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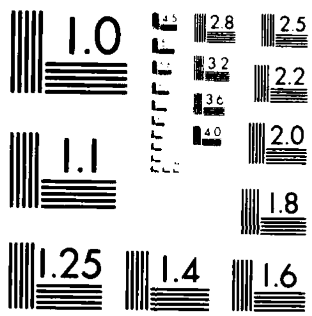
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Minimal Drag for Wings with Prescribed Lift, Roll Moment and Yaw Moment

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ABSTRACT

Let L, R, Y be arbitrary real constants. A wing with fixed span which produces lift L, roll moment R, yaw moment Y and which has minimal induced drag D is wanted. This problem arises in airplane engineering. It is solved by means of Prandtl's lifting line theory combined with computation.

AMS (MOS) Subject Classifications: 76B05, 39A10, 45E05, 49A34

Key Words: Prandtl's Wing Theory, Vertical Lift, Roll Moment, Adverse Yaw, Induced Drag

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SIGNIFICANCE AND EXPLANATION

"Adverse yaw" has been with us since the beginning of the era of heavier-than-air flight. The pioneers Lilienthal, the Wright brothers and others must have encountered this phenomenon. It consists of the fact that an airplane initiating a turn normally moves in the wrong (adverse) direction if only stick movement and no rudder deflection is used.

It appears that this phenomenon has not been studied very much by aerodynamicists or airplane engineers. There are several reasons behind this fact: The first may be that adverse yaw is normally more of a nuisance than a real problem. In most airplanes it can be controlled with the rudder. When teaching people how to fly, it is standard to demonstrate the adverse yaw and its manual correction.

Another reason may be that adverse yaw is most powerful for wings with high aspect ratio, e.g. sailplanes and airliners. Sailplane manufacturers, however, usually don't invest in scientific research. While the airline industry could make such an investment, it has not done so because the autopilot takes care of adverse yaw during flight.

Nevertheless, in both situations this correction for adverse yaw expends unneccessary energy. It is the aim of this paper to show how to reduce the energy required in the process of correcting for adverse yaw by minimizing the resulting additional drag.

The interest of the author in this subject comes from his work with tailless airplanes where adverse yaw is a serious problem. It is hoped that other kinds of airplanes may also benefit from the results presented in this report.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

MINIMAL DRAG FOR WINGS WITH PRESCRIBED LIFT, ROLL MOMENT AND YAW MOMENT

Karl L.E. Nickel

1. Explanation of the physical problem

Every airplane pilot is familiar with the following phenomenon: Suppose the pilot wants the airplane to roll. Then by using the ailerons as sketched in Figure 1 a roll moment R is produced. However, more than the (desired) roll around the longitudinal axis occurs. The airplane also turns its nose to the side as if a rudder deflection had been made, see Figure 2. This is known as "adverse yaw". In most cases it is not wanted and is regarded as a nuisance.

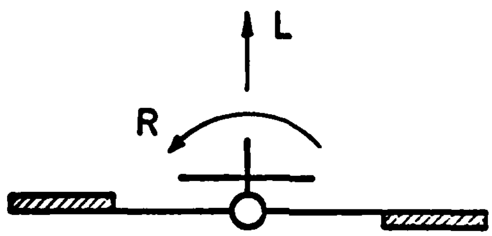

Figure 1. Sketch of airplane with lift L and roll moment R produced by aileron deflection.

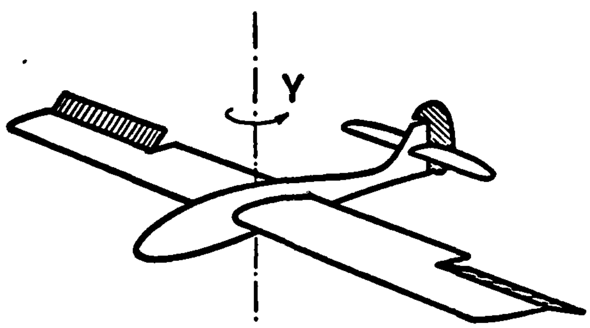

Figure 2. Sketch of airplane with yaw moment Y around the vertical axis produced either by the rudder deflection shown or by the aileron deflection shown.

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The physical reason for adverse yaw is that on the side with the aileron bent downward not only the (local) lift is increased, but that there is also an additional (local) drag. This drag is called (by Prandtl) the "induced" drag. It pulls this part of the wing back and therefore produces the yaw moment Y.

From this discussion, the following property is plausible: The adverse yaw is proportional to the lift L produced by the wing. Hence, no adverse yaw occurs if L is zero (vertical dive) and if the aileron configuration is symmetric. For negative lift L (upside-down flying; strong downward gust) but unchanged roll moment R, the sign of the yaw moment Y is obviously reversed.

In most airplanes of today (or of the past) the adverse yaw is annihilated by the (auto-) pilot just by steering against it with the rudder. There exist, however, at least two kinds of airplanes at present for which this is difficult to do or even impossible:

i) Tailless airplanes with no vertical tail fins and with no rudder. Most of today's hanggliders belong to that category.

ii) High efficency sailplanes with their usually very large wing span (24.5 m or 80 ft in 1983). In some of these airplanes, at maximum lift full rudder deflection is not sufficient (!) to fight adverse yaw. Hence, even though adverse yaw is a very old problem, it seems to have some modern significance.

There are several constructive devices to diminish or to annihilate adverse yaw. They will not be described here. Because of the properties mentioned above they are, however, normally effective only in a certain limited range of the lift L.

Since adverse yaw is generated by an asymetric distribution of the local induced drag one may suspect that it could, perhaps, be nullified by an adequate modification of that distribution. This leads to the

First Question : Are there special kinds of ailerons and/or wings for which adverse yaw does not exist for any values of lift L and roll moment R ?

It will be shown in this paper that there are infinitely many such configurations. This is a very "clean" solution to the problem of fighting adverse yaw since no additional devices are needed: The wing itself and its ailerons will be shown to suffice for the elimination of adverse yaw.

Obviously this solution will not be "for free". In order to counterbalance the one-sided induced drag one has to add some additional drag on the other side of the wing. Hence the total induced drag D of the airplane is increased. This leads to the

Next Question : Is it possible to select one solution to the First Question which produces minimal total drag D ?

It will be shown that there is exactly one such solution.

In most cases an airplane rolls if it wants to take a turn. In this case roll <u>and</u> positive yaw are wanted at the same time. Hence one may ask if one can replace rudder deflection (full or partial) by a favorable distribution of the induced drag on the wing. In extending and generalizing these Questions one comes therefore to the

> <u>Final Question</u> : Are there configurations of wing and ailerons with minimal drag D for arbitrarily prescribed values of lift L , roll moment R and yaw moment Y ?

It will be shown that this question has in general (i.e. for $R \neq 0$) exactly one solution. Examples will be presented. Moreover, explicit formulae will be given which bound the minimal drag from both sides within an error of less than 2.1% .

The approach used in this paper will be Prandtl's lifting line theory (see [4]). This theory is a first approximation to the lifting surface theory and normally cannot be used for general wing shapes, e.g. swept wings. For the problem considered here, however, it can be used without restriction due to Munks's shifting theorem [1]. This states that for the evaluation of the total induced drag only the projection of the lift distribution over the wing onto the Trefftz-plane is essential, regardless of sweep.

For the same reason, only the lift or circulation distribution over the wing is considered in what follows. There is (for $R \neq 0$) exactly one such optimal distribution. There are, however, infinitely many wing and aileron combinations which produce this desired distribution. The connection between wing and aileron shapes and circulation distribution is known and can be handled by standard methods.

Also not treated here is the following question of optimality: By using the final results of this paper, one can design an airplane with a very small rudder. This reduces drag in straight- ahead flight. During turns, however, a larger drag is produced due to the generation of positive yaw. What is the best combination of favorite yaw and small rudder ?

The results of the forthcoming theory have been field-tested. This was done by constructing, building and flying a specially designed ultralight airplane called "Falter 1" (Falter means "butterfly" in German but is also used for "foldable" - the pun was intended). The Falter 1 is a tailless aircraft with swept wings which has no vertical fins. It uses only ailerons for the steering around the three axes. It is quite obvious that both roll moment and pitch moment can be generated by using ailerons (the latter because of the swept back wings). But, on the other hand, it looks at first as if the yaw moment (which is necessary for turns) cannot be produced in this way. In the past, all known flying wings which were steered by aerodynamic means (that is, not by shifting weight, as the hanggliders do it) used either vertical fins or drag rudders for producing the yaw moment (Lippisch, Horten, Northrop et al.); not so with the Falter 1. It is steered around the vertical axis with only the ailerons which produce the desired yaw moment. By using the theory derived below, care was also taken to make the (unavoidable) additional induced drag as small as possible.

2. The Mathematical Problem

2.1 Definitions

The following notations and definitions are standard:

b = span of the wing,

S = surface of the wing,

$\lambda := b^2/S$ = aspect ratio,

v = velocity of the aircraft,

ρ = air density,

$q := \frac{\rho}{2}v^2$ = kinetic energy, sometimes called "dynamic pressure",

L = lift,

D = (induced) drag,

R = roll moment,

Y = (induced) yaw moment,

$c_L := L/qS$,

$c_D := D/qS$,

$c_R := R/qS\frac{b}{2}$,

$c_Y := Y/qS\frac{b}{2}$,

x,y = coordinates in wing direction. In what follows these coordinates are (by an affine transformation) restricted to the interval $-1 \leq x,y \leq +1$.

2.2 Prandtl's Lifting Line Theory. Definitions And Properties

$Z(y)$ = (local) circulation density

= (local) lift density at the wing,

$$\alpha_i(y) := \frac{1}{2\pi} \oint_{-1}^{+1} \frac{dZ(x)}{dx} \frac{dx}{y - x}$$ [1]

= (local) induced angle of attack.

With these two definitions one gets for the above defined coefficients the relations

$$c_L = \lambda \int_{-1}^{+1} Z(y)\, dy \,,$$

$$c_R = \lambda \int_{-1}^{+1} Z(y)\, y\, dy \,,$$ [2]

$$c_D = \lambda \int_{-1}^{+1} Z(y)\, \alpha_i(y)\, dy \,,$$

$$c_Y = \lambda \int_{-1}^{+1} Z(y)\, \alpha_i(y)\, y\, dy \,.$$ [2]

2.3 Transformation To An Algebraic Problem

With the transformation

$$y = \cos s \qquad \text{for} \quad 0 < s < \pi$$

[1] As usual, the letter *c* inside an integral sign means the cauchy principal value of that integral.

[2] Depending upon the orientation of the axis system used sometimes in the literature c_R and c_Y are replaced by $-c_R$ and by $-c_Y$.

and by developing Z into a Fourier series

$$Z(\cos s) = 2 \sum_{\nu=1}^{\infty} a_\nu \sin \nu s$$

one gets

$$c_L = \pi\lambda a_1 \ ,$$

$$c_R = \frac{\pi}{2}\lambda a_2 \ ,$$

$$c_D = \pi\lambda \sum_{\nu=1}^{\infty} \nu a_\nu{}^2 \ ,$$

$$c_Y = \frac{\pi}{2}\lambda \sum_{\nu=1}^{\infty} (2\nu+1) a_\nu a_{\nu+1} \ .$$

In order to get rid of the constants $\pi\lambda$ and $\frac{\pi}{2}\lambda$, it is useful to define new dimensionless forces and moments by

$$L' := L/\pi\lambda qS = c_L/\pi\lambda \ ,$$

$$R' := R/\frac{\pi}{2}\lambda qS\frac{b}{2} = 2c_R/\pi\lambda \ ,$$

$$D' := D/\pi\lambda qS = c_D/\pi\lambda \ ,$$

$$Y' := Y/\frac{\pi}{2}\lambda qS\frac{b}{2} = 2c_Y/\pi\lambda \ .$$

Then

$$L' = a_1,$$

$$R' = a_2,$$

$$D' = \sum_{\nu=1}^{\infty} \nu a_\nu^2 ,$$

$$Y' = \sum_{\nu=1}^{\infty} (2\nu+1) a_\nu a_{\nu+1} .$$

In what follows the abbreviation $a = \{a_\nu\}$ for the sequence $\{a_\nu\}$ will sometimes be used; moreover, the reduced drag D' will be regarded as a functional $D'(a)$ acting on a. This leads to the definition of the set A of admissible sequences by

$$A := \{a | D'(a) < \infty\} .$$

The Cauchy-Schwarz inequality implies

$$|\sum_{\nu=1}^{\infty} (2\nu+1) a_\nu a_{\nu+1}| < \infty \quad \text{for any} \quad a \in A .$$

With these definitions, the problem posed in the Final Question of Section 1 becomes :

Problem : **Given** are the real numbers L', R', Y'.

Wanted is an admissible sequence $a \in A$ which satisfies the conditions

(1) $a_1 = L'$,

(2) $a_2 = R'$,

(3) $\sum_{\nu=1}^{\infty} (2\nu+1)a_\nu a_{\nu+1} = Y'$

such that

(4) $D'(a) := \sum_{\nu=1}^{\infty} \nu a_\nu^2 = \text{minimum}$.

The existence, and (for $R' \neq 0$) the uniqueness, of a solution $\hat{a} \in A$ to this problem will be proven in Section 5.

3. Some Preliminary Results

The results presented in this section are all quite simple. Surprisingly enough, the author did not find them in papers or books on aerodynamics or airplane engineering (with the exception of Sections 3.1 and 3.2). It may, therefore, be worthwile to print them here.

3.1 Only The Lift L Is Prescribed.

In this case the solution $\hat{a}$ to the above problem is obviously

$\hat{a}_1 = L'$, $\hat{a}_\nu = 0$ for $\nu = 2, 3, \ldots$.

This problem and its solution goes back to Prandtl [4] in 1918. In physical terms it reads as

Theorem 1 (L.Prandtl): The optimal circulation distribution on a wing with prescribed lift L is elliptic and is given by

$$Z(y) := \frac{2c_L}{\pi\ \lambda}\sqrt{1 - y^2} \ . \tag{5}$$

The coefficient of the induced drag for this case is

$$c_D = \frac{c_L^{\ 2}}{\pi\ \lambda} \ . \tag{6}$$

3.2 Both Lift L And Roll Moment R Are Prescribed, But Nothing Else.

This is a special case of a more general problem, solved by the author in 1951, see [3]. The solution $\hat{a}$ to this problem can also be found quite easily. It is

$$\hat{a}_1 = L', \quad \hat{a}_2 = R', \quad \hat{a}_\nu = 0 \quad \text{for} \quad \nu = 3, 4, \ldots .$$

Expressed in physical terms it is stated as

<u>Theorem 2</u> (K.Nickel): The optimal circulation distribution on a wing with both prescribed lift L and roll moment R is

$$(7) \qquad z_{1,2}(y) := \frac{2c_L}{\pi\lambda}\sqrt{1-y^2} + \frac{8c_R}{\pi\lambda}\, y\sqrt{1-y^2} .$$

It is a linear combination of a (half) ellipse and a (half) lemniscate. The coefficient of the induced drag for this optimal distribution is

$$(8) \qquad c_D = \frac{c_L^{\,2}}{\pi\lambda} + \frac{8c_R^{\,2}}{\pi\lambda} .$$

Obviously the problem solved in Section 3.1 is a special case of this result if $c_R = 0$. The adverse yaw which is produced by this distribution can be found from equation (3). Its coefficient c_Y is

$$c_Y = \frac{3}{\pi\lambda}\, c_L\, c_R .$$

It shows that - at least for this case - the adverse yaw is proportional to both lift L and roll moment R, as remarked above.

3.3 Non-Optimal Distributions For Given Lift L And Roll Moment R With Vanishing Adverse Yaw (Y = 0).

There are, obviously, infinitely many sequences $a \in A$ for which $Y = 0$, i.e. with

$$(9) \qquad \sum_{\nu=1}^{\infty} (2\nu+1) a_\nu a_{\nu+1} = 0 \; .$$

Only two of them will be considered here.

3.3.1 $a_\nu = 0$ for $\nu = 4, 5, \ldots$.

In this case the equation (9) becomes

$3\, a_1\, a_2 + 5\, a_2\, a_3 = 0$.

Hence, by assuming $a_2 \neq 0$, w.l.o.g. one gets

$a_3 := -\frac{3}{5} a_1$.

This solves (9) for any value of a_2, i.e. of the roll moment R. The corresponding circulation distribution is

$$(10) \qquad z_{1.2.3}(y) = \frac{8c_L}{5\pi\lambda}(2-3y^2)\sqrt{1-y^2} + \frac{8c_R}{\pi\lambda}\, y\sqrt{1-y^2} \; .$$

The induced drag coefficient is given by

$$(11) \qquad c_D = 2.08\,\frac{c_L^2}{\pi\lambda} + 8\,\frac{c_R^2}{\pi\lambda} \; .$$

For R = 0 it exceeds the smallest possible induced drag for a given lift (see(6)) by the factor of 2.08 (!). This is the price paid for annihilating the adverse yaw. It will, however, be shown later that there are "better" distributions for which there is no additional drag for $R = 0$. A very simple example of this is shown in the next Section

3.3.2 $a_3 = 0$ and $a_\nu = 0$ for $\nu = 6,7, \ldots$.

From the equation (9) one gets the condition

$$a_4 \, a_5 = -\frac{1}{3} a_1 \, a_2 \, ,$$

with some freedom in choosing a_4 or a_5 .

By putting

$$a_4 = -\alpha a_5$$

and by determining α so that the induced drag becomes minimal one gets

$$a_4 := \pm \sqrt{\sqrt{\frac{5}{4}} \; \frac{|a_1 \, a_2|}{3}} \, ,$$

$$a_5 := \mp \sqrt{\sqrt{\frac{4}{5}} \; \frac{|a_1 \, a_2|}{3}}$$

and

$$c_D = \frac{1}{\pi\lambda} \left(c_L^{\,2} + 8 \, c_R^{\,2} + \frac{8}{3} \sqrt{5} \; | \, c_L \, c_R \, | \right) . \tag{12}$$

Obviously this distribution is much "better" than that given in 3.3.1 . For $R = 0$ (no roll moment) its drag coefficient reduces to that value given in (8). For $R = 0$ it exceeds the optimal value in (8) only by the term $8 \sqrt{5} \; | \, c_L \, c_R \, | \, / \, 3 \pi \lambda$ which is small for small c_L and/or c_R .

The corresponding circulation distribution is

(13) $\quad Z_{4.5}(\cos t) := \sqrt{\sqrt{\frac{5}{4}}\ \frac{1}{3}}\ \sin 4t - \sqrt{\sqrt{\frac{4}{5}}\ \frac{1}{3}}\ \sin 5t$.

It is sketched in Figure 3. By using the distribution $Z_{1.2}$ from equation (7) in section 3.2 together with $Z_{4.5}$ from (13) one gets the very interesting

Result: Both circulation distributions

$Z(y) := Z_{1.2}(y) \pm c_L\ c_R\ Z_{4.5}(y)$

have the same lift L, roll moment R, vanishing yaw moment Y = 0 and induced drag as given in (12).

See the Figures 3 and 4.

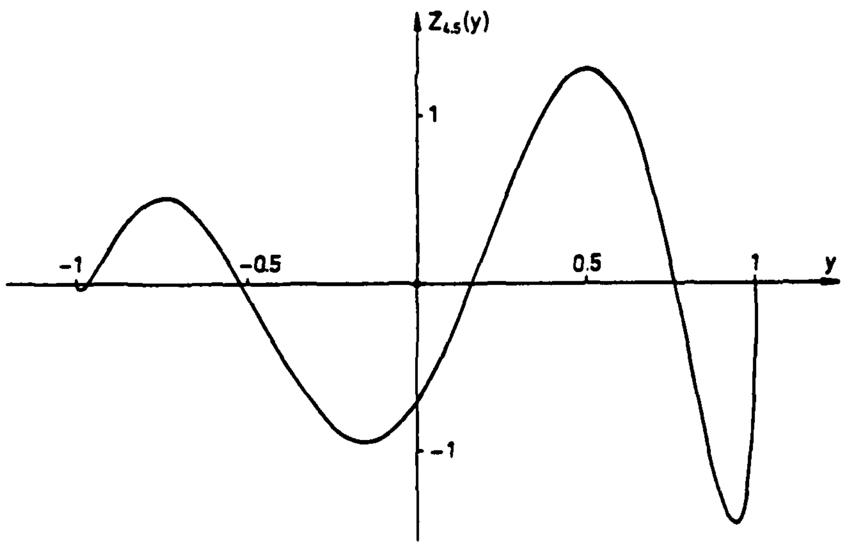


Figure 3. The circulation distribution $Z_{4.5}$.

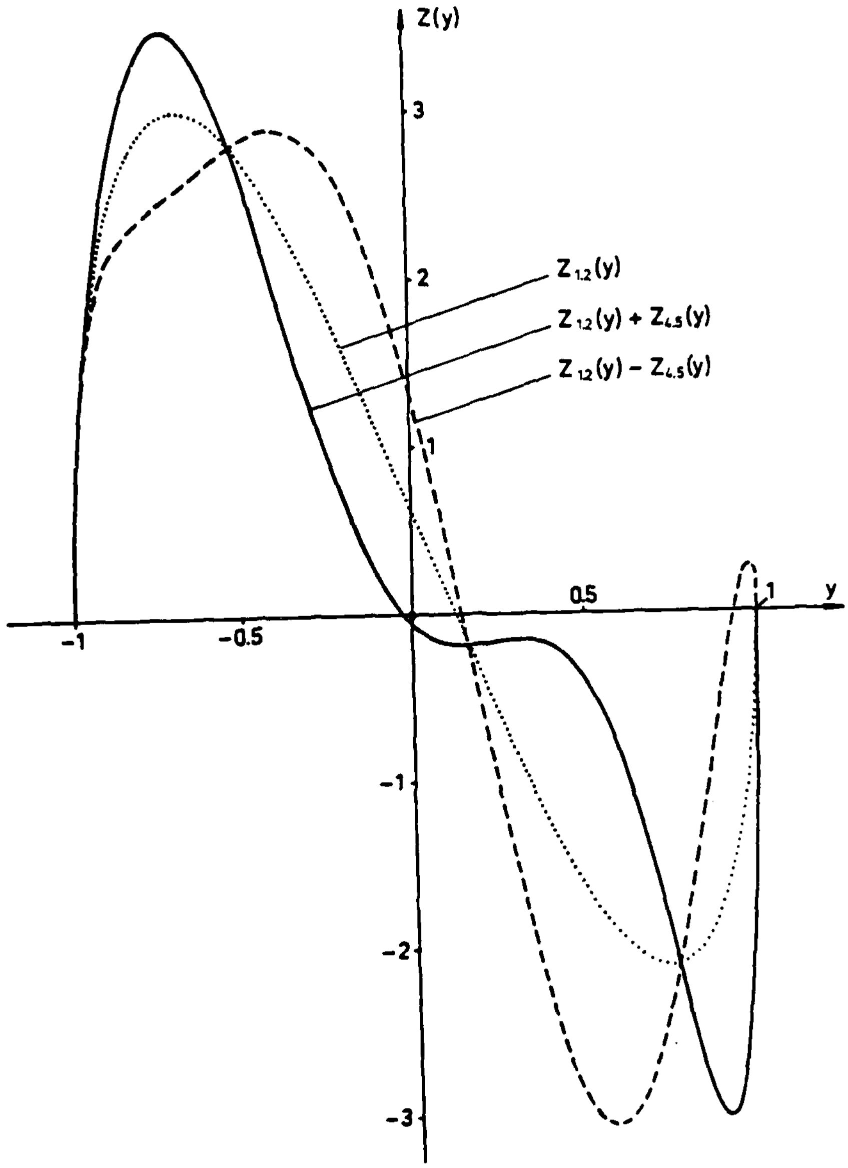
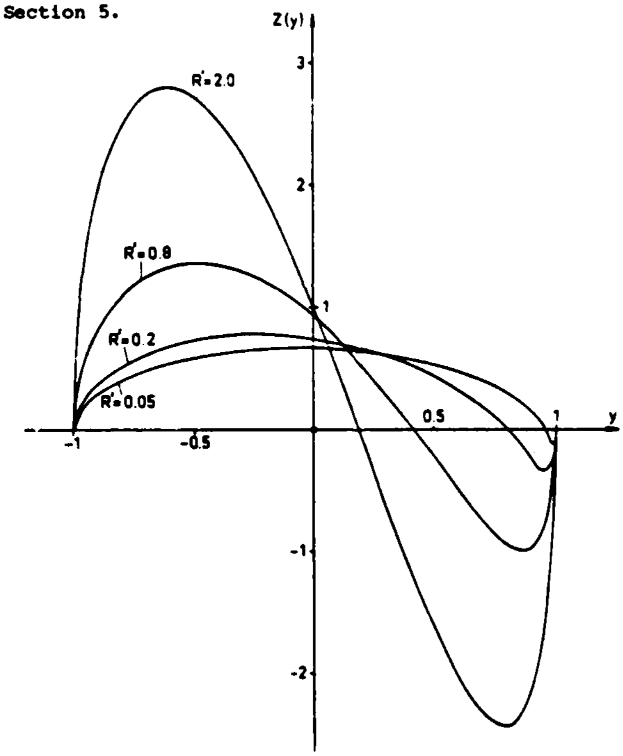


__Figure 4__. The three circulation distributions $Z_{1.2}$ and $Z_{1.2} \pm Z_{4.5}$. They have all the same lift L and roll moment R. The latter two have both vanishing yaw moment $Y = 0$ and attain the same induced drag D.

3.4 <u>Representation of Some Solutions.</u>

In Figures 5 to 8 some circulation distributions are presented which solve the problem of this paper. In all cases Y = 0, i.e. null adverse yaw was chosen. Because the problem is then homogenuous, without loss of generality L' := 1 was assumed. In these Figures four typical roll moments were selected. These results have been computed by Mrs. Norbert, and the Figures were drawn by Mrs. Sturm, both of Freiburg i.Br./GERMANY. The method used is sketched in Section 5.



<u>Figure 5</u>. Optimal circulation distributions for L' = 1, Y' = 0 and R' = 0.05, 0.2, 0.8, 2.0 .

In Figure 5 the four optimal circulation distributions are shown. In order to see more clearly the details, these distributions are broken up into the corresponding symmetric part Z_s and the antisymmetric part Z_a by the equation

$$Z = Z_s + Z_a .$$

These two parts are presented in Figures 6 and 7. As can bee seen from Figure 6, the optimal antisymmetric circulation distribution for the largest roll moment $R' = 2$ looks very similar to the lemniscate which is the optimal distribution if the condition $Y = 0$ were removed (see Section 3.2). It can be shown that this is typical, i.e. that

$$Z_a(y) \rightarrow \text{const. } y\sqrt{1 - y^2} \quad \text{as} \quad R \rightarrow \infty .$$

The proof of this will be given elsewhere.

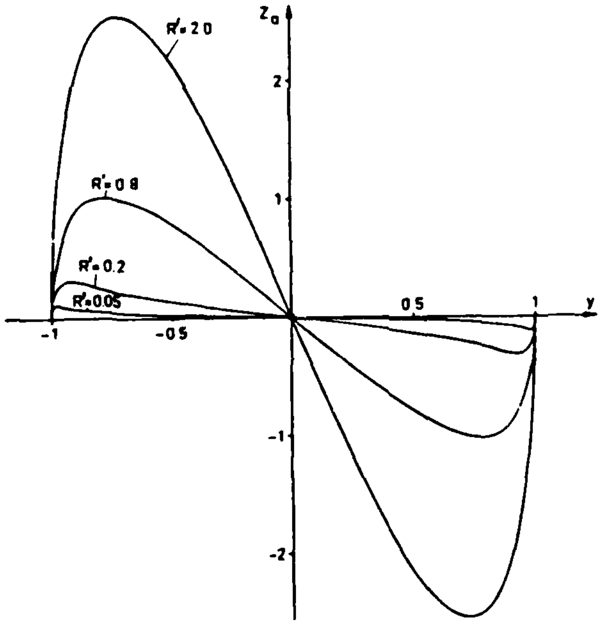


Figure 6. The antisymmetric part of the distributions of Figure 5.

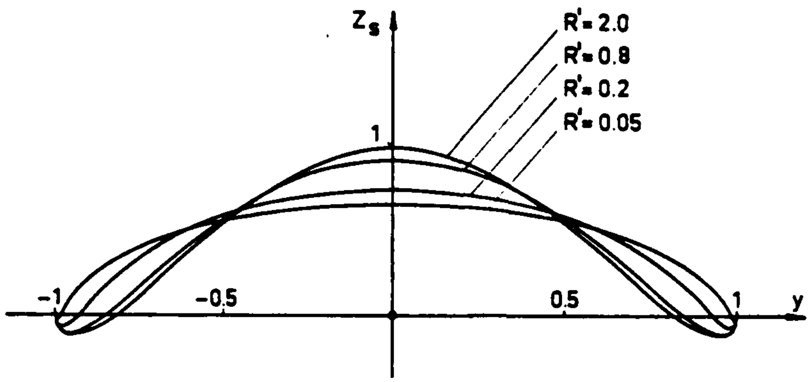


Figure 7. The symmetric part of the distributions of Figure 5.

In order to get a closer look at the symmetric circulation distribution in Figure 7 a new symmetric distribution with zero lift is defined by

$$Z_{s0}(y) := Z_s(y) - \frac{2c_L}{\pi \lambda} \sqrt{1 - y^2} \quad .$$

It is presented in Figure 8. Note that this distribution gives negative (!) additional load at the wingtips.

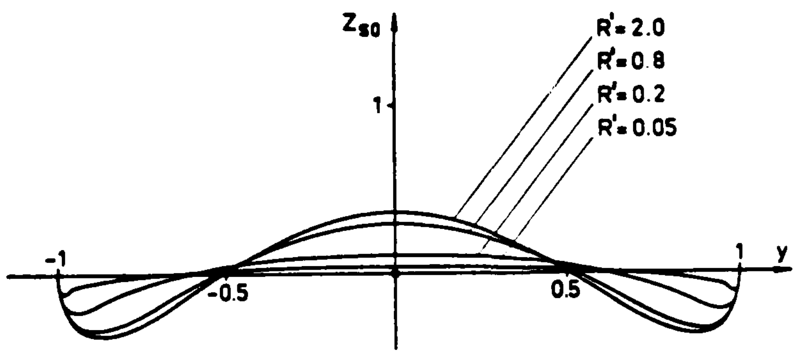


Figure 8. The zero-lift part of the distributions of Figure 7.

Similarly to the earlier result it can be shown that asymptotically

$$Z(y) \rightarrow Z_{1.2.3}(y) \quad \text{as} \quad R \rightarrow \infty$$

for <u>all</u> optimal circulation distributions, where $Z_{1.2.3}$ is defined in equation (10) .

3.5 Some Simple General Results.

Let again

$$Z(y) = Z_s(y) + Z_a(y)$$

with Z_s and Z_a the symmetric and the antisymmetric parts of Z, respectively. It is known since Prandtl [4] (see Section 3.1) that the elliptical symmetrical distribution

$$(5) \qquad Z_s(y) := \frac{2\, c_L}{\pi\, \lambda} \sqrt{1-y^2}$$

is optimal with respect to the drag, if no conditions on roll moment R and yaw moment Y are required. Hence, one may ask if, given the symmetric distribution in (5), an additional antisymmetric distribution Z_a can be found such that $Y = 0$, i.e. such that no adverse yaw occurs. By looking at (9) and bearing in mind that the Fourier coefficients $a_{2\nu}$ belong to the antisymmetric and the $a_{2\nu+1}$ to the symmetric distribution one sees immediately that <u>no</u> such distribution Z_a can be found, provided that $L \neq 0$ and $R \neq 0$. Hence, one has the

<u>Theorem 3</u>: Assume $L \neq 0$, $R \neq 0$. There is <u>no</u> circulation distribution $Z(y)$ with the symmetrical part Z_s as in equation (5) for which no adverse yaw occurs.

Now a somehow "dual" question arises : It is known (see Section 3.2) that the lemniscate shaped antisymmetric distribution

$$(14) \qquad z_a(y) := \frac{8\, c_R}{\pi\, \lambda}\; y \sqrt{1-y^2}$$

is optimal if there is no condition on the yaw moment. In analogy to the above one may ask, therefore, if there are symmetric distributions z_s corresponding to z_a from (14) such that $Y = 0$. By looking at equation (9) one sees easily - opposite to the above result in Theorem 3 - that there are infinitely many such distributions. One can, therefore, try to solve the basic problem of this paper under the side condition, that the antisymmetric part of the solution equals z_a in equation (14). Surprisingly enough the answer to this specialized problem is easy to get. By elementary operations one finds immediately the following

<u>Theorem 4</u> : Assume that the antisymmetrical part of the circulation distribution has the shape of a lemniscate, i.e. that z_a is given by equation (14). Then there is exactly one circulation distribution with prescribed lift L and roll moment R for which the induced yaw moment Y vanishes (no adverse yaw) and which has minimal induced drag D. It is given by $z_{1.2.3}$ from the equation (10). The corresponding (minimal) induced drag is given by equation (11) .

4. **Bounds For The Induced Drag.**

In this Section lower and upper bounds for the optimum $\hat{D}$ of the induced Drag D are given. To make these results as useful as possible to airplane engineers the bounds are presented for the optimal drag coefficient $\hat{c}_D$ as inequalities of the form

$$l_k \le \pi \lambda \hat{c}_D \le u_k \quad \text{for } k = 1, 2, \dots .$$

There are 3 lower and 4 upper bounds l_1, l_2, l_3 and u_1, u_2, u_3, u_4. They are defined as follows

$$l_1 := c_L^2 + 8c_R^2 ,$$

$$l_2 := c_L^2 + 3c_R^2 + |\pi\lambda c_Y - 3c_L c_R| ,$$

$$l_3 := c_L^2 + 4c_R^2 + \sqrt{16\, c_R^4 + \frac{24}{25}(\pi\lambda c_Y - 3c_L c_R)^2} ,$$

$$u_1 := c_L^2 + 8c_R^2 + 0.12\left(\frac{\pi\lambda c_Y}{c_R} - 3c_L\right)^2 \quad \text{for} \quad c_R \neq 0,$$

$$u_2 := c_L^2 + 8c_R^2 + 2\,|\pi\lambda c_Y - 3\, c_L\, c_R| ,$$

$$u_3 := c_L^2 + 8\, c_R^2 + 1.08 \frac{c_L^2}{\left(1 + \frac{0.168}{(0.3 - c_R)^2}\right)^2} + \frac{0.2304\, c_R^2}{(0.3 - c_R)^2}$$

for $c_Y = 0$ and $c_R \neq 0.3$,

$$u_4 := c_L^2 + 4\, c_R^2 + \sqrt{16\, c_R^4 + (\pi\, \lambda\, c_Y - 3\, c_L c_R)^2} .$$

The first bounds l_1, l_2, u_1, u_2, u_3 are sketched in Figure 9 for $Y = 0$ together with the solution which is represented by a dashed line. For $Y = 0$ the whole problem is homogenuous. Therefore $c_L := 1$ was chosen w.l.o.g..

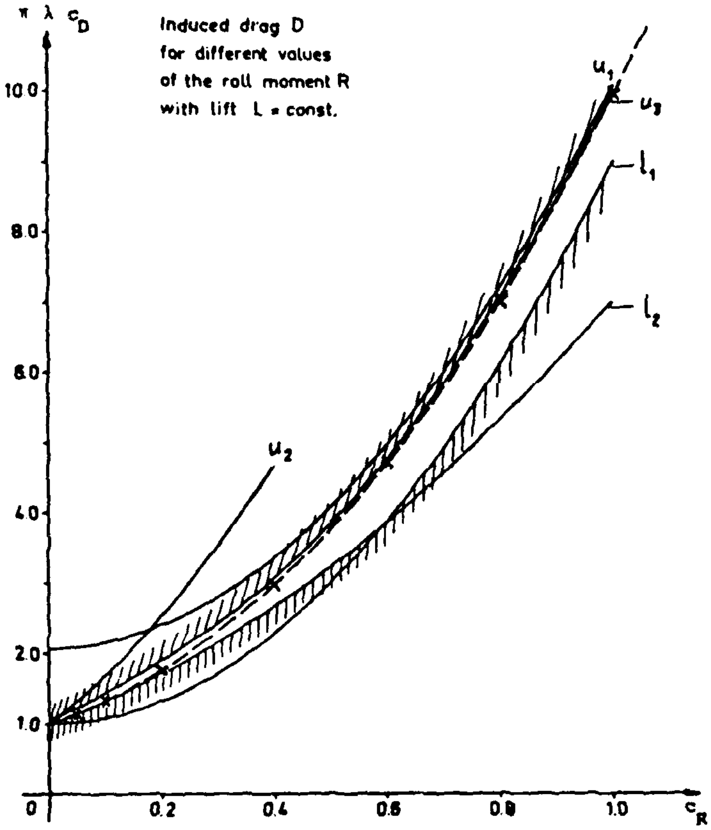


Figure 9. Upper and lower bounds u_1, u_2, u_3 and l_1, l_2 to the coefficient $\hat{c}_D$ of the optimal induced drag as functions of the coefficient c_R of the roll moment. The case $c_L := 1$, $c_Y := 0$ is treated. The dashed line represents $\hat{c}_D$.

The best bounds which were found in this paper are l_3 and u_4. They have both the same structure and differ only in one constant (24/25 in l_3 and 1 in u_4) . The relative error between them is less than 2.1% (!). Hence, for most purposes in aircraft engineering these easy-to-handle bounds are completely satisfactory. In most cases it will, therefore, not be necessary to solve the problem analytically [3]. The bounds l_3 and u_4 are not included in Figure 9 since they are both too close to the solution and would, therefore, interfere with the picture.

Proof Of The Bound Properties.

In this section the existence theory of Section 5 will not be used directly. It is assumed, however, that a (not necessarily unique) solution $\hat{a}$ to the problem of this paper exists. This existence is nearly trivial, since $D'(a)$ is a continuous functional of a, bounded by zero from below. For simplicity the notation $\hat{D} := D'(\hat{a})$ with the corresponding *optimal drag coefficient* $\hat{c}_D$ is introduced .

Lower Bounds

Ad l_1 : This bound is trivial since

$$\hat{D} := \sum_{\nu=1}^{\infty} \nu \, \hat{a}_\nu^2 \geq \hat{a}_1^2 + 2 \, \hat{a}_2^2 = l_1 / \pi^2 \lambda^2 .$$

Note that the first term in l_1 is the well known Prandtl formula (6) for the

[3]) Ludwig Prandtl once said: "Es gibt nichts Praktischeres als eine gute Theorie" ! (Translated: "There is nothing more practical than a good theory")

induced drag of an elliptical circulation distribution (see Theorem 1). It has been used by airplane engineers for more than 65 years. The second term in l_1 shows that it requires at least an induced drag coefficient of $8\, c_R^2$ to produce a roll moment coefficient with the value of c_R, regardless of the value of c_Y. The bound l_1 corresponds to the formula (8) from Theorem 2. The value of $\pi \lambda c_D$ equals l_1 in the special case where $a_1 = L'$, $a_2 = R'$. In this case $Y' = 3 L'R'$.

Ad l_2 : By rewriting D' one gets

$$4D' := 4 \sum_{\nu=1}^{\infty} \nu a_\nu^2$$

$$= \sum_{\nu=1}^{\infty} (2\nu+1)\, a_\nu^2 + \sum_{\nu=1}^{\infty} (2\nu-1)\, a_\nu^2$$

$$= a_1^2 + \sum_{\nu=1}^{\infty} (2\nu+1)(a_\nu^2 + a_{\nu+1}^2) .$$

By adding

$$\pm 2Y' = \pm \sum_{\nu=1}^{\infty} (2\nu+1)\; 2\, a_\nu\, a_{\nu+1}$$

one gets

$$4D' \pm 2Y' = a_1^2 + \sum_{\nu=1}^{\infty} (2\nu+1)(a_\nu \pm a_{\nu+1})^2$$

$$> a_1^2 + 3\, (a_1 \pm a_2)^2 .$$

A backtransformation then gives l_2. Obviously, as can be seen from Figure 9, the bound l_2 is better than l_1 for low values of c_R. For large values of c_R the opposite is true.

Ad l_3 : If one applies the Cauchy-Schwarz inequality to the identity

$$Y' - 3\, a_1\, a_2 = \sum_{\nu=2}^{\infty} (2\nu+1)\, a_\nu a_{\nu+1}$$

and uses the inequality

$$\frac{(2\nu + 1)^2}{\nu} < \frac{25}{6}(\nu + 1) \qquad \text{for} \quad \nu > 2$$

one gets

$$(Y' - 3\, a_1 a_2)^2 < \frac{25}{6} \sum_{\nu=2}^{\infty} \nu\, a_\nu^{\;2} \sum_{\nu=2}^{\infty} (\nu+1)\, a_{\nu+1}^{\;2}$$
$$= \frac{25}{6} (D'(a) - a_1^{\;2})\, (D'(a) - a_1^{\;2} - 2a_2^{\;2}) \quad .$$

By solving this quadratic inequality for $D'(a) - a_1^{\;2}$ and using the lower bound l_1 and by transforming it to the coefficients one gets the bound l_3.

Upper Bounds

To get upper bounds for $\hat{D}$ is quite easy. Since the solution $\hat{a}$ is defined by a minimum problem, any approximation $\bar{a}$ to $\hat{a}$ which satisfies the side conditions (1), (2) and (3) gives an upper bound $D'(\bar{a}) > \hat{D}$. In what follows for simplicity the sign $\bar{a}$ will be replaced by a.

Ad u_1 : Let a be defined by

$a_1 := L',\ a_2 := R',\ a_4 := a_5 := \ldots := 0.$

The condition (3) then reads

$$Y' = 3\, a_1\, a_2 + 5\, a_2\, a_3\ .$$

Hence for $R \neq 0$ one gets

$$a_3 \ := \ \frac{Y'}{5\ R'} \ - \ \frac{3}{5}\, L'\ .$$

Inserting these coefficients in $D'(a)$ then gives u_1.

<u>Ad u_2</u> : Here the coefficients of a are defined by

$$a_\nu := 0 \quad \text{for} \quad \nu \neq 1,\ 2,\ \mu,\ \mu{+}1 \quad \text{with} \quad \mu > 3\ .$$

The condition (3) reduces to

$$Y' = 3\, a_1 a_2 + \ (2\mu+1)\ a_\mu a_{\mu+1}\ ,$$

hence one of the coefficients a_μ or $a_{\mu+1}$ remains free. By choosing quite arbitarily

$$\sqrt{\mu}\ a_\mu + \ \sqrt{\mu+1}\ a_{\mu+1} = 0$$

one gets then

$$a_\mu \ := \ \sqrt{\sqrt{\frac{\mu+1}{\mu}}\ \frac{|3a_1a_2 \ - Y'|}{2\ \mu = 1}}$$

Inserting these coefficients into $D'(a)$ then gives

$$D'(a) \ = \ a_1^{\ 2} + 2\ a_2^{\ 2} \ + \ \frac{2\ \sqrt{\ \mu(\mu+1)}}{2\ \mu\ +\ 1}\ |3\ a_1a_2 - Y'|\ ,$$

where, because of $\mu > 4$, the coefficient of the third term is bounded by

$$0.9938\ldots \ = \ \frac{4\sqrt{5}}{9} \ < \ \frac{2\ \sqrt{\mu(\mu+1)}}{2\ \mu\ +\ 1} \ < \ 1\ .$$

Replacing it by the upper bound 1 and transforming back then yields u_2.

<u>Ad u_3</u> : In this case $a_\nu := 0$ for $\nu > 5$. The condition (3) then gives

$$a_3 := \frac{Y' - 3\,a_1 a_2}{5\,a_2 + 7\,a_4} ,$$

for arbitary a_4 (provided that the denominator does not vanish). The following suggestion for a_4 for the case $Y = 0$ has been given to me by Mrs. Norbert in Freiburg i.Br./ GERMANY:

$$\pi \lambda a_4 := - \frac{0.24\, c_R}{(0.3 - c_R)^2} \quad \text{for} \quad c_R \neq 0.3 .$$

Inserting everything in D'(a) then gives u_3.

<u>Ad u_4</u> : For finding the lower bound l_3 the Cauchy-Schwarz inequality was used. This inequality turns into an identity if the *different coefficients* are proportional to each other. This leads to the following "Ansatz" :

Define recursively

$$a_{\nu+1} := c \frac{2\nu}{2\nu+1} a_\nu \quad \text{for} \quad \nu > 2 ,$$

where the constant c is to be determined later. This gives

$$a_\nu = \beta_\nu c^{\nu-2} a_2 \quad \text{for} \quad \nu > 2$$

with

$$\beta_\nu := \frac{(2\nu-2)(2\nu-4) \dots 8\ 6\ 4}{(2\nu-1)(2\nu-3) \dots 9\ 7\ 5} .$$

It is easily proved that $\nu \beta_\nu^2 \leq 2$. With the aid of Stirling's formula it can also be shown that there exists a positive constant $\beta < \nu \beta_\nu^2$ for all $\nu > 2$. Using the recursion formula once more gives

$$(2\nu+1)\, a_\nu a_{\nu+1} = 2\, c\, \nu\, a_\nu^2 \quad \text{for} \quad \nu > 2$$

and therefore

$$Y' - 3\, a_1 a_2 = \sum_{\nu=2}^{\infty} (2\nu+1)\, a_\nu a_{\nu+1} = 2\, c \sum_{\nu=2}^{\infty} \nu\, a_\nu^2$$

$$= 2\, c\, D'(a) = 2\, a_2^2\, c \sum_{\nu=2}^{\infty} \nu\, \beta_\nu^2\, c^{2\nu-4} \; .$$

For any real value of $Y' - 3a_1a_2$ there is exactly one solution $\hat{c}$ to this equation with $c^2 < 1$. This comes from the fact that

$$c \sum_{\nu=2}^{\infty} \nu\, \beta_\nu^2\, c^{2\nu-4}$$

is a monotone function of c which goes to $\pm\infty$ as $c \to \pm 1$. This can be seen from the inequalities

$$\frac{\beta}{1 - c^2} < \sum_{\nu=2}^{\infty} \nu\, \beta_\nu^2\, c^{2\nu-4} < \frac{2}{1 - c^2} \; .$$

By solving the quadratic inequality

$$|Y' - 3\, a_1 a_2| < 4\, a_2^2 \frac{c}{1 - c^2}$$

one gets therefore a lower bound c^* for $|\hat{c}|$ with

$$\frac{1}{|\hat{c}|} < \frac{1}{c^*} = \frac{2\, a_2^2 + \sqrt{4\, a_2^4 + |Y' - 3\, a_1 a_2|^2}}{|\, Y' - 3\, a_1 a_2\, |} \; .$$

By inserting this in the above equation

$$D'(a) = \frac{Y' - 3\, a_1 a_2}{2\, \hat{c}}$$

and transforming it back, one gets finally u_4.

Comparison between u_4 and l_3

Using the abbreviations

$$r := 16\, c_R^4 \,, \quad s := (\pi\, \lambda\, c_Y - 3\, c_L c_R)^2$$

one finds for the relative error

$$\frac{u_4 - l_3}{l_3} = \frac{\sqrt{r + s} - \sqrt{r + 24\, s/25}}{c_L^2 + 4 c_R^2 + \sqrt{r + 24 s/25}}$$

$$\leq \frac{s\,(1 - 24/25)}{\sqrt{r + 24\, s/25}\,(\sqrt{r + s} + \sqrt{r + 24\, s/25})}$$

$$\leq \frac{1/25}{\sqrt{24/25}\,(1 + \sqrt{24/25})}$$

$$= \frac{1}{2\sqrt{6}\,(5 + 2\sqrt{6})} = 0.02062\ldots < 0.021 \mathrel{\hat{=}} 2.1\ \%$$

as predicted .

5. Existence, Uniqueness And Computation Of The Solution.

There are several approaches to a proof of the existence and uniqueness of the solution to the problem given in this paper. If these proofs are constructive, they also permit the computation of a solution. In the paper [2], the author used the theory of singular integral equations to prove existence and uniqueness of a solution. It was even possible to give an explicit expression for the optimal circulation distribution by using integrals. Some computation with these formulae has been performed in a Master's Thesis ("Diplom-Arbeit") at the University of Freiburg i.Br./Germany by Mr. Ramdane Kedache (unpublished). Unfortunately however, the numerical computation of these integrals (some of them singular; others elliptic integrals of first, second and third kind) was very strenuous work, even with today's fast computers.

Therefore, an algebraic approach is used in the present paper. The simplest method is to *replace the infinite series* in the equations (3) and (4) by finite sums. By using sufficiently many terms, any accuracy desired can be achieved. The computations for the results presented in Section 3.4 have been carried out by this method.

In what follows, however, the problem is treated in a different way by using the "Lagrange method" : It will be shown that the coefficients $\hat{a}_\nu$ in a solution $\hat{\gamma}$ satisfy a certain recurrence relation. With this result their theoretical structure is completely understood. What remains, is to determine a certain free constant in that relation. This can be computed from a one-dimensional transcendental equation.

5.1 <u>The Lagrange Method.</u>

In the problem of this paper three conditions (1), (2) and (3) have to be satisfied with arbitrary constants L, R, Y or L', R', Y' respectively. The first two of them are linear and of very simple structure. To solve them gives no difficulties at all. Hence, one has only to concentrate on the one remaining quadratic condition (3) for Y'.

In the Lagrange method the following fact is used: Suppose $\hat{a}$ is a solution to the problem (1) to (4) . Let β be an arbitrary real constant. Then not only

$$D'(a) := \sum_{\nu=1}^{\infty} \nu \, a_\nu^{\,2} = \text{minimum} \quad \text{for} \quad a = \hat{a},$$

but also the same is true for the following quadratic expression, namely that

$$Q(a) := \sum_{\nu=1}^{\infty} \nu \, a_\nu^{\,2} + \beta \sum_{\nu=1}^{\infty} (2\nu+1) \, a_\nu a_{\nu+1} = \text{minimum} \quad \text{for} \quad a = \hat{a} \, .$$

The constant β is called the Lagrange parameter. (Normally, the letter λ is used; but in the present paper the symbol λ is already the aspect ratio.) Now, by rearranging the terms in Q a necessary and sufficient condition for $\hat{a}$ is reached which guarantees that $\hat{a}$ is a solution.

Let $\hat{a} \in A$ be a solution and let $a = \hat{a} + b \in A$ be an arbitrary sequence which satisfies the conditions (1), (2), and (3). Hence, for the first two coefficients of b one sees immediately that $b_1 = b_2 = 0$. By inserting a into Q and after some transformations one gets the following identity

$$Q(a) - Q(\hat{a}) = T_1 + T_2 + T_3$$

with the three terms

$$T_1 := \sum_{\nu=3}^{\infty} b_\nu \, [\, \beta(2\nu+1)\, \hat{a}_{\nu+1} + 2\nu\, \hat{a}_\nu + \beta\, (2\nu-1)\, \hat{a}_{\nu-1}] \; ,$$

$$4\, T_2 := \sum_{\nu=2}^{\infty} (2\nu + 1)\, (b_\nu + 2\, \beta\, b_{\nu+1})^2 \; ,$$

$$4\, T_3 := (1 - 4\, \beta^2) \sum_{\nu=2}^{\infty} (2\nu + 1)\, b_{\nu+1}^2 \; .$$

Now, $\hat{a}$ is a solution to the problem if and only if $Q(a) - Q(\hat{a}) > 0$ for any $a \in A$ satisfying the conditions (1), (2) and (3) and that means for (almost) any b. The first term T_1 is linear in b. It, therefore, has to vanish for arbitrary b. This is true iff all the square brackets are zero. The next term T_2 is quadratic and obviously $T_2 \geq 0$ for all values of b. The same is true for the third term T_3 iff $\beta^2 < 1/4$. If one puts all these facts together one gets finally the

<u>Theorem 5</u> : Necessary and sufficient for $\hat{a} \in A$ to be a solution of the problem (1), (2), (3) and (4) are the following three conditions:

i) $\hat{a}$ satisfies (1), (2) and (3),

ii) there is a real constant β with $|\beta| < 1/2$ such that

iii) $\hat{a}$ satisfies the recurrence relation

(15) $\quad \beta\,(2\nu+1)\, a_{\nu+1} + 2\nu\, a_\nu + \beta\,(2\nu-1)\, a_{\nu-1} = 0 \quad$ for $\nu = 3, 4, \ldots$.

5.2 Solving The Recurrence Relation

If $\beta \in [-1/2,1/2]$ and if $a_2 := R'$ and a_3 is an arbitrary real constant, then (15) can be solved uniquely by recursion. Hence, there is no doubt about the existence of a solution of (15). The problem in question in Theorem 5, however, is different: Is it true that for any values of L', R' and Y' there exists a constant $\hat{\beta} \in [-1/2,1/2]$ and a starting value $\hat{a}_3$ such that the sequence $\hat{a}$ generated by (15) satisfies the conditions (1), (2) and (3) and that, moreover, $\hat{a} \in A$? If yes, are the constants $\hat{\beta}$ and $\hat{a}_3$ then uniquely determined ? It will be shown that all this is true. But it is useful first to treat

The Special Case $\beta = 0$:

In this case the recurrence relation (15) has for all starting values $\hat{a}_2$ the uniquely determined solution $\hat{a}_\nu = 0$ for $\nu > 3$. Therefore this case is equivalent to the problem treated in Section 3.2 with the solution given by Theorem 2. Hence, w.l.o.g. $\beta \neq 0$ can be assumed in what follows.

Obviously, the recurrence relation (15) is asymptotically equal to the similar relation

$$(16) \qquad \beta\,(2\nu+2)\, a_{\nu+1} + 2\nu\, a_\nu + \beta\,(2\nu-2)\, a_{\nu-1} = 0 \quad \text{for} \quad \nu = 3, 4, \dots .$$

As for any nondegenerate second order linear recursion relation both (15) and (16) have two linearily independent solutions. The two solutions of (16) for $0 < \beta^2 < 1/4$ are obviously

$$a_\nu := \alpha^\nu/\nu \quad \text{for} \quad \nu = 2, 3, \ldots ,$$

where α is one of the two solutions of the characteristic equation

(17) $\beta\,\alpha^2 + \alpha + \beta = 0$.

These solutions are

$$\alpha_1 := (-1 + \sqrt{1-4\beta^2})/2\beta ,$$

and

$$\alpha_2 := (-1 - \sqrt{1-4\beta^2})/2\beta .$$

From these expressions one sees easily that $|\alpha_1| < 1$, $|\alpha_2| > 1$ for $\beta^2 < 1/4$. Therefore only the solution with α_1 is a convergent sequence, moreover, only this one solution belongs (obviously) to the class of admissible sequences A. Because of the homogeneity of (16) the solution wanted is then

(18) $$a_\nu := 2\,a_2\,\frac{\alpha_1^{\nu-1}}{\nu} \quad \text{for } \nu = 2, 3, \ldots .$$

Inserting this into the condition (3) together with $a_1 = L'$ and $a_2 = R'$ gives the equation

$$Y' = 3\,L'R' + 4\,R'^2 \sum_{\nu=2}^{\infty} \frac{2\nu+1}{\nu(\nu+1)}\,\alpha_1^{2\nu-2}.$$

It can be shown rather easily that there is a unique solution $\hat{\alpha}_1$ to this equation for any real values of L', $R' \neq 0$ and Y'. Reinserting this solution $\hat{\alpha}_1$ into the characteristic equation (17) then gives the desired value

$$\hat{\beta} := \frac{-\hat{\alpha}_1}{1 + \hat{\alpha}_1^{\,2}} .$$

If $R' = 0$ and $Y' = 0$, then obviously the Prandtl solution (5) solves the problem of this paper. Hence, only the case $R' = 0$ and $Y' \neq 0$ remains to be treated. This will not be done here.

Up to this point only the asymptotic recurrence relation (16) has been treated. It can be solved explicitly by (18). It can be shown that (18) is also an asymptotic solution to the original equation (15). This will be done elsewhere. Hence one gets as a result the

Theorem 6: For any combination of real numbers L', $R' \neq 0$ and Y' there is exactly one solution $\hat{a} \in A$, $\hat{\beta}$ to the recurrence equation (15) such that the conditions i) and ii) of Theorem 5 are satisfied.

5.3 Uniqueness

It can be shown that there is a solution to the problem of this paper also in the case $R = 0$, $Y \neq 0$, which was not treated above. But in this case the solution is not unique, in opposition to the above result. This can bee seen very simply as follows:

Let $\hat{a} = \{ \hat{a}_1, 0, \hat{a}_3, \hat{a}_4, \ldots \}$ be a solution to the problem (1), (2), (3) and (4) with $R' = 0$, i.e. $\hat{a}_2 = 0$. Define

$a^* := \{ \hat{a}_1, 0, -\hat{a}_3, -\hat{a}_4, \ldots \}$.

Then a^* obviously also satisfies the conditions (1), (2) and (3) . Moreover $D'(\hat{a}) = D'(a^*)$ which shows that a^* is a solution of the problem, too. It can be shown, moreover, that these are the only two solutions, i.e. that no other solutions exist. The proof of this will not be given here.

5.4 Generating Function

(This part has been proposed to me by Professor Ben Noble, former director of the Mathematics Research Center.)

Let z be a complex variable and define the complex function f by the power series

$$f(z) := \sum_{\nu=2}^{\infty} a_\nu z^{2\nu} .$$

Such an fuction f is usually called a "generating function" of the sequence a . Then inside the disc of convergence one gets :

$$(f(z)/z)' = \sum_{\nu=1}^{\infty} (2\nu+1)\ a_{\nu+1}\ z^{2\nu} ,$$

$$z\ f'(z) = \sum_{\nu=2}^{\infty} 2\ \nu\ a_\nu\ z^{2\nu} ,$$

$$z^2(z\ f(z))' = \sum_{\nu=3}^{\infty} (2\nu-1)\ a_{\nu-1}\ z^{2\nu} .$$

Assume that the coefficients a_ν satisfy the recurrence relation (15). Then by the above three equations it follows immediately that the function f satisfies the following linear differential equation

$$z(\beta + z^2 + \beta z^4)f'(z) + \beta(z^4-1)f(z) = (a_2(3\beta + 4z^2) + 5\beta a_3 z^2)z^4 .$$

With the initial condition $f(0) = 0$ it has the unique solution

$$f(z) = z\ e^{H(z)} \int_0^z e^{-H(t)}\ k(t)\ dt$$

with

$$h(z) := -\frac{1 + 2\beta z^2}{\beta + z^2 + \beta z^4} ,$$

$$H(z) := \int_0^z h(t)\, dt ,$$

$$k(z) := \frac{a_2 z^2 (3\beta + 4z^2) + 5\beta a_3 z^4}{\beta + z^2 + \beta z^4} .$$

From this generating function the results of Theorem 6 can also be derived. This will not be shown here.

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Let L, R, Y be arbitrary real constants. A wing with fixed span which produces lift L, roll moment R, yaw moment Y and which has minimal induced drag D is wanted. This problem arises in airplane engineering. It is solved by means of Prandtl's lifting line theory combined with computation.

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