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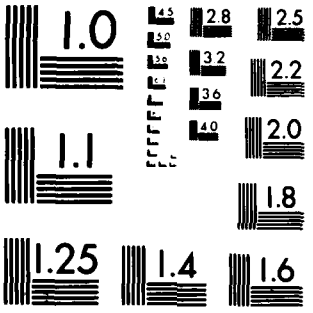

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THE FROUDE NUMBER FOR SOLITARY WAVES

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ABSTRACT

This paper is concerned with the problem of a solitary wave moving with constant form and constant velocity c on the surface of an incompressible, inviscid fluid over a horizontal bottom. The motion is assumed to be two-dimensional and irrotational, and if h is the depth of the fluid at infinity and g the acceleration due to gravity, then the Froude number F is defined by

$$F^2 = c^2/gh.$$

The result that $F > 1$ has recently been proved by Amick and Toland by means of a long and complicated argument. Here we give a short and simple one.

AMS (MOS) Subject Classifications: 35R35, 76B15, 76B25

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SIGNIFICANCE AND EXPLANATION

This paper is concerned with the behavior of a solitary wave moving with constant form and constant velocity c on the surface of an imcompressible, inviscid fluid over a horizontal bottom. The motion is assumed to be two-dimensional and irrotational, and if h is the depth of the fluid at infinity and g the acceleration due to gravity, then the Froude number F is defined by

$$F^2 = c^2/gh.$$

It has long been believed that it is necessary for the existence of such a wave that $F > 1$, but the only existing proof is long and complicated. The present paper gives a short and simple proof.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

THE FROUDE NUMBER FOR SOLITARY WAVES

J. B. McLeod

1. Introduction

The purpose of this paper is to give a very short and simple proof of a result in the theory of solitary waves for which the only existing proof is both complicated and lengthy. We are concerned with the problem of a solitary wave moving with constant form and constant velocity on the surface of an incompressible, inviscid fluid over a horizontal bottom. The motion is two-dimensional, and if we restrict ourselves to irrotational flow and assume that the form of the wave is symmetrical about the crest with the height steadily decreasing on either side of the crest, then it is known that the shape of the wave can be described by a solution of the equation

$$(1.1)\quad \theta(s) = \frac{1}{3\pi}\int_{-\pi}^{\pi} \frac{\sec\frac{1}{2}t\,\sin\theta(t)}{\nu^{-1}+\int_0^t \sec\frac{1}{2}u\,\sin\theta(u)du}\left\{\sum_{k=1}^{\infty}\frac{\sin ks\,\sin kt}{k}\right\}dt.$$

This equation is derived by Amick and Toland [1] from the original formulation of the problem as a free boundary problem (we will return to this formulation later), and it is also to be found in the book by Milne-Thomson [2]. It can be obtained by mapping the region under the wave conformally onto the unit disc cut along the negative real axis. The generic point on the circumference of the disc is e^{is}, with $-\pi < s < \pi$, negative values of s corresponding to the right-hand half of the wave and positive values to the left-hand half, and $\theta(s)$ gives the angle between the wave surface and the horizontal at the point on the surface which corresponds to the point e^{is} on the circumference of the disc. The constant ν is given by

$$\nu = \frac{3ghc}{\pi Q^3},$$

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where g is the acceleration due to gravity, h the depth of the fluid infinitely far from the crest, c the speed at which the wave form is progressing, and Q the speed of particles at the crest of the wave. In obtaining (1.1) it is assumed (as we have already mentioned) that the wave is symmetrical about the crest, and this is reflected in the fact that (1.1) certainly implies that $\theta(-s) = -\theta(s)$. Using this we can restrict attention to the interval $[0,\pi]$ and sum the series in (1.1) to give the equation

$$(1.2)\quad \theta(s) = \frac{1}{3\pi}\int_0^{\pi} \frac{\sec\frac{1}{2}t\,\sin\theta(t)}{\nu^{-1} + \int_0^t \sec\frac{1}{2}u\,\sin\theta(u)du}\,\log\frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)}\,dt\ .$$

We have already said that we are interested in waves that decrease steadily from the crest, so that $\theta(s) > 0$ for $s > 0$ (which corresponds to the left-hand half of the wave), and Amick and Toland have proved the existence of non-negative solutions of (1.1). A relevant parameter in the problem is the Froude number F, defined by

$$F^2 = c^2/gh,$$

and for their solutions Amick and Toland are able to show relatively easily that $F > 1$. Proof exist in the literature [3], [4], which purport to show that in fact $F > 1$, but Amick and Toland dismiss these on the grounds that they assume that the total mass of the fluid (with the height measured relative to the height at infinity) is finite. Indeed, for completeness, they give such a proof themselves, but they then take some sixteen pages and much complicated estimating of integrals to prove $F > 1$ without the assumption of finite mass.

In their criticism they are being somewhat unjust to the earlier authors (and also to themselves), for it is possible, essentially with just a couple of remarks, to adapt these proofs so that they prove not only that $F > 1$ without the added assumption of finite mass, but also that the added assumption is in any case necessarily satisfied. Our object is to give such an adapted proof.

2. Proof that $F > 1$

While Amick and Toland's long proof is based on the equation (1.1), the short proofs all make use of the free boundary formulation of the problem, and this we now give.

If we take a frame of reference moving with the velocity of the solitary wave, the flow is steady and occupies a fixed region. The free surface is stationary and the velocity of the flow as $x \to \pm\infty$ is $-c$. (We take the x-axis along the horizontal bottom and the y-axis through the crest.) The free surface of the fluid (which is apriori unknown) is given by the equation

$$y = H(x).$$

In accordance with the assumptions that lead to (1.1), $H(x)$ is an even function, strictly decreasing for $x > 0$, and we known that

$$H(x) \to h \quad \text{as} \quad x \to \pm\infty. \tag{2.1}$$

Since the flow is incompressible and irrotational, there exists a complex potential

$$w = \phi + i\psi$$

in the space occupied by the fluid, ϕ being the velocity potential and ψ the stream function, and $\phi + i\psi$ an analytic function of $z = x + iy$. Further, the velocities u, v in the directions of the coordinate axes are given by

$$(u,v) = (-\phi_x,-\phi_y) = (-\psi_y,\psi_x),$$

or

$$u - iv = -\frac{dw}{dz}.$$

The boundary conditions at infinity are

$$u(x,y) \to -c, \; v(x,y) \to 0, \quad \text{as } x \to \pm\infty. \tag{2.2}$$

The horizontal bottom and the free surface are stream lines, and so are curves $\psi =$ constant. We can normalise ψ so that the bottom is $\psi = 0$, and since

$$\psi_y(x,y) = \phi_x(x,y) \to c \quad \text{as} \quad x \to \infty,$$

we see that the free surface is given by $\psi = ch$.

The pressure $p(x,y)$ is given by

$$p = \frac{1}{2}c^2 - \frac{1}{2}(u^2+v^2) - g(y-h), \tag{2.3}$$

and Bernoulli's theorem states that this has to be a constant on the free surface. The

behavior as $x \to \pm\infty$ shows that this constant is in fact zero. The problem thus reduces to finding an analytic function w in the space occupied by the fluid, such that the above boundary conditions hold on the bottom, on the free surface, and at infinity. There are of course two boundary conditions on the free surface, that $\psi = ch$ and that $p = 0$, and it is the existence of this extra condition that in effect enables the free surface to be determined.

With this formulation we now want to show that $F > 1$. Specifically, we prove the following

Theorem. *Let* $w = \phi + i\psi$ *be a function of* $z = x + iy$, *analytic for* $0 < y < H(x)$, $-\infty < x < \infty$, *and continuously differentiable for* $0 \leq y \leq H(x)$, *where* $H(x)$ *is a non-constant function decreasing for* $x > 0$ *and increasing for* $x < 0$. *Suppose also that (2.1) holds, that (2.2) holds boundedly in* y, *where* $u - iv = -dw/dz$, *and that* $\psi(x,0) = 0$, $\psi(x,H(x)) = ch$, $p(x,H(x)) = 0$, *where* p *is given by (2.3). Then*

$$F^2 = c^2/gh > 1,$$

and

$$\int_{-\infty}^{\infty} \{H(x) - h\}\, dx < \infty.$$

(It should be remarked that all the hypotheses of the theorem are satisfied for the solutions whose existence Amick and Toland prove. This is either explicit in their Theorem 1.1 or readily derived from it.)

Proof. From (2.3) we have

$$p_x = -uu_x - vv_x = -\mathrm{div}(u^2, uv),$$

and if we integrate this over the region

$$\{(x',y') : x \leq x' \leq M,\ 0 \leq y' \leq H(x')\},$$

we have

$$\int_0^{H(x)} \{p(x,y) + u^2(x,y)\}\, dy = \int_0^{H(M)} \{p(M,y) + u^2(M,y)\}\, dy .$$

(We have used the facts that $p = 0$ in the free surface and that the free surface is a stream line, so that the normal component of the velocity vanishes on it.) Substituting for $p(M,y)$ from (2.3) and letting $M \to \infty$, we thus obtain

$$\int_0^{H(x)} \{p(x,y) + u^2(x,y)\}\, dy = \frac{1}{2} \int_0^{H(M)} \{u^2(M,y) - v^2(M,y) - 2g(y-h) + c^2\}\, dy$$

$$+ \int_0^h \{c^2 - g(y-h)\}\, dy,$$

so that, for all x,

$$\int_0^{H(x)} \{p(x,y) + u^2(x,y)\}\, dy = \int_0^h \{c^2 - g(y-h)\}\, dy.$$

Now integrate with respect to x over $(-M,M)$, and we have

$$\int_{-M}^{M} \int_0^{H(x)} \{p(x.y) + u^2(x,y) + g(y-h) - c^2\}\, dy\, dx$$

$$= \int_{-M}^{M} \int_h^{H(x)} \{g(y-h) - c^2\}\, dy\, dx \tag{2.4}$$

$$= \frac{1}{2} g \int_{-M}^{M} \{H(x) - h\}^2 dx - c^2 \int_{-M}^{M} \{H(x) - h\}\, dx.$$

Again from (2.3) we see that

$$\frac{\partial}{\partial y} \{yp + gy(y-h)\} = v^2 + p + g(y-h) - \operatorname{div}(yuv, yv^2).$$

Hence

$$\int_{-M}^{M} \int_0^{H(x)} \{v^2+p+g(y-h)\}\, dy\, dx = \int_{-M}^{M} H(x)\{p(x,H(x)) + g(H(x)-h)\}dx \tag{2.5}$$

$$+ \int_0^{H(M)} yu(M,y)v(M,y)dy - \int_0^{H(-M)} yu(-M,y)v(-M,y)\, dy,$$

and if we recall that $p(x,H(x)) = 0$, we can add (2.4) and (2.5) to obtain (using the definition of p in (2.3)) that

$$0 = \frac{3}{2} g \int_{-M}^{M} \{H(x) - h\}^2 dx + (gh - c^2) \int_{-M}^{M} \{H(x) - h\}\, dx$$

$$+ \int_0^{H(M)} yu(M,y)v(M,y)dy - \int_0^{H(-M)} yu(-M,y)v(-M,y)\, dy. \tag{2.6}$$

The last two terms on the right of (2.6) tend to zero as $M \to \infty$ by the boundary conditions (2.2), and $H(x) > h$ for all x. Hence if $gh > c^2$ we can let $M \to \infty$ and must conclude that

$$\int_{-M}^{M} \{H(x) - h\}^2 \, dx \to 0,$$

which implies the trivial solution $H(x) = h$ in which we are not interested. Thus $gh < c^2$, or $F > 1$, as required.

3. Proof that the mass is finite

Suppose for contradiction that the mass is infinite, so that

$$\int_{-M}^{M} \{H(x) - h\} \, dx \to \infty \quad \text{as} \quad M \to \infty.$$

Since $H(x) - h \to 0$ as $x \to \pm\infty$,

it is immediate that

$$\int_{-M}^{M} \{H(x) - h\}^2 dx = o\left\{ \int_{-M}^{M} \{H(x)-h\} \, dx \right\},$$

and so the right-hand side of (2.6) tends to $-\infty$ as $M \to \infty$, since $gh-c^2 < 0$. This gives the required contradiction.

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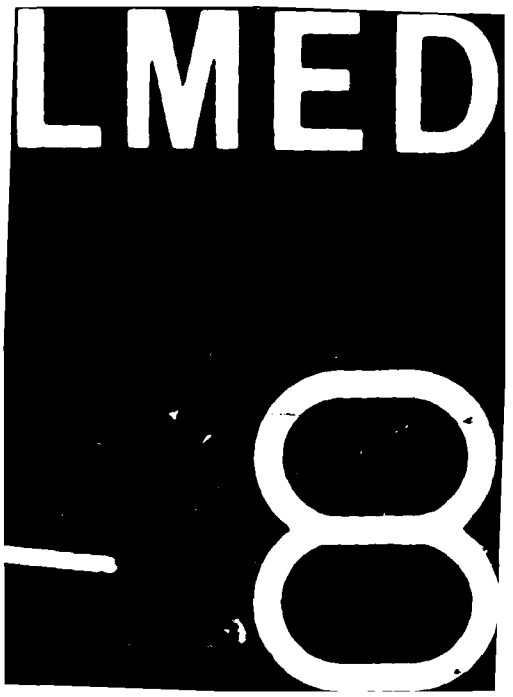
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