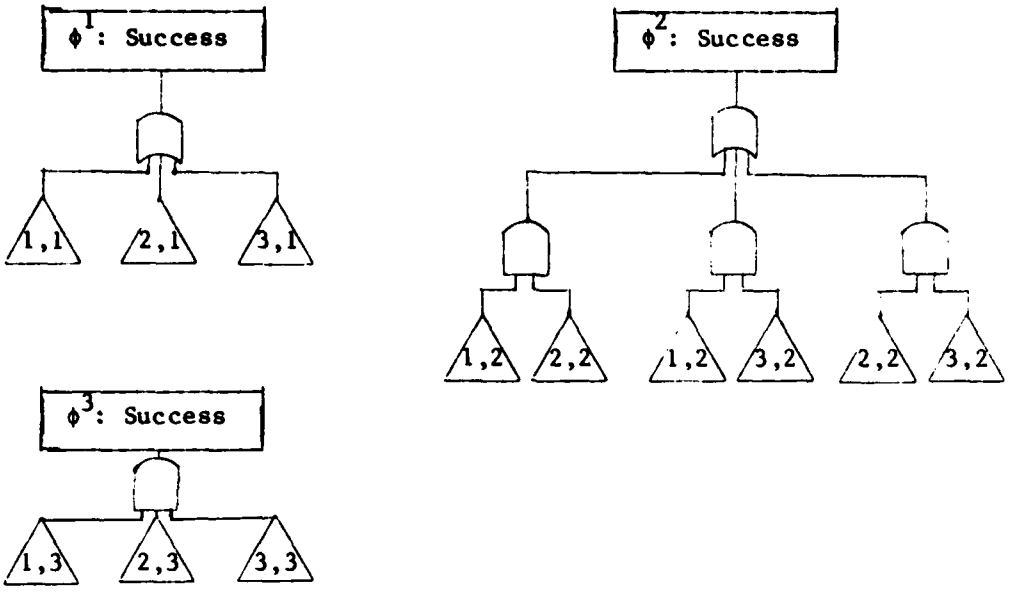

Figure 4.8 Multistate Event Trees

The special cases pertaining to multistate block diagrams have straightforward analogies to fault trees and event trees. In particular if the ϕ^k have identical block diagrams, the fault trees or event trees for each ϕ^k are also identical. This is very useful computationally since the min cut sets need to be found only once.

Example 4.8: The fault tree for the system contained in Example 4.5 is shown in Figure 4.9. The min cut sets for ϕ^k are $\{(4,k)\}$ and $\{(1,k),(2,k),(3,k)\}$. Thus,

$$\phi^k(\underline{X}) = X_{4k} \coprod_{i=1}^{3} X_{ik} \quad .$$

Assuming independent components,

$$E\phi^k(\underline{X}) = p_{4k} \coprod_{i=1}^{3} p_{ik} \quad . \square$$

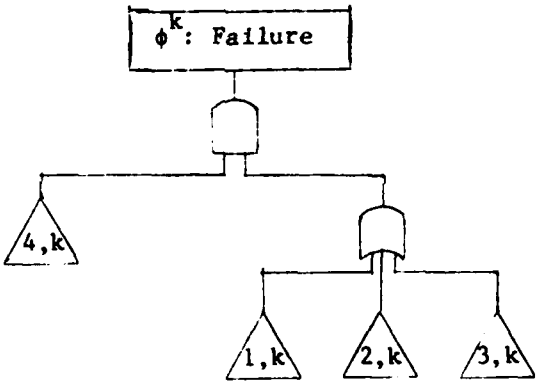

Figure 4.9 Multilevel Fault Tree

5. THE CONTINUOUS MODEL

In this chapter the state of each component and the system is allowed to be any real number in the unit interval. The restriction to the unit interval is used only for discussion purposes, and the results are valid for any finite segment (possibly different segments for each component) of the real line. The multistate model can be considered as a special case of the continuous model by restricting a component with $N + 1$ states to values $(0, 1/N, \ldots, (N-1)/N, 1)$ in the unit interval. This approach has not been taken since it is felt that the multistate model will be of more practical importance than the continuous model. Since any process requiring measurement has a number of states limited by the precision of the measurement device, there are no truly continuous processes in reliability engineering. For example, temperature can be measured only to the nearest degree or fraction of a degree. In current practice a continuous process is usually broken up into a finite number of qualitative states, e.g., low, normal, and high pressure.

All of the results previously presented for the multistate case have analogues in the continuous case. Coherence means that the structure function is increasing in the states of its components, that an infinitesimal change in any component state may cause an infinitesimal change in the system state, and that an increasing utility function is associated with the states of the system. Some concepts such as cut sets do not seem as relevant since there could be an uncountably infinite number of them, but the results presented in Chapters 2 and 3

are still valid. Block diagrams and fault trees can still be used to determine the probability that the process exceeds a certain level. Unfortunately, however, they cannot be used to determine expected system utility unless the utility function is piecewise constant since that would involve an infinite number of calculations.

5.1. Definitions

The definitions in this chapter are straightforward extensions of the definitions in Chapter 2. Let $X_i \in [0,1]$ be the state of the ith component ([0,1] is the unit interval), and let $\phi : [0,1]^n \to [0,1]$ be the state of the system.

Definition: $\phi(\underline{X})$ is a Continuous Structure Function (CSF) if

(1) $\phi(\underline{0}) = 0$, $\phi(\underline{1}) = 1$, and

(2) $\phi(\underline{X})$ is increasing in $\underline{X}$.

Definition: Component i is said to be

(A) Relevant if there exists $\underline{X}$ such that $\phi(1_i,\underline{X}) > \phi(0_i,\underline{X})$.

(B) Fully Relevant if $\forall\ 0 \leq Y < Z \leq 1$, there exists $\underline{X}$ such that $\phi(Z_i,\underline{X}) > \phi(Y_i,\underline{X})$.

Definition: Let $\phi(\underline{X})$ be a CSF, and let $U(\phi(\underline{X}))$ be the corresponding utility function which assigns utility $a(Y)$ to state Y. Let $a(0) = 0$.

$\phi(\underline{X})$ is called

(A) weakly coherent if

(1) every component is relevant, and

(2) $a(Z) \geq a(Y)\ \forall\ 0 \leq Y < Z \leq 1$.

(B) coherent if

(1) every component is fully relevant, and

(2) $a(Z) > a(Y)\ \forall\ 0 \leq Y < Z \leq 1$.

(C) strongly coherent if

(1) $\forall$ component i and state y, there exists $\underline{X}$ such that $\phi(Y_i,\underline{X}) = y$ but $\phi(Z_i,\underline{X}) \neq y\ \forall\ Z \neq y$, and

(2) $a(Z) > a(Y)\ \forall\ 0 \leq Y < Z \leq 1$.

Example 5.1: Let $a(X) = X$. Then $\phi(\underline{X})$ is coherent if it is any continuous monotonic mapping from $[0,1]^n$ onto $[0,1]$. For example,

$$\phi(X_1,X_2) = \frac{\exp(X_1^2\, X_2^{1/2}) - 1}{e - 1}$$

If $X_1 > Y_1$, then $\phi(X_1,1) > \phi(Y_1,1)$. If $X_2 > Y_2$, then $\phi(1,X_2) > \phi(1,Y_2)$. Thus, both components are fully relevant, and the system is coherent.□

To show that a CSF is coherent or strongly coherent could be very difficult since the entire unit interval must be considered. A situation such as the one in the previous example must exist.

Definition: A system represented by CSF ϕ is called

(1) Series if $\phi(\underline{X}) = \min_i (X_i)$.

(2) Parallel if $\phi(\underline{X}) = \max_i (X_i)$.

(3) k-out-of-n if $\phi(\underline{X}) = \max_y \{y : \sum_{i=1}^{n} 1_{(X_i \geq y)} \geq k\}$.

The extensions to series (parallel, k-out-of-n) at level y are similar.

Theorem 5.1: Let ϕ be a CSF.

(i) $\phi(\underline{X} \vee \underline{Y}) \geq \phi(\underline{X}) \vee \phi(\underline{Y})$ and if ϕ is parallel, equality holds.

(ii) $\phi(\underline{X} \wedge \underline{Y}) \leq \phi(\underline{X}) \wedge \phi(\underline{Y})$ and if ϕ is series, equality holds.

Notice that Theorem 5.1 is slightly different than Theorem 2.2, the multistate version. It is no longer true that $\phi(\underline{X} \vee \underline{Y}) = \phi(\underline{X}) \vee \phi(\underline{Y})$ implies that ϕ is a parallel structure as shown by Example 5.2.

Example 5.2: Let $\phi(X_1,X_2) = \max(X_1^2, X_2)$.

$$\phi(\underline{X} \vee \underline{Y}) = \max(X_1^2, X_2, Y_1^2, Y_2) = \max(X_1^2 \vee X_2,\ Y_1^2 \vee Y_2)$$
$$= \phi(\underline{X}) \vee \phi(\underline{Y}) \quad .$$

However, ϕ is not a parallel structure function. □

Definition: Let ϕ be a CSF. $\underline{X}$ is a path vector at level z if $\phi(\underline{X}) \geq z$. It is a path vector at maximal level z if $\phi(\underline{X}) = z$. If, in addition, $\phi(\underline{Y}) < z$ whenever $\underline{Y} < \underline{X}$, $\underline{X}$ is called a minimum path vector at maximum level z. Let $\underline{z} = (z_1,\ldots,z_n)$ be a vector such that if $\underline{X} \geq \underline{z}$, then $\phi(\underline{X}) \geq z$ and if $\underline{Y} < \underline{z}$, then $\phi(\underline{Y}) < z$. The vector $\underline{z}$ will be called a min path. The jth min path at system level z will be denoted $z^j = (z_1^j,\ldots,z_n^j)$. Also,
$\rho_j(\underline{X}) = 1_{\{\underline{X} \geq \underline{z}^j\}} = \prod_{i=1}^n 1_{\{X_i \geq z_i^j\}}$ is the jth minimum path structure function at level z.

If min cuts are defined as they were for MSFs, they do not exist in general since there is no $\underline{X}$ such that $\phi(\underline{X}) < z$ and $\phi(\underline{Y}) \geq z$ whenever $\underline{Y} > \underline{X}$ if ϕ is continuous in one of its components. In the

following definition, inequalities which occur in the multistate definition are changed to equalities so that min cuts will always exist.

Definition: Let ϕ be a CSF. $\underline{X}$ is a cut vector at level z if $\phi(\underline{X}) \geq z$. It is a cut vector at minimal level z if $\phi(\underline{X}) = z$. If, in addition, $\phi(\underline{Y}) > z$ whenever $\underline{Y} > \underline{X}$, $\underline{X}$ is a minimum cut vector at minimal level z. Let $\underline{\bar{z}} = (\bar{z}_1,\ldots,\bar{z}_n)$ be a vector such that if $\underline{X} \leq \underline{\bar{z}}$, $\phi(\underline{X}) \geq z$ and if $\underline{Y} > \underline{\bar{z}}$, $\phi(\underline{Y}) > z$. The vector $\underline{\bar{z}}$ is called a min cut. The jth min cut at system level z will be denoted $\underline{\bar{z}}^j = (\bar{z}^j_1,\ldots,\bar{z}^j_n)$. Also, $\kappa_j(\underline{X}) = 1_{\{\underline{X} \leq \underline{\bar{z}}\}} = 1 - \prod_{i=1}^{n} 1_{\{X_i \geq \bar{z}^j_i\}}$ is the jth minimum cut structure function at level z.

Example 5.3: Let $\phi(X_1,X_2) = .5(X_1+X_2)$. Then, $(X_1,1-X_1)$ is [illegible] a min cut and a min path at system level $1/2 \forall 0 \leq X_1 \leq 1$. Note that there are an uncountably infinite number of min cuts and min paths at an uncountably infinite number of system levels. There is no vector $\underline{z}$ such that $\phi(\underline{z}) < 1/2$ and $\phi(\underline{Y}) \geq 1/2 \forall \underline{Y} > \underline{z}$. □

The change in the definition of min cuts, although necessary for their existence as shown by Example 5.3, may create difficulties with respect to bounding system reliability. Bounds based on min paths will be bounds on $h^z \equiv P(\phi(\underline{X}) \geq z)$ while bounds based on min cuts will be bounds on $\bar{F}_\phi \equiv P(\phi(\underline{X}) > z)$. However, if $\bar{F}_\phi$ is continuous in a neighborhood of z, then $h^z = \bar{F}_\phi$, and this problem will not exist. Also, since $P(\phi(\underline{X}) > z) \leq P(\phi(\underline{X}) \geq z)$, lower bounds based on cuts still apply.

Theorem 5.2: Let ϕ be a CSF.

(i) $\phi(\underline{X}) \geq z \iff \coprod_{j=1}^{s} \rho_j(\underline{X}) = 1$.

(ii) $\phi(\underline{X}) > z \iff \coprod_{j=1}^{t} \kappa_j(\underline{X}) = 1$.

If the X_i are independent, then $P(\rho_j(\underline{X}) = 1) = \prod_{i=1}^{n} P(X_i \geq z_i^j)$ and $P(\kappa_j(\underline{X}) = 0) = \prod_{i=1}^{n} P(X_i \leq \bar{z}_i^j)$.

Definition: Let ϕ be a CSF. Its dual, ϕ^D, is defined by

$$\phi^D(\underline{X}) = 1 - \phi(\underline{1}-\underline{X}).$$

Theorems 2.4 and 2.5 which state that the dual of a MSF is a MSF with the same type of coherence and that $(\phi^D)^D = \phi$ are identical when CSF replaces MSF. Theorem 5.3 is the continuous analogue of Theorem 2.6.

Theorem 5.3: Let $\underline{X}$ be a path (cut) vector at level z for a CSF ϕ. Then $\underline{1} - \underline{X}$ is a cut (path) vector for ϕ^D at level $1 - z$. Furthermore, if $\underline{z}$ is a min path (cut) at maximum (minimum) level z for ϕ, then $\underline{1} - \underline{z}$ is a min cut (path) at minimum (maximum) level $1 - z$ for ϕ^D.

Proof: If $\underline{X}$ is a path vector at level z for ϕ, $\phi(\underline{X}) \geq z$. Thus, $\phi^D(\underline{1} - \underline{X}) = 1 - \phi(\underline{X}) \leq 1 - z$. If $\underline{z}$ is a min path for ϕ, $\phi(\underline{z}) = z$ and $\forall\, \underline{Y} < \underline{z}$, $\phi(\underline{Y}) < z$. Thus,

$$\phi^D(\underline{1} - \underline{z}) = 1 - \phi(\underline{z}) = 1 - z \quad \text{and} \quad \forall\, \underline{Y} < \underline{z},$$

$$\phi^D(\underline{1} - \underline{Y}) > 1 - z.$$

The results in parentheses hold by replacing ϕ with ϕ^D and $\underline{X}$ with $\underline{1} - \underline{X}$.□

Let (C,ϕ) denote a set of components C and a CSF ϕ.

Definition: (A,χ) is called a module of (C,ϕ) if $A \subseteq C$ and $\phi(\underline{X}) = \psi(\chi(\underline{X}^A), \underline{X}^{A^C})$ where ψ is a CSF. A modular decomposition is a set of modules $\{(\chi_1,A_1),\ldots(\chi_r,A_r)\}$ such that $\phi(\underline{X}) = \psi(\chi_1(\underline{X}^{A_1}),\ldots,\chi_r(\underline{X}^{A_r}))$ where the sets A_i partition C and ψ is the organizing structure.

Theorem 2.8 - which shows that if the modules and the organizing structure possess certain types of coherence, then so does the original MSF - is still valid with CSF replacing MSF. In particular, Example 5.4 shows that if χ and ψ are both weak coherent, ϕ is not necessarily weak coherent. The technique used in this example transforms a multistate system into a continuous system. This technique can be used to extend all multistate examples and counter-examples to the continuous case.

Example 5.4: The counter-example from Theorem 2.8 (iii) is extended to the continous case. That example contained ternary components and a ternary system.

Let state $0 \to$ states $[0,1/3]$, $1 \to (1/3,2/3]$, $2 \to (2/3,1]$.

$$\chi(X_1,X_2) = \phi(X_1,X_2) = 0 \quad \text{for} \quad 0 \le X_1 \le 2/3,\ 0 \le X_2 \le 1/3$$

$$\chi(X_1,X_2) = \phi(X_1,X_2) = 1/2 \quad \text{for} \quad 0 \le X_1 \le 2/3,\ 1/3 < X_2 \le 1$$

$$\chi(X_1,X_2) = \phi(X_1,X_2) = 1 \quad \text{for} \quad 2/3 < X_1 \le 1,\ 0 \le X_2 \le 1 \ .$$

Let $\phi(X_1,X_2,X_3) = \psi(\chi(X_1,X_2),X_3)$.

Then $\phi(X_1,0_2,X_3) = \phi(X_1,1_2,X_3)$ so component 2 is not relevant, and ϕ is not weakly coherent. □

Theorem 5.4: The dual of a module is a module in the dual, i.e., $\phi^D(\underline{X}) = \psi^D(\chi^D(\underline{X}^A),\underline{X}^{A^C})$.

Proof:

$$\begin{aligned}\psi^D(\chi^D(\underline{X}^A),\underline{X}^{A^C}) &= \psi^D(1-\chi(\underline{1}^A-\underline{X}^A),\underline{X}^{A^C}) \\ &= 1-\psi(\chi(\underline{1}^A-\underline{X}^A),\underline{1}^{A^C}-\underline{X}^{A^C}) \\ &= 1-\phi(\underline{1}-\underline{X}) = \phi^D(\underline{X}) \ . \ \square\end{aligned}$$

5.2. Extension of Stochastic Results

Throughout this section, it will be assumed that $\bar{F}_\phi(y) = P(\phi(\underline{X}) > y)$ is continuous in y. This implies that $P(\phi(\underline{X}) \ge y) = P(\phi(\underline{X}) > y)$, and the problem associated with the definition of min cuts does not exist. This could be generalized to handle random variables which have both discrete and continuous parts, but it does not seem worthwhile in a reliability context.

In the continuous case a utility function has a continuous domain, $\{a(X), 0 \le X \le 1\}$, rather than a discrete domain, $\{a_j, j=1,\ldots,M\}$. To get a continuous analogue of $b_j = a_j - a_{j-1}$, set $a(0) = 0$, assume $a(X)$ is continuously differentiable, and integrate by parts.

$$
\begin{aligned}
EU(\phi(\underline{X})) &= \int_0^1 a(Y)dF_\phi(Y) \\
&= a(Y)F_\phi(Y)\Big|_0^1 - \int_0^1 a'(Y)F_\phi(Y)dY \\
&= a(1) - \int_0^1 a'(Y)dY + \int_0^1 a'(Y)[1-F_\phi(Y)]dY \\
&= a(1) - a(1) + \int_0^1 a'(Y)\bar{F}_\phi(Y)dY \\
&= \int_0^1 a'(Y)P(\phi(\underline{X}) \geq Y)dY \quad .
\end{aligned}
$$

Thus, $a'(Y)$ is the desired analogue to b_j. Proper definition of $a'(Y)$ at discontinuities of $a(Y)$ would allow this situation to be slightly generalized, but for applications it seems easier to "smooth" $a(Y)$ than to be mathematically precise.

Theorem 5.5:

$$EU(\phi(\underline{X})) = \int_0^1 EU(\phi(Y_i,\underline{X})dF_i(Y) \quad .$$

Proof:

$$
\begin{aligned}
EU(\phi(X)) &= \int_0^1 E[U(\phi(X))|X_i=Y]dF_i(Y) \\
&= \int_0^1 E[U(\phi(Y_i,X))]dF_i(Y) \quad . \ \square
\end{aligned}
$$

Theorem 5.6: Let ϕ be a coherent CSF.

(i) $EU(\phi(\underline{X} \vee \underline{Y})) \geq [EU(\phi(\underline{X}))] \vee [EU(\phi(\underline{Y}))]$ and if ϕ is a parallel CSF, equality holds.

(ii) $EU(\phi(\underline{X} \wedge \underline{Y})) \leq [EU(\phi(\underline{X}))] \wedge [EU(\phi(\underline{Y}))]$ and if ϕ is a series CSF, equality holds.

Proof:

(i) $P(\phi(\underline{X} \vee \underline{Y}) \geq z) - P(\phi(\underline{X}) \geq z) \vee P(\phi(\underline{Y}) \geq z)$

$$= \int_{\underline{X}} \int_{\underline{Y}} [1_{\{\phi(\underline{x} \vee \underline{y}) \geq z\}} - 1_{\{\phi(\underline{x}) \vee \phi(\underline{y}) \geq z\}}]dF_{\underline{X}}(\underline{x})dF_{\underline{Y}}(\underline{y}) \geq 0$$

where equality holds if ϕ is a parallel CSF from Theorem 5.1. Multiplying by $a'(Y) > 0$ and integrating from 0 to 1 finishes the proof.

Part (ii) is similar. □

Theorem 5.7: Let $\underline{X}(\underline{Y})$ have distribution $\underline{q}(\underline{q}')$. If $\underline{q} \leq \underline{q}'$ and ϕ is a weakly coherent CSF, then $EU(\phi(\underline{X})) \leq EU(\phi(\underline{Y}))$.

Proof: Since $\phi(\underline{X})$ is increasing in $\underline{X}$, $P(\phi(\underline{X}) \geq z) \leq P(\phi(\underline{Y}) \geq z)$ $\forall$ z. Multiplying by $a'(z) \geq 0$ and integrating yields the result. □

Definition: The X,Y reliability importance of component i at level z is

$$I_h^{X,Y}(i) \equiv P(\phi(X_i,\underline{X}) \geq z) - P(\phi(Y_i,\underline{X}) \geq z) \quad \text{for} \quad 0 \leq Y_i < X_i \leq 1.$$

However, this measure of reliability importance is not as appealing as the multistate r,s reliability importance since there are an infinite number of X,Y, and z to choose from. Also, it is not so easy to change the distribution of some X_i and derive a nice expression for the change in reliability in terms of $I_h^{X,Y}(i)$. There is a stronger objection to extending the r,s criticality importance to the continuous case. The denominator of this measure is $(N_1+1) \cdots (N_n+1)$ representing the total number of component states. This denominator will be infinite in the continuous case. There are also problems with most of the other importance measures since they contain expressions such as $h^m((k+1)_i,\underline{P}) - h^m(k_i,\underline{P})$. It might be possible to consider derivatives and then integrate over the unit interval, but this does not seem very useful. However, the multistate extension of reliability

importance due to Griffith [1980] seems to generalize nicely to the continuous case.

Theorem 5.8:

$$EU(\phi(\underline{X})) = \int_0^1 a'(y)P(\phi(0_i,\underline{X}) \geq y)dy + \int_0^1 I(i)\bar{F}_i(z)dz$$

where

$$I(i) = \int_0^1 a'(y)[\frac{d}{dz}P(\phi(z_i,\underline{X}) \geq y)dy \quad .$$

Proof:

$$P(\phi(\underline{X}) \geq y) = \int_0^1 P(\phi(z_i,\underline{X}) \geq y)dF_i(z)$$

$$= P(\phi(z_i,\underline{X}) \geq y)F_i(z)\Big|_{z=0}^1 - \int_0^1 [\frac{d}{dz}P(\phi(z_i,\underline{X}) \geq y)]F_i(z)dz$$

$$= P(\phi(1_i,\underline{X}) \geq y) - \int_0^1 [\frac{d}{dz}P(\phi(z_i,\underline{X}) \geq y]dz$$

$$+ \int_0^1 [\frac{d}{dz}P(\phi(z_i,\underline{X}) \geq y](1-F_i(z))dz$$

$$= P(\phi(0_i,\underline{X}) \geq y) + \int_0^1 [\frac{d}{dz}P(\phi(z_i,\underline{X}) \geq y)]\bar{F}_i(z)dz \quad .$$

Multiplying both sides of the above equation by $a'(y)$ and integrating:

$$EU(\phi(\underline{X})) = \int_0^1 a'(y)P(\phi(0_i,\underline{X}) \geq y)dy$$

$$+ \int_0^1 a'(y) \int_0^1 [\frac{d}{dz}P(\phi(z_i,X) \geq y)]\bar{F}_i(z)dzdy \quad .$$

The result follows by interchanging the order of integration in the second term. □

Corollary 5.9: If component i is stochastically improved to a new distribution F_i^* with $\bar{F}_i^*(z) \geq \bar{F}_i(z)$ $\forall$ z, the change in expected utility is

$$\Delta EU(\phi(\underline{X})) = \int_0^1 I(i)[\bar{F}_i^*(z) - \bar{F}_i(z)]dz \quad .$$

The definitions of IFRA (IFR, etc.) stochastic processes apply to the continuous case as written. In fact the IFRA and NBU closure theorems (closure under formation of coherent systems) from Section 3.4.3 may be found in the literature for the continuous case rather than the multistate case. The other closure theorems and proofs apply immediately to the continuous case.

The bounds in Section 3.4.1 are upper and lower bounds on $P(\phi(\underline{X}) \geq k)$. These bounds apply immediately to the continuous case with k restricted to the unit interval. In some of the bounds there are products of min paths or min cuts. Since this number may now be uncountably infinite, these bounds may be difficult to compute. If this occurs, other bounds can be used or approximations can be made. The

Bonferroni bounds contain sums of a number of min paths or min cuts. Depending on the problem, it may be possible to transform these sums into integrals or to approximate the CSF. Example 5.6 shows how a CSF might be approximated by a MSF.

Example 5.6: Consider $\phi(X_1,X_2) = .5(X_1 + X_2)$. There are several ways to approximate this CSF by a MSF. As an example,

$$X_1' = [10X_1] \qquad X_2' = [10X_2]$$

$$\phi'(X_1,X_2) = [.5(X_1' + X_2')]$$

where $[X]$ is the largest integer less than or equal to X. Multistate techniques can then be applied to ϕ. On the other hand, if X_1 and X_2 are associated random variables with densities f_1 and f_2, a direct calculation is possible.

$$P(\phi(\underline{X}) \geq \tfrac{1}{2}) = P(X_1 + X_2 \geq 1)$$

$$= \int_0^1 \int_{1-y}^1 f_{12}(x,y)dxdy$$

$$\geq \int_0^1 \int_{1-y}^1 f_1(x)f_2(y)dxdy$$

$$= \int_0^1 f_1(y)\bar{F}_2(1-y)dy \quad . \square$$

Bounds on expected system utility can be obtained easily from the bounds on $h^z = P(\phi(\underline{X}) \geq z)$. Let

$$L(z) \leq P(\phi(\underline{X}) \geq z) \leq V(z) \quad .$$

Multiplying by $a'(z) \geq 0$ (coherent systems) and integrating yields

$$\int_0^1 a'(z)L(z)dz \leq EU(\phi(\underline{X})) \leq \int_0^1 a'(z)V(z)dz \quad .$$

This equation is the analogue of Theorem 3.28. To get an analogue of Theorem 3.29, let $Z^+ = \{z : a'(z) \geq 0\}$ and let $Z^- = \{z : a'(z) < 0\}$. Then

$$\int_{Z^+} a'(z)L(z)dz + \int_{Z^-} a'(z)V(z)dz \leq EU(\phi(\underline{X}) \leq \int_{Z^-} a'(z)L(z)dz + \int_{Z^+} a'(z)V(z)dz \quad .$$

Analogous results for the remainder of the results in Section 3.4 follow immediately by restricting the component and system states to the unit interval.

PART II

OPTIMAL MAINTENANCE OF MULTISTATE COMPONENTS

The theory presented in the first part of this thesis is mainly deterministic, meaning that time is either fixed or is not considered as a parameter in the reliability calculations and that no optimization is performed. This part of the thesis is stochastic in that it deals with selecting the best strategy for the maintenance of a system over an infinite time horizon. Determining the optimal maintenance policy for a system with failure and repair characterisitcs that vary in time could be a difficult task. Thus, the usual assumptions are that system operation is a Markov or semi-Markov process and that regeneration points exist. These assumptions allow Markovian decision processes to be utilized.

6. INTRODUCTION AND HISTORY

Most of the literature on optimal maintenance of multistate systems pertains to inspection models in which the state of the component is discovered at the end of each time period, and a decision must be made to replace the component or leave it in operation. The objective is to minimize costs which consist of a constant replacement fee and a penalty cost if the component enters its worst state. Since the transition laws are Markovian, the system is analyzed using discrete time Markov decision processes. The main results are control limit rules which call for replacement of the component when it reaches or drops below a certain state (called the control limit). This model and several variations of it are discussed in Section 6.2 after an introduction to Markov decision processes in Section 6.1. These models regard state 0 as the best state, and states 1, 2, ..., M as increasingly degraded. The opposite convention is used herein so that the notation in Parts I and II of this thesis is consistent.

The models in this thesis differ from previous models in that the components are assumed to be constantly monitored rather than periodically inspected. Thus, the decision to retain or replace a component may be made at any time instead of immediately following an inspection. In Chapter 7 the repair or replacement process occurs with the system inoperable. It is shown that an equivalence exists between the continuous time model and the discrete time models previously considered in the literature. Thus, the major results are control limit rules. Shock models are those models in which damage to a component accumulates via a

Poisson process, and a decision is made after a shock occurs to retain or replace the component. In Section 7.4 it is shown that an equivalence exists between shock models and the continuous time model presented herein. Thus, control limit rules which apply to shock models also apply to the continuous time model. In Chapter 8 it is assumed that the component or system continues to operate while the repair process is taking place. The problem is to decide from which states and to which states the process should be repaired. Theorems, including control limit rules, are presented to help resolve that issue. Choosing between repair and replacement of a component with the system inoperable is the subject of Chapter 9. The component may be repaired, replaced, or left alone, and it is shown that a type of control limit rule is optimal.

6.1. Markov Decision Processes

The main ingredients of discrete dynamic programming are a state space S, a set of actions A, rewards $r_i(a)$, and transition probabilities $p_{ij}(a)$. The process begins a period in a state $i \in S$, and an action $a \in A$ is chosen. A reward $r_i(a)$ is received, and the process jumps to state j to start the next period with probability $p_{ij}(a)$ (where $j = i$ is allowed). This series of events is repeated indefinitely. The process is a discrete time Markov decision process (DTMDP) if the rewards and transition probabilities depend only on the current state and chosen action. Thus, $p_{ij}(a)$ are Markov, i.e., $p_{ij}(a)$ $= P(X_{n+1} = j \mid X_n = i,$ action a is chosen) is independent of n and the past history of the process. It is assumed throughout this thesis

that S and A are finite. Rewards are also finite, and they will generally be considered costs rather than rewards since the optimization criterion used herein is minimizing cost rather than maximizing profit.

A policy R is a rule for selecting a decision in each state at each point in time. Let C be the set of all policies. By the Markovian assumption, it is sufficient to consider the subclass of policies that depend only on the current state and period. For an infinite horizon problem, the policies will depend only on the current state, i.e., they will be stationary. It can also be shown (see Derman [1962]) that it is sufficient to consider the class C_D of non-randomized policies, i.e., policies which assign only one action to each state in each period. Let $\{X_n, n=0,1,\ldots\}$ be the sequence of observed states, let $\{A_n, n=0,1,\ldots\}$ be the sequence of observed actions, and let $\{W_n, n=0,1,\ldots\}$ be a sequence of random variables such that $W_n = r_i(a)$ if $X_n = i$ and $A_n = a$. The problem is to minimize discounted or undiscounted costs over an infinite time horizon. When discounted costs are considered, a discount factor α will be used. Expected costs in a single period when policy R is used are

$$E_R W_n = \sum_{i \in S} \sum_{a \in A} P_R(X_n = i, A_n = a) r_i(a) .$$

The undiscounted costs for a finite horizon N using policy R and starting in state i is

$$S_R(i) \equiv E_R\left(\sum_{n=0}^{N} W_n\right) = \sum_{n=0}^{N} \sum_{j \in S} \sum_{a \in A} P_R(X_n = j, A_n = a \mid X_0 = i) r_j(a) .$$

The expected average cost is $\phi_R(i) \equiv \lim_{N\to\infty} S_R(i)/(N+1)$ when the limit exists. The expected discounted cost for a finite time horizon using policy R with initial state i is

$$\psi_R(i,\alpha,N) \equiv E_R\left(\sum_{n=0}^{N} \alpha^n W_n\right) ,$$

and the total expected discounted cost is

$$\psi_R(i,\alpha) \equiv \lim_{N\to\infty} \psi_R(i,\alpha,N) .$$

It can be shown that

$$\phi_R(i) = \lim_{\alpha\to 1} (1-\alpha)\ \psi_R(i,\alpha)$$

when the limit exists. Define

$$\psi(i,\alpha,N) \equiv \min_{R\in C_D} \psi_R(i,\alpha,N), \qquad \psi(i,\alpha) \equiv \min_{R\in C_D} \psi_R(i,\alpha) ,$$

and

$$\phi(i) \equiv \min_{R\in C_D} \phi_R(i) .$$

Continuous time Markov decision processes (CTMDP) are similar to DTMDP. Upon arriving in state $i \in S$, an action $a \in A$ is chosen, and a reward $r_i(a)$ is earned. The holding time in state i is exponentially distributed with parameter $\lambda_i(a)$, and the process then jumps to state j with probability $p_{ij}(a)$ ($p_{ii} > 0$ is permitted). The rewards, holding times, and transition probabilities depend only on the

current state and chosen action. S and A are assumed to be finite, and the objective is to minimize cost. In the discounted case, a discount rate β ($\alpha = e^{-\beta}$) will be used. Let $\{Y(t), t \geq 0\}$ be the state of the process at time t, let $Y(t) = Y_n$ if $T_n \leq t \leq T_{n+1}$ where T_n is the time of the nth jump, let A_n be the action chosen in state Y_n, and let $N_t = \max\{n: T_n \leq t\}$. The total expected discounted cost using (stationary) policy R with initial state i is

$$W_R(i,\beta) \equiv E_R \left(\sum_{n=0}^{\infty} e^{-\beta T_n} r_{Y_n}(A_n) \mid Y_0 = i \right) .$$

The expected average cost is

$$V_R(i) \equiv \lim_{t \to \infty} E_R \left(\frac{1}{t} \sum_{n=0}^{N_t} r_{Y_n}(A_n) \mid Y_0 = i \right)$$

when the limit exists.

The Markov decision processes in this thesis are usually derived from discrete time Markov chains (DTMC) and continuous time Markov chains (CTMC). Some results and notation from the theory of Markov chains will be useful. The Markov chains in this thesis will be irreducible and positive recurrent on a finite state space.

Let $\{X_n, n=0,1,\ldots\}$ be a DTMC.

Notation:

$$p_{ij} \equiv P(X_{n+1}=j \mid X_n=i) = P(X_{n+1}=j \mid X_n=i, X_{n-1}=k, \ldots, X_0=\ell) .$$

$$P \equiv (p_{ij}) \text{ is the matrix of transition probabilities .}$$

$$p_{ij}^{(n)} \equiv P(X_n=j \mid X_0=i) = \sum_{k \in S} p_{ik}^{(n-1)} p_{kj} .$$

$\pi_j \equiv \lim_{n\to\infty} p_{ij}^{(n)}$ are called the steady state probabilities.

$\pi P = \pi$, $\pi_i \geq 0 \ \forall\ i$, and $\sum_{k\in S} \pi_i = 1$.

Let $\{X(t), t\geq 0\}$ be a CTMC.

Notation: λ_{ij} is the transition rate from state i to state j.

$Q \equiv (\lambda_{ij})$ is the matrix of transition rates with diagonal elements $\lambda_i \equiv -\sum_{j\in S} \lambda_{ij}$.

$\lambda_{i\bullet} \equiv \sum_{j\in S} \lambda_{ij}$ is the parameter of exponential holding time in state i .

$m(i) = 1/\lambda_{i\bullet}$ is the mean holding time in state i .

$p_{ij} = \lambda_{ij}/\lambda_{i\bullet}$ is the transition probability from state i to state j .

μ_{ij} is the mean time from arrival in state i until the next arrival is state j, sometimes called a first passage time .

$\pi_j = m(j)/\mu_{jj}$ are the steady state probabilities .

6.2. Derman's Model and Extensions

The use of discrete dynamic programming to determine optimal maintenance policies for multistate systems was pioneered by Derman in the 1960's. Derman [1962] showed that only non-randomized decision rules need be considered, and in Derman [1963] it was shown that control limit rules are optimal for certain types of systems. The discussion herein follows Derman [1970].

Consider a component or system which is inspected at equally spaced points in time and classified into state $0,1,\ldots,M$ with state M

being the best state. At the beginning of each period, an operator may choose between two actions - to replace (which takes one period) or to retain the component. The component must be replaced when it reaches state 0. A control limit rule is a policy which replaces the component when the observed state is $0, 1, \ldots, i^*$ and retains the component when the observed state is $i^*+1,\ldots,M$ where state i^* is called the control limit. Replacing the component costs $C > 0$ units of money and there is an additional penalty cost $K \geq 0$ if the component is replaced from state 0. The states may be considered as the remaining capability of the component, and a penalty is assessed if the component ever becomes completely inoperative. Markovian transition probabilities p_{ij} are used. It is also assumed that $p_{i0}^{(n)} > 0$ for some $n \geq 1$ so that a component which is not replaced will eventually fail.

In the notation of dynamic programming, the problem becomes $S = \{0,1,\ldots M\}$, $A = \{0 = \text{do nothing}, 1 = \text{replace}\}$,

$$r_i(a) = \begin{cases} 0 & \text{if} \quad a = 0 \\ C & \text{if} \quad a = 1,\ i \geq 1 \\ C+K & \text{if} \quad a = 1,\ i = 0 \end{cases}$$

$$p_{ij}(a) = \begin{cases} p_{ij} & \text{if} \quad a = 0 \\ 1 & \text{if} \quad a = 1,\ j = M \\ 0 & \text{if} \quad a = 1,\ j \neq M \end{cases}$$

The standard dynamic programming recursion for the discounted cost case is

$$\phi(i,\alpha,N+1) = \min \{\alpha \sum_{j=0}^{M} p_{ij}\phi(j,\alpha,N);\ C + \alpha\phi(M,\alpha,N)\} \quad \text{for} \quad i \neq 0$$

$$= C + K + \alpha\phi(M,\alpha,N) \quad \text{for} \quad i = 0 .$$

This is slightly different from Derman's construction since he assumes that replacement is instantaneous. In that case $\psi(M,\alpha,N)$ is replaced by $\sum_{j=0}^{M} p_{Mj}\ \psi(j,\alpha,N)$ in the above equation. This might also be useful if the new component could be in any state due to damage in fabrication or shipping.

The main result in Derman [1970] is that if $\sum_{j=0}^{k} p_{ij}$ is nonincreasing in $i\ \forall\ k$, then a control limit rule is optimal for both the expected average cost criterion and the total expected discounted cost criterion. This means that there exist control limit policies R^* and R^{**} such that

$$R^* = \arg\min_{R \in C_D} [\psi_R(i,\alpha)] \quad \text{and} \quad R^{**} = \arg\min_{R \in C_D} [\phi_R(i)].$$

The policies R^* and R^{**} may be determined by solving linear programs, and they will generally be identical for small interest rates. The restriction that $\sum_{j=0}^{k} p_{ij}$ be nonincreasing in $i\ \forall\ k$ seems very reasonable since it means that if no replacement occurs and one component begins a period in a better state than another component, the first component will, on the average, also begin the next period in a better state.

There have been several extensions of the basic model. In Kolesar [1966] a state occupancy cost was added to the model so that cost A_i was charged each time the component was observed in state i. Derman's construction is a special case of this with $A_0 = K$ when replacement is the chosen action in state 0 and $A_i = 0$ otherwise. A control limit

rule is shown to be optimal under the additional assumption that A_i is nonincreasing in i. The costs A_i are not restricted to be nonnegative, and $U_i = -A_i$ may be thought of as a utility rather than a cost. The problem is then to maximize utility, and a control limit rule is optimal when U_i is nondecreasing in i. Thus, the analysis applies to coherent systems.

A generalization of Kolesar's model is contained in Ross [1969]. This model permits a continuous state space which is useful in inventory applications. A control limit rule is again shown to be optimal. The model also applies to the case in which several components may be ordered at once. The state of the system is (n,x) where n represents the number of spare components, and x represents the state of the component currently in operation. When the system is in state $(0,0)$, any number of components may be ordered. It is shown that a control limit rule applied to the component currently in operation is optimal.

The basic model has been extended to include more general cost structures and state transitions. In Kalymon [1972], the replacement cost is random, and there is a salvage value which depends on the replaced component's state and on the replacement cost. The total cost of replacement is then $C + r(C) + s(i)$ where C is a random variable and the salvage value is $-(r(C) + s(i))$. This leads to the following recursion.

$$\phi(i,\alpha,N+1) = \min \{A_i + \alpha \sum_{j=0}^{M} p_{ij}\phi(j,\alpha,N);$$

$$A_i + C + r(C) + s(i) + \alpha\phi(M,\alpha,N)\} \quad .$$

The following monotonicity assumptions are made.

(1) A_i and $s(i)$ are nonincreasing in i.

(2) $C + r(C)$ is nondecreasing in C.

(3) $\alpha \sum_{j=0}^{M} p_{i+1,j}[A_j - s(j)] - \alpha \sum_{j=0}^{M} p_{ij}[A_j - s(j)]$
$\geq s(i+1) - s(i) \quad \forall\ i \leq M - 2.$

The third condition seems rather strange, and no heuristic reason for its necessity is provided in the paper. If these monotonicity assumptions are added to the assumptions in the basic model, a control limit rule is optimal for both the discounted and average cost cases.

In Kao [1972] the transitions are semi-Markov rather than Markov. The holding time in a state can then depend on the current state and the target state, but only transitions to smaller states are allowed. Although the possibilities of variable replacement cost and variable replacement time are discussed, they are held constant. It is also assumed that the expected cost per occupancy in state i is nonincreasing in i. Under these conditions, a control limit rule is optimal.

In all the aforementioned models, it is assumed that inspections occur in every time period and that replacement provides a new unit. Adding inspection scheduling and repair/replacement considerations into the decision process further complicates the basic model. In Klein [1962] the decision space is enlarged to allow repair to any state and scheduling of the next inspection some number of periods later. Costs of repair, replacement, inspection, and penalty costs for failure are included in the model. It is shown that the problem may be formulated

as a linear program, but no discussion of control limit policies or the form of the optimal solution is included. Another model of this type is due to Luss [1976]. In this paper it is assumed that a control limit rule is optimal, and an algorithm is presented to find the optimal control limit. State transitions are governed by a continuous pure death process. This is converted to a discrete model in which opportunities for inspection occur at regular points in time. State occupancy costs as well as maintenance, inspection, and penalty costs are included in the model.

An interesting type of control limit rule is contained in Rosenfield [1976A]. Let C_j be the cost of repair from the jth state (includes penalty cost), let I be the constant inspection cost, and let A_j be the state occupancy cost. In each time period, the operator may choose to inspect the component, replace the component, or do nothing. The system state is (i,k) which means that the system was in state i at its last inspection, k periods ago. This leads to the following recursion.

$$\phi(i,k,\alpha,N+1) = \min \{\text{inspect; replace; do nothing}\}$$

$$= \min \{I + \sum_{j=0}^{M} p_{ij}^{(k)} A_j + \alpha \sum_{j=0}^{M} p_{ij}^{(k+1)} \phi(j,0,\alpha,N);$$

$$\sum_{j=0}^{M} p_{ij}^{(k)} C_j + \alpha\phi(M,0,\alpha,N);$$

$$\sum_{j=0}^{M} p_{ij}^{(k)} A_j + \alpha\phi(i,j+1,\alpha,N)\} \qquad \text{for } i > 0 \ .$$

The following assumptions are made:

(1) C_j, A_j, and $C_j - A_j$ are nonincreasing in j.

(2) $\sum_{j=0}^{k} p_{ij}$ is nonincreasing in $i \; \forall \; k$, and $p_{i0}^{(n)} > 0$ for some $n \geq 1$.

(3) P is upper triangle, i.e., $p_{ij} = 0 \; \forall \; j < i$.

(4) P is totally positive of order 2 (TP_2), i.e., $p_{ik}p_{j\ell} \geq p_{i\ell}p_{jk}$ $\forall \; i \geq j, \; k \geq \ell$.

If these assumptions hold, Rosenfield shows that a 4-region policy is optimal. A 4-region policy is one in which for fixed i and increasing k, it is optimal to first do nothing, then to inspect, then to again do nothing, and finally to replace. Figure 6.1 is an illustration of this type of policy.

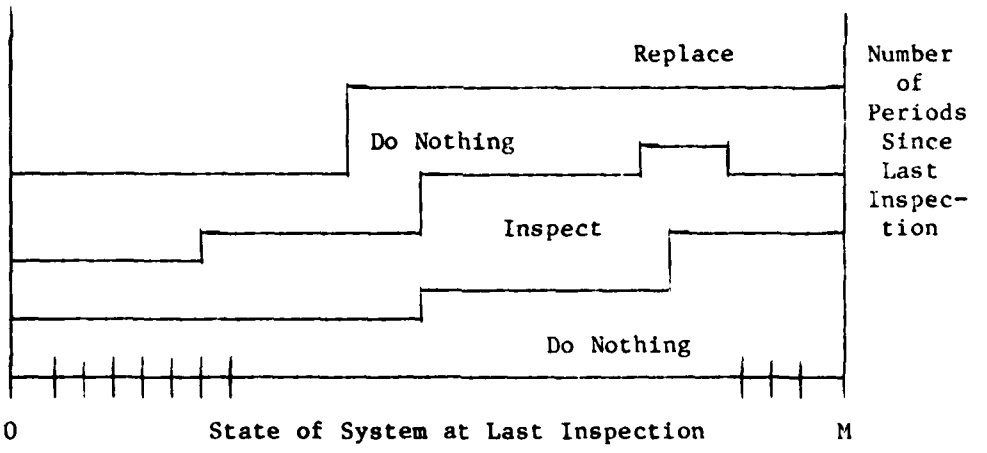

Figure 6.1. 4-Region Policy

In Rosenfield [1976B] it is shown that if assumptions (1), (2), and a slightly weaker version of (3) hold, then a type of control limit rule is optimal. There is a state $i^*(k)$ for which it is optimal to replace

if $i \leq i^*(k)$ and to either inspect or do nothing if $i > i^*(k)$. With the weaker assumptions, it is not possible to decide whether to inspect or to do nothing for $i > i^*(k)$.

The previous models generally assume that restoration returns the system to its best state. A model constructed by Eppen [1965] allows the system to return to any higher state at a cost $C(k)$ where k = new state - old state, i.e., k is the number of states by which the system is improved. The following assumptions are made:

(1) $C(k) = Ck$, $C > 0$, $k > 0$.

(2) A_j, the one period operating cost in state j, is convex and positive with $A_1 - A_0 < -C \Rightarrow$ some kind of maintenance will occur in state 0.

(3) $p_{ii} > 0$, $p_{i,i-1} = 1 - p_{ii}$, and $p_{ij} = 0 \; \forall \; j \neq i, i-1$. Also, p_{ii} is concave in $i \; \forall \; i \geq 1$.

The only optimality criterion considered is minimized discounted cost for a finite time horizon. The standard recursion becomes:

$$\phi(i,\alpha,N+1) = \min_{j \geq i} \{C(j-i) + A_j + \alpha \sum_{j=0}^{M} p_{ij} \phi(j,\alpha,N)\} .$$

The optimal policy is similar to a control limit rule. If $i < i_N^*$ at the beginning of the first time period, then the system should be repaired to state i_N^*. If $i \geq i_N^*$, do nothing. It is also shown that $i_1^* \geq 1$ and $i_{N+1}^* \geq i_N^*$.

It must be noted that even if a control limit rule is the optimal way to operate a multistate component, that may not be the best way to run a system composed of multistate components. This is shown in Example 4 of Denardo [1967].

7. RESTORATION WITH THE SYSTEM INOPERABLE

This chapter pertains to the optimal restoration of constantly monitored equipment with the system inoperable. In Section 6.2 the components were periodically inspected which allowed the system to be treated as a discrete time Markov decision process. In this chapter assumptions are made which allow the optimal operation of a multistate system to be treated as a continuous time Markov decision process. A decision is made in each state to either continue operation or to perform a restoration activity. It is shown that, when restoration costs are constant, the restoration process should always return the system to its best state. Control limit rules are optimal in most cases.

Section 7.1 extends the discrete inspection model by permitting the restoration activity to fail. Thus, it may require more than one time period to perform the intended restoration. If the state occupation costs are not paid while the system is being restored, a control limit rule is shown to be optimal. However, if the state occupation costs are paid while the system is being restored, then the restoration epoch must be shorter than the exponential holding time in any state for a control limit rule to be optimal. In Section 7.2 the continuous time model for constantly monitored components is described. Using a known equivalence between discrete and continuous time Markov decision processes, it is shown that this problem is equivalent either to the discrete time model contained in Section 7.1 or, with different assumptions, to the basic model discussed in Section 6.2. This equivalence establishes the

optimality of a control limit rule in the continuous time case. Some examples are presented in Section 7.3. The last section of Chapter 7 is devoted to showing that the continuous time model can be considered as a special case of a shock model. The shock model has some generality not contained in the continuous model, and its optimal operation is described by a control limit rule. However, the shock model does not incorporate a utility function into its cost structure nor is it obvious how that would be accomplished. Thus, Section 7.2 is necessary to show control limit rule optimality for more complicated cost functions.

7.1 Failure to Replace Model

In this section a model similar to the one in Kolesar [1966] is analyzed. The difference between this model and previous models of this type is that restoration is allowed to fail, i.e., there is a certain probability that the attempted system restoration will not be completed in a single period. This model is interesting by itself, and it will turn out to be the appropriate generalization when a continuous model is analyzed using discrete methods. The main result is that a control limit rule with return state M is optimal when the state occupation costs are not paid during the restoration process. The return state is defined as the state in which the process will be immediately following a successful repair or replacement. An example is given to show that a control limit rule is not necessarily optimal when state occupation costs must be paid during restoration.

Note: The words repair, replace, and restore are used interchangeably throughout Part II of this thesis. An effort has been made to use the

word restoration when talking in general terms, to use the word repair when the system is not returned to its best state or continues to operate during restoration, and to use the word replace when the system is returned to its best state and is rendered inoperable by restoration. The words system and component are also used interchangeably. The results herein apply to any process with state space $\{0,1,\ldots,M\}$, which could be either a single multistate component or a system. They do not necessarily apply to a system consisting of multistate components.

Let $C \geq 0$ be the cost of replacing a component with a penalty cost of $K \geq 0$ for replacement from state 0, and let A_i be the one-period state occupancy costs in state i ($U_i = -A_i$ is interpreted as the utility in state i). Transitions among states are Markovian, and α is the discount factor. Replacement must always be attempted from state 0. Let $\tilde{p}$ be the probability that a planned replacement succeeds. It is assumed that if a failure to replace occurs in any period, the probability that a planned replacement succeeds in the next period is still $\tilde{p}$. This may not be entirely realistic, but it is necessary for a simple model. First assume that state occupancy costs are not paid during replacement. The standard dynamic programming recursion is:

$$\phi(i,\alpha,N+1) = \min\{A_i + \alpha \sum_{j=0}^{n} p_{ij}\phi(j,\alpha,N);$$

$$C + \alpha\tilde{p}\phi(M,\alpha,N) + \alpha(1-\tilde{p})\phi(i,\alpha,N)\} \text{ for } i > 0$$

$$= C + K + \alpha\tilde{p}\phi(M,\alpha,N) + \alpha(1-\tilde{p})\phi(0,\alpha,N) \text{ for } i = 0. \quad (7.1)$$

Equation (7.1) is valid when replacement cost is paid even though the attempted replacement fails. If replacement cost is paid only when the replacement is successful, the second term in the minimization would become $\tilde{p}C + \alpha\tilde{p}\psi(M,\alpha,N) + \alpha(1-\tilde{p})\psi(i,\alpha,N)$. The analysis will be identical regardless of what option is selected.

The following assumptions are necessary.

(1) A_i is nonincreasing in i.

(2) $\sum_{j=0}^{k} p_{ij}$ is nonincreasing in i $\forall$ k.

(3) $p_{i0}^{(n)} > 0$ for some $n \geq 1$.

From condition (2) it can be shown (see Derman [1970], page 123) that $\sum_{j=0}^{M} p_{ij} f(j)$ is nonincreasing in i for every nonincreasing function $f(\cdot)$. It can also be shown that a random variable with density $P(X_i = j) = p_{ij}$ has an IFR distribution. The third assumption ensures that replacement will eventually occur; thus, system operation is a regenerative process. As discussed in Section 6.1, only policies which are deterministic and which do not depend on the past history of the process (class C_D) need be considered. The inductive argument contained in Theorem 7.1 will often be used to prove theorems concerning control limit rules and restoration to state M.

Theorem 7.1: If conditions (1), (2), and (3) hold and total discounted cost is the optimality criterion, then state M is the optimal return state.

Proof: Since restoration to any state k is allowed, Equation (7.1) becomes

$$\psi(i,\alpha,N+1) = \min\{A_i + \alpha \sum_{j=0}^{M} p_{ij}\psi(j,\alpha,N);$$
$$C + \alpha\tilde{p}\min_{\ell\in S}[\psi(\ell,\alpha,N)] + \alpha(1-\tilde{p})\psi(i,\alpha,N)\} \quad \text{for } i > 0$$
$$= C + K + \alpha\tilde{p}\min_{\ell\in S}[\psi(\ell,\alpha,N)] + \alpha(1-\tilde{p})\psi(0,\alpha,N) \quad \text{for } i \geq 0 .$$

$\psi(i,\alpha,0) = \min\{A_i,C\}$ is clearly nonincreasing in i, and it will be shown that is true for every N. Assume inductively that $\psi(i,\alpha,n)$ is nonincreasing in i $\forall$ $n = 0, 1, \ldots, N$. Then $\sum_{j=0}^{M} p_{ij}\psi(j,\alpha,N)$ is nonincreasing in i from assumption (2) and A_i is nonincreasing in i from assumption (1). Thus, every term in the above minimization is nonincreasing in i, so $\psi(i,\alpha,N+1)$ is nonincreasing in i. By induction, $\psi(i,\alpha,N)$ is nonincreasing in i $\forall$ N. Then $\psi(i,\alpha) = \lim_{N\to\infty} \psi(i,\alpha,N)$ is nonincreasing in i (see Derman [1970], page 37, for a proof). Thus, the optimal return state is given by $\psi(M,\alpha) = \min_{\ell\in S}[\psi(\ell,\alpha)]$. □

The proof of Theorem 7.1 applies anytime $\psi(i,a,N)$ is nonincreasing in i for every N. This will be the case in all the theorems pertaining to control limit rule optimality. To avoid unnecessary repetition, whenever a theorem states that state M is the optimal return state using the discounted cost criterion, Theorem 7.1 will be referenced. For simplicity, $\psi(M,\alpha)$ will replace $\min_{\ell\in S}[\psi(\ell,\alpha)]$ in the proofs of these theorems. The same remarks apply to Theorem 7.3 when the average cost criterion is being used in place of the discounted cost criterion. Lemma 7.2 is a well known Abelian theorem (see Derman [1970], page 144).

Lemma 7.2: Let $\{a_n, n=0,1,\ldots\}$ be a sequence of real numbers, and let $f(X) = \sum_{n=0}^{\infty} a_n X^n$, $0 \leq X \leq 1$. If $\lim_{N\to\infty} (1/N) \sum_{n=0}^{N} a_n = A < \infty$, then $\lim_{X\to 1}(1-X)f(X) = A$.

Let W_n be the (finite) expected reward earned in period n when using policy R with initial state i. Then $\phi_R(i) = \lim_{N\to\infty} 1/(N+1) \sum_{n=0}^{N} W_n$ and $\psi_R(i,\alpha) = \sum_{n=0}^{\infty} \alpha^n W_n$. Letting $X = \alpha$ and $a_n = W_n$ in Lemma 7.2, $\phi_R(i) = \lim_{\alpha\to 1}(1-\alpha)\psi_R(i,\alpha)$.

Theorem 7.3: If conditions (1), (2), and (3) hold, and expected average cost is the optimality criterion, then state M is the optimal return state.

Proof: From Theorem 7.1 the policy $R(\alpha)$ which minimizes $\psi(i,\alpha)$ must have return state M. Since S and A are finite and only non-randomized policies need be considered, there are a finite number of policies. Thus, there must be a policy R^* which has return state M and a sequence of discount factors $\{\alpha_n, n=1,2,\ldots\}$ such that $\lim_{n\to\infty} \alpha_n = 1$ and $R(\alpha_1) = R(\alpha_2) = \cdots = R^*$ is the optimal policy for each α_n. For any other policy R and any α_n, $(1-\alpha_n)\psi_R(i,\alpha_n) \geq (1-\alpha_n)\psi_{R^*}(i,\alpha_n)$. From Lemma 7.2,

$$\phi_R(i) = \lim_{n\to\infty}(1-\alpha_n)\psi_R(i,\alpha_n) \geq \lim_{n\to\infty}(1-\alpha_n)\psi_{R^*}(i,\alpha_n) = \phi_{R^*}(i).$$

Thus, R^* is optimal. □

Normally, R^* will be optimal for all sufficiently small interest rates as well as being optimal in the average cost case.

Theorem 7.4: If conditions (1), (2), and (3) hold, there is a control limit policy which minimizes $\psi(i,\alpha)$.

Proof: From Theorem 7.1, $\psi(i,\alpha)$ is nonincreasing in i and can be written

$$\psi(i,\alpha) = \min\{A_i + \alpha \sum_{j=0}^{M} p_{ij}\psi(j,\alpha);$$
$$C + \alpha\tilde{p}\psi(M,\alpha) + \alpha(1-\tilde{p})\psi(i,\alpha) \qquad \text{for } i > 0$$
$$= C + K + \alpha\tilde{p}\psi(M,\alpha) + (1-\tilde{p})\psi(0,\alpha) \qquad \text{for } i = 0 \ .$$

Assume there exists an $i^* > 0$ such that replacement is better than inaction (if not, replacing only in state 0 is trivially a control limit rule). Then:

$$\psi(i^*,\alpha) = C + \alpha\tilde{p}\psi(M,\alpha) + \alpha(1-\tilde{p})\psi(i^*,\alpha) \leq A_{i^*} + \alpha \sum_{j=0}^{M} p_{i^*j}\psi(j,\alpha) \ .$$

Solving for $\psi(i^*,\alpha)$ yields:

$$\psi(i^*,\alpha) = [C+\alpha\tilde{p}\psi(M,\alpha)]/[1-\alpha(1-\tilde{p})] \ .$$

Since $\psi(i^*,\alpha)$ is independent of i^*, set $\psi(i,\alpha) = \psi(i^*,\alpha)$ $\forall$ $i \leq i^*$. Since $\psi(i,\alpha)$ is nonincreasing in i, this must be the optimal policy. Thus, it is optimal to replace $\forall$ $i \leq i^*$ which is a control limit rule. □

Theorem 7.5: If conditions (1), (2), and (3) hold, there is a control limit rule which minimizes $\phi(i)$.

Proof: From Theorem 7.4 there is a control limit policy $R(\alpha)$ which minimizes $\psi(i,\alpha)$. Since there are a finite number of control limit policies, there must be a control limit policy R^* and a sequence of discount factors $\{\alpha_n, n=0,1,\ldots\}$ such that $\lim_{n\to\infty} \alpha_n = 1$ and $R(\alpha_1) = R(\alpha_2) = \cdots = R^*$ is the optimal policy for each α_n. For any other policy R and any α_n, $(1-\alpha_n)\psi_R(i,\alpha_n) \geq (1-\alpha_n)\psi_{R^*}(i,\alpha_n)$. From Lemma 7.2,

$$\phi_R(i) = \lim_{n\to\infty}(1-\alpha_n)\psi_R(i,\alpha_n) \geq \lim_{n\to\infty}(1-\alpha_n)\psi_{R^*}(i,\alpha_n) = \phi_{R^*}(i) \quad .$$

Thus, R^* is optimal. □

The preceding theorems have shown that a control limit policy is optimal when state occupancy costs are not paid during replacement. If those costs are paid during replacement, the standard recursion becomes

$$\psi(i,\alpha) = \min\{A_i + \alpha \textstyle\sum_{j=0}^{M} p_{ij}\psi(j,\alpha);$$
$$C + A_i + \alpha\tilde{p}\psi(M,\alpha) + \alpha(1-\tilde{p})\psi(i,\alpha)\} \quad \text{for} \quad i > 0$$

$$= C + K + A_0 + \alpha\tilde{p}\psi(M,\alpha) + \alpha(1-\tilde{p})\psi(0,\alpha) \quad \text{for} \quad i = 0 \quad .$$

If $\psi(i^*,\alpha)$ is determined as in Theorem 7.4, then

$$\psi(i^*,\alpha) = [C+A_{i^*}+\alpha\tilde{p}\psi(M,\alpha)]/[1-\alpha(1-\tilde{p})] \quad .$$

This is no longer independent of i^* because of the A_{i^*} term on the right hand side, and the analysis used in Theorem 7.4 fails. The following example shows that a control limit rule may not be optimal in this case.

Example 7.1: Consider the 4-state DTMC shown in Figure 7.1.

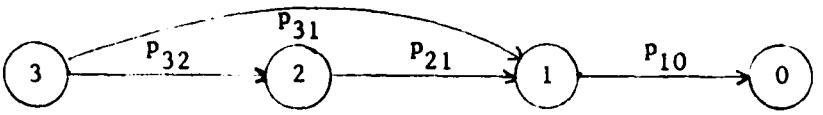

Figure 7.1. 4-state DTMC

Let:

$$\alpha = .5,\ C = 5,\ K = 0,\ A_3 = A_2 = -10,\ A_1 = A_0 = 0,\ \tilde{p} = .5,$$

$$p_{31} = .5,\ p_{32} = .5,\ p_{21} = 1,\ p_{10} = 1 \ .$$

$$\phi(3,\alpha) = A_3 + \alpha p_{32}\phi(2,\alpha) + \alpha p_{31}\phi(1,\alpha)$$

$$\phi(2,\alpha) = \min\{A_2 + \alpha\phi(1,\alpha);\ C + A_2 + \alpha\tilde{p}\phi(3,\alpha) + \alpha(1-\tilde{p})\phi(2,\alpha)\}$$

$$\phi(1,\alpha) = \min\{A_1 + \alpha\phi(0,\alpha);\ C + A_1 + \alpha\tilde{p}\phi(3,\alpha) + \alpha(1-\tilde{p})\phi(1,\alpha)\}$$

$$\phi(0,\alpha) = C + A_0 + \alpha\tilde{p}\phi(3,\alpha) + \alpha(1-\tilde{p})\phi(0,\alpha)$$

$$\phi(3,\alpha) = -12.38$$

$$\phi(2,\alpha) = \min\{-9.37;\ -10.79\} = -10.79 \text{ (replace)}$$

$$\phi(1,\alpha) = \min\{1.27;\ 2.22\} = 1.27 \text{ (do nothing)}$$

$$\phi(0,\alpha) = 2.54.$$

Thus, it is optimal to do nothing in states 1 and 3 and to replace from states 0 and 2 which is not a control limit rule. □

In Example 7.1 it is the high discount rate $(1-\alpha = .5)$ which causes the non-optimality of control limit rules. It can be shown, for $\alpha = .9$, that it is optimal to replace in states 0,1, and 2 which is a control limit rule. Thus, there is still hope that a control limit rule is optimal in the average cost case. Unfortunately, that is not true either as shown by Example 7.2 in Section 7.3.

7.2. Equivalence of the Continuous and Discrete Models

The subject of this section is the equivalence between a continuous time Markov decision process (CTMDP) and a discrete time Markov decision process (DTMDP). By equivalence it is meant that the two processes have identical optimal policies and identical minimum cost. This equivalence is due to Serfozo [1979]. It is useful since it allows a constantly monitored system to be treated as a periodically inspected system.

Let $X = (S,A,r,p,\alpha)$ be a DTMDP, and let $Y = (S,A,\hat{r},\lambda,\hat{p},\beta)$ be a CTMDP. The notation presented in Section 6.1 is used throughout this Chapter. Let $\gamma = \sup_{i,a} \lambda_i(a)$. An equivalence between X and Y is given by Theorem 7.6.

Theorem 7.6 (Serfozo 1979]): Let $Y = (S,A,\hat{r},\lambda,\hat{p},\beta)$ be a CTMDP with $\gamma < \infty$ and countable S and A. Let $X = (S,A,r,p,\alpha)$ be a DTMDP with

$$\alpha = \gamma/(\gamma+\beta)$$

$$r_i(a) = \begin{cases} \hat{r}_i(a)[\beta+\lambda_i(a)]/(\beta+\gamma) & \text{for discounted rewards} \\ \hat{r}_i(a)\lambda_i(a)/\gamma & \text{for average rewards} \end{cases}$$

$$p_{ij}(a) = \begin{cases} \lambda_i(a)\,\hat{p}_{ij}(a)/\gamma & \text{if } j \neq i \\ 1 - \lambda_i(a)[1-\hat{p}_{ii}(a)]/\gamma & \text{if } j = i \end{cases} .$$

If Y and X are both controlled by stationary policy R, then $W_R(i,\beta) = \phi_R(i,\alpha)$ and $V_R(i) = \gamma\phi_R(i)$.

Remarks:

(1) The requirement that S and A be countable is necessary only to avoid technical details. They are finite for the cases considered herein.

(2) The condition $\gamma < \infty$ means that the exponential holding time parameters must be uniformly bounded which is equivalent to saying that the Markov process is regular. This is trivially satisfied in this thesis since S is finite.

(3) For $\beta = 0$ ($\alpha=1$), the rewards in the discounted case become the rewards in the average value case. Because of this, only the discounted rewards will be expressly written when applying Theorem 7.6.

(4) Since both Y and X have the same state and action spaces, the same policies apply to both. Only stationary policies need be considered, and there are a finite number of those. Theorem 7.6 says that if a stationary policy is optimal for one system, then the same stationary policy is optimal in the other system. Thus, if a CTMDP is converted to a DTMDP and something is proven regarding the optimal policy for the DTMDP, then that is also true of the optimal policy for the CTMDP.

(5) The rewards for the CTMDP in Theorem 7.6 are lump rewards rather than reward rates. If $\hat{r}(a)$ are reward rates, the appropriate DTMDP rewards are $r_i(a) = \hat{r}_i(a)/(\beta+\gamma)$.

Consider a constantly monitored system with $S = \{0,1,\ldots,M\}$ and $A = \{0 = \text{do nothing}, 1 = \text{repair}\}$. Let $C \geq 0$ be the replacement cost rate with a penalty cost rate $K \geq 0$ for replacement from state 0, and

let A_i be the state occupancy cost rate. Let $\lambda_{iM} \equiv \lambda$ be the replacement rate from all states. Normally replacement takes a specified time T in which case $\lambda \equiv 1/T$. This won't affect the results since they are in terms of expected values. Let λ_{ij}, $j < i$, be the transition rates when replacement is not transpiring. Now consider a DTMDP with S and A as above, and

$$p_{ij}(0) = \begin{cases} \lambda_{ij}/\gamma & \text{if } j \neq i \\ 1 - \lambda_{i\bullet}/\gamma & \text{if } j = i \end{cases}$$

$$p_{ij}(1) = \begin{cases} 0 & \text{if } j \neq i, M \\ \lambda/\gamma & \text{if } j = M \\ 1 - \lambda/\gamma & \text{if } j = i \end{cases}$$

$$r_i(0) = A_i/(\beta+\gamma)$$

$$r_i(1) = \begin{cases} C/(\beta+\gamma) & \text{for } i > 0 \\ (C+K)/(\beta+\gamma) & \text{for } i = 0 \end{cases} .$$

The rewards $r_i(1)$ were calculated assuming that state occupancy costs are not collected during replacement. If those costs are collected during replacement,

$$r_i(1) = \begin{cases} C/\gamma + A_i/\gamma & \text{for } i > 0 \\ (C + K)/\gamma + A_i/\gamma & \text{for } i = 0 \end{cases} .$$

It is always true that $\sum_{j=0}^{k} p_{ij}(1)$ is nonincreasing in i since transitions occur only when $j = i$ and $j = M$. Obviously, one condition which ensures that $\sum_{j=0}^{k} p_{ij}(0)$ is nonincreasing in $i \forall k$ is $\sum_{j=0}^{k} \lambda_{ij}$ nonincreasing in $i \forall k$. This is very restrictive,

however. One of the implications of this condition is that, with $k = M$, $\lambda_{10} \geq \sum_{j=0}^{M} \lambda_{Mj}$. A more natural condition which ensures that $\sum_{j=0}^{k} p_{ij}(0)$ is nonincreasing in $i \; \forall \; k$ is given in Lemma 7.7.

Lemma 7.7: If $\lambda_{ij} \geq \lambda_{\ell j} \; \forall \; j \leq i \leq \ell$, then $\sum_{j=0}^{k} p_{ij}(0)$ is nonincreasing in $i \; \forall \; k$.

Proof: Consider any $\sum_{j=0}^{k} p_{\ell j}(0)$ and $\sum_{j=0}^{k} p_{ij}(0)$ with $\ell \geq i$. If $k \geq i$, then $\sum_{j=0}^{k} p_{ij}(0) = 1$, and the result holds trivially. If $k < i$, then

$$\sum_{j=0}^{k} p_{\ell j}(0) = \sum_{j=0}^{k} \lambda_{\ell j}/\gamma \leq \sum_{j=0}^{k} \lambda_{ij}/\gamma = \sum_{j=0}^{k} p_{ij}(0). \; \square$$

The following assumptions, which are the continuous equivalents of the assumptions in Section 7.1, are necessary.

(1) A_i is nonincreasing in i.

(2) $\sum_{j=0}^{k} \lambda_{ij}$ is nonincreasing in $i \; \forall \; k$, or $\lambda_{ij} \geq \lambda_{\ell j} \; \forall \; j \leq i \leq \ell$.

(3) $\lambda_{i,i_1} \lambda_{i_1,i_2} \cdots \lambda_{i_n,0} > 0$ for some sequence $i_1, \ldots, i_n$. This assumption simply assures that state 0 will eventually be reached if no replacement is performed.

Theorem 7.8: If (1), (2), and (3) hold and costs A_i are not paid during replacement, then a control limit rule with return state M is optimal for both the discounted and average cost criteria.

Proof: In the discounted case, the recursion is

$$\phi(i,\alpha,N+1) = \min\{A_i/(\beta+\gamma) + \alpha \sum_{j=0}^{i} p_{ij}(0)\phi(j,\alpha,M);$$

$$C/(\beta+\gamma) + \alpha\lambda\phi(M,\alpha,N)/\gamma + \alpha(1-\lambda/\gamma)\phi(i,\alpha,N)\} \text{ for } i > 0$$

$$= (C+K)/(\beta+\gamma) + \alpha\lambda\phi(M,\alpha,N)/\gamma + \alpha(1-\lambda/\gamma)\phi(0,\alpha,N) \text{ for } i = 0.$$

With $\tilde{p} = \lambda/\gamma$, Theorems 7.1, 7.3, 7.4, and 7.5 of Section 7.1 are applicable. The limiting argument is still valid since the average rewards are the limit as $\alpha \to 1$ ($\beta \to 0$) of the discounted rewards. □

Theorem 7.9: If (1), (2), and (3) hold, if costs A_i are paid during replacement, and if $\lambda \geq \lambda_i$. $\forall$ i, then a control limit rule with return state M is optimal for both the discounted and average cost criteria.

Proof: If $\lambda \geq \lambda_i$. $\forall$ i, then $\gamma \equiv \sup_{i,a} \lambda_i(a) = \lambda$, and $p_{iM}(1) = 1$ $\forall$ i. The recursion in the discounted case then becomes

$$\phi(i,\alpha,N+1) = \min\{A_i/(\beta+\gamma) + \alpha \sum_{j=0}^{i} p_{ij}(0)\phi(j,\alpha,M);$$

$$C/(\beta+\gamma) + A_i/(\beta+\gamma) + \alpha\phi(M,\alpha,N)\} \qquad \text{for } i > 0$$

$$= (C+K)/(\beta+\gamma) + A_0/(\beta+\gamma) + \alpha\phi(M,\alpha,N) \qquad \text{for } i = 0 .$$

Since there is no possibility of replacement failure, this recursion has the same form as the recursion in Kolesar [1966]. The result follows from Theorem 1 of that paper and Theorem 7.1 of Section 7.1. □

7.3. Examples

In Theorem 7.9 it was necessary to assume $\lambda \geq \lambda_{i\bullet}$ $\forall$ i, i.e., the replacement rate is larger than the sum of the failure rates from any state. This is a reasonable assumption since most components will be quickly replaced; otherwise, their availability could be very poor. However, if $\lambda < \lambda_{i\bullet}$ for some i and state occupancy costs are paid during replacement, then a control limit rule is not necessarily optimal as shown by Example 7.2.

Example 7.2: Consider the 4-state CTMC shown in Figure 7.2.

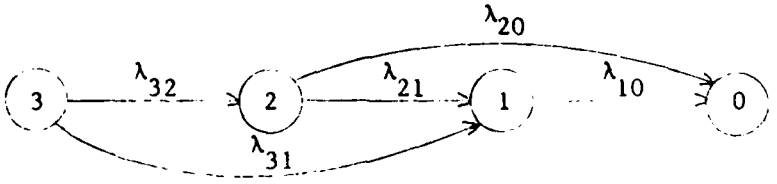

Figure 7.2. CTMC

Let $\lambda_{32} = .1$, $\lambda_{31} = .9$, $\lambda_{21} = 4.5$, $\lambda_{20} = .5$, $\lambda_{10} = .5$, $\lambda = 2$ (or $T = .5$), $C = 1.5$, $K = 0$, $A_0 = 0$, $A_1 = -1$, $A_2 = -2$, and $A_3 = -2.01$. Average expected cost is the optimality criterion. From the theory of regenerative processes, average expected cost can be calculated from

$$E(\text{average cost}) = E(\text{cost per cycle})/E(\text{cycle length}).$$

State 3 is considered as the return state. Replacement may occur from the following groups of states:

$$\{3,2,1,0\},\ \{2,1,0\},\ \{2,0\},\ \{1,0\},\ \text{and}\ \{0\}.$$

Calculations are shown below. The results appear in Table 7.1. Let

E(cost) ≡ E(cost per cycle), and E(cycle) ≡ E(cycle length).

(1) {3,2,1,0} $E(\text{cost}) = C + A_3T = .495$

$E(\text{cycle}) = T = .5$

(2) {2,1,0} $$E(\text{cost}) = C + \frac{A_3}{\lambda_{32}+\lambda_{31}} + \frac{\lambda_{32}}{\lambda_{32}+\lambda_{31}} A_2T + \frac{\lambda_{31}}{\lambda_{32}+\lambda_{31}} A_1T = -1.06$$

$$E(\text{cycle}) = T + 1/(\lambda_{32}+\lambda_{31}) = 1.5$$

(3) {2,0} $$E(\text{cost}) = C + \frac{A_3}{\lambda_{32}+\lambda_{31}} + \frac{\lambda_{32}}{\lambda_{32}+\lambda_{31}} A_2T + \frac{\lambda_{31}}{\lambda_{31}+\lambda_{31}} \cdot \left(\frac{A_1}{\lambda_{10}} + A_0T\right) = -2.41$$

$$E(\text{cycle}) = T + \frac{1}{\lambda_{32}+\lambda_{31}} + \frac{\lambda_{31}}{\lambda_{32}+\lambda_{31}} \left(\frac{1}{\lambda_{10}}\right) = 3.3$$

(4) {1,0} $$E(\text{cost}) = C + \frac{A_3}{\lambda_{32}+\lambda_{31}} + \frac{\lambda_{32}}{\lambda_{32}+\lambda_{31}} \left[\frac{A_2}{\lambda_{21}+\lambda_{20}} + \frac{\lambda_{21}}{\lambda_{21}+\lambda_{20}} A_1T + \frac{\lambda_{20}}{\lambda_{21}+\lambda_{20}} A_0T\right] + \frac{\lambda_{31}}{\lambda_{32}+\lambda_{31}} A_1T = -1.045$$

$$E(\text{cycle}) = T + \frac{1}{\lambda_{32}+\lambda_{31}} + \frac{\lambda_{32}}{\lambda_{32}+\lambda_{31}}\left(\frac{1}{\lambda_{21}+\lambda_{20}}\right) = 1.52$$

(5) {0}
$$E(\text{cost}) = C + \frac{A_3}{\lambda_{32}+\lambda_{31}} + \frac{\lambda_{32}}{\lambda_{32}+\lambda_{31}}\left[\frac{A_2}{\lambda_{21}+\lambda_{20}} + \frac{\lambda_{21}}{\lambda_{21}+\lambda_{20}}\left(\frac{A_1}{\lambda_{10}}\right)\right] + \frac{\lambda_{31}}{\lambda_{32}+\lambda_{31}}\left(\frac{A_1}{\lambda_{10}}\right) + A_0 T = -2.53$$

$$E(\text{cycle}) = T + \frac{1}{\lambda_{32}+\lambda_{31}} + \frac{\lambda_{32}}{\lambda_{32}+\lambda_{31}}\left[\frac{1}{\lambda_{21}+\lambda_{20}} + \frac{\lambda_{21}}{\lambda_{21}+\lambda_{20}}\left(\frac{1}{\lambda_{10}}\right)\right] + \frac{\lambda_{31}}{\lambda_{32}+\lambda_{31}}\left(\frac{1}{\lambda_{10}}\right) = 3.5$$

Set of States	{3,2,1,0}	{2,1,0}	{2,0}	{1,0}	{0}
E(cost)/E(cycle)	.99	-.71	-.73	-.69	-.72

Table 7.1 Cost Comparison

The minimum expected average cost is achieved by inaction in states 1 and 3 and replacement in states 0 and 2. This is clearly not a control limit rule. It can be shown that, for $\lambda \geq 5 = \max_i \lambda_{i\bullet}$, a control limit rule is optimal. The results of this example are also valid for the discounted case with a small discount rate. □

It may seem unreasonable that the cost of replacing a component is constant except for a penalty cost when replacing from state 0. The component might have a salvage value so that replacement cost C_i is nonincreasing in i. Unfortunately, this does not necessarily lead to a control limit rule as shown in Example 7.3.

Example 7.3: Consider the DTMC shown in Figure 7.3.

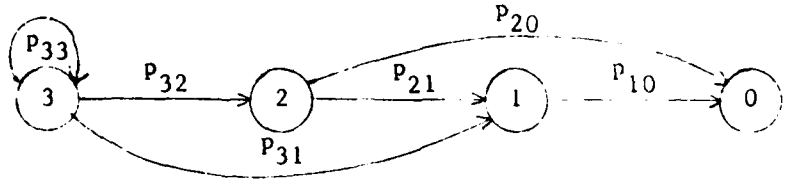

Figure 7.3 DTMC

Let $C_3 = C_2 = C$, and let $C_1 = C_0 = C + K$ with $C = 1$ and $K = 10$. This might occur if states 1 and 0 were very undesirable. Let $A_i = 0 \; \forall \; i$, $p_{33} = .9$, $p_{32} = .09$, $p_{31} = .01$, $p_{21} = .5$, $p_{20} = .5$, and $p_{10} = 1$. Compute E(average cost) = E(cost)/E(cycle) as in Example 7.2 with state 3 as the return state. Possible groups of states to replace from are {3,2,1,0}, {2,1,0}, {2,0}, {1,0}, and {0}. Calculations are shown below, and the results are contained in Table 7.2.

(1) {3,2,1,0} E(cost) = C = 1

E(cycle) = 1

(2) {2,1,0} E(cost) = $C + p_{31}K = 1.1$

E(cycle) = $1 + p_{31} + p_{32} = 1.1$

(3) {2,0} E(cost) = $C + p_{31}K = 1.1$

E(cycle) = $1 + 2p_{31} + P_{32} = 1.11$

(4) {1,0} $E(\text{cost}) = C + (p_{31} + p_{32})K = 2$

$E(\text{cycle}) = 1 + p_{31} + 2p_{32} = 1.19$

(5) {0} $E(\text{cost}) = C + (p_{31} + p_{32})K = 2$

$E(\text{cycle}) = 1 + 2p_{31} + p_{32}(2p_{20} + 3p_{21}) = 1.245$

Set of States	{3,2,1,0}	{2,1,0}	{2,0}	{1,0}	{0}
E(cost)/E(cycle)	1	1	.99	1.68	1.6

Table 7.2 Cost Comparison

The only policy which yields an average cost less than one is inaction in states 1 and 3 and replacement in states 0 and 2. Thus, a control limit rule is not optimal. □

The main reason a control limit rule is not optimal in Example 7.3 is that once the system is in a state in which it will have to pay the penalty cost, it must be optimal to try and stretch out the cycle length. Average cost is minimized by allowing the system to operate as long as possible before replacement. In Example 7.4 the system is profitable to run. The objective is to maximize utility instead of minimizing cost. The replacement cost is constant, but the replacement rate, λ_{iM}, is increasing in i. This might transpire if a repairman was always at the site to fix the system. Then there would be no additional cost involved in restoring the system, but there might be additional time involved for an increasingly degraded system. A control limit rule is not necessarily optimal in this case either.

Example 7.4: Consider the 4-state CTMC in Figure 7.4 which is identical to the system in Example 7.1.

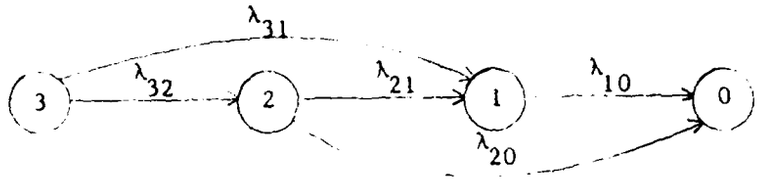

Figure 7.4 CTMC

Let $C = 2$, $K = 0$, $U_i = i \; \forall \; i$ (linear utility function), $\lambda_{32} = .1$, $\lambda_{31} = .9$, $\lambda_{21} = .5$, $\lambda_{10} = .5$, $\lambda_{20} = .5$, $\lambda_{33} = \lambda_{23} = 1$ ($T_{23} = 1$), $\lambda_{13} = .02$ ($T_{13} = 50$), and $\lambda_{03} = .01$ ($T_{03} = 100$). The optimality criterion is again expected average value, but this time the objective is to maximize utility. It is assumed that utility is not collected during restoration. Possible groups of states to replace from are {3,2,1,0}, {2,1,0}, {2,0}, {1,0}, and {0}. E(average utility) = E(util)/E(cycle) where E(util) ≡ E(utility per cycle) and E(cycle) ≡ E(cycle length).

Calculations are shown below, and the results are in Table 7.3.

(1) {3,2,1,0} $E(\text{util}) = -C = -2$

$E(\text{cycle}) = 1/\lambda_{33} = 1$

(2) {2,1,0} $E(\text{util}) = U_3/(\lambda_{32}+\lambda_{31}) - C = 1$

$$E(\text{cycle}) = \frac{1}{\lambda_{32}+\lambda_{31}} + \frac{\lambda_{32}}{\lambda_{32}+\lambda_{31}} T_{23} + \frac{\lambda_{31}}{\lambda_{32}+\lambda_{31}} T_{13} = 46.1$$

(3) {2,0} $E(util) = \frac{U_3}{\lambda_{32}+\lambda_{31}} + \frac{\lambda_{31}}{\lambda_{32}+\lambda_{31}} \left(\frac{U_1}{\lambda_{10}}\right) - C = 3.8$

$$E(cycle) = \frac{1}{\lambda_{32}+\lambda_{31}} + \frac{\lambda_{32}}{\lambda_{32}+\lambda_{31}} T_{23}$$

$$+ \frac{\lambda_{31}}{\lambda_{32}+\lambda_{31}} \left(\frac{1}{\lambda_{10}} + T_{03}\right) = 92.9$$

(4) {1,0} $E(util) = \frac{U_3}{\lambda_{32}+\lambda_{31}} + \frac{\lambda_{32}}{\lambda_{32}+\lambda_{31}} \left(\frac{U_2}{\lambda_{21}+\lambda_{20}}\right) - C = 1.2$

$$E(cycle) = \frac{1}{\lambda_{32}+\lambda_{31}} + \frac{\lambda_{32}}{\lambda_{31}+\lambda_{32}} \left[\frac{1}{\lambda_{21}+\lambda_{20}} + \frac{\lambda_{21}}{\lambda_{21}+\lambda_{20}} T_{13}\right.$$

$$\left. + \frac{\lambda_{20}}{\lambda_{21}+\lambda_{20}} T_{03}\right] + \frac{\lambda_{31}}{\lambda_{32}+\lambda_{31}} T_{13} = 98.6$$

(5) {0} $E(util) = \frac{U_3}{\lambda_{32}+\lambda_{31}} + \frac{\lambda_{32}}{\lambda_{32}+\lambda_{31}} \left[\frac{U_2}{\lambda_{21}+\lambda_{20}} + \frac{\lambda_{21}}{\lambda_{21}+\lambda_{20}} \left(\frac{U_1}{\lambda_{10}}\right)\right]$

$$+ \frac{\lambda_{31}}{\lambda_{32}+\lambda_{31}} \left(\frac{U_1}{\lambda_{10}}\right) - C = 4.1$$

$$E(cycle) = \frac{1}{\lambda_{32}+\lambda_{31}} + \frac{\lambda_{32}}{\lambda_{32}+\lambda_{31}} \left[\frac{1}{\lambda_{21}+\lambda_{20}}\right.$$

$$\left. + \frac{\lambda_{21}}{\lambda_{21}+\lambda_{20}} \left(\frac{1}{\lambda_{10}}\right)\right]$$

$$+ \frac{\lambda_{31}}{\lambda_{32}+\lambda_{31}} \left(\frac{1}{\lambda_{10}}\right) + T_{03} = 103$$

Set of States	{3,2,1,0}	{2,1,0}	{2,0}	{1,0}	{0}
E(util)/E(cycle)	-2	.0217	.0409	.0122	.0398

Table 7.3 Cost Comparison

Maximum expected utility is achieved by inaction in states 1 and 3 and replacement in states 0 and 2. Again, a control limit rule is not optimal. □

7.4. Equivalence of the Continuous and Shock Models

Shock models are models in which a random amount of damage occurs at a random point in time. The times when damage occurs are called shocks. The damage to the component is cumulative and eventually causes component failure. Shock models are of interest in reliability since the lifetime of a component in such a model is governed by one of the distributions discussed in Section 1.1. The appropriate life distribution naturally depends upon assumptions about shock interarrival times and the damage accumulation process. Optimal maintenance policies for shock models can also be considered. Shock models mimic Derman's model in that the component can be replaced at a cost $C \geq 0$ and a penalty $K \geq 0$ is levied if failure occurs. They have not been extended to include state occupation costs or variable maintenance costs. Since it is assumed that the state of the system is always known, these models apply to constantly monitored systems. A control limit rule in this setting means that the component will be replaced either upon failure or after accumulating a certain amount of damage.

The first shock model which will be useful in this thesis is Bergman [1978]. Let $\{X(t), t \geq 0\}$ be a decreasing stochastic process with state space $\{0,1,\ldots,M\}$. There is also a nonincreasing failure rate function $\nu(X(t))$ for which $P(T > t|X(s), s \geq 0)$ $= \exp(-\int_0^t \nu(X(s))ds$ where T is the component lifetime. The failure rate function is the rate at which the process goes from a positive state to state 0. Bergman shows that if this rate is nonincreasing in $X(t)$, then a control limit rule is optimal. Another model of this type is Gottlieb [1982]. In this paper failure can only occur when a shock arrives. The probability that a component in state j will survive the next shock is denoted $R(j)$ and is assumed to be nonincreasing in j. If the sojourn time (time before the next shock) in state j is nonincreasing in j, then a control limit rule is optimal. A later paper will show that this result remains valid when partial repair of the component between shocks is permitted.

A CTMDP can be considered as a special case of a shock model when shock interarrival times are exponential with parameter $\lambda_{i\bullet}$, and the damage distribution is $p_{ij} = \lambda_{ij}/\lambda_{i\bullet}$. Decisions are made after a shock occurs. From Bergman [1978], if λ_{j0} is increasing in j, a control limit rule is optimal. This result weakens the hypothesis that $\sum_{j=0}^{k} \lambda_{ij}$ be nonincreasing in $i \nu k$, and it also weakens the discrete hypothesis that $\sum_{j=0}^{k} p_{ij}$ be nonincreasing in $i \nu k$. This change makes sense because the objective is to minimize restoration cost, and the only additional cost is incurred when the process is in state 0. The result from an extension of Gottlieb [1982] is that control limit rules are still optimal when transition rates λ_{ij} with

$j > i$ are permitted. Unfortunately, the shock models do not incorporate utility functions nor is it obvious how that would be accomplished. Thus, Section 7.2 is necessary to establish the optimality of control limit rules in the CTMDP with state occupancy costs.

8. REPAIR DURING SYSTEM OPERATION

The cost and risk associated with system failure makes highly reliable systems very desirable. One way to achieve high reliability without sacrificing availability is to allow multistate components or systems to continue operation during repair. Only one model addressing this issue has been found in the literature (Smith [1978]). It is assumed throughout this chapter that the failure and repair rates of a component are constant in time. This assumption allows the use of continuous time Markov chains in modeling the system.

In Section 8.1 a model identical to the model in Section 7.2, except that repair occurs during system operation, is considered. It is shown that a control limit rule is optimal. The remainder of the chapter is devoted to the following problem. Assume that a component is in state J and a decision must be made to repair either to state $J + I$ or to state $J + H$ where $H > I$. This is similar to determining whether it is better to repair all at once or in stages when failures may occur during the repair process. When certain assumptions are valid, it is optimal to repair to state $J + H$, the better state. These assumptions are (1) an increasing utility function, (2) the failure or repair rate into a state increases as the distance to the current state decreases, and (3) the repair rate from J to $J + H$ must be at least as large as the repair rate from J to $J + I$. Examples are given to show that it is not necessarily optimal to repair to the better state when any of these assumptions fail to hold. Surprisingly, it is not necessarily optimal to repair in stages even if the mean repair time when

repairing in stages is shorter than the mean repair time when repairing all at once. However, it may be better to repair in stages even if the mean repair time is longer than when repairing all at once because of the increased utility associated with higher states.

8.1. Description of the Model

The model considered in this section is identical to the model considered in the previous chapter except that repair occurs during system operation. This means that failures may occur during the repair process. Let $C \geq 0$ be the repair cost rate with a penalty of $K \geq 0$ assessed for repair from state 0, and let A_i be the occupancy cost rate of state i. The failure process is a CTMC with transitions λ_{ij}, $j < i$, and repair rate $\lambda_{iM} = \lambda \ \forall \ i$. If the chosen action in state i is to repair, then the transition from state i to state M is added while if the chosen action is inaction, the transition from i to M does not exist. In this section, it is shown that the optimal policy is a control limit rule with return state M.

Let $S = \{0,1,\ldots,M\}$, let $A = \{0 = \text{do nothing}, 1 = \text{repair}\}$, and let $\gamma = \sup_{i,a} \lambda_i(a)$. The rewards and transitions for the equivalent DTMDP are calculated according to Theorem 7.6.

$$p_{ij}(0) = \begin{cases} \lambda_{ij}/\gamma & \text{for } j \neq i \\ 1 - \lambda_{i\bullet}/\lambda & \text{for } j = i \end{cases}$$

$$p_{ij}(1) = \begin{cases} \lambda_{ij}/\gamma & \text{for } j \neq i,M \\ \lambda/\gamma & \text{for } j = M \\ 1 - (\lambda_{i\bullet}+\lambda)/\gamma & \text{for } j = i \end{cases}$$

$$r_i(0) = A_i/(\beta+\gamma)$$

$$r_i(1) = \begin{cases} C/(\beta+\gamma) + A_i/(\beta+\gamma) & \text{for } i > 0 \\ (C+K)/(\beta+\gamma) + A_0/(\beta+\gamma) & \text{for } i = 0 \end{cases} .$$

The following assumptions were made in Section 7.2 and are also necessary here.

(1) A_i is nonincreasing in i.

(2) $\sum_{j=0}^{k} \lambda_{ij}$ is nonincreasing in $i \; \forall \; k$,

or $\lambda_{ij} \geq \lambda_{\ell j} \; \forall \; j \leq i \leq \ell$.

(3) $\lambda_{i,i_1} \cdots \lambda_{i_{n,0}} > 0$ for some sequence $i_1, \ldots, i_n$.

Lemma 8.1: If condition (2) holds, then $\sum_{j=0}^{k} p_{ij}(0)$ and $\sum_{j=0}^{k} p_{ij}(1)$ are nonincreasing in $i \; \forall \; k$.

Proof: Let $\ell > i$ and consider $\sum_{j=0}^{k} p_{ij}(1)$ and $\sum_{j=0}^{k} p_{\ell j}(1)$ for any k. If $k = M$, both sums are 1. If $i \leq k < M$, then

$$\sum_{j=0}^{k} p_{ij}(1) = 1 - \frac{\lambda}{\gamma} \geq \sum_{j=0}^{k} p_{\ell j}(1) .$$

If $k < i$, then

$$\sum_{j=0}^{k} p_{ij}(1) = \sum_{j=0}^{k} \frac{\lambda_{ij}}{\gamma} \geq \sum_{j=0}^{k} \frac{\lambda_{\ell j}}{\gamma} = \sum_{j=0}^{k} p_{\ell j}(1) .$$

In all cases

$$\sum_{j=0}^{k} p_{ij}(1) \geq \sum_{j=0}^{k} p_{\ell j}(1) .$$

The result for $\sum_{j=0}^{k} p_{ij}(0)$ follows by setting $\lambda = 0$. □

Theorem 8.2: If conditions (1), (2), and (3) hold, then a control limit rule with return state M is optimal in both the discounted and average cost cases.

Proof: In the discounted case, the standard dynamic programming recursion is

$$\phi(i,\alpha,N+1) = \min \left\{ \frac{A_i}{(\beta+\gamma)} + \alpha \sum_{j=0}^{M} p_{ij}(0)\phi(j,\alpha,N);\ \frac{C}{(\beta+\gamma)} + \frac{A_i}{(\beta+\gamma)} + \alpha \sum_{j=0}^{M} p_{ij}(1)\phi(j,\alpha,N) \right\} \qquad \text{for } i > 0$$

$$= \frac{(C+K+A_0)}{(\beta+\gamma)} + \alpha \sum_{j=0}^{M} p_{0j}(1)\phi(j,\alpha,N) \qquad \text{for } i = 0$$

Since A_i is nonincreasing in i, $\phi(i,\alpha,0)$ is nonincreasing in i. Assume inductively that $\phi(i,\alpha,N)$ is nonincreasing in i. Then by the hypotheses and Lemma 8.1, $\phi(i,\alpha,N+1)$ is nonincreasing in i, and $\phi(i,\alpha) = \lim_{N\to\infty} \phi(i,\alpha,N)$ is nonincreasing in i. Theorem 7.1 then applies to show that state M is the optimal return state. The recursion for $\phi(i,\alpha)$ for $i > 0$ is

$$\phi(i,\alpha) = \min \left\{ \frac{A_i}{\beta+\gamma} + \frac{\alpha}{\gamma} \sum_{j=0}^{i-1} \lambda_{ij}\phi(j,\alpha) + \alpha \frac{1-\lambda_{i\bullet}}{\gamma} \phi(i,\alpha);\right.$$

$$\left. \frac{C}{\beta+\gamma} + \frac{A_i}{\beta+\gamma} + \frac{\alpha}{\gamma} \sum_{j=0}^{i-1} \lambda_{ij}\phi(j,\alpha) + \alpha \frac{1-\lambda_{i\bullet}-\lambda}{\gamma} \phi(i,\alpha) + \alpha \frac{\lambda}{\gamma} \phi(M,\alpha) \right\}$$

$$= \frac{A_i}{(\beta+\gamma)} + \frac{\alpha}{\gamma} \sum_{j=0}^{i-1} \lambda_{ij} \phi(j,\alpha) + \alpha(1-\lambda_{i\cdot}/\gamma)\phi(i,\alpha)$$

$$+ \min \left\{0; \frac{C}{(\beta+\gamma)} + \alpha \frac{\lambda}{\gamma} \phi(M,\alpha) - \alpha \frac{\lambda}{\gamma} \phi(i,\alpha)\right\} \quad .$$

Since $\phi(i,\alpha)$ is nonincreasing in i, if the minimum is obtained by repair in state $i^* > 0$, then it will also be obtained by repair in all states $k \leq i^*$. Thus, a control limit rule is optimal in the discounted case. The result follows for the average cost case from $\phi(i) = \lim_{\alpha \to 1} (1-\alpha)\phi(i,\alpha)$. □

8.2. Theorem on Optimal Repair Policies

This section differs from the preceding sections in that no cost structure is assumed. System operation is modeled by a positive recurrent CTMC with state space $\{0,1,\ldots,M\}$. The system operates during repair, and the decision of interest is the designation of a target state for the repair process. There is a choice of repairing from state J to state $J + H$ or from J to $J + I$ with $I \leq H$. This may be thought of as repairing a system all at once $(J \to J+H)$ or in stages $(J \to J+I \to J+H)$. It is necessary to assume $\lambda_{J,J+I} \leq \lambda_{J,J+H}$ which seems counterintuitive since a repair process which returns the system to a higher state would be expected to take longer. Unfortunately, an assumption such as $1/\lambda_{J,J+I} + 1/\lambda_{J+I,J+H} \geq 1/\lambda_{J,J+H}$, which says that mean repair time is shorter when repairing in stages than when repairing all at once, is not sufficient. Examples are given in section 8.3 to illustrate the problem.

Throughout this section, system (1) will refer to repair from J to J+I while system (2) will refer to repair from J to J+H. Superscripts (1) and (2) will be used in this capacity, e.g., $\lambda^{(1)}_{J,J+I} > 0$, $\lambda^{(1)}_{J,J+H} = 0$, $\lambda^{(2)}_{J,J+I} = 0$, $\lambda^{(2)}_{J,J+H} > 0$.

Lemmas 8.3, 8.9, and 8.11 are results from CTMC theory. Their proofs are contained in the Appendix since they are lengthy and do not add to the results of this section.

Lemma 8.3: Assume $\lambda_{ij} \leq \lambda_{\ell j}$ and $\lambda_{ji} \leq \lambda_{\ell i}$ $\forall$ $j < \ell < i$. Let set $A \equiv \{k, k+1, \ldots, M\}$. Then $\mu_{jA} \geq \mu_{iA}$ $\forall$ $j < i < k$ where equality holds if and only if (i) $\lambda_{j\ell} = \lambda_{i\ell}$ $\forall$ $\ell = 0, \ldots, j-1$ and $\ell = i+1, \ldots, k-1$, and (ii) $\lambda_{jA} = \lambda_{iA}$. If (i) and (ii) hold, then $\mu_{jA} = \mu_{j+1,A} = \cdots = \mu_{iA}$. Also, if set $B = \{0, 1, \ldots, k\}$, then $\mu_{jB} \leq \mu_{iB}$ $\forall$ $k < j < i$ where equality holds if and only if (i') $\lambda_{j\ell} = \lambda_{i\ell}$ $\forall$ $\ell = k+1, \ldots, j-1$ and $\ell = i+1, \ldots, M$, and (ii') $\lambda_{jB} = \lambda_{iB}$. If (i') and (ii') hold, then $\mu_{jB} = \mu_{j+1,B} = \cdots = \mu_{iB}$.

Notation: $\mu_{AB|C}$ = E(time from arrival in state or set of states A until the next arrival in state or set of states B, given that condition C holds).

Theorem 8.4: If $\lambda_{ij} \leq \lambda_{\ell j}$ and $\lambda_{ji} \leq \lambda_{\ell i}$ $\forall$ $j < \ell < i$, then

$$\sum_{i=k}^{M} \pi_i^{(1)} \leq \sum_{i=k}^{M} \pi_i^{(2)} \quad \forall\ k = 0, 1, \ldots, M \ .$$

(The superscripts (1) and (2) refer to systems (1) and (2) as discussed above).

Proof: Let set $A \equiv \{k,k+1,\ldots,M\}$, and consider the case $J < k$. $\pi_A = \sum_{i=k}^{M} \pi_i = m(A)/\mu_{AA}$. The two systems are identical except for transitions from state J. Thus, $m^{(1)}(A) = m^{(2)}(A)$.

$$\mu_{AA} = \mu_{AA|(noJ)} P(noJ) + [\mu_{AJ|(J)} + \mu_{JA}] P(J)$$

where (J) ≡ hitting state J before returning to set A, and (noJ) ≡ not hitting state J before returning to set A. The only difference between $\mu_{AA}^{(1)}$ and $\mu_{AA}^{(2)}$ is $\mu_{JA}^{(1)}$ and $\mu_{JA}^{(2)}$. It will be shown that $\mu_{JA}^{(1)} \geq \mu_{JA}^{(2)}$ which implies $\mu_{AA}^{(1)} \geq \mu_{AA}^{(2)}$ which implies $\pi_A^{(1)} \leq \pi_A^{(2)}$.

Conditioning on the first jump from state J yields

$$(1) \quad \mu_{JA}^{(1)} = m^{(1)}(J) + \sum_{i<k} p_{Ji}^{(1)} \mu_{iA}^{(1)}$$

$$= \frac{1}{\lambda_{J*} + \lambda_{J,J+I}^{(1)}} + \frac{\lambda_{J,J+I}^{(1)}}{\lambda_{J*} + \lambda_{J,J+I}^{(1)}} \mu_{J+I,A}^{(1)} + \sum_{\substack{i<k \\ i \neq J+I}} \frac{\lambda_{Ji}}{\lambda_{J*} + \lambda_{J,J+I}^{(1)}}$$

$$\times [\mu_{iA|(noJ)} P_i(noJ) + (\mu_{iJ|(J)} + \mu_{JA}^{(1)}) P_i(J)]$$

where $\lambda_{J*} = \sum_{\substack{i \neq J+I, \\ J+H}} \lambda_{Ji}$.

Rewriting the equation:

$$(\lambda_{J*} + \lambda_{J,J+I}^{(1)}) \mu_{JA}^{(1)} = 1 + \lambda_{J,J+I}^{(1)} \mu_{J+I,A}^{(1)} + \mu_{JA}^{(1)} \sum_{\substack{i<k \\ i \neq J+I}} \lambda_{Ji} P_i(J)$$

$$+ \sum_{\substack{i<k \\ i \neq J+I}} \lambda_{Ji} [\mu_{iA|(noJ)} P_i(noJ) + \mu_{iJ|(J)} P_i(J)]$$

$$(\lambda_{J*}+\lambda^{(1)}_{J,J+I}-C_\lambda)\mu^{(1)}_{JA} = 1 + \lambda^{(1)}_{J,J+I}\mu^{(1)}_{J+I,A} + \sum_{\substack{i<k \\ i\neq J+I}} \lambda_{Ji}\left[\mu_{iA|(noJ)}P_i(noJ) + \mu_{iJ|(J)}P_i(J)\right] \quad (8.1)$$

where

$$C_\lambda = \sum_{\substack{i<k \\ i\neq J+I,J+H}} \lambda_{Ji}P_i(J)$$

(2) Performing a similar calculation for $\mu^{(2)}_{JA}$ yields

$$(\lambda_{J*}+\lambda^{(2)}_{J,J+H}-C_\lambda)\mu^{(2)}_{JA} = 1 + \lambda^{(2)}_{J,J+H}\mu^{(2)}_{J+H,A} + \sum_{\substack{i<k \\ i\neq J+H}} \lambda_{Ji}\left[\mu_{iA|(noJ)}P_i(noJ) + \mu_{iJ|(J)}P_i(J)\right] \quad (8.2)$$

Subtracting Equation (8.2) from (8.1):

$$(\lambda_{J*}+\lambda^{(1)}_{J,J+I}-C_\lambda)\mu^{(1)}_{JA} - \lambda^{(1)}_{J,J+I}\mu^{(1)}_{J+I,A} = (\lambda_{J*}+\lambda^{(2)}_{J,J+H}-C_\lambda)\mu^{(2)}_{JA} - \lambda^{(2)}_{J,J+H}\mu^{(2)}_{J+H,A} \quad (8.3)$$

Now assume $\mu^{(1)}_{JA} < \mu^{(2)}_{JA}$. This will lead to a contradiction.

With

$$\mu^{(1)}_{JA} < \mu^{(2)}_{JA} \quad \text{and} \quad \lambda_{J*} - C_\lambda = \sum_{\substack{i<k \\ i\neq J+I,J+H}} \lambda_{Ji}(1-P_i(J)) > 0$$

(since otherwise the process never returns to A), Equation (8.3) becomes

$$\lambda^{(1)}_{J,J+I}\mu^{(1)}_{JA} - \lambda^{(1)}_{J,J+I}\mu^{(1)}_{J+I,A} > \lambda^{(2)}_{J,J+H}\mu^{(2)}_{JA} - \lambda^{(2)}_{J,J+H}\mu^{(2)}_{J+H,A} \quad .$$

From Lemma 8.3, $\mu^{(2)}_{J+H,A} \leq \mu^{(2)}_{J+I,A}$, so:

$$\lambda^{(1)}_{J,J+I}\left(\mu^{(1)}_{JA} - \mu^{(1)}_{J+I,A}\right) > \lambda^{(2)}_{J,J+H}\left(\mu^{(2)}_{JA} - \mu^{(2)}_{J+I,A}\right) \quad . \tag{8.4}$$

From Lemma 8.3, both quantities in parentheses in Equation (8.4) are nonnegative. Conditioning on the first jump from state $J+I$ in either system yields:

$$\mu^{(1,2)}_{J+I,A} = \mu_{J+I,A|(noJ)}P_{J+I}(noJ) + \left(\mu_{J+I,J|(J)} + \mu^{(1,2)}_{JA}\right)P_{J+I}(J)$$

$$= D_\lambda + \mu^{(1,2)}_{JA}\,P_{J+I}(J)$$

where

$$D_\lambda \equiv \mu_{J+I,A|(noJ)}P_{J+I}(noJ) + \mu_{J+I,J|(J)}P_{J+I}(J) \quad .$$

Using $\mu^{(1,2)}_{J+I,A}$ in Equation (8.4) yields:

$$\lambda^{(1)}_{J,J+I}\left[\mu^{(1)}_{JA}\left(1-P_{J+I}(J)\right)-D_\lambda\right] > \lambda^{(2)}_{J,J+H}\left[\mu^{(2)}_{JA}\left(1-P_{J+I}(J)\right)-D_\lambda\right] \quad .$$

However, this is impossible since $\lambda^{(1)}_{J,J+I} \leq \lambda^{(2)}_{J,J+H}$, the quantities in the brackets are nonnegative, and $\mu^{(1)}_{JA} < \mu^{(2)}_{JA}$ was assumed. This is the desired contradiction.

Now consider the case for which $k \leq J$. Let $B = \{0,1,\ldots,k-1\}$. $\pi^{(1)}_A \leq \pi^{(2)}_A$ is equivalent to $\pi^{(1)}_B \geq \pi^{(2)}_B$. As before, $\pi_B = m(B)/\mu_{BB}$, and the only difference between systems (1) and (2) is $\mu^{(1)}_{JB}$ and $\mu^{(2)}_{JB}$.

(1) $$\mu_{JB}^{(1)} = m^{(1)}(J) + \sum_{i \geq k} P_{Ji}\mu_{iB}^{(1)}$$

$$= \frac{\{1+\lambda_{J,J+I}^{(1)}\mu_{J+I,B}^{(1)} + \sum_{\substack{i \geq k \\ i \neq J+I}} \lambda_{Ji}[\mu_{iB|(noJ)}P_i(noJ)+(\mu_{iJ|(J)}+\mu_{JB}^{(1)})P_i(J)]\}}{\lambda_{J*} + \lambda_{J,J+I}^{(1)}}$$

where

$$\lambda_{J*} = \sum_{\substack{i \neq J+I, \\ J+H}} \lambda_{Ji} \quad , \quad C_\lambda = \sum_{\substack{i \geq k \\ i \neq J+I, J+H}} \lambda_{Ji}P_i(J) \quad .$$

$$(\lambda_{J*} + \lambda_{J,J+I}^{(1)} - C_\lambda)\mu_{JB}^{(1)} = 1 + \lambda_{J,J+I}^{(1)}\mu_{J+I,B}^{(1)}$$

$$+ \sum_{\substack{i \geq k \\ i \neq J+I}} \lambda_{Ji}[\mu_{iB|(noJ)}P_i(noJ) + \mu_{iJ|(J)}P_i(J)] \quad . \quad (8.5)$$

(2) A similar calculation for $\mu_{JB}^{(2)}$ yields

$$(\lambda_{J*}+\lambda_{J,J+H}^{(2)}-C_\lambda)\mu_{JB}^{(2)} = 1 + \lambda_{J,J+H}^{(2)}\mu_{J+H,B}^{(2)}$$

$$+ \sum_{\substack{i \geq k \\ i \neq J+H}} \lambda_{Ji}[\mu_{iB|(noJ)}P_i(noJ) + \mu_{iJ|(J)}P_i(J)] \quad . \quad (8.6)$$

Subtracting Equation (8.6) from (8.5) yields:

$$(\lambda_{J*}+\lambda_{J,J+I}^{(1)}-C_\lambda)\mu_{JB}^{(1)} - \lambda_{J,J+I}^{(1)}\mu_{J+I,B}^{(1)} = (\lambda_{J*}+\lambda_{J,J+H}^{(2)}-C_\lambda)\mu_{JB}^{(2)}$$

$$- \lambda_{J,J+H}^{(2)}\mu_{J+H,B}^{(2)} \quad . \quad (8.7)$$

Now assume $\mu_{JB}^{(1)} > \mu_{JB}^{(2)}$. This will lead to a contradiction. With $\mu_{JB}^{(1)} > \mu_{JB}^{(2)}$ and $\lambda_{J*} - C_\lambda > 0$, Equation (8.7) becomes

$$\lambda_{J,J+I}^{(1)}\mu_{JB}^{(1)} - \lambda_{J,J+I}^{(1)}\mu_{J+I,B}^{(1)} < \lambda_{J,J+H}^{(2)}\mu_{JB}^{(2)} - \lambda_{J,J+H}^{(2)}\mu_{J+H,B}^{(2)} \quad .$$

From Lemma 8.3, $\mu_{J+H,B}^{(2)} \geq \mu_{J+I,B}^{(2)}$, so

$$\lambda_{J,J+I}^{(1)}(\mu_{JB}^{(1)} - \mu_{J+I,B}^{(1)}) < \lambda_{J,J+H}^{(2)}(\mu_{JB}^{(2)} - \mu_{J+I,B}^{(2)}) \qquad \text{or}$$

$$\lambda_{J,J+I}^{(1)}(\mu_{J+I,B}^{(1)} - \mu_{JB}^{(1)}) > \lambda_{J,J+H}^{(2)}(\mu_{J+I,B}^{(2)} - \mu_{JB}^{(2)}) \qquad (8.8)$$

where both quantities in parentheses in Equation (8.8) are nonnegative by Lemma 8.3. Conditioning on the first jump from state J+I in either system:

$$\mu_{J+I,B}^{(1,2)} = \mu_{J+I|(noJ)}P_{J+I}(noJ) + (\mu_{J+I,J|(J)} + \mu_{JB}^{(1,2)})P_{J+I}(J)$$

$$= D_\lambda + \mu_{JB}^{(1,2)}P_{J+I}(J)$$

where:

$$D_\lambda = \mu_{J+I,J|(noJ)}P_{J+I}(noJ) + \mu_{J+I,J|(J)}P_{J+I}(J) \quad .$$

Using $\mu_{J+I,B}^{(1,2)}$ in Equation 8.8 yields:

$$\lambda_{J,J+I}^{(1)}[D_\lambda - \mu_{JB}^{(1)}(1-P_{J+I}(J))] > \lambda_{J,J+H}^{(2)}[D_\lambda - \mu_{JB}^{(2)}(1-P_{J+I}(J))] \quad .$$

However, this equation cannot hold since $\lambda^{(1)}_{J,J+I} \leq \lambda^{(2)}_{J,J+H}$, the quantities in the brackets are nonnegative, and $\mu^{(1)}_{JB} > \mu^{(2)}_{JB}$ was assumed. This contradiction means

$\mu^{(1)}_{JB} \leq \mu^{(2)}_{JB} \Rightarrow \mu^{(1)}_{BB} \leq \mu^{(2)}_{BB} \Rightarrow \pi^{(1)}_{B} \geq \pi^{(2)}_{B}$. □

The following corollaries help relate Theorem 8.4 to the optimal operation of a system.

Corollary 8.5: If there is a nondecreasing utility function corresponding to the states of the CTMC, then under the conditions of Theorem 8.4, system (2) has higher expected long-run utility.

Proof: Let the utility of state j be a_j with $a_j \geq a_{j-1}$.

$$\sum_{i=0}^{M} a_i \pi^{(1)}_i = \sum_{i=0}^{M} b_i \Big(\sum_{j=i}^{M} \pi^{(1)}_j \Big) \leq \sum_{i=0}^{M} b_i \Big(\sum_{j=i}^{M} \pi^{(2)}_j \Big) = \sum_{i=0}^{M} a_i \pi^{(2)}_i$$

where

$$b_i \equiv a_i - a_{i-1} \quad \text{for} \quad i \geq 1 \quad \text{and} \quad b_o \equiv a_0 \ . \ \square$$

Corollary 8.6: If there is a nondecreasing utility function corresponding to the states of the CTMC, if the hypotheses of Theorem 8.4 hold, and if $\lambda_{Ji} = \lambda_{J\ell}$ $\forall$ $i > J$, $\ell > J$, then it is optimal to repair to state M.

Proof: In System (2) set $H = M-J$, and in System (1) set $I = 0, 1, \ldots, M-J-1$. From Theorem 8.4,

$$\sum_{i=k}^{M} \pi^{(1)}_i \leq \sum_{i=k}^{M} \pi^{(2)}_i \quad \forall \ k,$$

and from Corollary 8.5, this is optimal. □

Corollary 8.7: Under the conditions of Theorem 8.4, increasing a repair rate increases $\sum_{i=k}^{M} \pi_i \; \forall \; k$.

Proof: This is the case $I = H$ in Theorem 8.4. □

It can also be shown that decreasing a failure rate increases $\sum_{i=k}^{M} \pi_i$.

Corollary 8.8: If Theorem 8.4 describes the steady-state operation of one component in a coherent MSF, System (2) has higher expected utility then System (1).

Proof: All min paths at all levels contain the component at a certain level or higher. Since

$$\sum_{i=k}^{M} \pi_i^{(1)} \leq \sum_{i=k}^{M} \pi_i^{(2)} \; \forall \; k,$$

$h^k(\underline{P})$ is larger in System (2) $\forall$ k. □

The hypotheses of Theorem 8.4 can be changed to other hypotheses which were used in the previous section.

Lemma 8.9: Assume $\sum_{j=0}^{\ell} \lambda_{ij}$ is nonincreasing in $i \; \forall \; \ell < i$ and $\sum_{j=\ell}^{M} \lambda_{ij}$ is nondecreasing in $i \; \forall \; \ell > i$. Let $A \equiv \{k, k+1, \ldots, M\}$. Then $\mu_{jA} \geq \mu_{iA} \; \forall \; j < i < k$ where equality holds if and only if (i) $\lambda_{j\ell} = \lambda_{j+1,\ell} = \cdots = \lambda_{i\ell} \; \forall \; \ell = 0, 1, \ldots, j-1$ and $\ell = i+1, \ldots, k-1$, and (ii) $\lambda_{jA} = \lambda_{j+1,A} = \cdots = \lambda_{iA}$. If (i) and

(ii) hold, $\mu_{jA} = \cdots = \mu_{iA}$. Also, if $B \equiv \{0,1,\ldots,k\}$, then $\mu_{jB} \leq \mu_{iB}$ $\forall$ $k < j < i$ where equality holds if and only if (i') $\lambda_{j\ell} = \lambda_{j+1,\ell} = \cdots = \lambda_{i\ell}$ $\forall$ $\ell = k+1, \ldots, j$ and $\ell = i+1, \ldots, M$, and (ii') $\lambda_{jB} = \lambda_{j+1,B} = \cdots = \lambda_{iB}$. If (i') and (ii') hold, $\mu_{jB} = \cdots = \mu_{iB}$.

Theorem 8.10: If $\sum_{j=0}^{\ell} \lambda_{ij}$ is nonincreasing in i $\forall$ $\ell < i$ and $\sum_{j=\ell}^{M} \lambda_{ij}$ is nondecreasing in i $\forall$ $\ell > i$, then $\sum_{i=k}^{M} \pi_i^{(1)} \leq \sum_{i=k}^{M} \pi_i^{(2)}$ $\forall$ $k = 0, 1, \ldots, M$.

Proof: The proof is identical to Theorem 8.4 with Lemma 8.9 replacing Lemma 8.3. □

In Theorem 8.4 several types of restoration may occur simultaneously from a single state. In fact it was not necessary to have $\lambda_{J,J+H}^{(1)} = 0$ and $\lambda_{J,J+I}^{(2)} = 0$. *The weaker hypotheses*

$$\lambda_{J,J+I}^{(1)} + \lambda_{J,J+H}^{(1)} \leq \lambda_{J,J+I}^{(2)} + \lambda_{J,J+H}^{(2)} \quad \text{and} \quad \lambda_{J,J+H}^{(1)} \leq \lambda_{J,J+H}^{(2)}$$

would have sufficed, but this lengthens the proof. In Theorem 8.12 only one restoration activity from each state is permitted. The hypotheses are slightly different from Theorem 8.4, but the result is the same.

Lemma 8.11: Assume $\lambda_{ij} \leq \lambda_{\ell j}$ $\forall$ $j < \ell < i$ and $\lambda_{ji} > 0$ for a single $i > j$ for each j. Assume $\lambda_{ji} \leq \lambda_{\ell m}$ and $i \leq m$ when $j < \ell$. Let $A = \{k,\ldots,M\}$. Then $\mu_{iA} \leq \mu_{jA}$ $\forall$ $j < i < k$ where equality holds if and only if (i) $\lambda_{j\ell} = \lambda_{i\ell}$ $\forall$ $\ell = 0,1,\ldots,j-1$ and $\ell = i+1, \ldots, k-1$, and (ii) $\lambda_{jA} = \lambda_{iA}$. If (i) and (ii) hold, then $\mu_{jA} = \mu_{j+1,A} = \cdots = \mu_{iA}$. Also, if $B = \{0,1,\ldots,k\}$, then $\mu_{iB} \geq \mu_{jB}$ $\forall$ $k < j < i$

where equality holds if and only if (i') $\lambda_{j\ell} = \lambda_{i\ell}$ $\forall$ $\ell = k+1, \ldots, j-1$ and $\ell = i+1, \ldots, M$, and (ii') $\lambda_{jB} = \lambda_{iB}$. If (i') and (ii') hold, then $\mu_{jB} = \cdots = \mu_{iB}$.

Theorem 8.12: Assume $\lambda_{ij} \leq \lambda_{\ell j}$ $\forall$ $j < \ell < i$ and $\lambda_{ji} > 0$ for a single $i > j$ for each j. Assume $\lambda_{ji} \leq \lambda_{\ell m}$ and $i \leq m$ when $j < \ell$. Then $\sum_{i=k}^{M} \pi_i^{(1)} \leq \sum_{i=k}^{M} \pi_i^{(2)}$ $\forall$ k.

Proof: Let $A = \{k, k+1, \ldots, M\}$ and consider the case $J < k$. As in Theorem 8.4, the only difference between Systems (1) and (2) is μ_{JA}. It will be shown that $\mu_{JA}^{(1)} \geq \mu_{JA}^{(2)}$ which implies $\mu_{AA}^{(1)} \geq \mu_{AA}^{(2)}$ which implies $\pi_A^{(1)} \leq \pi_A^{(2)}$. Conditioning on the first jump from state J yields:

$$(1) \quad \mu_{JA}^{(1)} = m^{(1)}(J) + \sum_{i<k} p_{Ji}^{(1)} \mu_{iA}^{(1)}$$

$$= \frac{1}{\lambda_{J*}+\lambda_{J,J+I}^{(1)}} + \frac{\lambda_{J,J+I}^{(1)}}{\lambda_{J*}+\lambda_{J,J+I}^{(1)}} \mu_{J+I,A}^{(1)}$$

$$+ \sum_{i<J} \frac{\lambda_{Ji}}{\lambda_{J*}+\lambda_{J,J+I}^{(1)}} [\mu_{iA|(noJ)} P_i(noJ) + (\mu_{iJ|(J)} + \mu_{JA}^{(1)}) P_i(J)]$$

where $\lambda_{J*} \equiv \sum_{i<J} \lambda_{Ji}$ and $(J) \equiv$ hitting state J before returning to set A (as in Theorem 8.4).

$$(\lambda_{J*} + \lambda_{J,J+I}^{(1)} - C_\lambda)\mu_{JA}^{(1)} = 1 + \lambda_{J,J+I}^{(1)} \mu_{J+I,A}^{(1)}$$

$$+ \sum_{i<J} \lambda_{Ji} [\mu_{iA|(noJ)} P_i(noJ) + \mu_{iJ|(J)} P_i(J)] \qquad (8.9)$$

where $C_\lambda = \sum_{i<J} \lambda_{Ji} P_i(J)$.

(2) A similar calculation for $\mu_{JA}^{(2)}$ yields:

$$(\lambda_{J*} + \lambda_{J,J+H}^{(2)} - C_\lambda)\mu_{JA}^{(2)} = 1 + \lambda_{J,J+H}^{(2)}\mu_{J+H,A}^{(2)}$$

$$+ \sum_{i<J} \lambda_{Ji}[\mu_{iA|(noJ)}P_i(noJ) + \mu_{iJ|(J)}P_i(J)] \quad . \qquad (8.10)$$

Subtracting Equation (8.9) from (8.10):

$$(\lambda_{J*} + \lambda_{J,J+I}^{(1)} - C_\lambda)\mu_{JA}^{(1)} - \lambda_{J,J+I}^{(1)}\mu_{J+I,A}^{(1)} = (\lambda_{J*} + \lambda_{J,J+H}^{(2)} - C_\lambda)\mu_{JA}^{(2)}$$

$$- \lambda_{J,J+H}^{(2)}\mu_{J+H,A}^{(2)} \quad . \qquad (8.11)$$

Now assume $\mu_{JA}^{(1)} < \mu_{JA}^{(2)}$. This will lead to a contradiction.

Since $\lambda_{J*} - C_\lambda > 0$, Equation (8.11) becomes

$$\lambda_{J,J+I}^{(1)}\mu_{JA}^{(1)} - \lambda_{J,J+I}^{(1)}\mu_{J+I,A}^{(1)} > \lambda_{J,J+H}^{(2)}\mu_{JA}^{(2)} - \lambda_{J,J+H}^{(2)}\mu_{J+H,A}^{(2)}$$

From Lemma 8.11, $\mu_{J+I,A}^{(1)} \geq \mu_{J+H,A}^{(1)}$, so

$$\lambda_{J,J+I}^{(1)}(\mu_{JA}^{(1)} - \mu_{J+H,A}^{(1)}) > \lambda_{J,J+H}^{(2)}(\mu_{JA}^{(2)} - \mu_{J+H,A}^{(2)}) \qquad (8.12)$$

where the quantities in parentheses in Equation (8.12) are nonnegative by Lemma 8.11. Conditioning on the first jump from state J+H:

$$\mu_{J+H,A}^{(1,2)} = \mu_{J+H,A|(noJ)}P_{J+H}(noJ) + [\mu_{J+H,J|(J)} + \mu_{JA}^{(1,2)}]\ P_{J+H}(J)$$

$$= \mu_{JA}^{(1,2)}\ P_{J+H}(J) + D_\lambda$$

where

$$D_\lambda \equiv \mu_{J+H,A|(noJ)}P_{J+H}(noJ) + \mu_{J+H,J|(J)}P_{J+H}(J) \quad .$$

Using $\mu^{(1,2)}_{J+H,A}$ in Equation (8.12):

$$\lambda^{(1)}_{J,J+I}[\mu^{(1)}_{JA}(1-P_{J+H}(J)) - D_\lambda] > \lambda^{(2)}_{J,J+H}[\mu^{(2)}_{JA}(1-P_{J+H}(J)) - D_\lambda] \quad .$$

However, $\lambda^{(1)}_{J,J+I} \leq \lambda^{(2)}_{J,J+H}$, the quantities in the brackets are nonnegative, and $\mu^{(1)}_{JA} < \mu^{(2)}_{JA}$ was assumed, so this is the desired contradiction, and $\mu^{(1)}_{JA} \geq \mu^{(2)}_{JA}$.

Now let $B = \{0,1,\ldots,k-1\}$ and consider the case $k \leq J$. As in Theorem 8.4, the only difference between $\pi^{(1)}_B$ and $\pi^{(2)}_B$ is $\mu^{(1)}_{JB}$ and $\mu^{(2)}_{JB}$.

$$(1) \quad \mu^{(1)}_{JB} = m^{(1)}(J) + \sum_{i \geq k} P^{(1)}_{Ji}\, \mu^{(1)}_{iB}$$

$$= \frac{1}{\lambda_{J*}+\lambda^{(1)}_{J,J+I}} + \frac{\lambda^{(1)}_{J,J+I}}{\lambda_{J*}+\lambda^{(1)}_{J,J+I}}\, \mu^{(1)}_{J+I,B} + \sum_{k \leq i < J} \frac{\lambda_{Ji}}{\lambda_{J*}+\lambda^{(1)}_{J,J+I}}$$

$$\times [\mu_{iB|(noJ)} P_i(noJ) + (\mu_{iJ|(J)} + \mu^{(1)}_{JB}) P_i(J)]$$

where $\lambda_{J*} = \sum_{i<J} \lambda_{Ji}$.

$$(\lambda_{J*}+\lambda^{(1)}_{J,J+I} - C_\lambda)\mu^{(1)}_{JB} = 1 + \lambda^{(1)}_{J,J+I}\mu^{(1)}_{J+I,B}$$

$$+ \sum_{k \leq i < J} \lambda_{Ji}[\mu_{iB|(noJ)} P_i(noJ) + \mu_{iJ|(J)} P_i(J)] \quad (8.13)$$

where $C_\lambda = \sum_{k \leq i < J} \lambda_{Ji} P_i(J)$.

(2) A similar calculation for System (2) yields:

$$(\lambda_{J*} + \lambda^{(2)}_{J,J+H} - C_\lambda)\mu^{(2)}_{JB} = 1 + \lambda^{(2)}_{J,J+H}\mu^{(2)}_{J+H,B} + \sum_{k\le i<J} \lambda_{Ji}[\mu_{iB|(noJ)}P_i(noJ) + \mu_{iJ|(J)}P_i(J)] \quad . \quad (8.14)$$

Subtracting Equation (8.14) from (8.13):

$$(\lambda_{J*} + \lambda^{(1)}_{J,J+I} - C_\lambda)\mu^{(1)}_{JB} - \lambda^{(1)}_{J,J+I}\mu^{(1)}_{J+I,B} = (\lambda_{J*} + \lambda^{(2)}_{J,J+H} - C_\lambda)\mu^{(2)}_{JB} - \lambda^{(2)}_{J,J+H}\mu^{(2)}_{J+H,B} \quad . \quad (8.15)$$

Now assume $\mu^{(1)}_{JB} > \mu^{(2)}_{JB}$. This will lead to a contradiction.

Since $\lambda_{J*} - C_\lambda > 0$, Equation (8.15) becomes:

$$\lambda^{(1)}_{J,J+I}\mu^{(1)}_{JB} - \lambda^{(1)}_{J,J+I}\mu^{(1)}_{J+I,B} < \lambda^{(2)}_{J,J+H}\mu^{(2)}_{JB} - \lambda^{(2)}_{J,J+H}\mu^{(2)}_{J+H,B} \quad .$$

From Lemma 8.11, $\mu^{(2)}_{J+I,B} \le \mu^{(2)}_{J+H,B}$, so

$$\lambda^{(1)}_{J,J+I}[\mu^{(1)}_{J+I,B} - \mu^{(1)}_{JB}] > \lambda^{(2)}_{J,J+H}[\mu^{(2)}_{J+I,B} - \mu^{(2)}_{JB}] \quad . \quad (8.16)$$

where both quantities in brackets in Equation (8.16) are nonnegative by Lemma 8.11. Conditioning on the first jump from state J+I:

$$\mu^{(1,2)}_{J+I,B} = \mu_{J+I,B|(noJ)}P_{J+I}(noJ) + (\mu_{J+I,J|(J)} + \mu^{(1,2)}_{JB})P_{J+I}(J)$$

$$= \mu^{(1,2)}_{JB}P_{J+I}(J) + D_\lambda$$

where

$$D_\lambda = \mu_{J+I,B|(noJ)}P_{J+I}(noJ) + \mu_{J+I,J|(J)}P_{J+I}(J) \quad .$$

Using $\mu_{JB}^{(1,2)}$ in Equation (8.16) yields:

$$\lambda_{J,J+I}^{(1)}[D_\lambda - \mu_{JB}^{(1)}(1-P_{J+I}(J))] > \lambda_{J,J+H}^{(2)}[D_\lambda - \mu_{JB}^{(2)}(1-P_{J+I}(J))] \quad .$$

However, this is a contradiction since $\lambda_{J,J+I}^{(1)} \leq \lambda_{J,J+H}^{(2)}$, the quantities in brackets are nonnegative, and $\mu_{JB}^{(1)} > \mu_{JB}^{(2)}$ was assumed. Thus, $\mu_{JB}^{(1)} \leq \mu_{JB}^{(2)} \Rightarrow \pi_B^{(1)} \geq \pi_B^{(2)}$. □

Theorem 8.13 differs from the preceding theorems in that any repair from a state smaller than state J+H must reach J+H before proceding to any state larger than J+H. This provides a slightly different result without assumptions on the failure rates.

Theorem 8.13: Let $\lambda_{ij} = 0$ whenever $i < J+H$, $j > J+H$ or $i > J+H$, $j < J+H$. Then $\pi_k^{(1)} \leq \pi_k^{(2)}$ $\forall$ $k \geq J+H$.

Proof: Conditioning on the first jump from state $k \geq J+H$:

$$\mu_{kk} = \mu_{kk|(noJ+H)}\, P_k(noJ+H) + [\mu_{k,J+H|(J+H)} + \mu_{J+H,k}]\, P_k(J+H) \quad .$$

The only difference between Systems (1) and (2) is $\mu_{J+H,k}^{(1,2)}$. It will be shown that $\mu_{J+H,k}^{(1)} \geq \mu_{J+H,k}^{(2)}$ which implies $\mu_{kk}^{(1)} \geq \mu_{kk}^{(2)}$ which implies $\pi_k^{(1)} \leq \pi_k^{(2)}$.

$$\begin{aligned}\mu_{J+H,k} &= \mu_{J+H,k|(noJ)}\, P_{J+H}(noJ) + [\mu_{J+H,J|(J)} + \mu_{J,J+H} + \mu_{J+H,k}]\, P_{J+H}(J) \\ &= [\mu_{J+H,k|(noJ)}\, P_{J+H}(noJ) + (\mu_{J+H,J|(J)} + \mu_{J,J+H})\, P_{J+H}(J)] \\ &\qquad / [1-P_{J+H}(J)] \quad . \qquad (8.17)\end{aligned}$$

The only difference between $\mu^{(1)}_{J+H,k}$ and $\mu^{(2)}_{J+H,k}$ is $\mu^{(1)}_{J,J+H}$ and $\mu^{(2)}_{J,J+H}$.

Conditioning on the first jump from state J:

$$(1)\quad \mu^{(1)}_{J,J+H} = m^{(1)}(J) + \sum_{i<J+H} P^{(1)}_{Ji}\ \mu^{(1)}_{i,J+H}$$

$$= \frac{1}{\lambda^{(1)}_{J\bullet}} + \frac{\lambda^{(1)}_{J,J+I}}{\lambda^{(1)}_{J\bullet}}\ \mu^{(1)}_{J+I,J+H} + \sum_{\substack{i<J+H \\ i\neq J+I}} \frac{\lambda_{Ji}}{\lambda_{J\bullet}}$$

$$\times\ [\mu_{i,J+H|(noJ)} P_i(noJ) + (\mu_{iJ|(J)} + \mu^{(1)}_{J,J+H}) P_i(J)]$$

$$\mu^{(1)}_{J,J+H} = (1 + \lambda^{(1)}_{J,J+I}\mu^{(1)}_{J+I,J+H} + L)/(\lambda^{(1)}_{J\bullet} - C_\lambda) \qquad (8.18)$$

where

$$C_\lambda = \sum_{\substack{i<J+H \\ i\neq J+I}} \lambda_{Ji} P_i(J),$$

$$L = \sum_{\substack{i<J+H \\ i\neq J+I}} \lambda_{Ji}[\mu_{i,J+H|(noJ)} P_i(noJ) + \mu_{iJ|(J)} P_i(J)]\ .$$

(2) A similar calculation for $\mu^{(2)}_{J,J+H}$ yields:

$$\mu^{(2)}_{J,J+H} = (1+L)/(\lambda^{(2)}_{J\bullet} - C_\lambda)\ . \qquad (8.19)$$

Since

$$\lambda^{(1)}_{J\bullet} = \lambda^{(1)}_{J,J+I} + \sum_{\substack{i\neq J+I,\\ J+H}} \lambda_{Ji} \leq \lambda^{(2)}_{J,J+H} + \sum_{\substack{i\neq J+I,\\ J+H}} \lambda_{Ji} = \lambda^{(2)}_{J\bullet}$$

and

$$\lambda^{(1)}_{J,J+I}\mu^{(1)}_{J+I,J+H} \geq 0,$$

comparing Equations (8.18) and (8.19) shows that $\mu^{(1)}_{J,J+H} \geq \mu^{(2)}_{J,J+H}$. Then, from Equation (8.17), $\mu^{(1)}_{J+H,k} \geq \mu^{(2)}_{J+H,k}$ as required. □

8.3. Examples

The first two examples in this section are complementary. Example 8.1 contains a system with mean repair time that is shorter when repairing all at once than when repairing in stages. However, the optimal repair strategy is repairing in stages. This means that the assumption $\lambda^{(1)}_{J,J+I} \leq \lambda^{(2)}_{J,J+H}$ cannot be replaced by $1/\lambda^{(2)}_{J,J+H} \leq 1/\lambda^{(1)}_{J,J+I} + 1/\lambda^{(2)}_{J+I,J+H}$. In Example 8.2 it is optimal to repair all at once even though mean repair time when repairing in stages is shorter than when repairing all at once.

Example 8.1: Consider the 4-state CTMC shown in Figure 8.1.

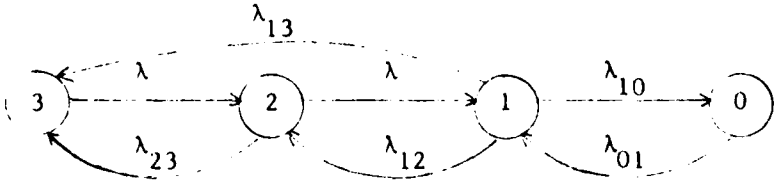

Figure 8.1. CTMC

In state 1 there is a choice between repairing all at once: $\lambda_{13} > 0$, $\lambda_{12} = 0$, or repairing in stages: $\lambda_{13} = 0$, $\lambda_{12} > 0$. Let $\lambda_{32} = \lambda_{21} = \lambda = .1$, $\lambda_{10} = 1$, $\lambda_{01} = 1$, $\lambda_{12} = 5$, $\lambda_{13} = \lambda_{23} = 2$. To find the steady-state probabilities, the following equations are solved.

(1) $\lambda\pi_3 = \lambda_{13}\pi_1 + \lambda_{23}\pi_2$

(2) $(\lambda+\lambda_{23})\pi_2 = \lambda_{12}\pi_1 + \lambda\pi_3$

(3) $(\lambda_{10}+\lambda_{12}+\lambda_{13})\pi_1 = \lambda_{01}\pi_0 + \lambda\pi_2$

(4) $\lambda_{01}\pi_0 = \lambda_{10}\pi_1$

(5) $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$

Case 1: repair in stages, $\lambda_{13} = 0$

$\bar{\lambda} = \lambda^2\lambda_{10} + \lambda^2\lambda_{01} + \lambda\lambda_{01}\lambda_{12} + \lambda_{01}\lambda_{12}\lambda_{23} = 10.52$

$\pi_0 = \lambda^2\lambda_{10}/\bar{\lambda} = .0010$

$\pi_1 = \lambda^2\lambda_{01}/\bar{\lambda} = .0010$

$\pi_2 = \lambda\lambda_{01}\lambda_{12}/\bar{\lambda} = .0474$

$\pi_3 = \lambda_{01}\lambda_{12}\lambda_{23}/\bar{\lambda} = .9506$

Case 2: repair all at once, $\lambda_{12} = 0$

$\bar{\lambda} = \lambda^2\lambda_{10} + \lambda^2\lambda_{01} + 2\lambda\lambda_{01}\lambda_{13} + \lambda_{01}\lambda_{13}\lambda_{23} = 4.42$

$\pi_0 = \lambda^2\lambda_{10}/\bar{\lambda} = .0023$

$\pi_1 = \lambda^2\lambda_{01}/\bar{\lambda} = .0023$

$\pi_2 = \lambda\lambda_{01}\lambda_{13}/\bar{\lambda} = .0452$

$\pi_3 = (\lambda\lambda_{01}\lambda_{13}+\lambda_{01}\lambda_{13}\lambda_{23})/\bar{\lambda} = .9502$

$\sum_{i=k}^{3} \pi_i$ is larger in case 1 $\forall$ k so it is optimal to repair in stages. This is true even though expected repair time when repairing all at once, $1/\lambda_{13} = .5$, is shorter than expected repair time when

repairing in stages, $1/\lambda_{12} + 1/\lambda_{23} = .7$. The reason for this is that $1/\lambda_{13} = .5 > 1/\lambda_{12} = .25$, so the system exits from state 1 more quickly when repairing in stages. This means that the transition from state 1 to state 0 will occur less frequently in case 1. The example may seem somewhat contrived since $\lambda_{10} = 10\lambda_{21}$. However, it is still optimal to repair in stages with $\lambda_{10} = \lambda_{21} = \lambda_{32} = 1$ when a linear utility function is used as shown in Table 8.1. Repairing in stages is not uniformly superior for all nondecreasing utility functions in this case because $\pi_3^{(1)} < \pi_3^{(2)}$. □

	$\bar{\lambda}$	π_0	π_1	π_2	π_3	$U = \sum_{i=0}^{3} i\pi_i$
Case 1	17	.059	.059	.294	.588	2.41
Case 2	10	.1	.1	.2	.6	2.3

Table 8.1. Repair Comparison

Example 8.2: Consider the 5-state CTMC shown in Figure 8.2.

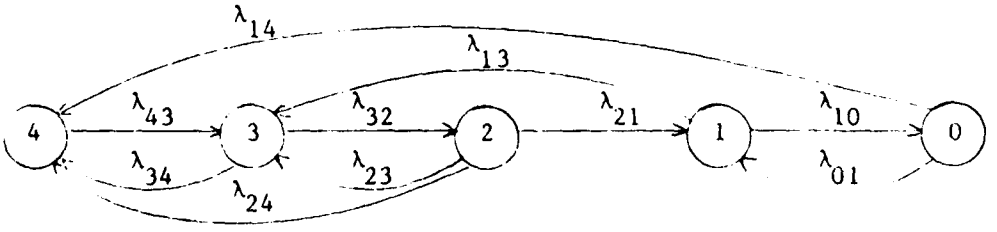

Figure 8.2. CTMC

In state 1 there is a choice between repairing to state 3 or state 4. Let $\lambda_{43} = .1$, $\lambda_{32} = 1$, $\lambda_{34} = .2$, $\lambda_{24} = .1$, $\lambda_{23} = .25$, $\lambda_{21} = 1$,

$\lambda_{14} = .1$, $\lambda_{13} = .25$, $\lambda_{10} = .1$, $\lambda_{01} = 1$. The following equations need to be solved to determine the steady-state probabilities.

(1) $\lambda_{43}\pi_4 = \lambda_{14}\pi_1 + \lambda_{24}\pi_2 + \lambda_{34}\pi_3$

(2) $(\lambda_{32}+\lambda_{34})\pi_3 = \lambda_{13}\pi_1 + \lambda_{23}\pi_2 + \lambda_{43}\pi_4$

(3) $(\lambda_{21}+\lambda_{23}+\lambda_{24})\pi_2 = \lambda_{32}\pi_3$

(4) $(\lambda_{10}+\lambda_{13}+\lambda_{14})\pi_1 = \lambda_{01}\pi_0 + \lambda_{21}\pi_2$

(5) $\lambda_{01}\pi_0 = \lambda_{10}\pi_1$

(6) $\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$

Case 1: repair in stages, $\lambda_{14} = 0$

$$\bar{\lambda} = \lambda_{10}\lambda_{21}\lambda_{32}\lambda_{43} + \lambda_{01}\lambda_{21}\lambda_{32}\lambda_{43} + \lambda_{01}\lambda_{13}\lambda_{32}\lambda_{43} + \lambda_{01}\lambda_{13}\lambda_{43}(\lambda_{21}+\lambda_{23}+\lambda_{24}) + \lambda_{01}\lambda_{13}(\lambda_{24}\lambda_{32}+\lambda_{21}\lambda_{34}+\lambda_{23}\lambda_{34}+\lambda_{24}\lambda_{34})$$

$$\pi_0 = \lambda_{10}\lambda_{21}\lambda_{32}\lambda_{43}/\bar{\lambda}$$

$$\pi_1 = \lambda_{01}\lambda_{21}\lambda_{32}\lambda_{43}/\bar{\lambda}$$

$$\pi_2 = \lambda_{01}\lambda_{13}\lambda_{32}\lambda_{43}/\bar{\lambda}$$

$$\pi_3 = \lambda_{01}\lambda_{13}\lambda_{43}(\lambda_{21}+\lambda_{23}+\lambda_{24})/\bar{\lambda}$$

$$\pi_4 = \lambda_{01}\lambda_{13}(\lambda_{24}\lambda_{32}+\lambda_{21}\lambda_{34}+\lambda_{23}\lambda_{34}+\lambda_{24}\lambda_{34})/\bar{\lambda}$$

Case 2: repair all at once, $\lambda_{13} = 0$

$$\bar{\lambda} = \lambda_{10}\lambda_{21}\lambda_{32}\lambda_{43} + \lambda_{01}\lambda_{21}\lambda_{32}\lambda_{43} + \lambda_{01}\lambda_{14}\lambda_{43}(\lambda_{21}+\lambda_{23}+\lambda_{24}+\lambda_{32}) + \lambda_{01}\lambda_{14}(\lambda_{21}\lambda_{32} + \lambda_{24}\lambda_{32} + \lambda_{21}\lambda_{34} + \lambda_{23}\lambda_{34} + \lambda_{24}\lambda_{34})$$

$$\pi_0 = \lambda_{10}\lambda_{21}\lambda_{32}\lambda_{43}/\bar{\lambda}$$

$$\pi_1 = \lambda_{01}\lambda_{21}\lambda_{32}\lambda_{43}/\bar{\lambda}$$

$$\pi_2 = \lambda_{01}\lambda_{14}\lambda_{32}\lambda_{43}/\bar{\lambda}$$

$$\pi_3 = \lambda_{01}\lambda_{14}\lambda_{43}(\lambda_{21}+\lambda_{23}+\lambda_{24})/\bar{\lambda}$$

$$\pi_4 = \lambda_{01}\lambda_{14}(\lambda_{21}\lambda_{32}+\lambda_{24}\lambda_{32}+\lambda_{21}\lambda_{34}+\lambda_{23}\lambda_{34}+\lambda_{24}\lambda_{34})/\bar{\lambda}$$

The results of the calculations are shown in Table 8.2.

	$\bar{\lambda}$	π_0	π_1	π_2	π_3	π_4
Case 1	.2612	.038	.383	.096	.129	.354
Case 2	.2705	.037	.370	.037	.050	.506

Table 8.2. Steady-state Probabilities

From Table 8.2 $\sum_{i=k}^{3} \pi_i$ is larger in Case 2 $\forall$ k, so repairing all at once is optimal even though expected repair time when repairing all at once, $1/\lambda_{14} = 10$, is larger than expected repair time when repairing in stages, $1/\lambda_{13} + 1/\lambda_{34} = 9$. The reason for this is that the transition from state 3 to state 2 may occur when repairing in stages. It can be shown that $\mu_{14}^{(2)} = 11 < \mu_{14}^{(1)} = 22.6$. □

The next example is a system for which it is optimal to repair from state J to state J+H and from state J+1 to state J+I even though I < H.

Example 8.3: Consider the 4-state CTMC shown in Figure 8.3.

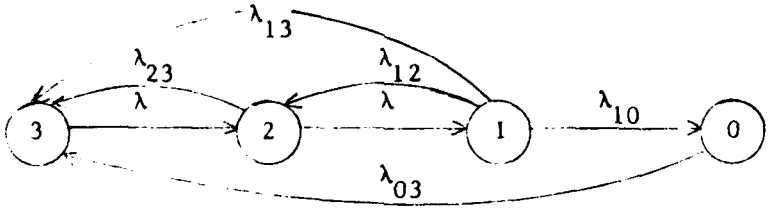

Figure 8.3. CTMC

In state 1 there is a choice between repairing all at once: $\lambda_{13} > 0$, $\lambda_{12} = 0$, or repairing in stages: $\lambda_{13} = 0$, $\lambda_{12} > 0$. Let $\lambda_{32} = \lambda_{21} = \lambda = .1$, $\lambda_{10} = 1$, $\lambda_{03} = 1$, $\lambda_{23} = 2$, $\lambda_{12} = 4$, and $\lambda_{13} = 2$. The following equations are solved to determine the steady-state probabilities.

(1) $\lambda\pi_3 = \lambda_{03}\pi_0 + \lambda_{13}\pi_1 + \lambda_{23}\pi_2$

(2) $(\lambda+\lambda_{23})\pi_2 = \lambda_{12}\pi_1 + \lambda\pi_3$

(3) $(\lambda_{10}+\lambda_{12}+\lambda_{13})\pi_1 = \lambda\pi_2$

(4) $\lambda_{03}\pi_0 = \lambda_{10}\pi_1$

(5) $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$

Case 1: repair in stages, $\lambda_{13} = 0$

$$\bar{\lambda} = \lambda^2\lambda_{10} + \lambda^2\lambda_{03} + 2\lambda\lambda_{03}\lambda_{10} + \lambda\lambda_{03}\lambda_{13} + \lambda_{03}\lambda_{10}\lambda_{23} + \lambda_{03}\lambda_{12}\lambda_{23}$$

$$\pi_0 = \lambda^2\lambda_{10}/\bar{\lambda}$$

$$\pi_1 = \lambda^2\lambda_{03}/\bar{\lambda}$$

$$\pi_2 = (\lambda\lambda_{03}\lambda_{10} + \lambda\lambda_{03}\lambda_{13})/\bar{\lambda}$$

$$\pi_3 = (\lambda\lambda_{03}\lambda_{10} + \lambda_{03}\lambda_{10}\lambda_{23} + \lambda_{03}\lambda_{12}\lambda_{23})/\bar{\lambda}$$

Case 2: repair all at once, $\lambda_{12} = 0$

$$\bar{\lambda} = \lambda^2\lambda_{10} + \lambda^2\lambda_{03} + 2\lambda\lambda_{03}\lambda_{10} + 2\lambda\lambda_{03}\lambda_{13} + \lambda_{03}\lambda_{10}\lambda_{23} + \lambda_{03}\lambda_{13}\lambda_{23}$$

$$\pi_0 = \lambda^2\lambda_{10}/\bar{\lambda}$$

$$\pi_1 = \lambda^2\lambda_{03}/\bar{\lambda}$$

$$\pi_2 = (\lambda\lambda_{03}\lambda_{10} + \lambda\lambda_{03}\lambda_{13})/\bar{\lambda}$$

$$\pi_3 = (\lambda\lambda_{03}\lambda_{10}+\lambda\lambda_{03}\lambda_{13}+\lambda_{03}\lambda_{10}\lambda_{23}+\lambda_{03}\lambda_{13}\lambda_{23})/\bar{\lambda}$$

The results of the calculations are shown in Table 8.3.

	$\bar{\lambda}$	π_0	π_1	π_2	π_3
Case 1	10.62	.001	.001	.047	.951
Case 2	6.62	.0015	.0015	.0755	.9215

Table 8.3. Steady-state Probabilities

From Table 8.3, $\sum_{i=k}^{3} \pi_i$ is larger in Case $1 \forall k$, so it is optimal to repair in stages. This is true even though expected repair time when repairing all at once, $1/\lambda_{13} = .5$, is less than expected repair time when repairing in stages, $1/\lambda_{12} + 1/\lambda_{23} = .75$. The reason for this is that $1/\lambda_{13} = .5 > 1/\lambda_{12} = .25$, so the system exists from state 1 more quickly when repairing in stages, and the transition from state 1 to state 0 will not occur as frequently. Note, however, this example does not satisfy the hypotheses of Theorem 8.4 since $\lambda_{03} > \lambda_{13}$ when repairing in stages. □

The result of Example 8.3 is that it is optimal to repair from state 0 to state 3 and from state 1 to state 2, even though it is possible to repair from state 1 to state 3. A simple replacement model is considered in Example 8.4, and it is shown that an increase in a failure rate could actually increase expected system utility.

Example 8.4: Consider the CTMC shown in Figure 8.4.

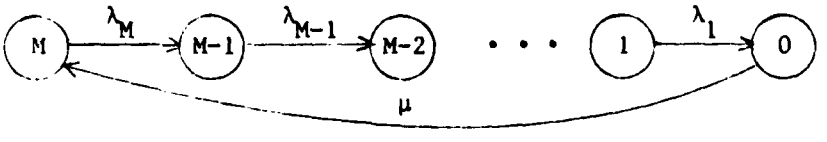

Figure 8.4. CTMC

The system modeled by the CTMC shown in Figure 8.4 is a system which is run until complete failure and then replaced. By a standard renewal arguement with $\lambda_0 \equiv \mu$, $\pi_k = (1/\lambda_k)/\sum_{i=0}^{M}(1/\lambda_i)$. If λ_i increases, π_i decreases, and π_k increases $\forall\ k \neq i$. Consider $\sum_{j=k}^{M} \pi_j$ $= 1 - \sum_{j=0}^{k-1} \pi_j$. If λ_i increases, then $\sum_{j=k}^{M} \pi_j$ *increases for* $k > i$ but decreases for $k \leq i$ (since $\sum_{j=0}^{k-1} \pi_j$ increases). Assume the usual nondecreasing utility function. If λ_M increases, then $\sum_{j=k}^{M} \pi_j$ decreases $\forall\ k \geq 1$, and the expected utility of the system decreases. If $\lambda_0 = \mu$ increases, then $\sum_{j-k}^{M} \pi_j$ increases $\forall\ k \geq 1$, and the expected system utility increases. However, a change in any other λ_i may increase or decrease expected system utility depending upon the utility function and the other transition rates. □

9. REPAIR VS. REPLACEMENT

A discrete time model is considered in this chapter. In each state the operator has a choice to repair the multistate component, replace the component, or do nothing. It is shown that the optimal policy is to either repair or do nothing until the component reaches a certain state, and to replace whenever the component drops below that state. This is a type of control limit rule. The control limit may be 0 in which case it is optimal to never replace. It was hoped that a 4-region policy - a policy for which the optimal actions are inaction, repair, inaction, and replacement as the system state decreases - would be optimal as it is in a similar model. However, an example is presented in which a 5-region policy is optimal, and it is conjectured that no limit on the number of possible regions exists.

9.1. Control Limit Rule Optimality

Consider a general discrete model in which the operator has a choice of inaction or returning the system to any state. This model differs from previous ones in that a decision must be made not only whether to restore the system, but also how much to restore the system. Replacement returns the system to its best state while repair improves the system but does not necessarily return it to its best state. Let A_i be the 1-period operating cost in state i, let C be the constant 1-period cost of replacement, and let C_{ij} be the 1-period cost of repair from state i to state j $(j > i)$. The following assumptions are made relative to the cost structure.

(1) A_i is nonincreasing in i.

(2) $C_{i,i+k}$ is nondecreasing in k and nonincreasing in i.

The second assumption means that it costs more to do more total repair ($C_{i,i+k}$ nondecreasing in k), and it costs more to do the same amount of repair starting in a worse state ($C_{i,i+k}$ nonincreasing in i). It is also expected that $C_{i,i+1} < C$ for some i and $C_{i0} > C$ for some i so that there is a nontrivial choice between repair and replacement. If $C_{i,i+1} \geq C \; \forall \; i$, replacement is always better than repair. If $C_{i0} \leq C \; \forall \; i$, repair is always better than replacement.

As in previous chapters, p_{ij} is the 1-period transition probability from state i to state j given that no restoration takes place during the period. The usual assumptions pertaining to transition probabilities are made, namely

(3) $\sum_{j=0}^{k} p_{ij}$ is nonincreasing in $i \; \forall \; k$.

(4) $p_{i0}^{(k)} > 0$ for some k.

Assume that state occupancy costs are not paid during restoration and that doing nothing is not a possible action in state 0. Optimal system operation in the discounted case is described by the following recursion.

$$\phi(i,\alpha,N+1) = \min \{A_i + \alpha \sum_{j=0}^{M} p_{ij}\phi(j,\alpha,N) \; ; \; C + \alpha\phi(M,\alpha,N) \; ;$$

$$C_{i,i+1} + \alpha\phi(i+1,\alpha,N) \; ; \; \cdots \; ; \; C_{iM} + \alpha\phi(M,\alpha,N)\} \text{ for } i>0$$

$$= \min \{C + \alpha\phi(M,\alpha,N) \; ; \; C_{01} + \alpha\phi(1,\alpha,N) \; ;$$

$$\cdots \; ; \; C_{0M} + \alpha\phi(M,\alpha,N)\} \qquad \text{for } i = 0 \; . \quad (9.1)$$

Theorem 9.1 shows that a special type of control limit rule is optimal for this system.

Theorem 9.1. If assumptions (1)-(4) hold, then there exists an i^* for which it is optimal to replace whenever $i \leq i^*$, and to either repair or do nothing whenever $i > i^*$ for both the discounted and average cost optimality criteria.

Proof: Since A_i and $C_{i,i+k}$ are nonincreasing in i, it is clear that $\psi(i,\alpha,0)$ is nonincreasing in i from Equation 9.1. Assume, inductively, that $\psi(i,\alpha,N)$ is nonincreasing in i. Then, since $\sum_{j=0}^{M} p_{ij}\psi(i,\alpha,N)$ is nonincreasing in i by assumption (3), $\psi(i,\alpha,N)$ is nonincreasing in i $\forall$ N. Thus, $\psi(i,\alpha) = \lim_{N\to\infty} \psi(i,\alpha,N)$ is nonincreasing in i.

$$\psi(i,\alpha) = \min \{A_i + \alpha \sum_{j=0}^{M} p_{ij}\psi(i,\alpha) \; ; \; C + \alpha\psi(M,\alpha) \; ;$$

$$\cdots \; C_{iM} + \alpha\psi(M,\alpha,N)\} \qquad \text{for } i > 0$$

$$= \min \{C + \alpha\psi(M,\alpha) \; ; \; \cdots \; ; \; C_{0M} + \alpha\psi(M,\alpha)\} \qquad \text{for } i = 0$$

Note that, in the above recursion, $C + \alpha\psi(M,\alpha)$ is constant in i while all other terms are nonincreasing in i. Thus, if $\psi(i,\alpha) = C + \alpha\psi(M,\alpha)$ *for some* $i \equiv i^*$, then $\psi(i,\alpha) = C + \alpha\psi(M,\alpha)$ $\forall$ $i \leq i^*$. If $C + \alpha\psi(M,\alpha)$ never minimizes $\psi(i,\alpha)$, set $i^* = -1$. This completes the proof in the discounted case. In the average cost case, take $\phi(i) = \lim_{\alpha\to 1} (1-\alpha)\psi(i,\alpha)$, and apply the usual limiting argument.□

The ideas contained in Chapter 7 can be used to extend this result as shown by the following corollaries.

Corollary 9.2.: If costs A_i are paid during repair and replacement, the control rule described in Theorem 9.1 is still optimal.

Proof: The recursion for $i > 0$ becomes

$$\phi(i,\alpha,N+1) = A_i + \min \{\alpha \sum_{j=0}^{M} p_{ij}\phi(j,\alpha,N) \; ; \; C + \alpha\phi(M,\alpha,N) \; ; \; \cdots \; ; \; C_{iM} + \alpha\phi(M,\alpha,N)\}$$

Since the minimization term is Equation (9.1) with $A_i = 0 \; \forall \; i$, the result follows from Theorem 9.1.□

Corollary 9.3: Let $\tilde{p}_k$ be the probability that repair $C_{i,i+k}$ succeeds, and let $\tilde{p}$ be the probability that replacement succeeds. If costs A_i are not paid during replacement, then the control limit rule described in Theorem 9.1 is still optimal for both $\psi(i,\alpha)$ and $\phi(i)$.

Proof: The standard recursion becomes:

$$\begin{aligned}\phi(i,\alpha,N+1) = \min \{&A_i + \alpha \sum_{j=0}^{M} p_{ij}\phi(j,\alpha,N) \; ; \; C + \alpha\tilde{p}\phi(M,\alpha,N) \\ &+ \alpha(1-\tilde{p})\phi(i,\alpha,N) \; ; \; C_{i,i+1} + \alpha\tilde{p}_1\phi(i+1,\alpha,N) \\ &+ \alpha(1-\tilde{p}_1)\phi(i,\alpha,N) \; ; \; \cdots \; ; \; C_{iM} + \alpha\tilde{p}_{M-i}\phi(M,\alpha,N) \\ &+ \alpha(1-\tilde{p}_{M-i})\phi(i,\alpha,N)\} \qquad i > 0\end{aligned}$$

$$= \min \{C + \alpha \tilde{p}\, \phi(M,\alpha,N) + \alpha(1-\tilde{p})\phi(0,\alpha,N) \ ;$$

$$\ldots; \ C_{0M} + \alpha\tilde{p}_{M+1}\phi(M,\alpha,N) + \alpha(1-\tilde{p}_{M+1})\phi(0,\alpha,N)\} \quad i = 0.$$

As in Theorem 9.1, everything is nonincreasing in i so, $\phi(i,\alpha)$ is nonincreasing in i. As in Theorem 7.4, if the replacement action is optimal for state i^*, $\phi(i^*,\alpha) = [C + \alpha\tilde{p}\phi(M,\alpha,N)]/[1-\alpha(1-\tilde{p})]$. Since $\phi(i,\alpha)$ is nonincreasing in i and $\phi(i^*,\alpha)$ is independent of i, $\phi(i,\alpha) = \phi(i^*,\alpha)$ $\forall$ $i \leq i^*$. If replacement is never optimal, set $i^* = -1$. The usual limiting argument yields the result for $\phi(i)$. If restoration costs are not paid when restoration fails, replace C by $\tilde{p}C$ and replace C_{ij} by $\tilde{p}_{j-i}C_{ij}$, and the analysis is unchanged. □

With $C_{i,i+k} = \infty$ $\forall$ i,k, the model in this section is the same as those in Chapter 7. Thus, from Example 7.2, a control limit rule is not necessarily optimal in the failure to replace case when state occupancy cost A_i are paid during replacement.

The repair/replace model can also be extended to the continuous case. The problem is considered as a CTMDP. Costs C, C_{ij}, and A_i become cost rates. It is assumed that the system does not operate during restoration and that state occupancy costs are not paid during restoration. The action set is {0 = do nothing, 1 = replace, 2 = repair $i \to i+1$, $\cdots$, $M-i+1$ = repair $i \to M$}. Transitions from state i to state j occur at rate $\lambda_{ij}(j < i)$ when no restoration occurs. The replacement rate is λ, and the repair rate from state i to state j is $\lambda_{ij}(j > i)$. Some kind of restoration must be performed in state 0. Let $\gamma = \sup_{i,a} \lambda_i(a)$. The rewards and transition probabilities for an equivalent DTMDP are as follows (from Theorem 7.6).

$$r_i(0) = A_i/(\beta+\gamma)$$

$$r_i(1) = C/(\beta+\gamma)$$

$$r_i(j) = C_{i,i+j-1}/(\beta+\gamma) \qquad \text{for } j > 1$$

$$p_{ij}(0) = \begin{cases} \lambda_{ij}/\gamma & \text{for } j < i \\ 1 - \lambda_{i\cdot}/\gamma & \text{for } j = i \\ 0 & \text{for } j > i \end{cases}$$

$$p_{ij}(1) = \begin{cases} \lambda/\gamma & \text{for } j = M \\ 1 - \lambda/\gamma & \text{for } j = i \\ 0 & \text{for } j \neq i, M \end{cases}$$

$$p_{ij}(k) = \begin{cases} \lambda_{i,i+k-1}/\gamma & \text{for } j = i+1; k > 1 \\ 1 - \lambda_{i,i+k-1}/\gamma & \text{for } j = i; k > 1 \\ 0 & \text{for } j \neq i, i+1; k > 1 \end{cases}$$

Corollary 9.4: Let assumptions (1) and (2) remain valid, and let assumptions (3) and (4) be replaced by

(3') $\lambda_{ij} \geq \lambda_{\ell j}$ $\forall$ $j < i < \ell$ or $\sum_{j=0}^{k} \lambda_{ij}$ is nonincreasing in i $\forall$ k,

(4') $\lambda_{i,i_1}\, \lambda_{i_1,i_2} \cdots \lambda_{i_n,0} > 0$ for some sequence $i_1, \cdots, i_n$.

Then the control limit rule described in Theorem 9.1 is optimal in both the discounted and average cost cases.

Proof: The recursion for the DTMDP with $i > 0$ is

$$\phi(i,\alpha,N+1) = \min \{A_i/(\beta+\gamma) + \alpha \sum_{j=0}^{M} p_{ij}(0)\phi(j,\alpha,N) \; ; \; C/(\beta+\gamma)$$

$$+ \alpha\lambda\phi(M,\alpha,N)/\gamma + \alpha(1-\lambda/\gamma)\phi(i,\alpha,N) \; ; \; C_{i,i+1}/(\beta+\gamma)$$

$$+ \alpha\lambda_{i,i+1}\phi(i+1,\alpha,N)/\gamma + \alpha(1-\lambda_{i,i+1}/\gamma)\phi(i,\alpha,N) \; ;$$

$$\cdots \; ; \; C_{iM}/(\beta+\gamma) + \alpha\lambda_{iM}\phi(M,\alpha,N)/\gamma$$

$$+ \alpha(1-\lambda_{iM}/\gamma)\phi(i,\alpha,N)\}$$

From Lemma 7.7, $\sum_{j=0}^{M} p_{ij}(0)\phi(j,\alpha,N)$ is nonincreasing in i, and assumption (4') means that $p_{i0}^{(k)}(0) > 0$ for some k. Thus, with $\tilde{p}_k = \lambda_{i,i+k}/\gamma$ and $\tilde{p} = \lambda/\gamma$, this recursion is the same as that in Corollary 9.3.□

9.2. Examples

The control limit rule in Section 9.1 differentiates only between replacement and the other alternatives. It would be nice to show that there is a region of the state space for which repair is optimal and a region for which inaction is optimal. In state M it must be correct to do nothing since otherwise the system would be in a constant state of repair. From the previous section, it is optimal to replace in states $0,1,\ldots,i^*$. Thus, in states $i^*+1,\ldots,M-1$, the optimal action will alternate between repair and inaction. A "nice" policy is one in which the optimal action alternates infrequently such as the 3-region and 4-region policies shown in Figure 9.1.

States	0,...,i*	i*+1,...,i'-1	i',...,M
Optimal Action for 3-Region Policy	Replace	Repair	Inaction
Optimal Action for 4-Region Policy	Replace	Inaction for $i \leq i_0$ Repair for $i > i_0$ $i^* \leq i_0 \leq i-1$	Inaction

Figure 9.1. 3-Region and 4-Region Policies

Example 9.1 is a 4-region policy. Policies of this type were found to be optimal in Rosenfield [1976A] when the possible actions were replacement, inspection, and inaction. Unfortunately, as shown in Example 9.2, a 4-region policy is not necessarily optimal for the repair/replace model. Since Example 9.2 contains a linear utility function, a totally positive transition matrix, and repair only from state i to state i+1, it is difficult to think of hypotheses which would make a 4-region policy optimal. It is hypothesized that a counter-example can be devised for any n-region policy where n is a finite number.

Example 9.1: Consider the 4-state DTMC shown in Figure 9.2.

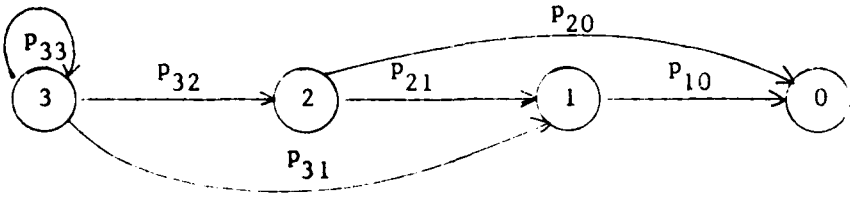

Figure 9.2. DTMC

Repair is restricted to repair from state i to state $i+1$. Let $A_3 = -3$, $A_2 = A_1 = -2$, $A_0 = 0$, $C = 6$, $C_{i,i+1} = 4 * i$, $p_{33} = .8$, $p_{32} = .1$, $p_{31} = .1$, $p_{21} = .1$, $p_{20} = .9$, $p_{10} = 1$, and $\alpha = .9$. The recursions in the discounted case are shown below.

$$\phi(3,\alpha) = A_3 + \alpha[p_{33}\phi(1,\alpha) + p_{32}\phi(2,\alpha) + p_{31}\phi(1,\alpha)]$$

$$\phi(2,\alpha) = \min \{A_2 + \alpha[p_{21}\phi(1,\alpha) + p_{20}\phi(0,\alpha)] \; ; \; C_{23} + \alpha\phi(3,\alpha) \; ; \; C + \alpha\phi(3,\alpha)\}$$

$$\phi(1,\alpha) = \min \{A_1 + \alpha p_{10}\phi(0,a) \; ; \; C_{12} + \alpha\phi(2,\alpha) \; ; \; C + \alpha\phi(3,\alpha)\}$$

$$\phi(0,\alpha) = \min\{C_{01} + \alpha\phi(1,\alpha) \; ; \; C + \alpha\phi(3,\alpha)\}$$

Solving the recursions yields:

$$\phi(3,\alpha) = (A_3+\alpha p_{32}C_{23}+\alpha p_{31}A_1+\alpha^2 p_{31}C)/(1-\alpha p_{33}-\alpha^2 p_{32}-\alpha^3 p_{31}) = -18.51$$

$$\phi(2,\alpha) = \min \{-11.68; \; -12.66; \; -10.66\} = -12.66 \text{ (repair)}$$

$$\phi(1,\alpha) = \min \{-11.6; \; -7.4; \; -10.66\} = -11.6 \text{ (do nothing)}$$

$$\phi(0,\alpha) = \min \{-6.44; \; -10.66\} = -10.66 \text{ (replace)}$$

Thus the optimal actions are inaction in states 1 and 3, repair in state 2, and replacement in state 0 as shown in Table 9.1. This is a 4-region policy. □

State	3	2	1	0
Optimal Action	Inaction	Repair	Inaction	Replace

Table 9.1. 4-Region Policy

AD-A134 589 MULTISTATE RELIABILITY(U) STANFORD UNIV CA A P WOOD
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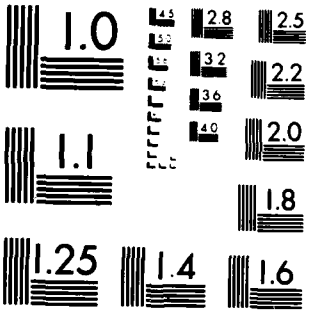

Example 9.2: Consider the 5-state DTMC shown in Figure 9.2.

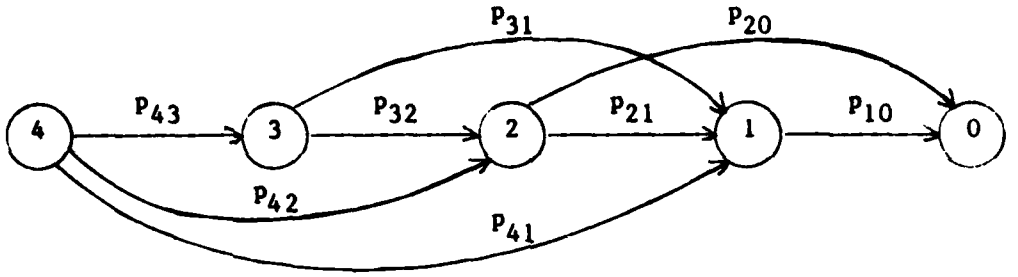

Figure 9.3. DTMC

The only allowable repair is from state i to state $i + 1$. Let

$$A_i = -i \vee i,\ C = 7,\ C_{i,i+1} = 1.95 \vee i,$$

and

$$P = \begin{bmatrix} 0 & .01 & .899 & .1 & 0 \\ 0 & 0 & .01 & .99 & 0 \\ 0 & 0 & 0 & .99 & .1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The transition matrix is obviously upper triangular and can be shown to be totally positive. The optimality criterion is expected average cost which is computed from the equation E(average cost) = E(cost per cycle)/ E(cycle length).

The return state is the largest state that the process will return to infinitely often (normally state 4). In Table 9.2 all permissable policies are considered. The table has been abbreviated by using the

control limit rule in Section 9.1, always doing nothing in state 4, and choosing only between repair and replacement in state 0. Dashes appear in the table whenever the chosen action is immaterial to the calculation of expected average cost. It can be seen from Table 9.2 that the policy yielding the lowest expected average cost is inaction in states 2 and 4, repair in states 1 and 3, and replacement in state 0. This might be called a 5-region policy.□

State	4	3	2	1	0	E(Average Cost)
	N	RR	RR	RR	-	.03
	N	RR	RR	RL	-	.072
	N	N	RR	RR	-	.294
	N	N	RR	RL	-	.54
	N	RL	RL	RL	-	1.75
	N	N	RL	RL	-	1.498
Actions	N	RR	RL	RL	-	1.497
	N	N	N	RL	RL	.413
	N	RR	N	RL	RL	.413
	N	N	N	N	RL	.053
	N	N	N	RR	RL	-.018677
	N	N	RR	N	RL	.413
	N	RR	N	RR	RL	-.018681
	N	RR	N	N	RL	.053
	-	-	-	N	RR	.475
	-	-	N	RR	RR	-.015
	N	RR	RR	N	RL	.031

N = Do Nothing
RR = Repair
RL = Replace

Table 9.2. Policy Comparison

APPENDIX

Proof of Lemma 8.3:

First it is shown that if (i) and (ii) hold, then $\mu_{jA} = \cdots = \mu_{iA}$. From the inequalities for $\lambda_{j\ell}$, $\lambda_{j\ell} = \lambda_{i\ell} \Rightarrow \lambda_{j\ell} = \lambda_{j+1,\ell} = \cdots = \lambda_{i\ell}$ and $\lambda_{jA} = \lambda_{iA} \Rightarrow \lambda_{jA} = \lambda_{j+1,A} = \cdots = \lambda_{iA}$. For each state r where $j \leq r \leq i$,

$$\mu_{rA} = \frac{1}{\lambda_{r\bullet}} + \sum_{\ell<k} \lambda_{r\ell}\mu_{\ell A}/\lambda_{r\bullet}$$

$$(\lambda_{r*} + \sum_{\ell=j}^{i} \lambda_{r\ell})\mu_{rA} = 1 + \sum_{\substack{0<\ell<j \\ i<\ell<k}} \lambda_{r\ell}\mu_{\ell A} + \sum_{\ell=j}^{i} \lambda_{r\ell}\mu_{\ell A}$$

where

$$\lambda_{r*} = \sum_{\substack{0<\ell<j \\ i<\ell}} \lambda_{r\ell} \ .$$

One solution to these equations is

$$\mu_{rA} = \Big(1 + \sum_{\substack{0<\ell<j \\ i<\ell<k}} \lambda_{r\ell}\mu_{\ell A}\Big)/\lambda_{r*} \ .$$

This solution is constant in $r \ \forall \ j \leq r \leq i$ and must be unique since the system is a positive recurrent irreducible CTMC. Thus, $\mu_{jA} = \cdots = \mu_{iA}$.

Now assume that at least one of (i) and (ii) does not hold. It will be shown by induction on the size of the state space that

$\mu_{0A} > \mu_{1A} > \cdots > \mu_{k-1,A}$ $\forall$ A. For M = 1, the lemma is obvious. Assume $\mu_{0A} > \cdots > \mu_{k-1,A}$ $\forall$ A and $\forall$ M = 1, 2, ..., N. Consider a set A = {k, ..., N+1} with $k \leq N$ as a single state by setting $\lambda_{iA} = \sum_{\ell=k}^{N+1} \lambda_{i\ell}$ for all $i \notin A$. The state space then has fewer than N states, so the induction hypothesis implies that $\mu_{0A} > \cdots > \mu_{k-1,A}$. Thus, it suffices to show that $\mu_{0,N+1} > \cdots > \mu_{N,N+1}$. Each $\mu_{j,N+1}$ is a continuous function of $\lambda_{\ell m}$ $\forall$ ℓ,m when $\lambda_{\ell m} > 0$. From the inequalities on the transition rates, $\lambda_{N,N+1} > 0$. Let A = {N, N+1}, and let $\lambda_{N,N+1} \to \infty$. Then $\mu_{jA} = \mu_{j,N+1}$ $\forall$ j, and $\mu_{0A} > \cdots > \mu_{N-1,A}$ or $\mu_{0,N+1} > \cdots > \mu_{N-1,N+1}$ from the induction hypothesis. Also, $\mu_{N,N+1} \to 0$ as $\lambda_{N,N+1} \to \infty$, so $\mu_{N,N+1} < \mu_{N-1,N+1}$.

By continuity, since the $\mu_{j,N+1}$'s are strictly ordered for $\lambda_{N,N+1} \to \infty$, they must also be strictly ordered for some large finite value of $\lambda_{N,N+1}$. Start with $\lambda_{N,N+1}$ at that large value and decrease it toward its original value. If it reaches its original value with the $\mu_{j,N+1}$'s still strictly ordered, the proof is finished. If not, let $\lambda'_{N,N+1}$ be the largest value of $\lambda_{N,N+1}$ for which an equality occurs. Let ϵ be a small positive number. By continuity, for some j and $i > j$:

$$\lambda_{N,N+1} = \lambda'_{N,N+1} + \epsilon \Rightarrow \mu_{j-1,N+1} > \mu_{j,N+1} > \cdots > \mu_{i,N+1} > \mu_{i+1,N+1}$$

$$\lambda_{N,N+1} = \lambda'_{N,N+1} \Rightarrow \mu_{j-1,N+1} > \mu_{j,N+1} = \cdots = \mu_{i,N+1} > \mu_{i+1,N+1}$$

$$\lambda_{N,N+1} = \lambda'_{N,N+1} - \epsilon \Rightarrow \mu_{i+1,N+1} < \mu_{j,N+1} < \cdots < \mu_{i,N+1} < \mu_{j-1,N+1}.$$

Conditioning on the first jump from state j yields:

$$(1)\mu_{j,N+1} = m(j) + \sum_{\ell \leq N} p_{j\ell}\,\mu_{\ell,N+1}$$

$$= \left(1 + \lambda_{j,j+1}\mu_{j+1,N+1} + \sum_{\substack{\ell \leq N \\ \ell \neq j,j+1}} \lambda_{j\ell}\mu_{\ell,N+1}\right)/\left(\lambda_{j*}+\lambda_{j,j+1}+\lambda_{j,N+1}\right)$$

where

$$\lambda_{j*} = \sum_{\ell \neq j+1,N+1} \lambda_{j\ell} \cdot$$

$$(\lambda_{j*} + \lambda_{j,j+1} + \lambda_{j,N+1})\mu_{j,N+1}$$

$$= 1 + \lambda_{j,j+1}\mu_{j+1,N+1} + \sum_{\substack{\ell \leq N \\ \ell \neq j,j+1}} \lambda_{j\ell}\,\mu_{\ell,N+1} \cdot \qquad (A.1)$$

(2) A similar calculation for state $j+1$ yields:

$$(\lambda_{j+1,*} + \lambda_{j+1,j} + \lambda_{j+1,N+1})\mu_{j+1,N+1}$$

$$= 1 + \lambda_{j+1,j}\mu_{j,N+1} + \sum_{\substack{\ell \leq N \\ \ell \neq j,j+1}} \lambda_{j+1,\ell}\,\mu_{\ell,N+1} \qquad (A.2)$$

where

$$\lambda_{j+1,*} = \sum_{\ell \neq j,N+1} \lambda_{j+1,\ell} \cdot$$

Subtracting Equation (A.2) from (A.1):

$$(\lambda_{j*} + \lambda_{j,j+1} + \lambda_{j,N+1} + \lambda_{j+1,j})\mu_{j,N+1}$$

$$= (\lambda_{j+1,*} + \lambda_{j+1,j} + \lambda_{j+1,N+1} + \lambda_{j,j+1})\mu_{j+1,N+1}$$

$$+ \sum_{\substack{\ell \le N \\ \ell \neq j, j+1}} (\lambda_{j\ell} - \lambda_{j+1,\ell})\mu_{\ell,N+1} \quad .$$

With $\lambda_{N,N+1} = \lambda'_{N,N+1}$ so that $\mu_{j,N+1} = \mu_{j+1,N+1}$, this equation becomes

$$(\lambda_{j*} + \lambda_{j,N+1})\mu_{j,N+1}$$

$$= (\lambda_{j+1,*} + \lambda_{j+1,N+1})\mu_{j+1,N+1} + \sum_{\substack{\ell \le N \\ \ell \neq j, j+1}} (\lambda_{j\ell} - \lambda_{j+1,\ell})\mu_{\ell,N+1} \quad .$$

Since $\lambda_{j,N+1} \le \lambda_{j+1,N+1}$,

$$\lambda_{j*}\ \mu_{j,N+1} \ge \lambda_{j+1,*}\ \mu_{j+1,N+1} + \sum_{\substack{\ell \le N \\ \ell \neq j, j+1}} (\lambda_{j\ell} - \lambda_{j+1,\ell})\mu_{\ell,N+1} \qquad \text{(A.3)}$$

where equality holds if and only if $\lambda_{j,N+1} = \lambda_{j+1,N+1}$.

Rewriting Equation (A.3) using the definitions of λ_{j*} and $\lambda_{j+1,*}$:

$$\sum_{\substack{\ell \le N \\ \ell \neq j, j+1}} \lambda_{j\ell}(\mu_{j,N+1} - \mu_{\ell,N+1}) \ge \sum_{\substack{\ell \le N \\ \ell \neq j, j+1}} \lambda_{j+1,\ell}(\mu_{j+1,N+1} - \mu_{\ell,N+1}) \quad \text{or:}$$

$$\sum_{\ell < j} \lambda_{j\ell}(\mu_{j,N+1} - \mu_{\ell,N+1}) + \sum_{i < \ell \le N} \lambda_{j\ell}(\mu_{j,N+1} - \mu_{\ell,N+1})$$

$$\ge \sum_{\ell < j} \lambda_{j+1,\ell}(\mu_{j+1,N+1} - \mu_{\ell,N+1}) + \sum_{i < \ell \le N} \lambda_{j+1,\ell}(\mu_{j+1,N+1} - \mu_{\ell,N+1})$$

(A.4)

where $\mu_{j,N+1} = \cdots = \mu_{i,N+1}$ was used to derive Equation (A.4). For $\ell < j$, $\mu_{j,N+1} - \mu_{\ell,N+1} = \mu_{j+1,N+1} - \mu_{\ell,N+1} < 0$ and $\lambda_{j\ell} \geq \lambda_{j+1,\ell}$. For $\ell > i$, $\mu_{j,N+1} - \mu_{\ell,N+1} = \mu_{j+1,N+1} - \mu_{\ell,N+1} > 0$ and $\lambda_{j\ell} \leq \lambda_{j+1,\ell}$. Thus, each term on the left hand side of Equation (8.4) is smaller than or equal to the corresponding term on the right hand side. The only way that Equation (A.4) can be valid is if $\lambda_{j\ell} = \lambda_{j+1,\ell}$ $\forall$ $0 \leq \ell \leq j-1$, $i+1 \leq \ell \leq N$, in which case it holds at equality. From Equation (A.3), $\lambda_{j,N+1} = \lambda_{j+1,N+1}$. Thus, conditions (i) and (ii) hold with $i = j+1$, $A = \{N+1\}$.

Now repeat the above calculations for pairs of states $(j+1, j+2); \cdots ; (i+1,i)$. For Equations (A.3) and (A.4) (appropriately modified) to remain valid, it must be that $\lambda_{j\ell} = \cdots = \lambda_{i\ell}$ $\forall$ $0 \leq \ell \leq j-1$ and $i+1 \leq \ell \leq N+1$. However, this is equivalent to (i) and (ii) with $A = \{N+1\}$ which is a contradiction since it was assumed that at least one of (i) and (ii) did not hold. Thus, there does not exist $\lambda'_{N,N+1}$ such that $\mu_{j,N+1} = \cdots = \mu_{i,N+1}$, and continuity implies that $\mu_{0,N+1} > \cdots > \mu_{N,N+1}$.

To prove the result for set $B = \{0, 1, \ldots, k\}$, let state i correspond to state $M-i$ in the previous argument. Then set B becomes set A, and the result follows. □

<u>Lemma A.1</u>: Assume $\sum_{j=0}^{\ell} \lambda_{ij}$ is nonincreasing in i $\forall$ $\ell < i$ and $\sum_{j=\ell}^{M} \lambda_{ij}$ is nondecreasing in i $\forall$ $\ell > i$. Let $0 \leq a_0 \leq \cdots \leq a_M$ and $b_0 \leq b_1 \leq \cdots \leq b_M \leq 0$. Then $\sum_{j=\ell}^{M} \lambda_{ij} a_j$ and $\sum_{j=0}^{\ell} \lambda_{ij} b_j$ are nondecreasing in i $\forall$ $\ell > i$ and $\forall$ $\ell < i$, respectively. If $\{a_i\}$ is a strictly increasing sequence, then

$\forall\ \ell > i$, $\sum_{j=\ell}^{M} \lambda_{i+1,j} a_j = \sum_{j=\ell}^{M} \lambda_{ij} a_j \iff \lambda_{i+1,j} = \lambda_{ij}$
$\forall\ j = \ell, \ldots, M$. If $\{b_i\}$ is a strictly increasing sequence, then
$\forall\ \ell < i$, $\sum_{j=0}^{\ell} \lambda_{ij} b_j = \sum_{j=0}^{\ell} \lambda_{i+1,j} b_j \iff \lambda_{ij} = \lambda_{i+1,j}$
$\forall\ j = 0, \ldots, \ell$.

Proof: Let $A_i \equiv a_i - a_{i-1} \geq 0\ \forall\ i > \ell$, and let $A_\ell \equiv a_\ell \geq 0$,

$$\sum_{j=\ell}^{M} (\lambda_{i+1,j} - \lambda_{ij}) a_j = \sum_{k=\ell}^{M} \sum_{j=k}^{M} (\lambda_{i+1,j} - \lambda_{ij}) A_k = \sum_{k=\ell}^{M} A_k \sum_{j=k}^{M} (\lambda_{i+1,j} - \lambda_{ij}) . \tag{A.5}$$

By hypothesis, $\sum_{j=k}^{M} (\lambda_{i+1,j} - \lambda_{ij}) \geq 0$, so since $A_k \geq 0$, $\sum_{j=\ell}^{M} \lambda_{ij} a_j$ is nondecreasing in i. If $A_k > 0$, equality in Equation (A.5) can hold if and only if $\sum_{j=k}^{M} (\lambda_{i+1,j} - \lambda_{ij}) = 0\ \forall\ k = \ell, \ldots, M$ or $\lambda_{i+1,j} = \lambda_{ij}\ \forall\ j = \ell, \ldots, M$.

Let $B_i \equiv b_i - b_{i+1} \leq 0\ \forall\ i < \ell$, and let $B_\ell \equiv b_\ell \leq 0$,

$$\sum_{j=0}^{\ell} (\lambda_{i+1,j} - \lambda_{ij}) b_j = \sum_{k=0}^{\ell} \sum_{j=0}^{k} (\lambda_{i+1,j} - \lambda_{ij}) B_k = \sum_{k=0}^{\ell} B_k \sum_{j=0}^{k} (\lambda_{i+1,j} - \lambda_{ij}) . \tag{A.6}$$

By hypothesis $\sum_{j=0}^{k} (\lambda_{i+1,j} - \lambda_{ij}) \leq 0$, so since $B_k \leq 0$, $\sum_{j=0}^{\ell} \lambda_{ij} b_j$ is nondecreasing in i. If $B_k < 0$, equality in Equation (A.6) can hold if and only if $\sum_{j=0}^{k} (\lambda_{i+1,j} - \lambda_{ij}) = 0\ \forall\ k = 0, \ldots, \ell$ or $\lambda_{i+1,j} = \lambda_{ij}\ \forall\ j = 0, \ldots, \ell$. □

Proof of Lemma 8.9: The proof of Lemma 8.9 is identical to the proof of Lemma 8.3 except at the places where hypotheses on the λ_{ij}'s are invoked. The necessary changes are shown herein, but the entire proof is not repeated.

Since $\lambda_{j\ell} = \lambda_{i\ell}$ for $j < i$ does not necessarily imply $\lambda_{j\ell} = \cdots = \lambda_{i\ell}$ in Lemma 8.9, it is assumed that $\lambda_{j\ell} = \cdots = \lambda_{i\ell}$ in condition (i) and $\lambda_{j,A} = \cdots = \lambda_{iA}$ in condition (ii). Note that $\lambda_{j,N+1} \leq \lambda_{i,N+1}$ $\forall$ $j < i$.

Consider Equation (A.4). For $\ell < j$, $\mu_{j,N+1} - \mu_{\ell,N+1} = \mu_{j+1,N+1} - \mu_{\ell,N+1} < 0$, and by Lemma (A.1)

$$\sum_{\ell<j} \lambda_{j\ell}(\mu_{j,N+1}-\mu_{\ell,N+1}) \leq \sum_{\ell<j} \lambda_{j+1,\ell}(\mu_{j+1,N+1}-\mu_{\ell,N+1}) ,$$

where equality holds if and only if $\lambda_{j\ell} = \lambda_{j+1,\ell}$ $\forall$ $\ell = 0, \ldots, j-1$. For $\ell > i$, $\mu_{j,N+1} - \mu_{\ell,N+1} = \mu_{j+1,N+1} - \mu_{\ell,N+1} > 0$, and by Lemma (A.1)

$$\sum_{i<\ell\leq N} \lambda_{j\ell}(\mu_{j,N+1}-\mu_{\ell,N+1}) \leq \sum_{i<\ell\leq N} \lambda_{j+1,\ell}(\mu_{j+1,N+1}-\mu_{\ell,N+1}) ,$$

where equality holds if and only if $\lambda_{j\ell} = \lambda_{j+1,\ell}$ $\forall$ $\ell = i+1,\ldots,N$. Thus, Equation (A.4) is not valid unless conditions (i) and (ii) hold, and the same contradiction exists that exists in the proof of Lemma 8.3. □

Proof of Lemma 8.11. The proof of this lemma is identical to the proof of Lemma 8.3 to the point at which the calculations of $\mu_{j,N+1}$ and $\mu_{j+1,N+1}$ begin (Equations (A.1) and (A.2)). This proof begins at that point with $\lambda_{jr} > 0$ and $\lambda_{j+1,s} > 0$ for $j < r \leq s$.

(1) $$\mu_{j,N+1} = m(j) + \sum_{\ell<k} \lambda_{j\ell}\,\mu_{\ell,N+1}$$

$$= \left(1 + \lambda_{jr}\mu_{r,N+1} + \sum_{\ell<j} \lambda_{r\ell}\mu_{\ell,N+1}\right)/(\lambda_{j*}+\lambda_{jr})$$

where

$$\lambda_{j*} \equiv \sum_{\ell<j} \lambda_{j\ell} \quad .$$

$$(\lambda_{j*}+\lambda_{jr})\mu_{j,N+1} = 1 + \lambda_{jr}\mu_{r,N+1} + \sum_{\ell<j} \lambda_{j\ell}\mu_{\ell,N+1} \quad . \qquad (A.7)$$

(2) A similar calculation for $j+1$ yields:

$$(\lambda_{j+1,*}+\lambda_{j+1,j}+\lambda_{j+1,s})\mu_{j+1,N+1}$$
$$= 1 + \lambda_{j+1,j}\mu_{j,N+1} + \lambda_{j+1,s}\mu_{s,N+1} + \sum_{\ell<j} \lambda_{j+1,\ell}\mu_{\ell,N+1} \qquad (A.8)$$

where

$$\lambda_{j+1,*} = \sum_{\ell<j} \lambda_{j+1,\ell} \quad .$$

Subtracting Equation (A.8) from (A.7):

$$(\lambda_{j*}+\lambda_{jr}+\lambda_{j+1,j})\mu_{j,N+1} - \lambda_{jr}\mu_{r,N+1}$$
$$= (\lambda_{j+1,*}+\lambda_{j+1,s}+\lambda_{j+1,j})\mu_{j+1,N+1} - \lambda_{j+1,s}\mu_{s,N+1}$$
$$+ \sum_{\ell<j} \mu_{\ell,N+1}(\lambda_{j\ell}-\lambda_{j+1,\ell}) \quad . \qquad (A.9)$$

Let $\lambda_{N,N+1} = \lambda'_{N,N+1}$ so that $\mu_{j,N+1} = \mu_{j+1,N+1}$.
Rewriting Equation (A.9) using the definitions of λ_{j*} and $\lambda_{j+1,*}$ yields:

$$\sum_{\ell<j} \lambda_{j\ell}(\mu_{j,N+1}-\mu_{\ell,N+1}) + \lambda_{jr}(\mu_{j,N+1}-\mu_{r,N+1})$$
$$= \sum_{\ell<j} \lambda_{j+1,\ell}(\mu_{j+1,N+1}-\mu_{\ell,N+1}) + \lambda_{j+1,s}(\mu_{j+1,N+1}-\mu_{s,N+1}) \quad . \qquad (A.10)$$

For $\ell < j$, $\mu_{j,N+1} = \mu_{j+1,N+1} < \mu_{\ell,N+1}$, and $\lambda_{j\ell} \geq \lambda_{j+1,\ell}$, so

$$\sum_{\ell<j} \lambda_{j\ell}(\mu_{j,N+1}-\mu_{\ell,N+1}) \leq \lambda_{j+1,\ell}(\mu_{j+1,N+1}-\mu_{\ell,N+1}) < 0 ,$$

where equality holds if and only if $\lambda_{j\ell} = \lambda_{j+1,\ell}$ for $\ell < j$. Since $\lambda_{jr} \leq \lambda_{j+1,s}$ and $\mu_{r,N+1} \geq \mu_{s,N+1}$,

$$\lambda_{jr}(\mu_{j,N+1}-\mu_{r,N+1}) \leq \lambda_{j+1}(\mu_{j+1,N+1}-\mu_{s,N+1})$$

where equality holds if and only if $s \leq i$ or $\lambda_{jr} = \lambda_{j+1,s}$ and $= s$. Thus, Equation (A.10) can hold if and only if $\lambda_{j\ell} = \lambda_{j+1,\ell}$ $\forall$ $\ell = 0, \ldots, j-1$ and $\ell = i+1, \ldots, N+1$. Repeating the argument pairs of states $(j+1,j+2); \ldots; (i-1,i)$, the appropriate modifications of Equations (A.9) and (A.10) hold if and only if $\lambda_{j\ell} = \lambda_{i\ell}$ $\forall$ $\ell = 0, \ldots, j-1$ and $\ell = i+1, \ldots, n+1$. Thus, $\mu_{0,N+1} > \cdots > \mu_{N,N+1}$ when conditions (i) and (ii) do not simultaneously hold.

To prove the result for set $B = \{0, 1, \ldots, k\}$, let state i correspond to state $M-i$ in the previous argument. □

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ABSTRACT: MULTISTATE RELIABILITY

by Alan P. Wood

In modern society the consequences of system failure can be catastrophic. The study of reliability has evolved from the desire to prevent, or at least mitigate the consequences of, failure. A reliability analysis is performed to determine the probability that a component or system is able to perform its specified function. Two major topics useful in that endeavor are considered herein. The first topic is the extension of coherent structure function theory to components and systems with several states. The second topic is the optimal maintenance of multistate components.

Coherent structure function theory is an axiomatic approach to reliability in which the components and systems are binary, i.e., they have two states - operational and failed. The first part of the thesis extends the theory to components and systems with multiple states. This is useful for modeling systems in which partial failure may occur. Multistate coherent structure functions are defined, and it is shown that most of the binary results have multistate analogs. These results deal with duals, modules, minimum cut and path sets, reliability importance, reliability bounds, closure theorems, fault trees, and block diagrams. The theory is further extended to allow each component and the system to have a continuum of states.

Optimal maintenance policies for periodically inspected multistate components have previously appeared in the literature. The second part of the thesis extends those policies to continuously monitored equipment by using Markov decision processes and continuous time Markov chains. The main theorems are in the form of control limit rules which state that it is optimal to repair or replace a component whenever it has degraded to a certain level. It is shown that under certain assumptions the optimal policy is to repair the component as much as possible. Equivalences between shock models, continuous time models, and discrete time models are discussed.


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