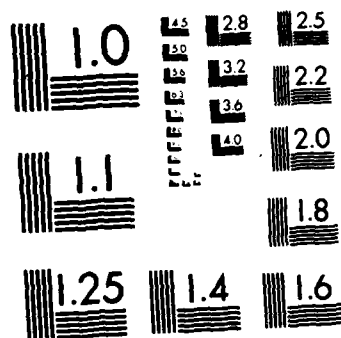AD-A134 255   ESTIMATING CRITICAL PATH AND ARC PROBABILITIES IN   1/1
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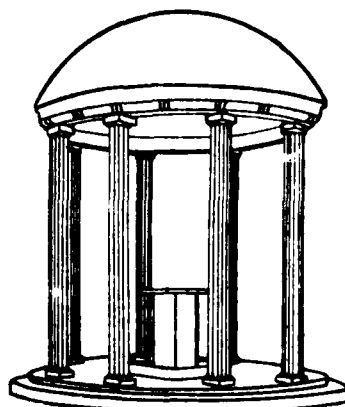IN STOCHASTIC ACTIVITY NETWORKS

George S. Fishman

Technical Report No. UNC/ORSA/TR-83/5

August 1983

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Abstract

This paper describes a new procedure for estimating parameters of a stochastic activity network of N arcs. The parameters include the probability that path m is the longest path, the probability that path m is the shortest path, the probability that arc i is on the longest path and the probability that arc i is on the shortest path. The proposed procedure uses quasirandom points together with information on a cutset H of the network to produce an upper bound of $O((\log K)^{N-|H|+1}/K)$ on the absolute error of approximation where K denotes the number of replications. This is a deterministic bound and is more favorable than the convergence rate of $1/K^{1/2}$ that one obtains for the standard error for K independent replications using random sampling. It is also shown how series reduction can improve the convergence rate by reducing the exponent on $\log K$. The technique is illustrated using a Monte Carlo sampling experiment for a network of 16 relevant arcs with a cutset of $H=7$ arcs. The illustration shows the superior performance of using quasirandom points with a cutset (plan A) and the even better performance of using quasirandom points with the cutset together with series reduction (plan B) with regard to mean-square error. However, it also shows that computation time considerations favor plan A when K is small and plan B when K is large.

KEY WORDS

Longest path

Monte Carlo methods

Network analysis

Quasirandom points

Shortest path

Introduction

Although comprehensive procedures have been developed for analyzing an activity network with deterministic arc completion times (e.g., Elmaghraby 1977), no comparable set of comprehensive procedures exists for a network in which the arc completion times are random. In particular, a stochastic activity network is properly characterized by distributional information. For example, on independent passes through such a network from source to sink, the network completion times on each pass usually differ, as do the paths identified as the critical paths. Here, the distribution function (d.f.) of network completion time and the frequency with which each path is the critical path provide useful characterizations for describing network behavior.

Whereas conventional search algorithms have been the principal tools of analysis for the deterministic case, Monte Carlo methods have been the principal tool for the stochastic case. For example, see Elmaghraby (1977) and Sullivan, Hayya and Schaul (1982). However, the performance of the Monte Carlo method has not been exceptional until recently when Sigal, Pritsker and Solberg (1979) showed how one could use information on a cutset of a network to improve the accuracy of estimate for a given cost. More recently, Fishman (1983) showed how the cutset information can be used together with quasirandom points and series reduction to estimate the d.fs. of network completion time and shortest path time with upper bounds on the errors of estimate that converge more rapidly than $O(1/K^{1/2})$, the rate for the standard error in a Monte Carlo experiment using random sampling on K independent replications.

The purpose of this paper is to extend this combined cutset/quasirandom points/series reduction approach to the estimation of the probability that path m is the longest path, the probability that path m is the shortest

path, the probability that arc i is on the longest path and the probability that arc i is on the shortest path. Section 1 introduces the necessary notation. Section 2 describes the use of <u>cutset</u> information, as proposed by Sigal, Pritsker and Solberg (1979), to estimate a parameter by Monte Carlo methods with less variance than would obtain for the same number of replications when using crude Monte Carlo techniques. Section 3 then describes how one can use <u>quasirandom points</u> with the cutset information on each replication to make the error of estimate converge more rapidly than in the case of random sampling. Section 4 extends the analysis to the remaining estimators of interest. Section 5 describes how <u>series reduction</u> in the network can work to advantage by reducing the number of dimensions sampled. Section 6 then describes an experimental design for evaluating the benefits to be derived from using cutset information together with quasirandom points and series reduction. Section 7 presents an example which assesses the benefits of each sampling plan versus its relative computation time.

1. <u>Definitions</u>

Consider an acyclic directed network with a single source, single sink, N arcs and L paths. Let $X_1,\ldots,X_N$, the passage times for arcs $1,\ldots,N$, be independent random variables where X_i has distribution function (d.f.) F_i on $[0,\infty)$ and inverse distribution function $G_i(u) = \min[x: F_i(x) \geq u$, $0 \leq u < 1]$. The completion time of path m is

$$T_m = \sum_{i=1}^{N} a_{im}X_i = \sum_{i \in I_m} X_i \tag{1}$$

where $a_{im}=1$ if arc i is on path m, $a_{im}=0$ otherwise and I_m denotes the set of arcs on path m. Consider the network completion time

$$T^* = \max(T_1,\ldots,T_L) \tag{2}$$

and the shortest path time

$$T_* = \min(T_1,\ldots,T_L) \quad . \tag{3}$$

Characterizations of the network include:

a. $\mathrm{pr}(T^* \le t)$ $\qquad\qquad$ $0 \le t < \infty$

b. $E\, T^*$

c. $g_m = \mathrm{pr}(m$ is the longest path) $\qquad$ $m=1,\ldots,L$

d. $r_i = \mathrm{pr}$ (arc i is on the longest path) $\quad$ $i=1,\ldots,N$

e. $\mathrm{pr}(T_* \le t)$ $\qquad\qquad$ $0 \le t < \infty$

f. $E\, T_*$

g. $h_m = \mathrm{pr}(m$ is the shortest path) $\qquad$ $m=1,\ldots,L$

h. $s_i = \mathrm{pr}(\mathrm{arc}\ i$ is on the shortest path) $\quad$ $i=1,\ldots,N$.

Fishman (1983) describes a highly efficient method of estimating a, b, e and f based on cutset information, quasirandom points, and series reduction. The present paper extends the method to the estimation of the probabilities $\{g_m ; m=1,\ldots,L\}$, $\{h_m; m=1,\ldots,L\}$, $\{r_i ; i=1,\ldots,N\}$ and $\{s_i; i=1,\ldots,N\}$.

2. Cutsets

Observe that

$$g_m = \mathrm{pr}[T_m > T_n; n \ne m, n=1,\ldots,L] \quad m=1,\ldots,L \quad . \tag{4}$$

Because of the shared arcs among paths, $T_1,\ldots,T_L$ are usually not independent and therefore g_m generally has no readily accessible analytical form. However, one can make use of the cutset information as in Sigal, Pritsker and Solberg (1979) to derive a useful expression for estimating g_m .

Let H denote a cutset of the network. As used here, a cutset denotes a set of arcs that connects a set of nodes W containing the source with

a set of nodes $\bar{\omega}$ containing the sink. Also assume that each path has only one arc in H. If each arc in H points from ω to $\bar{\omega}$, H is called a <u>uniformly</u> <u>directed</u> <u>cutset</u> (UDC). We assume that H is a UDC.

Let

$$S_m = T_m - X_{i_m} \qquad\qquad m=1,\ldots,L$$

and

$$Y_i = \sup_{\substack{m=1,\ldots,L \\ a_{i_m}=1}} S_m \qquad\qquad i \in H \qquad\qquad (5)$$

where $i_m = I_m \cap H$ denotes the arc on path m that is in the cutset. Then $X_i + Y_i$ is the maximal time of all paths that use arc i $i \in H$ and

$$\text{pr}(m \text{ is the longest path} | S_1, \ldots, S_L)$$

$$= \int_0^\infty \left[\pi_{\substack{i \in H \\ i \neq i_m}} F_i(t-Y_i) \right] dF_{i_m}(t-S_m) \qquad \text{if} \quad Y_{i_m} = S_m \qquad (6)$$

$$= 0 \qquad\qquad\qquad\qquad \text{otherwise.}$$

Therefore,

$$A_m = \pi_{\substack{i \in H \\ i \neq i_m}} F_i(T_m - Y_i) \qquad \text{if} \quad T_m = Y_{i_m} + X_{i_m} \qquad (7)$$

$$= 0 \qquad\qquad\qquad \text{otherwise}$$

has expectation g_m. Let the subscript j denote replication j. Then for K replications

$$\hat{g}_{mK} = \frac{1}{K} \sum_{j=1}^{K} \delta(T_{mj} - Y_{i_m j} - X_{i_m j}) \pi_{\substack{i \in H \\ i \neq i_m}} F_i(T_{mj} - Y_{ij}) , \qquad (8)$$

where

$$\delta(x) = 1 \qquad\qquad \text{if } x=0$$

$$= 0 \qquad\qquad \text{otherwise,}$$

is an unbiased estimator of g_m . If the replications are statistically independent, then var $\hat{g}_{mK}$ is proportional to $1/K$.

In principle, (8) is the estimator of g_m that Sigal, Pritsker and Solberg propose. They note that it should give smaller variance for K independent replications than the more conventional estimator

$$\tilde{g}_{mK} = \frac{1}{K} \sum_{j=1}^{K} \delta(T_j^* - T_{mj})$$

where

$$T_j^* = \max(T_{1j}, \ldots, T_{Lj}) .$$

However, both var $\hat{g}_{mK}$ and var $\tilde{g}_{mK}$ converge as $1/K$. In this paper we propose the use of _quasirandom_ _points_ to speed up the convergence of the error of estimate. See Niederreiter (1978) for a comprehensive review of the associated theory.

3. Quasirandom Points

Suppose one wants to estimate the multivariable integral

$$\theta = \int_0^1 \ldots \int_0^1 f(x_1, \ldots, x_N) \, dx_1 \ldots dx_N \tag{9}$$

by the summation

$$\theta_K = \frac{1}{K} \sum_{j=1}^{K} f(u_{1j}, \ldots, u_{Nj})$$

where $\{u_{1j}, \ldots, u_{Nj}: j=1, \ldots, K\}$ is a sequence of points uniformly distributed in the N-dimensional unit hypercube $\mathcal{Q}_N = [0,1]^N$. Let $A(R;K)$ denote the number of points $\underset{\sim}{u}_j = (u_{1j}, \ldots, u_{Nj})$ $j=1, \ldots, K$ that fall in the region $R \subseteq \mathcal{Q}_N$. Then $\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_K$ are said to be uniformly distributed in $\mathcal{Q}_N$ if

$$\lim_{K \to \infty} \frac{A(R;K)}{K\lambda(R)} = 1$$

for all $R = \{(x_1, \ldots, x_N): \alpha_i \leq x_i \leq \beta_i ; i=1, \ldots, N\}$. Here $\lambda(R)$ is the

measure of volume of R in Q_N . We define the <u>extreme</u> <u>discrepancy</u> of $u=\underset{\sim}{u}_1,\ldots,\underset{\sim}{u}_K$ as

$$D_K = D_K(N,u) = \sup_{R \subseteq Q_N} \left| \frac{A(R;K)}{K} - \lambda(R) \right| \tag{10}$$

for $R=\{(x_1,\ldots,x_N): 0 \le x_i \le \beta_i \; ; \; i=1,\ldots,N\}$. If f is of bounded variation in the sense of Hardy and Krause, then

$$|\theta_K - \theta| < V(f,N)D_K . \tag{11}$$

where $V(f,N)$ is a function of the bounded variation of f in N and lower dimensions. Most importantly, uniform sequences exist for which

$$D_K \le O((\log K)^N/K) . \tag{12}$$

This is an impressive result, for in the special case in which $\{u_{ij}: i=1,\ldots,N; \; j=1,\ldots,K\}$ is a sequence of i.i.d. random variables uniformly distributed on $[0,1]$, one has only

$$\lim_{K \to \infty} \sup \frac{\sqrt{2K} \, D_K}{\sqrt{\log \log K}} = 1 \qquad \text{w.p. 1,} \tag{13}$$

a result due to Chung (1949) for $N=1$ and to Kiefer (1961) for $N>1$.

Halton (1960) describes a uniformly distributed sequence for which (12) holds. If $R \ge 2$ is an integer, then every non-negative integer n has an expansion of the form

$$n = \sum_{i=0}^{m} a_i R^i \qquad \begin{aligned} &a_i \in \{0,1,\ldots,R-1\} \\ &0 \le i \le m \text{ and } m=\lfloor \log_R n \rfloor . \end{aligned} \tag{14}$$

Corresponding to this expansion one has the <u>radical inverse</u> function

$$\phi_R(n) = \sum_{i=0}^{m} a_i R^{-i-1} . \tag{15}$$

Then Halton shows that the sequence $\{\phi_{R_1}(j),\ldots,\phi_{R_N}(j); \; j=0,1,\ldots,K-1\}$, where $R_1,\ldots,R_N$ are pairwise relatively prime, gives the bound (12).

We now specialize these results to the network analysis at hand.

Define for $j=1,\ldots,K$

$$X_{ij} = G_i(u_{ij}) \qquad\qquad i=1,\ldots,N \qquad\qquad (16)$$

$$T_{mj} = \sum_{i=1}^{N} a_{im} X_{ij} \qquad\qquad m=1,\ldots,L \qquad\qquad (17)$$

$$Y_{ij} = \sup_{\substack{m=1,\ldots,L \\ a_{im}=1}} T_{mj} - X_{i_m j} \qquad i \in H \quad . \qquad\qquad (18)$$

Then it is not difficult to see that $\cdot$ can write g_m as a
$(N-|H|+1)$-dimensional integral over $($ $'_{|H|+1}$ with integrand of bounded
variation and that (8), using (16), ('d (18), is an approximation
to this integral. As a result, one has for a given network using the
Halton sequence the deterministic bound

$$|\hat{g}_{mK} - g_m| \leq O((\log K)^{N-|H|+1}/K) \quad , \qquad\qquad (19)$$

which makes clear the desirability of choosing as H the cutset of
maximal cardinality.

4. Estimating $\{r_i\}$, $\{h_m\}$ and $\{s_i\}$

This section describes estimators analogous to (8) for

$$r_i = \text{pr (arc } i \text{ is on the longest path)} \quad i=1,\ldots,N \qquad\qquad (20)$$

$$h_m = \text{pr (}m \text{ is the shortest path)} \qquad m=1,\ldots,L \qquad\qquad (21)$$

and

$$s_i = \text{pr (arc } i \text{ is on the shortest path)} \quad i=1,\ldots,N. \qquad\qquad (22)$$

Before doing so, we explain the motivation for studying $\{r_i\}$ and $\{s_i\}$.
If, say, r_i turns out to be considerably greater than the remaining arc
probabilities on the longest path, then there is good reason to believe
that the completion time associated with arc i is a principal determinant

of the longest path time. This knowledge of the relative importance of arc i may lead to a modification of the activity associated with this arc for the purpose of shortening the longest path time. Analogous considerations apply to $\{s_i\}$. Since

$$r_i = \sum_{m=1}^{L} a_{im} g_m \qquad (23)$$

one has the estimator

$$\hat{r}_{iK} = \sum_{m=1}^{L} a_{im} \hat{g}_{mK} . \qquad (24)$$

Moreover, it is easily seen that $|\hat{r}_{iK} - r_i| \le O(\log K)^{N-|H_i|+1}/K)$ for the Halton sequence.

The estimation of $\{h_m\}$ follows closely that for $\{g_m\}$. Let

$$Z_i = \inf_{\substack{m=1,\ldots,L \\ a_{im}=1}} S_m \qquad i \in H \qquad (25)$$

so that $X_i + Z_i$ gives the minimal time of all paths that use arc $i \in H$. Then

pr(m is the shortest path $|S_1,\ldots,S_L)$

$$= \int_0^{\infty} \{\prod_{\substack{i \in H \\ i \ne i_m}} [1-F_i(t-Z_i)]\} dF_{i_m}(t-S_m) \qquad \text{if} \quad Z_{i_m} = S_m$$

$$= 0 \qquad \qquad \text{otherwise.}$$

Again, let the subscript j denote replication j . Also let

$$B_{mj} = \prod_{\substack{i \in H \\ i \ne i_m}} [1-F_i(T_{mj}-Z_{ij})] \qquad \text{if} \quad T_{mj} = Z_{i_m j} + X_{i_m j}$$

$$= 0 \qquad \qquad \text{otherwise.}$$

Then

$$\hat{h}_{mK} = \frac{1}{K} \sum_{j=1}^{K} \delta(T_{mj} - Z_{i_m j} - X_{i_m j}) B_{mj} \tag{26}$$

is our estimator of h_m and again for the Halton sequence one has
$|\hat{h}_{mK} - h_m| \leq O((\log K)^{N-|H|+1}/K)$.

For $\{s_i; i=1,\ldots,N\}$ note that

$$s_i = \sum_{m=1}^{L} a_{im} h_m \tag{27}$$

so that our estimator has the form

$$\hat{s}_{iK} = \sum_{m=1}^{L} a_{im} \hat{h}_{mK} \quad , \tag{28}$$

with error $|\hat{s}_{iK} - s_i| \leq O((\log K)^{N-|H|+1}/K)$ for the Halton sequence.

5. Series Reduction

As shown in Section 2, the rate of convergence of the error of approximation (10) is bounded by a function $O((\log K)^N/K)$. Using the cutset H enables one to bound this convergence more tightly by $O((\log K)^{N-|H|+1}/K)$ and, presumably, using the cutset H with maximal cardinality is best. In fact, any preanalysis that leads to a smaller exponent for log K is desirable from the viewpoint of a tighter error bound. The technique of series reduction is one such approach to further reduction.

Figure 1 shows an activity network due to Battersby (1970) for the partial overhaul of a unit in an oil refinery. Table 1 shows the corresponding incidence matrix. Note that arcs 17 and 18 are both on all eight paths.

Insert Fig. 1 and Table 1 about here.

Therefore, we can ignore these arcs and reduce the number of relevant arcs for analysis from 18 to N=16 arcs without any effect on our results.

Observe that e_2, e_3, e_5, e_6, e_9, e_{10} and e_{12} form a cutset H for the network. Since $N=16$ arcs and $|H|=7$ the exponent for log K is $N-|H|+1=10$. Now note that arcs e_1 and e_4 occur only on paths 1 and 2, e_6 and e_{15} occur only on path 5, e_7 and e_{16} occur only on paths 6, 7 and 8 and e_8 and e_{11} occur only on paths 7 and 8. Suppose we now replace these arc pairs by single arcs e_1+e_4, e_6+e_{15}, e_7+e_{16} and e_8+e_{11}. Then the network has $N'=12$ arcs and the cutset H' with e_2, e_3, e_5, e_6+e_{15}, e_9, e_{10} and e_{12} has cardinality 7 so that $N'-|H'|+1=6$ is the resulting exponent for log K in the bound on $|\hat{g}_{mK}-g_m|$ and $|\hat{h}_{mK}-h_m|$. Note that if one adopts this series reduction it is not possible to estimate r_i and s_i for those original arcs in the reduction.

The feasibility of implementing this series reduction in practice depends on whether or not the d.fs. of the summed arc times are convenient for computation and on the relative computation costs with and without series reduction. Suppose $W_1,\ldots,W_n$ are independent exponentially distributed random variables with distinct rates $\beta_1,\ldots,\beta_n$. Then for $n>1$ $V_n=W_1+\ldots+W_n$ has the d.f.

$$F_{V_n}(t) = 1- \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j\neq i}}^{n} (1 \; \beta_j/\beta_i)^{-1} e^{-\beta_i t} . \tag{29}$$

For summed arcs in the cutset one need only replace their individual d.fs. in (8) and (26) by the corresponding d.f. from (29). For summed arcs not in the cutset one needs to solve the equation

$$v = F_{V_n}(T') \tag{30}$$

for T' , v being the appropriate element of the Halton sequence. This can be done readily by the Newton-Raphson method. A suggested starting point for the iteration is

$$T_0' = -[\ln(1-v)]\sum_{i=1}^{n} \frac{1}{\beta_i} \quad . \tag{31}$$

We evaluate this approach in our example where we study the relative benefits of the cutset information, quasirandom points and series reduction. As we show, the relative benefit of series reduction becomes more apparent as the sample size K increases. In fact, for small K series reduction may be less preferable.

6. Experimental Design

This section describes the layout for an experiment designed to determine the extent to which quasirandom points lead to accelerated accuracy when estimating $\{g_m\}$ and $\{h_m\}$ for the network in Fig. 1. In particular, we introduce a degree of randomness into the experiment in order to compute estimates of the mean-square errors of our point estimates. We then study the behavior of these sample mean-square errors as K increases. Note that we choose this method of evaluating our proposed method merely because it is necessary to find a common basis for comparing results based on a deterministic method with baseline results using random sampling.

Consider an experiment consisting of Q statistically independent blocks or macroreplications each of K microreplications. Let $\{v_{im}:$ i=1,...,N; m=1,...,Q}$ denote a sequence of i.i.d. random variables from $U[0,1)$ and define

$$C_{im} = (c: \phi_{R_i}(c) = v_{im}) \quad i=1,...,N \quad m=1,...,Q \quad . \tag{32}$$

Then on macroreplication m we use the quasirandom point

$\phi_{R_1}(C_{1m} + j-1),\ldots,\phi_{R_N}(C_{Nm} + j-1)$ on microreplication j for $j=1,\ldots,K$.

Let $\hat{\theta}_{jm}$ denote an estimate of a particular quantity θ

computed on microreplication j on macroreplication m . Then

$\bar{\theta}_1,\ldots,\bar{\theta}_Q$, where

$$\bar{\theta}_m = \frac{1}{K} \sum_{j=1}^{K} \hat{\theta}_{jm} \qquad m=1,\ldots,Q , \qquad (33)$$

are i.i.d. random variables with sample mean-square error

$$s_K^2 = \frac{1}{Q-1} \sum_{m=1}^{Q} (\bar{\theta}_m - \bar{\bar{\theta}}_Q)^2 \qquad\qquad (34)$$

where

$$\bar{\bar{\theta}}_Q = \frac{1}{Q} \sum_{m=1}^{Q} \bar{\theta}_m .$$

It is the behavior of s_K^2 versus K that interests us. Note again that
the only reason for randomizing the seeds $\{v_{im} ; i=1,\ldots,N\}$ on each of
the Q macroreplications is to enable us to derive the estimates of
mean-square error.

To provide an instructive normalization we also run M statistically
independent macroreplications each of $K=1$ microreplication. Let $\hat{\theta}_1,\ldots,\hat{\theta}_M$
denote the resulting estimates of θ each with sample mean-square error

$$w_M^2 = \frac{1}{M-1} \sum_{m=1}^{M} (\hat{\theta}_m - \tilde{\theta}_M)^2 \qquad\qquad (35)$$

where

$$\tilde{\theta}_M = \frac{1}{M} \sum_{m=1}^{M} \hat{\theta}_m .$$

Then the quantity $w_M^2 / K s_K^2$ should increase as K increases if accelerated
convergence is occurring.

7. Example

To illustrate the proposed procedures we use the network of Fig. 1 for which we estimate $\{g_m; m=1,\ldots,8\}$ and $\{h_m; m=1,\ldots,8\}$. Monte Carlo experiments were performed using quasirandom points and the cutset $H'=\{e_2, e_3, e_5, e_6 + e_{15}, e_9, e_{10}, e_{12}\}$ with

Plan A

No series reduction on the remaining arcs

$$\{e_1, e_4, e_7, e_8, e_{11}, e_{13}, e_{14}, e_{16}\}$$

and then with

Plan B

Series reduction on the remaining arcs

$$\{e_1 + e_4, e_7 + e_{16}, e_8 + e_{11}, e_{13}, e_{14}\}$$

Motivation for this choice of plans arose from the observation that plan A requires (29) to be evaluated once per microreplication whereas plan B additionally requires evaluation of (29) and its derivative iteratively (by the Newton-Raphson method) for each of the three reductions $e_1 + e_4$, $e_7 + e_{16}$ and $e_8 + e_{11}$. While it is true that plan B also requires three less quasirandom points per micro-replication, the relative cost for an iterative computation needs to be considered explicitly.

Expression (8) and (26) were used to estimate each g_m and h_m respectively. For the baseline case of pure random sampling g_m and h_m were estimated by

$$\tilde{g}_{mM} = \frac{1}{M} \sum_{j=1}^{M} \delta(T_j^\star - T_{mj})$$

and

$$\tilde{h}_{mM} = \frac{1}{M} \sum_{j=1}^{M} \delta(T_{\star j} - T_{mj}) \; ,$$

respectively, where

$$T_{*j} = min(T_{1j}, \ldots, T_{Lj})$$

and

$$T_j^* = max(T_{1j}, \ldots, T_{Lj}) .$$

Note that estimates of $\{r_i\}$ and $\{s_i\}$ could in principle also be computed at small marginal cost.

Let $w_M^2(\rho)/K$ and $s_K^2(\rho)$ denote the sample mean-square errors for the estimators of ρ . Then Table 2 lists for selected K the averages

$$\beta_h(W) = \frac{1}{8} \sum_{m=1}^{8} [w_M^2(h_m)/Ks_K^2(h_m)]$$

for the shortest path using sampling plans $W=A$ and $W=B$

Insert Table 2 about here.

---

in columns 3 and 4, respectively, and the averages

$$\beta_g(W) = \frac{1}{8} \sum_{m=1}^{8} [w_M^2(g_m)/Ks_K^2(g_m)]$$

for the longest path in columns 5 and 6 respectively.

These results are most encouraging for they indicate substantive increases in these ratios as the number of microreplications K increases. Also, they show the relative desirability of using series reduction as K increases. Note that computation time considerations limited the number of macroreplications for $K > 2^{11}$.

When evaluating a proposed accuracy accelerating technique one also needs to study its cost relative to the cost of crude Monte Carlo sampling. One way to make this assessment is in terms of the CPU times required per observation with each sampling plan. To simplify the discussion,

let us concentrate on the estimation of, say, g_m . Suppose that a
single observation using pure random sampling has mean-square error V .
Then over K_1 independent observations or microreplications the resulting
estimate of g_m has mean-square error $V_{K_1} = V/K_1$. Suppose that for K
observations of the quasirandom point approach the estimator of g_m has
mean-square error Ω_K . Let ω_W denote the ratio of cost per
microreplication for plan W using quasirandom points over the cost
per microreplication using independent random sampling. If we begin
by taking K observations using plan A and $K_1 = \omega_A K$ observations using
independent random sampling, then each experiment takes exactly the
same amount of time. Moreover, the ratio of their mean-square errors is

$$\frac{V_{K_1}}{\Omega_K} = \frac{V/K_1}{\Omega_K} = \frac{1}{\omega_A} \cdot \frac{V/K}{\Omega_K} \ .$$

Let $\hat{\theta}_i$ denote the value of the estimate on independent observation
$i=1,\ldots,M$. To estimate V we use w_M^2 so that w_M^2/K is the estimate of
mean-square error V/K for the case of pure random sampling for any K .
Also $\{s_K^2 : K=2^j ; j=7,\ldots,14\}$ are the estimates of mean-square errors
$\{\Omega_K\}$ for plan A for the selected values of K .

To estimate the average ratios of mean-square errors for equal run
times (using $K_1 = \omega_W K$) we use $\beta_h(W)/\omega_W$ and $\beta_g(W)/\omega_W$ for sampling plans
W=A,B . For the case of pure random sampling one microreplication
required an average of 7.37 microseconds on an IBM 4341/M2 whereas
plan A required an average of 12.25 microseconds per microreplication.
Plan B required an average of 14.85 microseconds per microreplication.
Therefore, $1/\omega_A = 7.37/12.25 = .60$ when comparing random sampling to
plan A and $1/\omega_B = 7.37/14.85 = .49$ when comparing random sampling to

plan B.   Columns 7 through 10 show $\beta_h(A)/\omega_A$, $\beta_h(B)/\omega_B$, $\beta_g(A)/\omega_A$

and $\beta_g(B)/\omega_B$, respectively for selected numbers of micro-

replications K .

These results are sobering for they show that the relative desirability

of series reduction in the noncutset arcs does not set in until $K=2^{11}$

for $\{h_m\}$ and for $\{g_m\}$. Let us now put this cost analysis

into perspective. Since ω_A and ω_B are essentially independent of K

and generally $\omega_B > \omega_A$, there is always a crossover point K* for each

network at which one prefers plan A to plan B whenever K < K* and

prefers plan B to plan A whenever K > K* . Since for an arbitrary network

one does not know K* , the choice between the two plans is not trivial.

Therefore, the advice to do series reduction whenever possible cannot be

accepted unqualifiedly unless the planned number of microreplications K
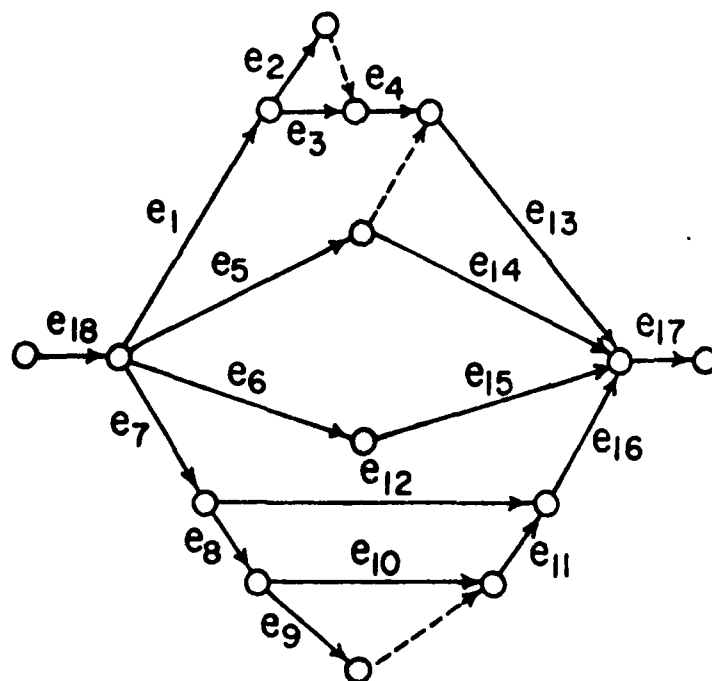
is substantially large.

Figure 1. Battersby Network

Table 1

Incidence Matrix for Battersby Network

a_{im}

Arc i	λ_i	Path m 1	2	3	4	5	6	7	8
e_1	16	1	1						
e_2	16	1							
e_3	8		1						
e_4	6	1	1						
e_5	16			1	1				
e_6	40					1			
e_7	24						1	1	1
e_8	16							1	1
e_9	16								1
e_{10}	24							1	
e_{11}	8							1	1
e_{12}	4						1		
e_{13}	36	1	1	1					
e_{14}	12				1				
e_{15}	8					1			
e_{16}	24						1	1	1
e_{17}	8	1	1	1	1	1	1	1	1
e_{18}	24	1	1	1	1	1	1	1	1

Table 2

Experimental Results

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(1) No. of macro-replications Q	(2) No. of micro-replications K	(3) $\beta_h(A)$	(4) $\beta_h(B)$	(5) $\beta_g(A)$	(6) $\beta_g(B)$	(7) $\beta_h(A)/\omega_A$	(8) $\beta_h(B)/\omega_B$	(9) $\beta_g(A)/\omega_A$	(10) $\beta_g(B)/\omega_B$
10^5	1	4.24	2.68	2.72	2.68	2.54	1.31	1.61	1.31
2^7	2^7	3.39	2.85	4.43	4.67	2.03	1.40	2.66	2.29
2^7	2^8	4.76	3.66	6.90	6.26	2.86	1.79	4.14	3.07
2^7	2^9	8.19	8.37	7.02	8.80	4.91	4.10	4.21	4.31
2^7	2^{10}	10.84	10.66	11.58	12.12	6.50	5.22	6.95	5.94
2^7	2^{11}	13.95	17.61	12.54	20.31	8.37	8.63	7.52	9.95
2^6	2^{12}	29.06	44.42	16.72	33.01	17.44	21.77	10.03	16.17
2^5	2^{13}	37.20	49.98	15.31	45.71	22.32	24.49	9.19	22.40
2^4	2^{14}	45.42	70.42	24.52	48.78	27.25	34.51	14.71	23.90

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This paper describes a new procedure for estimating parameters of a stochastic activity network of N arcs. The parameters include the probability that path m is the longest path, the probability that path m is the shortest path, the probability that arc i is on the longest path and the

probability that arc i is on the shortest path. The proposed procedure uses quasirandom points together with information on a cutset H of the network to produce an upper bound of $O((\log K)^{N-|H|+1}/K)$ on the absolute error of approximation where K denotes the number of replications. This is a deterministic bound and is more favorable than the convergence rate of $1/K^{1/2}$ that one obtains for the standard error for K independent replications using random sampling. It is also shown how series reduction can improve the convergence rate by reducing the exponent on $\log K$. The technique is illustrated using a Monte Carlo sampling experiment for a network of 16 relevant arcs with a cutset of $H=7$ arcs. The illustration shows the superior performance of using quasirandom points with a cutset (plan A) and the even better performance of using quasirandom points with the cutset together with series reduction (plan B) with regard to mean-square error. However, it also shows that computation time considerations favor plan A when K is small and plan B when K is large.

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