DRSMC/SA/MR-2

STATISTICAL PROPERTIES OF AN INTEGRATED STATIONARY STOCHASTIC PROCESS

GEORGE J. SCHLENKER

SEPTEMBER 1983

Distribution Statement

Approved for public release; distribution unlimited.

US ARMY ARMAMENT, MUNITIONS AND CHEMICAL COMMAND
SYSTEMS AND OPERATIONS ANALYSIS DIRECTORATE
ROCK ISLAND, ILLINOIS 61299

DISPOSITION

DISCLAIMER

WARNING

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER DRSMC/SA/MR-2	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Statistical Properties of an Integrated Stationary Stochastic Process		5. TYPE OF REPORT & PERIOD COVERED Report-Final 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) George J. Schlenker		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS US Army Armament, Munitions and Chemical Command Systems and Operations Analysis Directorate Rock Island, IL 61299		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS US Army Armament, Munitions and Chemical Command Systems and Operations Analysis Directorate Rock Island, IL 61299		12. REPORT DATE September 1983 13. NUMBER OF PAGES 45
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Inquiries may be forwarded to Commander, US Army Armament, Munitions and Chemical Command, ATTN: DRSMC-SAS (R), Rock Island, IL 61299. AUTOVON 793-5041/6370

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Statistics, Stochastic Processes, Time Series, Operations Research, Numerical Analysis, Correlation Analysis, Spectrum Analysis

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report examines some statistical properties of a first-order, stationary time series $\{x_t\}$ and of a series $\{y_t\}$ consisting of disjoint sums of $\{x_t\}$. A connection between continuous stochastic processes and discrete time series is made via the concept of correlation time interval. Expressions for the variances of the sum and average of $\{x_t\}$ for n terms are derived. An asymptotic variance estimate of the average is derived and is shown to be a reasonable upper bound. (Cont'd)

General expressions are derived for the variance and autocovariance of the integrated process $\{y_t\}$ in terms of the same properties for any stationary process $\{x_t\}$. These results are particularized for the case in which $\{x_t\}$ is first-order. Expressions for the spectral densities of x_t and y_t are given when $\{x_t\}$ is first-order. A computer program to calculate the autocovariance and autospectrum of y_t is attached.

DRSMC/SA/MR-2

STATISTICAL PROPERTIES OF AN INTEGRATED STATIONARY STOCHASTIC PROCESS

GEORGE J. SCHLENKER

September 1983

ABSTRACT

This report examines some statistical properties of a first-order, stationary time series $\{x_t\}$ and of a series $\{y_t\}$ consisting of disjoint sums of $\{x_t\}$. A connection between continuous stochastic processes and discrete time series is made via the concept of correlation time interval. Expressions for the variances of the sum and average of $\{x_t\}$ for n terms are derived. An asymptotic variance estimate of the average is derived and is shown to be a reasonable upper bound.

General expressions are derived for the variance and autocovariance of the integrated process $\{y_t\}$ in terms of the same properties for any stationary process $\{x_t\}$. These results are particularized for the case in which $\{x_t\}$ is first-order. Expressions for the spectral densities of x_t and y_t are given when $\{x_t\}$ is first-order. A computer program to calculate the autocovariance and autospectrum of y_t is attached.

EXECUTIVE SUMMARY

This report is addressed to analysts with a good background in statistics and with some exposure to time series analysis. Applications of time series are found in a host of fields including the physical and biological sciences, in economics, and in industrial processes. The principal products of this study are derivations of certain equations and numerical illustrations of their use. A computer program is provided to implement the calculations. These results relate statistical properties of one time series to another series whose elements are subsummed from those of the first. Special attention is given the case in which the first series is related to itself with first-order dynamics. The fluctuating behavior of the series studied is assumed to be steady-state. The study results can assist in building statistical models thru identification of proper functional forms and in parameter estimation.

CONTENTS

DRSMC-SAS (R) September 1983

MEMORANDUM REPORT

SUBJECT: Statistical Properties of an Integrated, Stationary Stochastic Process

1. Background

In some applications where a random time series is studied, the questions of interest are: What is the variance of the series sum? and What is the variance of the average? In this kind of application a member of the time series is viewed as the rate of change of the quantity of interest. Such might be the case, for example, in a series of production output rates. The focus in this case is on the output of the process over some long time interval. The latter is, of course, the integral (or numerical sum) of the rate over this time period. Perhaps one cannot be assured that the rates are statistically independent. Or perhaps -- as is often the case in a real manufacturing process -- the rates are known to be autocorrelated. In this case one might ask how precise is the estimate of the quantity of interest. An example of an autocorrelated industrial process of military interest is the production of batches of explosive Composition B [1]. Because of RDX-batch mixing operations, successive batches of Comp B exhibit batch viscosities which are not statistically independent.

Other applications may require the identification of an appropriate stochastic model for a time series $\{y_t\}$ subsummed from another $\{x_t\}$. Successive members of $\{y_t\}$ do not share members of $\{x_t\}$. For example, weekly totals are obtained from daily quantities. Questions of interest in this regard might be: How does the autocorrelation function of y_t compare with that of x_t? or How do the spectra of these related processes compare?

---

[1] "Stochastic Model of the Batch Viscosity Generation Process in Sequential Batches of Composition B ...", DRSAR/PE/N-88, (AD B057395), April 1981.

Paraphrasing, one might ask how independent are $\{y_t\}$ relative to $\{x_t\}$ or, relatively, how variable are the fluctuations of a particular duration in the two series. Another challenging problem arises when only data for subsummed series are available but one wishes to model $\{x_t\}$. An example of a stationary autocorrelated time series, in which subsummed data is often published, is found in ballistic tests of large caliber artillery, [2]. In some instances evidence of "ballistic memory" exists.

2. Objectives

One purpose of this report is to derive an expression for the variance of the integral of a stationary stochastic process which is taken to be first-order. This brief exposition will serve to quantify the effect of auto-correlation in the rate process on the precision of sums (or integrals) and averages of that process. Quantitative insights will be provided in the discussion. Another objective is to derive expressions for the autocorrelation function and the autospectrum of a first-order process $\{x_t\}$ and of the subsummed process $\{y_t\}$. These are compared in specific numerical examples.

3. Notation

The notational conventions of Jenkins and Watts [3] are followed here. Consider the time series $\{x_t\}$ with integer index t. The time interval separating successive members of the series is a constant Δ. If this

---

[2] "Statistical Evidence of a Memory Process in Cannon Artillery Weapons and Its Implication for Testing," DRSAR/SA/N-76, (AD C013498), January 1978.

[3] Jenkins, G.M. and Watts, D.G. Spectral Analysis and Its Applications, Holden-Day, San Francisco, c. 1969.

series is derived from a continuous process, Δ is the sampling interval. This process is assumed covariance stationary. Without loss of generality the mean of the process is taken as zero. Thus, with E denoting the ensemble expectation operator,

$$E[x_t] = 0 \quad . \tag{1}$$

The variance, $V[x_t]$, is denoted by $\gamma_{xx}(0)$.

$$\gamma_{xx}(0) = E[x_t^2] \quad . \tag{2}$$

The process studied satisfies the first-order difference equation:

$$x_t = bx_{t-1} + n_t \quad , \tag{3a}$$

with constant b and where

$\{n_t\}$ is a white noise.

Thus,

$$E[n_t] = 0 \tag{3b}$$

$$\gamma_{nn}(k) = E[n_t \; n_{t+k}] = 0 \quad , \quad k \neq 0 \quad , \tag{3c}$$

$$\gamma_{nn}(0) = \sigma^2 \quad . \tag{3d}$$

Notationally, let

$$\gamma_{xn}(k) = E[x_{t+k} n_t] \quad . \tag{4}$$

Since future values of the noise in (3a) are uncorrelated with x_t,

$$\gamma_{xn}(k) = 0 \quad , \quad k<0 \quad . \tag{5}$$

The autocorrelation of $\{x_t\}$ is denoted by $\rho_{xx}(k)$, where

$$\rho_{xx}(k) = \gamma_{xx}(k)/\gamma_{xx}(0) \quad . \tag{6}$$

4. <u>Autocorrelation of x_t</u>

From (3a),

$$E[x_{t+k}\, x_t] = E[(bx_{t+k-1} + n_{t+k})(bx_{t-1} + n_t)] \;, \qquad k \geq 1 \;.$$

Using the results of equations (3) with the above,

$$\gamma_{xx}(k) = \frac{b}{1-b^2}\, \gamma_{xn}(k-1) \;. \tag{7}$$

Clearly, for stability of the $\{x_t\}$ process $|b| < 1$.

By multiplying both sides of (3a) by n_{t-k} and taking expectations,

$$\gamma_{xn}(k) = b\, \gamma_{xn}(k-1) + \gamma_{nn}(k) \;. \tag{8}$$

Using (3c) and (8) ,

$$\gamma_{xn}(k) = b^k\, \gamma_{xn}(0) \;. \tag{9}$$

With (5), equation (8) implies that

$$\gamma_{xn}(0) = \gamma_{nn}(0) = \sigma^2 \;. \tag{10}$$

Hence, from (9)

$$\gamma_{xn}(k) = b^k\, \sigma^2 \;. \tag{11}$$

Finally, (7) and (11) yield

$$\gamma_{xx}(k) = \frac{b^k}{1-b^2}\, \sigma^2 \;, \tag{12}$$

from which

$$\rho_{xx}(k) = b^k \;, \tag{13a}$$

with the variance (σ_x^2) of $\{x_t\}$ given by

$$\gamma_{xx}(0) = \frac{1}{1-b^2}\, \sigma^2 \;. \tag{13b}$$

5. Correlation Time Interval

A concept of considerable significance in time series analysis is correlation time, τ^*. This notion is discussed at length by Fishman and Kiviat* in application to analysis of stochastic simulations. Any stationary stochastic process must be observed for a time interval of $2\tau^*$ in order to derive information that is equivalent to one observation of an uncorrelated random variable. Thus, the number of equivalent degrees of freedom obtained by observing a stationary process for time T is $T/(2\tau^*)$. For a continuous 1st-order process, i.e., one that is defined at every point in time -- say, z(t) -- the autocorrelation function is shown by Fishman and Kiviat to have the form

$$\rho_{zz}(t) = e^{-\lambda t} \quad , \qquad (14)$$

where λ is a rate constant. Now, by definition, for a continuous process

$$\tau_z^* = \int_0^\infty \rho_{zz}(t)\, dt \ . \qquad (15)$$

Then, for the first-order continuous stochastic process

$$\tau_z^* = 1/\lambda \quad . \qquad (16)$$

Similarly, for a discrete process, the correlation time is defined as

$$\tau_x^* = \Delta\Sigma_{k=0}^{\infty}\, \rho_{xx}(k) \quad , \qquad (17)$$

where neighboring members of $\{x_t\}$ are separated by duration Δ. Using (13a) and (17), the discrete first-order process has correlation time

$$\tau_x^* = \Delta/(1-b) \quad . \qquad (18)$$

It is worthwhile to compare the results of (16) and (18).
Suppose that the continuous process $\{z(t)\}$ is time-sampled with sampling interval Δ. In this case the process is observed only at times $t = k\Delta$, with k an integer.

---

* Fishman, G.S. and Kiviat, P.J. Spectral Analysis of Time Series Generated by Simulation Models, RM-4393-PR, Rand Corp., Santa Monica, CA, February 1965.

Then, the correlation function in (14) would be defined discretely as

$$\rho_{zz}(k) = b^k \qquad (19a)$$

with

$$b = e^{-\lambda\Delta} \quad . \qquad (19b)$$

The constant b in the discrete process is related to the rate parameter λ of the continuous process by

$$\lambda = -\ln b/\Delta \quad . \qquad (20)$$

Using (16) and (20) ,

$$\tau_z^*/\Delta = -1/\ln b \quad . \qquad (21)$$

The last equation should be compared with its counterpart for the discrete process, equation (18). To give a numerical example, suppose that b = 0.9. Then, the correlation time of the discrete process, from (18), is 10Δ. For a continuous process having the same autocorrelation, the correlation time, from (21), is 9.491Δ. One observes that the value of τ^* is slightly larger for the discrete process. This would require a longer observation period for the discrete (time-sampled) process than for the underlying continuous stochastic process to achieve the same degree of statistical precision. This result quantifies the information loss associated with coarse time-sampling of a continuous process.

6. Variance of the Sum of $\{x_t\}$

After the slight digression in paragraph 5, we return to the main task of deriving an expression for the variance of the sum of the discrete time series $\{x_t\}$.

Let

$$y_t = \Sigma_{k=1}^{t} \; x_k \quad . \qquad (22)$$

Since $\{x_k\}$ has zero mean,

$$E[y_t] = 0 \qquad (23a)$$

and

$$V[y_t] = E[y_t^2] \quad . \qquad (23b)$$

The variance of $\{y_t\}$ is also denoted by σ_y^2. With (22) and (23b),

$$V[y_t] = E[\Sigma_{i=1}^{t} x_i^2] + 2E[\Sigma_{i=1}^{t-1} \Sigma_{k=1}^{t-i} x_i x_{i+k}] \quad . \tag{24}$$

Also

$$V[y_{t+1}] = E[\Sigma_{i=1}^{t} x_i^2] + E[x_{t+1}^2]$$

$$+ 2E[\Sigma_{i=1}^{t-1} \Sigma_{k=1}^{t-i} x_i x_{i+k}] + 2\Sigma_{i=1}^{t} E[x_i x_{t+1}] \quad . \tag{25}$$

By comparing (24) and (25)

$$V[y_{t+1}] = V[y_t] + \gamma_{xx}(0) + 2\Sigma_{k=1}^{t} \gamma_{xx}(k) \quad , \quad t \geq 1 \quad . \tag{26}$$

With (12), (13), and (26),

$$V[y_{t+1}] = V[y_t] + \sigma_x^2 (1+2\beta_t) \quad , \quad t \geq 1 \quad , \tag{27a}$$

where

$$\beta_t = \Sigma_{k=1}^{t} b^k \tag{27b}$$

or

$$\beta_t = b(1-b^t)/(1-b) \quad . \tag{27c}$$

From (22),

$$V[y_1] = \sigma_x^2 \tag{28}$$

From (27a) and (28), by induction,

$$V[y_t]/\sigma_x^2 = t + 2\Sigma_{k=1}^{t-1} \beta_k \quad , \quad t \geq 2 \quad . \tag{29}$$

Using (27c) the last term on the right-hand side of (29) can be written as

$$S_t = \frac{2b}{1-b} [t - \frac{(1-b^t)}{1-b}] \quad . \tag{30}$$

Finally,

$$\sigma_y^2/\sigma_x^2 = t + S_t \quad . \tag{31}$$

This is one of the expressions of interest. Note that an uncorrelated series, with b=0, would produce the familiar result that the variance of the sum is the sum of the variances, i.e.,

$$\sigma_y^2 = t\sigma_x^2 \quad . \tag{32}$$

The effect of autocorrelation on the ratio σ_y^2/σ_x^2 is to increase the effective series length relative to an uncorrelated series by S_t. As noted in pgf. 4, the magnitude of b must be less than unity.

7. Variance of the Average of $\{x_t\}$

Suppose that the variable of interest is the estimated mean $(\bar{x}_t)$ of the time series $\{x_t\}$, where

$$\bar{x}_t = y_t/t \quad . \tag{33}$$

Since t is a constant in (33),

$$V[\bar{x}_t] = V[y_t]/t^2 \quad . \tag{34}$$

From (31) and (34),

$$V[\bar{x}_t] = \sigma_x^2\,(t + S_t)/t^2 \quad . \tag{35}$$

If the $\{x_t\}$ process is uncorrelated, one has the conventional result

$$V[\bar{x}_t] = \sigma_x^2/t \quad . \tag{36}$$

However, for long correlation intervals and relatively short sampling intervals, one may expect S_t to be larger than t. In this case the variance of the average should be calculated from (30) and (35) using the best estimate of b.

8. An Asymptotic Result

As t becomes large, the second term on the right in (30) becomes small relative to the first. Asymptotically, the variance of the estimated mean, given in (35), becomes

$$V[\bar{x}_t] = (\sigma_x^2/t)(1+b)/(1-b) \quad . \tag{37}$$

Thus, for a positively autocorrelated first-order time series, the variance of the average is amplified by a factor of at most $(1+b)/(1-b)$ relative to that of an uncorrelated time series of the same length.

9. A Numerical Example

To provide a feeling for magnitude relative to this problem, consider the following special case. Suppose that $\{x_t\}$ is a first-order time series obtained by sampling a continuous stochastic process every minute. The continuous process is claimed to have a correlation time of 20 minutes. This implies that every 40 samples are "worth" one degree of freedom. From (18), solving for b gives

$$b = 0.95 .$$

Then, the variance of the estimated mean is given by (30) and (35) as

$$\sigma_x^2 \, [t + 38 \, (t - 20 \, (1 - 0.95^t))]/t^2 \quad .$$

The variance of an uncorrelated series of length t is, by contrast,

$$\sigma_x^2/t \quad .$$

The asymptotic variance estimate for this example, from (37), is

$$39\sigma_x^2/t \quad .$$

To yield a reasonable approximation, the asymptotic result should only be used when time exceeds, say, 20 correlation time intervals. These variance values are shown in Table 1.

10. Autocovariance of an Integrated Process

The variable (y_t) characterizing the integral of a stationary stochastic process is given in discrete form by equation (22). Suppose that successive values of y_t are formed from disjoint sums of n successive values of the $\{x_t\}$ series thus

$$y_j = \Sigma_{i=1}^{n} x_{i+(n-1)j} \quad . \tag{38}$$

TABLE 1

VARIANCES FOR THE ESTIMATE OF THE MEAN
OF A FIRST-ORDER STOCHASTIC PROCESS

Parameters:

Correlation time (τ^*) 20 minutes

Sampling interval (Δ) 1 minute

Autocorrelation constant (b) 0.95

Time, $t\Delta$ (min)	$t\Delta/\tau^*$	$V[\bar{x}_t]/\sigma_x^2$	Uncor. $V[\bar{x}_t]/\sigma_x^2$	Asymp. $V[\bar{x}_t/\sigma_x^2]$
2	0.1	0.9750	0.5000	—
20	1.0	0.7311	0.0500	—
120	6.0	0.2723	0.0083	—
240	12.0	0.1493	0.0042	0.1625
480	24.0	0.0780	0.0021	0.0812
960	48.0	0.0398	0.0010	0.0406

Note that y_j is not a conventional moving average of the x-series. Since $\{x_t\}$ is covariance stationary with mean zero, $\{y_t\}$ also has these properties. Using previous notation, the autocovariance of the $\{y_t\}$ process is given as

$$\gamma_{yy}(k) = E[y_{j+k}\, y_j] \quad . \qquad (39)$$

For notational simplicity the number (n) of members of $\{x_t\}$ in each y_t is not noted explicitly here. For the case in which n=2, from (38),

$$y_1 = x_1 + x_2$$
$$y_2 = x_3 + x_4 \qquad (40)$$

and so forth.

Then,

$$E[y_1^2] = E[x_1^2 + 2x_1x_2 + x_2^2]$$

or

$$\gamma_{yy}(0) = 2\gamma_{xx}(0) + 2\gamma_{xx}(1) \quad . \qquad (41)$$

If $\{x_t\}$ is a first-order process, from (13),

$$\gamma_{xx}(1) = \gamma_{xx}(0)b \quad .$$

Then,

$$\gamma_{yy}(0)/\gamma_{xx}(0) = 2 + 2b, \qquad (42)$$

which agrees with (29) for the special case of t = 2.

Pursuing this special case where n = 2,

$$E[y_1y_2] = E[x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4]$$

or

$$\gamma_{yy}(1) = \gamma_{xx}(1) + 2\gamma_{xx}(2) + \gamma_{xx}(3) \quad . \qquad (43)$$

Similarly,

$$\gamma_{yy}(2) = \gamma_{xx}(3) + 2\gamma_{xx}(4) + \gamma_{xx}(5) \qquad (44)$$

and

$$\gamma_{yy}(3) = \gamma_{xx}(5) + 2\gamma_{xx}(6) + \gamma_{xx}(7) \quad . \qquad (45)$$

By induction, one infers that

$$\gamma_{yy}(j) = \gamma_{xx}(2j-1) + \gamma_{xx}(2j) + \gamma_{xx}(2j+1) \quad , \quad j\geq 1 \ , \qquad (46)$$

for this case (n=2).

11. Consider another special case for the $\{y_t\}$ process in which n=3 in equation (38). Following the same procedure used in developing (46), one finds

$$\gamma_{yy}(j) = \gamma_{xx}(3j-2) + 2\gamma_{xx}(3j-1)$$

$$+ 3\gamma_{xx}(3j) + 2\gamma_{xx}(3j+1) + \gamma_{xx}(3j+2) \quad . \qquad (47)$$

One can generalize the results presented this far by applying mathematical induction to the case in which n is not assigned a specific value in equation (38). The general result is

$$\gamma_{yy}(j) = \Sigma_{k=1}^{n} k \ \gamma_{xx}(n(j-1) + k)$$

$$+ \Sigma_{k=1}^{n-1} k \ \gamma_{xx}(n(j+1) - k) \quad , \quad j\geq 1 \quad . \qquad (48)$$

A compact expression, which is equivalent to (48), is

$$\gamma_{yy}(j) = \Sigma_{k=0}^{2n-1} c_{nk} \ \gamma_{xx}(n(j-1)+k+1) \ ,$$

with

$$c_{nk} = |n - |\mathrm{int}[(n+1)/2] - k|| \ , \ j\geq 1 \ . \qquad (49)$$

Note that the expressions in (48) and (49) are not restrictive regarding the form of the autocovariance of $\{x_t\}$. Only the assumption of stationarity is involved.

12. <u>A Variance Relationship for $\{y_t\}$</u>

The procedure followed in deriving the relationship between the autocovariance of an integrated $\{y_t\}$ process and the autocovariance of the $\{x_t\}$, or rate-, process involved mathematical induction at several places. A similar procedure is followed in obtaining the textbook formula for the variance of $\{y_t\}$.

$$\gamma_{yy}(0) = n\gamma_{xx}(0) + 2\Sigma_{k=1}^{n-1} (n-k) \ \gamma_{xx}(k) \quad . \qquad (50)$$

13. An Autocorrelation Relationship

Often the analysis of an autocorrelated time series involves a comparison of an empirical autocorrelation function with several theoretical alternatives. To facilitate the comparison, autocorrelations are used instead of autocovariances. The following autocorrelation function relating the $\{y_t\}$ and $\{x_t\}$ processes is, in general, rather complicated. Calculation of $\rho_{yy}(j)$ is best done via computer. The computer program which does this is in Annex 1.

Since

$$\rho_{yy}(j) = \gamma_{yy}(j)/\gamma_{yy}(0) \quad ,$$

$$\rho_{yy}(j) = \frac{\gamma_{yy}(j)\ \gamma_{xx}(0)}{\gamma_{xx}(0)\ \gamma_{yy}(0)} \quad . \tag{51}$$

But, from (50),

$$\gamma_{yy}(0)/\gamma_{xx}(0) = n + 2\Sigma_{k=1}^{n-1}\ (n-k)\rho_{xx}(k) \quad . \tag{52}$$

And, from (48),

$$\gamma_{yy}(j)/\gamma_{xx}(0) = T(n,j) + U(n,j) \quad , \tag{53a}$$

with

$$T(n,j) = \Sigma_{k=1}^{n}\ k\ \rho_{xx}(n(j-1) + k) \tag{53b}$$

$$U(n,j) = \Sigma_{k=1}^{n-1}\ k\ \rho_{xx}(n(j+1) - k) \quad . \tag{53c}$$

Finally, from (51, 52, 53),

$$\rho_{yy}(j) = \frac{T(n,j) + U(n,j)}{n + 2\Sigma_{k=1}^{n-1}\ (n-k)\rho_{xx}(k)} \quad . \tag{54}$$

This result is also one of the study objectives.

14. To corroborate the expression for the variance of an integrated first-order process -- equation (31) -- we substitute

$$\gamma_{xx}(k) = \sigma_x^2 \, b^k \tag{55}$$

into (50), obtaining

$$\gamma_{yy}(0)/\sigma_x^2 = n + 2b[\frac{n}{1-b} - \frac{(1-b^n)}{(1-b)^2}] \quad . \tag{56}$$

Apart from notational differences, equations (31) and (56) are identical.

15. Autocorrelation of an Integrated First-Order Process

A general equation for the autocorrelation function of an integrated first-order process is obtained by substituting b^k for $\rho_{xx}(k)$ in equations (53) and (54). This operation yields

$$\rho_{yy}(j) = \frac{C(n,b)\ b^{n(j-1)} + D(n,b)\ b^{nj}}{n + 2D(n,b)} \quad , \quad j \geq 1 \quad , \tag{57a}$$

where

$$C(n,b) = \Sigma_{k=1}^{n}\ k\ b^k \tag{57b}$$

$$C(n,b) = \frac{b(1-b^n)}{(1-b)^2} - \frac{n\ b^{n+1}}{(1-b)}$$

and

$$D(n,b) = \Sigma_{k=1}^{n-1}\ (n-k)\ b^k$$

$$D(n,b) = \frac{nb}{1-b} - \frac{b(1-b^n)}{(1-b)^2} \quad . \tag{57c}$$

As an example, for n=2,

$$\rho_{yy}(j) = \frac{(1+b)}{2b}\ b^{2j} \quad , \quad j \geq 1 \quad . \tag{58}$$

16. The Autospectrum

The discussion of autocovariance and autocorrelation concerns the temporal relationship between members of a time series. An alternative means of studying the dynamics of a stochastic process is set in the frequency domain. The variance (or power) of a process is decomposed into the variance components associated with each of its Fourier frequencies. This form of expressing process dynamics is calculated via the autospectrum of the process. The kind of question answered by spectrum analysis is the following: How does the variance associated with process fluctuations of one temporal period (or reciprocal frequency) compare with those of other periods? Altho the autospectrum of a continuous stochastic process x(t) can be defined in terms of the Fourier integral of x(t), an equivalent ("indirect") method is used here, which involves the autocovariance of x(t).

17. Notationally, the autospectrum of x(t) is expressed as a function of angular frequency ω as

$\Gamma'_{xx}(\omega)$:

$$\Gamma'_{xx}(\omega) = \frac{2}{\pi} \int_0^\infty \gamma_{xx}(t) \cos(\omega t) dt \quad . \tag{59}$$

The autospectrum is often expressed in terms of the natural frequency ν in cycles per unit time:

$$\Gamma_{xx}(\nu) = 2\pi \, \Gamma'_{xx}(\omega) \quad . \tag{60}$$

From p. 259, Jenkins and Watts (1968), [3], the smoothed (or estimated) autospectrum of the discrete process $\{x_t\}$ is

$$\Gamma_{xx}(\nu) = \Delta \Sigma_{k=-\infty}^{\infty} w(k) \, \gamma_{xx}(k) e^{-j2\pi\nu k\Delta} \quad ,$$

using complex notation with $j = \sqrt{-1}$.
Alternatively,

$$\Gamma_{xx}(\nu) = 2\Delta[\gamma_{xx}(0) + 2 \, \Sigma_{k=1}^{\infty} w(k) \, \gamma_{xx}(k) \cos 2\pi\nu k\Delta] \quad , \quad 0 \le \nu \le 1/(2\Delta) \quad . \tag{61}$$

The weight function w(k), of lag k, is bounded between zero and one. When w(k) is set to unity for all k, the unsmoothed autospectrum is obtained. To smooth out the random variation occuring in the estimate of $\gamma_{xx}(k)$, a variety of forms for w(k) may be used. Some are suggested in [3] and

---

[3] Jenkins, G.M. and Watts, D.G. Spectral Analysis and Its Applications, c. 1968.

[4]. All w(k) have the property:

$$w(k) = 0 \quad , \quad |k| > m ,$$

for some integer m.

However, we are only concerned with the theoretical autospectrum here, since the exact expression for $\gamma_{xx}(k)$ is available. Hence, w(k) will be set to unity in analytic applications. When calculating a spectral density via direct numerical evaluation of (61), it is necessary to truncate the infinite sum at some maximum lag m. This procedure is equivalent to assigning w(k) the boxcar function:

$$w(k) = 1 \quad , \quad |k| \leq m ,$$

$$w(k) = 0 \quad , \quad |k| > m .$$

High frequency perturbations are superposed on the spectrum due to this truncation -- the Gibbs' phenomenon (p. 73, [4]). To minimize this effect, the boxcar weight function (or lag window) should not be used. A better choice, used in the annexed computer program, is the Tukey lag window.

18. To compare the autospectra of time series having different variance values, it is convenient to use a normalized autospectrum, the spectral density:

$$\bar{\Gamma}_{xx}(\nu) = \Gamma_{xx}(\nu)/\gamma_{xx}(0) \quad . \tag{62}$$

From (61) and (62) ,

$$\bar{\Gamma}_{xx}(\nu) = 2\Delta[1+2\Sigma_{k=1}^{\infty} w(k)\rho_{xx}(k)\cos k\lambda] \quad , \quad 0 \leq k\lambda \leq \pi \tag{63}$$

where

$$\lambda = 2\pi\nu\Delta \quad . \tag{64}$$

---

[4] Hamming, R.W. Digital Filters, Prentice-Hall, Englewood Cliffs, NJ, c. 1977.

19. Autospectrum of a First-Order Process

Consider $\{x_t\}$ to be first-order so that (13) applies. Then, with (63),

$$\overline{\Gamma}_{xx}(\nu) = 2\Delta[1+2\Sigma_{k=1}^{\infty} b^k \cos k\lambda] \quad . \tag{65}$$

To obtain the sum in (65), one can use the following result from Gradshteyn and Ryzhik (1965), [5]:

$$\Sigma_{k=0}^{n-1} p^k \cos kx = \frac{1-p\cos x - p^n \cos nx + p^{n+1} \cos (n+1)x}{1 - 2p \cos x + p^2} \quad . \tag{66}$$

For $|p| < 1$, as n approaches ∞,

$$\Sigma_{k=1}^{\infty} p^k \cos kx = \frac{p(\cos x - p)}{1 - 2p \cos x + p^2} \quad . \tag{67}$$

Using (67), equation (65) becomes

$$\overline{\Gamma}_{xx}(\nu) = \frac{2\Delta\,(1 - b^2)}{1 + b^2 - 2b \cos \lambda} \quad . \tag{68}$$

Parenthetically, the last result could have been obtained by another method. This method uses the fact that a first-order time series is a first-order digital filter of white noise. Thus,

$$x_{t+1} = b\, x_t + n_t \quad .$$

Taking Fourier transforms on both sides of this equation gives the transfer function $H(\omega)$, whose square is given by

$$H^2(\omega) = 1/[1 + b^2 - 2\, b \cos \omega\Delta] \quad .$$

One then uses the following theorem relating the spectra of the $\{x_t\}$ and $\{n_t\}$ processes to obtain equation (68):

$$\Gamma'_{xx}(\omega) = H^2(\omega)\Gamma'_{nn}(\omega) \quad , \quad 0 \le \omega \le \pi/\Delta \quad ,$$

with

$$\Gamma'_{nn}(\omega) = \sigma^2\Delta/\pi \quad .$$

Equation (13b) is also employed to obtain the spectral density in (68).

---

[5] Gradshteyn, I.S. and Ryzhik, I.M. Tables of Integrals, Series, and Products, Academic Press, N.Y., c. 1965.

20. Autospectrum of an Integrated First-Order Process

One can substitute the expression for ρ_{yy}, (57) in lieu of ρ_{xx}, into the general equation for a spectral density (equation 63). This process yields an equation for the autospectrum of an integrated first-order process. Except for special values of n, this process is messy. The computer program for obtaining numerical results is found in Annex 1. The particular case of an integrated first-order process for n=2 is tractable. In this case, ρ_{yy} is given by (58). Substitution of this expression for the autocorrelation into (63), with w(k) set to unity, yields

$$\overline{\Gamma}_{yy}(\nu) = 2\Delta[1 + \frac{(1+b)}{b} \Sigma_{k=1}^{\infty} b^{2k} \cos k\lambda] \ . \tag{69}$$

Using (67) with (69),

$$\overline{\Gamma}_{yy}(\nu) = 2\Delta[1 + \frac{(1+b)b(\cos \lambda - b^2)}{1-2b^2 \cos \lambda + b^4}] \ . \tag{70}$$

21. Numerical Results

A numerical example will facilitate the comparison of properties of first-order $\{x_t\}$ with integrated first-order $\{y_t\}$ time series. Let the parameters b = 0.9 and Δ = 1 minute. Equations (13a) and (58) are used to calculate autocorrelation functions, and (68) and (70) are used for spectral densities. Autocorrelations are tabulated in Table 2 and plotted in Figure 1. The autocorrelation for the integrated (n=2) process displays a much more rapid decrease with increasing lag. For lags $k \geq 1$, the autocorrelation $\rho_{yy}(k)$ for the integrated process is linear on a log-uniform plot. However, unlike $\rho_{xx}(k)$, $\rho_{yy}(k)$ has a change of slope on this plot at k=1. The autospectra are compared in Figure 2. Notice that $\overline{\Gamma}_{yy}(\nu)$ is much flatter than $\overline{\Gamma}_{xx}(\nu)$. This illustrates the approach of the integrated process toward white noise. However, for this degree of integration (n=2) and range of frequency ν, the two autospectra have quite similar shapes. With a change in parameter b to 0.82 in $\overline{\Gamma}_{xx}(\nu)$, the spectral densities are nearly conformable. In practical terms, this implies that the estimated autospectrum of an integrated (n=2) first-order process would be nearly indistinguishable from that of a first-order process for certain parameters.

TABLE 2

COMPARISON OF AUTOCORRELATIONS OF A
FIRST-ORDER AND AN INTEGRATED*
FIRST-ORDER TIME SERIES

Autocorrelation parameter for $\{x_t\}$: $b = 0.9$

Lag, k	$\rho_{xx}(k)$	$\rho_{yy}(k)$
0	1.0000	1.0000
1	0.9000	0.8550
2	0.8100	0.6926
3	0.7290	0.5610
4	0.6561	0.4544
5	0.5905	0.3680
6	0.5314	0.2981
10	0.3487	0.1283
20	0.1216	0.0156

* Two terms are in the sum.

$y_j = x_{j+1} + x_{j+2}$, $j \geq 0$.

$\rho_{xx}(k) = b^k$

$\rho_{yy}(k) = ((1+b)/2)b^{2k-1}$

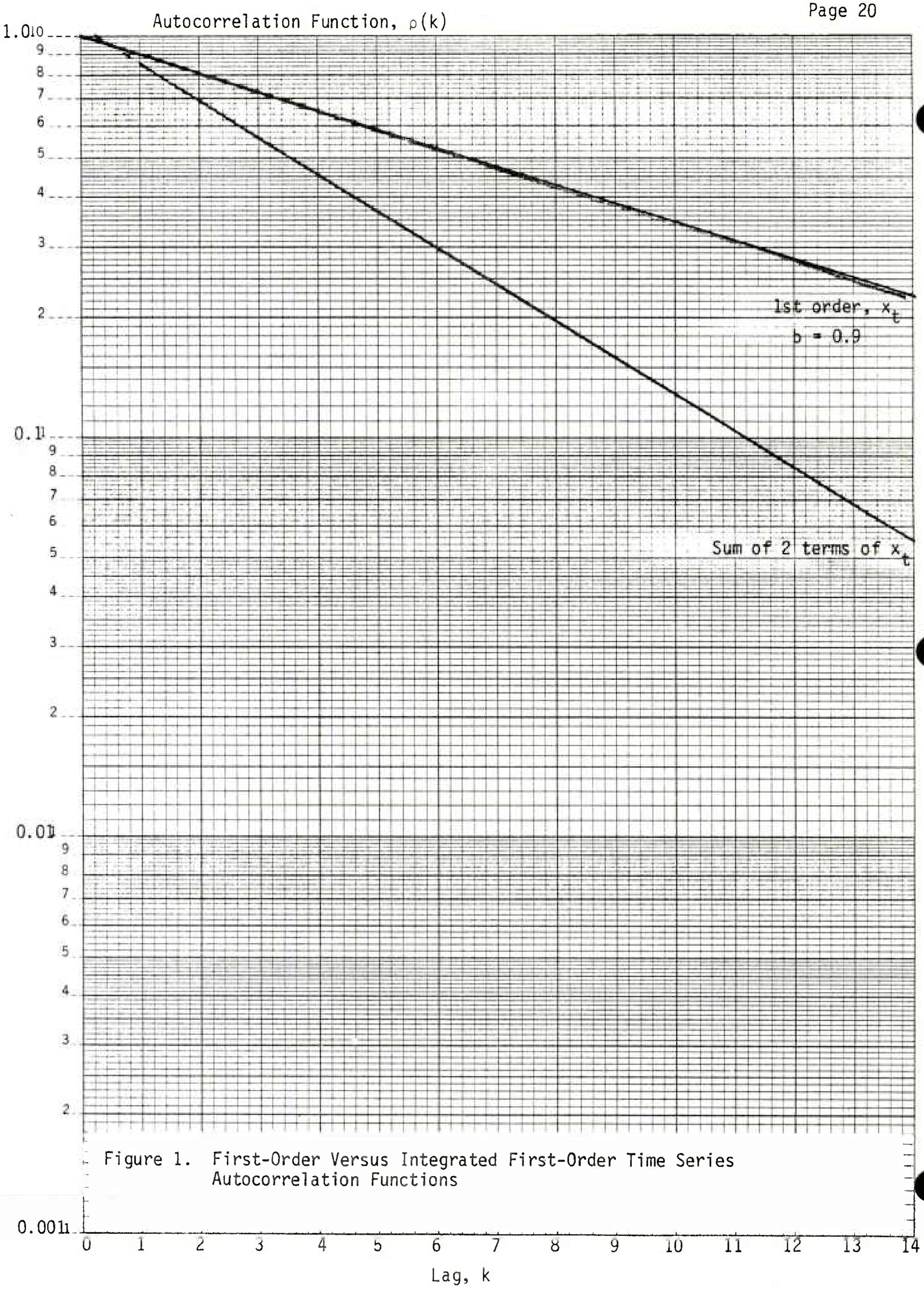

Figure 1. First-Order Versus Integrated First-Order Time Series Autocorrelation Functions

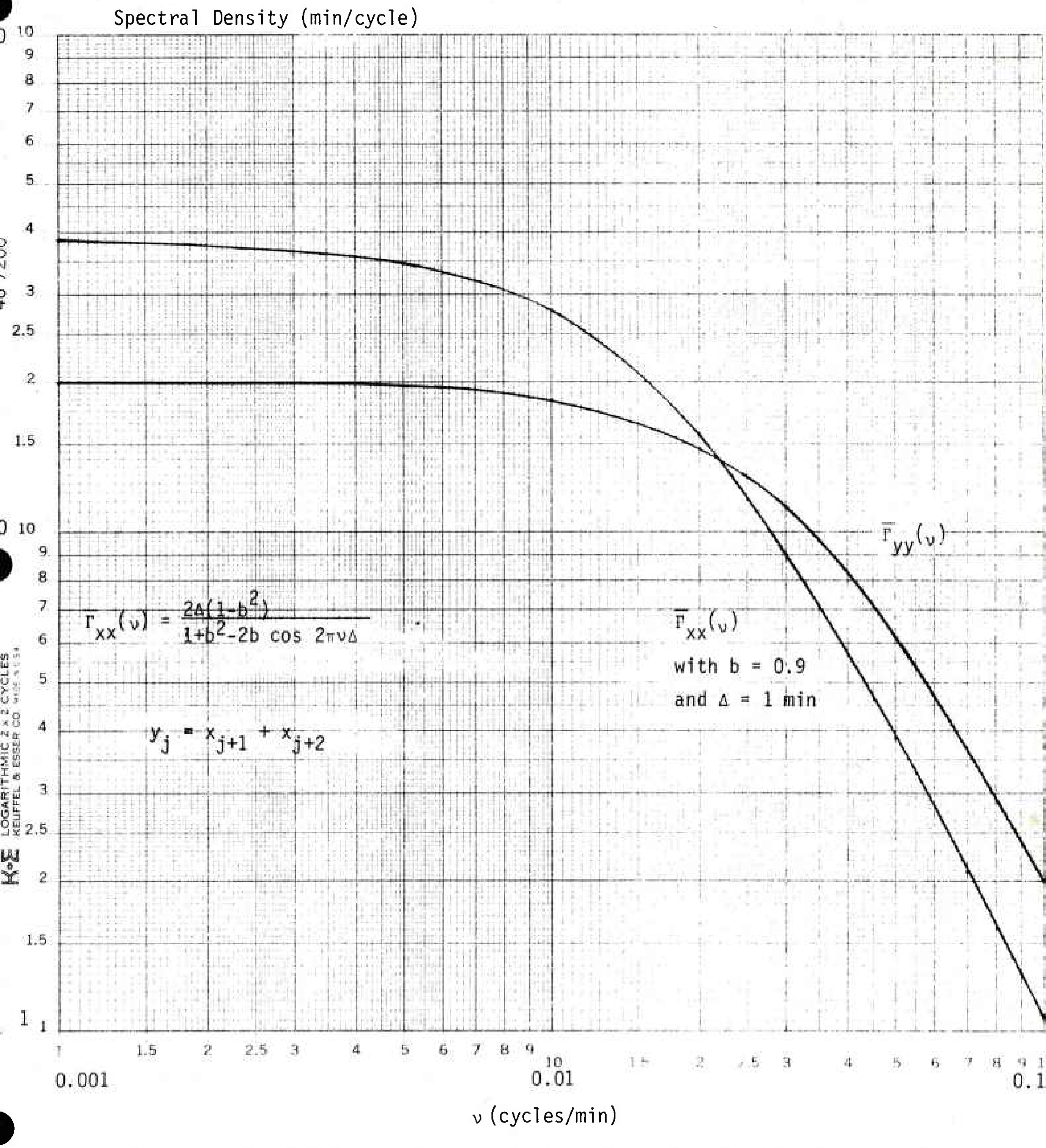

Figure 2. Comparison of Spectral Densities for a First-Order and an Integrated First-Order Time Series

22. Summary

This report has examined and compared statistical properties of stationary, first-order time series and integrated first-order series. A connection between continuous stochastic processes and time series was made thru the concept of correlation time. Expressions for the variance of the sum and average of a first-order series were derived. Asymptotic results for large samples were shown to be reasonable. When $\{y_t\}$ consists of disjoint sums of any stationary process $\{x_t\}$, the autocovariances and autospectra of these processes are related. These realtionships were derived, and were particularized for the case in which $\{x_t\}$ is first-order. Numerical examples were provided and discussed. A computer program is provided to evaluate special cases.

DISTRIBUTION LIST

NO. OF COPIES	
1	HQDA (DAMO-ZA) WASH DC
1	HQDA (DALO-SMZ) WASH DC
	Commander USADARCOM 5001 Eisenhower Ave. Alexandria, VA 22333
1	ATTN: DRCRE-IP
1	DRCPA-S
1	DRCDP
	Director, US AMSAA Aberdeen PG, MD 21005 ATTN:
1	Messrs Gilbert DRXSY-DL
1	Cohen DRXSY-MP
1	Ron Simmon DRXSY-R
	Commander USA Communications and Electronics Command Ft Monmouth, NJ 07703
1	ATTN: DRSEL-PL-SA
	Commander CECOM (R&D) Ft Monmouth, NJ 07703
1	ATTN: DRSEL-SAD
	Commander USAMICOM Redstone Arsenal AL 35809
1	ATTN: DRSMI-DS
	Commander USATACOM Warren, MI 48090
1	ATTN: DRSTA-V
	Office of Project Mgr
1	Cannon Arty Wpns, Dover DRCPM-CAWS NJ 07801

NO. OF COPIES	
	Commander, US Army Logistics Center
1	Fort Lee, VA 23801 ATTN: ATCL-S
	Director, AMCCOM
1	Ammo Center SARAC-DO
	Savanna, IL 61074
	Commander USATECOM
	Aberdeen PG, MD 21005
1	ATTN: DRSTE-SY
1	Director, US Army
	Inventory Research Off DRXMC-IRO
	Room 800, Custom House
	2nd & Chestnut Streets
	Philadelphia, PA 19106
12	Defence Technical Information Center
	Cameron Station
	Alexandria, VA 22314
	Commander US AMCCOM (D)
	ATTN:
1	DRSMC-LC (D)
1	DRSMC-SC (D)
1	DRSMC-SE (D)
1	DRSMC-QA (D)
	Dover, NJ 07801
1	Commander
	US Army Production Base
	Modernization Agency
	Dover, NJ 07801
	Commander US AMCCOM (R)
	Rock Island, IL 61299
	ATTN:
1	DRSMC-AS (R)
1	DRSMC-IR (R)
1	DRSMC-LE (R)
1	DRSMC-PC (R)
1	DRSMC-PD (R)
1	DRSMC-QA (R)
1	DRSMC-OP (R)
2	Technical Library DRSMC-LEP-L

DISTRIBUTION LIST (CONT)

NO. OF COPIES	
1	Commander Watervliet Arsenal ATTN: DRSMC-LCB-TL Watervliet, NY 12189
1	Commander Chemical R and D Center Aberdeen Proving Ground (Edgewood Area), MD 21010 ATTN: DRSMC-CLJ-IA (A)
1	Director US AMETA ATTN: DRXOM-QA Rock Island, IL 61299
1	Director Naval Postgraduate School ATTN: Dept of Operations Anal. Monterey, CA 93940
1	Director Advanced Research Projects Agency 1400 Wilson Blvd Arlington, VA 22209
1	Director USA TRASANA ATTN: ATAA-SL (Tech Lib) White Sands Missile Range White Sands, NM 88002
1	Commander Defense Logistics Studies Information Exchange Fort Lee, VA 23801

ANNEX 1

COMPUTER PROGRAMS

This annex contains the computer program used to calculate the autocorrelation function and spectral density of a general, integrated first-order stochastic process. The main program (SA.DRIVER) and subroutines are written in SIMSCRIPT 2.5. They have been run on the PR1ME 550 minicomputer in an interactive mode. However, no commands specific to PR1ME are used. The program listing is quite English-like and, with internal comments, is essentially self-documenting. Conversion to Fortran is straightforward. The autocorrelation functions of both the x-series -- $\rho_{xx}(k)$ -- and integrated x-series (or y-series) are sent to the terminal for output. At the user's option the spectral densities of the x- and y-series are also calculated and printed. By changing only the formula for $\rho_{xx}(k)$, one can obtain these statistical properties for integrals of other-than-first-order processes.

Options = SEQUENCE,IE,SUBCHK,XREF,NOEXPLIST,TRACE3

```
1  PREAMBLE
2  NORMALLY MODE IS REAL
3  DEFINE SPK.WT AS A REAL FUNCTION GIVEN 2 ARGUMENTS
4  END ''PREAMBLE FOR SA.DRIVER
```

C R O S S - R E F E R E N C E

NAME	TYPE	MODE	LINE NUMBERS OF REFERENCES
SPK.WT	ROUTINE	DOUBLE	3

```
MAIN ''SA.DRIVER
''
''DRIVER FOR AUTOCORRELATION FUNCTION OF AN INTEGRATED, 1 ST-ORDER SERIES.
''SPECTRA FOR THE RATE PROCESS AND ITS INTEGRAL ARE OPTIONALLY CALCULATED.
''
DEFINE FLAG.S, FLAG.W, I, J, K, MLAG, N AS INTEGER VARIABLES
DEFINE ANSWER AS A TEXT VARIABLE
DEFINE RHOXV AND RHOYV AS REAL, 1-DIMENSIONAL ARRAYS
''
''GET PARAMETERS FROM THE TERMINAL.
''
'L1'PRINT 1 LINE THUS
INPUT THE INTEGER NUMBER OF TERMS IN THE SUM.  ZERO STOPS.
        READ N
        IF N < 2
            STOP
        OTHERWISE
        PRINT 1 LINE THUS
INPUT THE MAX LAG IN THE AUTOCORRELATION FUNCTION.
        READ MLAG
        LET MLAG=MAX.F(MLAG,N)
        RESERVE RHOXV(*) AND RHOYV(*) AS MLAG
        PRINT 1 LINE THUS
INPUT THE FIRST-ORDER CORRELATION PARAMETER (B).
        READ B
        LET DELTA=1.0
        PRINT 1 LINE THUS
ARE SPECTRA DESIRED?  (YES OR NO).
        READ ANSWER
        IF SUBSTR.F(ANSWER,1,LENGTH.F(ANSWER)) NE "Y"
            LET FLAG.S=0
            GO TO L2
        OTHERWISE
        LET FLAG.S=1
        PRINT 1 LINE THUS
DO YOU WANT SPECTRAL SMOOTHING?  (YES OR NO).
        READ ANSWER
        IF SUBSTR.F(ANSWER,1,LENGTH.F(ANSWER))="Y"
            LET FLAG.W=1
        OTHERWISE
            LET FLAG.W=0
        ALWAYS
'L2'SKIP 4 LINES
''
''ECHO INPUTS.
''
        PRINT 7 LINES WITH N, MLAG, B, DELTA THUS
INPUT PARAMETERS FOR AUTOCORRELATIONS AND AUTOSPECTRA

NUMBER OF TERMS IN THE SERIES SUM ____   *
MAXIMUM LAG IN THE AUTOCORRELATION ____  *
CORRELATION PARAM IN 1 ST-ORD PROCESS    *.***
SAMPLING INTERVAL (MINUTES) __________ **.***

''
''PRINT HEADINGS.
''
```

```
        SKIP 2 LINES
        PRINT 6 LINES THUS
AUTOCORRELATION FUNCTIONS FOR 1 ST-ORDER- AND INTEGRATED 1 ST-ORDER SERIES

-----------------------------------------------------------------------------
LAG     TM LAG      X-AUTO-     Y-AUTO-
INDEX   (MIN)       CORREL      CORREL
-----------------------------------------------------------------------------
     FOR K=1 TO MLAG DO
        LET TM.LAG=DELTA*K
        LET RHOX=B**K
        LET RHOXV(K)=RHOX
        CALL S.AUTOCOR GIVEN B, N, K YIELDING RHOY
        LET RHOYV(K)=RHOY
        PRINT 1 LINE WITH K, TM.LAG, RHOX, RHOY THUS
 **       **.*       *.******    *.******
     LOOP ''OVER LAGS
        PRINT 2 LINES THUS
-----------------------------------------------------------------------------

        LET VAR.RATIO=N
     FOR K=1 TO N-1 DO
        ADD 2.0*(N-K)*RHOXV(K) TO VAR.RATIO
     LOOP ''OVER K
        PRINT 1 LINE WITH VAR.RATIO THUS
VARIANCE(Y)/VARIANCE(X) ----  ***.****
        IF FLAG.S NE 1
            GO TO L1
        OTHERWISE
        SKIP 4 LINES
        IF FLAG.W=1
            PRINT 4 LINES WITH N, MLAG, DELTA THUS
SMOOTHED SPECTRAL DENSITIES OF X- AND INTEGRATED X-SERIES WITH N = **

  TUKEY LAG WINDOW WITH *** LAGS OF ***.*** TIME UNITS EACH

        OTHERWISE
            PRINT 2 LINES WITH N THUS
  SPECTRAL DENSITIES OF X- AND INTEGRATED X-SERIES WITH N = **

        ALWAYS
        CALL SPK.DENS(DELTA, MLAG, FLAG.W, RHOXV(*), RHOYV(*))
        GO TO L1
     END ''SA.DRIVER
```

```
                    C R O S S - R E F E R E N C E

 NAME                  TYPE                             MODE      LINE NUMBERS OF REFERENCES

ANSWER       RECURSIVE VARIABLE    WORD  8          TEXT       7   25   26*  32   33*
B            RECURSIVE VARIABLE    WORD 22          DOUBLE    22   42   50   52
DELTA        RECURSIVE VARIABLE    WORD 24          DOUBLE    23   42   49   67   71
FLAG.S       RECURSIVE VARIABLE    WORD  1          INTEGER    6   27   30   62
FLAG.W       RECURSIVE VARIABLE    WORD  2          INTEGER    6   34   36   66   71
I            RECURSIVE VARIABLE    WORD  3          INTEGER    6
J            RECURSIVE VARIABLE    WORD  4          INTEGER    6
K            RECURSIVE VARIABLE    WORD  5          INTEGER    6   48*  49   50   51   52   53
                                                              54   58*  59*
L1           LABEL                                            12   63   72
L2           LABEL                                            28   38
LENGTH.F     ROUTINE                                INTEGER   26   33
MAX.F        ROUTINE                                INTEGER   19
MLAG         RECURSIVE VARIABLE    WORD  6          INTEGER    6   18   19*  20*  42   48   67
                                                              71
N            RECURSIVE VARIABLE    WORD  7          INTEGER    6   13   14   19   42   52   57
                                                              58   59   67   69
RHOX         RECURSIVE VARIABLE    WORD 30          DOUBLE    50   51   54
RHOXV        RECURSIVE VARIABLE    WORD  9  (1-D)   DOUBLE     8   20*  51   59   71
RHOY         RECURSIVE VARIABLE    WORD 32          DOUBLE    52   53   54
RHOYV        RECURSIVE VARIABLE    WORD 10  (1-D)   DOUBLE     8   20*  53   71
S.AUTOCOR    ROUTINE                                INTEGER   52
SPK.DENS     ROUTINE                                INTEGER   71
SUBSTR.F     ROUTINE                                TEXT      26   33
TM.LAG       RECURSIVE VARIABLE    WORD 28          DOUBLE    49   54
UIB.W        IMPLIED SUBSCRIPT     SYS   4          INTEGER   38   46   65
VAR.RATIO    RECURSIVE VARIABLE    WORD 34          DOUBLE    57   59*  61
```

```
Options = SEQUENCE,IO.SUBCHK,XREF,NOEXPLIST,TRACE3

  1   ROUTINE FOR S.AUTOCOR GIVEN B, N, LAG YIELDING RHO
  2   ''
  3   ''ROUTINE CALCULATES THE AUTOCORRELATION FUNCTION AT GIVEN LAG FOR
  4   ''AN INTEGRATED (SUMMED) FIRST-ORDER TIME SERIES, HAVING N TERMS
  5   ''IN THE SUM, WITH CORRELATION PARAMETER B.
  6   ''
  7   DEFINE N AND LAG AS INTEGER VARIABLES
  8       IF N < 2
  9           PRINT 1 LINE WITH N THUS
 INPUT ERROR TO S.AUTOCOR.  N =    *.
 10           STOP
 11       OTHERWISE
 12       LET BEN=B**N
 13       LET OMB=1.0-B
 14       LET OMB2=OMB**2
 15       LET CTERM=B*(1.0-BEN)/OMB2
 16       LET C=CTERM-N*B*BEN/OMB
 17       LET D=N*B/OMB-CTERM
 18       LET RHO=(C*B**(N*(LAG-1))+D*B**(N*LAG))/(N+2.0*D)
 19       RETURN
 20   END ''S.AUTOCOR
```

C R O S S - R E F E R E N C E

NAME	TYPE			MODE	LINE NUMBERS OF REFERENCES						
B	ARGUMENT	NO.	1	DOUBLE	1	12	13	15	16	17	18*
BEN	RECURSIVE VARIABLE	WORD	12	DOUBLE	12	15	16				
C	RECURSIVE VARIABLE	WORD	20	DOUBLE	16	18					
CTERM	RECURSIVE VARIABLE	WORD	18	DOUBLE	15	16	17				
D	RECURSIVE VARIABLE	WORD	22	DOUBLE	17	18*					
LAG	ARGUMENT	NO.	3	INTEGER	1	7	18*				
N	ARGUMENT	NO.	2	INTEGER	1	7	8	9	12	16	17
					18*						
OMB	RECURSIVE VARIABLE	WORD	14	DOUBLE	13	14	16	17			
OMB2	RECURSIVE VARIABLE	WORD	16	DOUBLE	14	15					
RHO	ARGUMENT	NO.	4	DOUBLE	1	18					
S.AUTOCOR	ROUTINE			INTEGER	1						

```
ROUTINE FOR SPK.DENS(DELTA, MLAG, FLAG.W, RHOXV, RHOYV)
''
''ROUTINE CALCULATES AND PRINTS THE SPECTRAL DENSITIES (OR SMOOTHED
''SPECTRAL DENSITIES) OF TWO TIME SERIES: X-SERIES AND Y-SERIES, WHERE
''Y-SERIES CONSISTS OF N-ELEMENT SUBSUMS OF X-SERIES.  THE CALCULATION
''USES THE AUTOCORRELATION FUNCTIONS OF THESE TIME SERIES.
''
''INPUTS:
''DELTA    ------------ TIME SAMPLING INTERVAL.
''MLAG     ------------ MAX LAG OF THE AUTOCORRELATION FUNCTIONS.
''FLAG.W   ------------ INTEGER FLAG TO SIGNAL USE OF A WEIGHT FUNCTION (=1)
''                      OR NOT (=0) ON THE AUTOCORRELATIONS.
''RHOXV    ------------ VECTOR OF AUTOCORRELATIONS OF THE X-SERIES.
''RHOYV    ------------ VECTOR OF AUTOCORRELATIONS OF THE Y-SERIES.
''
''OUTPUT:
''THE SPECTRAL DENSITIES ARE PRINTED TO THE TERMINAL FROM THIS ROUTINE.
''
DEFINE FLAG.W, I, K, MLAG, NF AS INTEGER VARIABLES
DEFINE RHOXV AND RHOYV AS REAL, 1-DIMENSIONAL ARRAYS
     SKIP 2 LINES
     IF FLAG.W=1
          PRINT 4 LINES THUS
---------------------------------------------------------------------
NATURAL     SMOOTHED    DENSITIES  FOR
FREQUENCY   X-SERIES    Y-SERIES (TIME UNITS/CYCLE)
---------------------------------------------------------------------
     OTHERWISE
          PRINT 4 LINES THUS
---------------------------------------------------------------------
NATURAL     UNSMOOTHED  DENSITIES  FOR
FREQUENCY   X-SERIES    Y-SERIES (TIME UNITS/CYCLE)
---------------------------------------------------------------------
     ALWAYS
     LET NF=100 ''FREQUENCY POINTS EXCLUSIVE OF ZERO
     LET F.FREQ=0.5/DELTA
     LET DFREQ=F.FREQ/NF
     LET TPID=2.0*PI.C*DELTA
     LET W=1.0
 FOR I=1 TO NF+1 DO
     LET FREQ=(I-1)*DFREQ
     LET LAMBDA=TPID*FREQ
     LET SUMX=0.0
     LET SUMY=0.0
     FOR K=1 TO MLAG DO
          LET COSKL=COS.F(K*LAMBDA)
          IF FLAG.W=1
               LET W=SPK.WT(K,MLAG)
          ALWAYS
          ADD W*RHOXV(K)*COSKL TO SUMX
          ADD W*RHOYV(K)*COSKL TO SUMY
     LOOP ''OVER K
     LET DENSX=2.0*DELTA*(1.0+2.0*SUMX)
     LET DENSY=2.0*DELTA*(1.0+2.0*SUMY)
     PRINT 1 LINE WITH FREQ, DENSX, DENSY THUS
***.****     ***.****     ***.****
 LOOP ''OVER (I) FREQUENCIES
```

```
  49       PRINT 2 LINES THUS
-------------------------------------------------------------------------------
  50       RETURN
  51   END ''SPK.DENS
```

C R O S S - R E F E R E N C E

NAME	TYPE				MODE	LINE NUMBERS OF REFERENCES					
COS.F	ROUTINE				DOUBLE	38					
COSKL	RECURSIVE VARIABLE	WORD	32		DOUBLE	38	42	43			
DELTA	ARGUMENT	NO.	1		DOUBLE	1	28	30	45	46	
DENSX	RECURSIVE VARIABLE	WORD	34		DOUBLE	45	47				
DENSY	RECURSIVE VARIABLE	WORD	36		DOUBLE	46	47				
DFREQ	RECURSIVE VARIABLE	WORD	18		DOUBLE	29	33				
F.FREQ	RECURSIVE VARIABLE	WORD	16		DOUBLE	28	29				
FLAG.W	ARGUMENT	NO.	3		INTEGER	1	19	22	39		
FREQ	RECURSIVE VARIABLE	WORD	24		DOUBLE	33	34	47			
I	RECURSIVE VARIABLE	WORD	1		INTEGER	19	32*	33			
K	RECURSIVE VARIABLE	WORD	2		INTEGER	19	37*	38	40	42	43
LAMBDA	RECURSIVE VARIABLE	WORD	26		DOUBLE	34	38				
MLAG	ARGUMENT	NO.	2		INTEGER	1	19	37	40		
NF	RECURSIVE VARIABLE	WORD	3		INTEGER	19	27	29	32		
PI.C	PERMANENT ATTRIBUTE	SYS	33		DOUBLE	30					
RHOXV	ARGUMENT	NO.	4	(1-D)	DOUBLE	1	20	42			
RHOYV	ARGUMENT	NO.	5	(1-D)	DOUBLE	1	20	43			
SPK.DENS	ROUTINE				INTEGER	1					
SPK.WT	ROUTINE				DOUBLE	40					
SUMX	RECURSIVE VARIABLE	WORD	28		DOUBLE	35	42*	45			
SUMY	RECURSIVE VARIABLE	WORD	30		DOUBLE	36	43*	46			
TPID	RECURSIVE VARIABLE	WORD	20		DOUBLE	30	34				
UIB.W	IMPLIED SUBSCRIPT	SYS	4		INTEGER	21					
W	RECURSIVE VARIABLE	WORD	22		DOUBLE	31	40	42	43		

```
Options = SEQUENCE,ID,SUBCHK,XREF,NOEXPLIST,TRACE3

  1  FUNCTION FOR SPK.WT (K, MAX)
  2  ''
  3  ''ROUTINE CALCULATES THE WEIGHT FUNCTION (OR LAG WINDOW), AS A FUNCTION OF
  4  ''INTEGER LAG (K), USED IN OBTAINING THE SMOOTHED AUTOSPECTRUM VIA THE
  5  ''INDIRECT METHOD OF CALCULATING SPECTRA.  REFERENCE:  P. 259, JENKINS AND
  6  ''WATTS, SPECTRAL ANALYSIS AND ITS APPLICATIONS, C. 1968.  THE TUKEY LAG
  7  ''WINDOW (PAGE 244) IS USED HERE.
  8  ''
  9  DEFINE K AND MAX AS INTEGER VARIABLES
 10       IF K GE MAX
 11            RETURN WITH 0.0
 12       OTHERWISE
 13       LET W=0.5*(1.0+COS.F(PI.C*K/MAX))
 14       RETURN WITH W
 15  END ''SPK.WT
```

C R O S S - R E F E R E N C E

NAME	TYPE			MODE	LINE NUMBERS OF REFERENCES			
COS.F	ROUTINE			DOUBLE	13			
K	ARGUMENT	NO.	1	INTEGER	1	9	10	13
MAX	ARGUMENT	NO.	2	INTEGER	1	9	10	13
PI.C	PERMANENT ATTRIBUTE	SYS	33	DOUBLE	13			
SPK.WT	ROUTINE			DOUBLE	1			
W	RECURSIVE VARIABLE	WORD	1	DOUBLE	13	14		