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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The characteristic function of general second-order processors with nonstationary nonzero mean Gaussian inputs is derived in closed form. Three classes of processors are considered: in the first, the decision variable is the sum of K independent terms, each of second-order form involving two statistically dependent Gaussian random variables; the second class of processor is a narrowband cross-correlator of arbitrary Gaussian processes, with accumulation of K independent low pass filter output samples; in the third class, the decision variable is a general quadratic-plus-linear form		

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of M random variables, all statistically dependent on each other. Specializations to various forms of weighted energy detectors and correlators are made. Also, the characteristic function for the first class of processor subject to fading is evaluated.

Programs for evaluating the cumulative and exceedance distribution functions of all three classes of processors are given and have been used to plot representative examples of performance. A comparison with a simulation result corroborates the analysis and program of the first class of processor.

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Exact Performance of General Second-Order Processors for Gaussian Inputs

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Preface

This research was conducted under NUSC Project No. A75205, Subproject No. ZR0000101, "Applications of Statistical Communication Theory to Acoustic Signal Processing," Principal Investigator Dr. Albert H. Nuttall (Code 3302), Program Manager CAPT Z. L. Newcomb, Naval Material Command (MAT-05B).

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LIST OF SYMBOLS

s,t	jointly-Gaussian random variables
a,b,c,d,e	constant weights
x	output random variable of second-order processor
$f_x(\xi)$	characteristic function of random variable x
overbar	statistical average
D_1, D_2	denominator constants in characteristic function
N_0, N_1, N_2	numerator constants in characteristic function
μ_x	mean of random variable x
σ_x^2	variance of random variable x
K	number of independent terms in random variables x and v
M_{ij}	mean parameters, (12)
r	power scale factor, (18)-(19)
$s_1(t)$, $s_2(t)$	input signals to narrowband cross-correlator
$n_1(t)$, $n_2(t)$	input noises to narrowband cross-correlator
$z(t)$	lowpass filter output
v	narrowband cross-correlator system output random variable
$a_j(t)$, $b_j(t)$	in-phase and quadrature components of $s_j(t)$; $j = 1,2$
$A_j(t)$, $P_j(t)$	envelope and phase modulation of $s_j(t)$
$x_j(t)$, $y_j(t)$	in-phase and quadrature components of $n_j(t)$
σ_j^2	variance of $x_j(t)$ and $y_j(t)$, (27)-(28)
γ	complex correlation coefficient of noises, (27)
ρ,λ	real and imaginary parts of γ, (27)
$w(k)$	weight in narrowband cross-correlator
$\mu_z(s+n)$	mean of random variable z with signal and noise present
R_z	signal-to-noise ratio of random variable z
M	size of Hermitian and quadratic forms
X	random complex vector, Mx1
A	constant complex linear weight matrix
B	constant complex Hermitian weight matrix

LIST OF SYMBOLS (Con'd)

a_m	elements of A
b_{mn}	elements of B
q	Hermitian and quadratic form outputs
X^H	transpose and conjugate of X
E	mean of random vector X, (46)
$\tilde{X}$	ac component of X, (46)
K	covariance matrix of X, (46)
W	linearly transformed random variables, (54)
C	modal matrix of BK, (58)
Λ	eigenvalue matrix of BK, (58)
λ_m	elements of Λ
D	auxiliary vector, (59)
d_m	elements of D, (68)
ν_m	elements of $\overline{W}$, (68)
$\chi_q(n)$	n^{th} cumulant of random variable q
U,V	components of X, (71)
N	number of elements in U and V; M=2N
B^{ij}	partitioned matrices of B
$A^{(j)}$	partitioned vectors of A
u_n, v_n	components of U, V respectively
K_{uv}	partitioned matrices of K, (73)
K_o	common covariance matrix in special case, (89)
Q	modal matrix of K_o, (92)
Γ	eigenvalue matrix of K_o, (92)
γ_n	elements of Γ

EXACT PERFORMANCE OF GENERAL SECOND-ORDER PROCESSORS FOR GAUSSIAN INPUTS

INTRODUCTION

The performance of weighted energy detectors and correlators for processing deterministic and/or random signals in the presence of nonstationary noise is a topic of frequent interest. Most often, a second-moment approach is adopted, whereby the means and variances of the decision variable under the various hypotheses are evaluated and employed in a central limit assumption to get approximate false alarm and/or detection probabilities. This approach is suspect for small false alarm probabilities or for cases where the decision variable is not the sum of a large number of independent random variables all of comparable variance.

A recent technical report [1] has presented an accurate and efficient method for evaluating cumulative and exceedance distribution functions directly from characteristic functions. This approach is very fruitful for determining the performance of general time-varying second-order processors with nonstationary nonzero mean Gaussian inputs, since the characteristic function of the decision variable can be evaluated in closed form in these cases.

We will consider three classes of processors and derive the characteristic functions for all three decision variables in closed form. The first two classes are special cases of the third, but are of interest in their own right, since they include and immediately reduce to many practical processors in current use. Also there is no need to solve for the eigenvalues and eigenvectors of a general symmetric matrix that is encountered in the third more-general class of processors. Rather, the characteristic functions are given directly in terms of specified processor weights and input statistics.

There has been considerable effort on this problem in the past; for example, see [2,3] and the references listed therein. Most of the lengthy analytical derivations and results have been aimed at getting workable expressions for the probability density function and/or cumulative distribution function. Here, when we consider our three classes of processors, we encounter characteristic functions which are more general than that given in the recent paper for a filtered analog processor [3, eq. 5]; thus specialization of our results will yield those of [3] and the references listed therein. The technique employed here to proceed directly to the cumulative and exceedance distribution functions is a numerical one, as given in [1], and does not require any series expansions or analytical manipulations at all. The asymptotic behaviors of the cumulative and exceedance distribution functions on both tails are easily observed and will be found to corroborate the comment made in [3, p. 673] that these tails are generally exponential rather than Gaussian; however, there can be a considerable transition region.

The programs listed in the appendices require the user merely to input his processor weights, signal constants, and noise statistical parameters in a series of data statements at the top of the program, and to select values for

- L, limit on integral of characteristic function,
- Δ, sampling increment on characteristic function,
- b, additive constant, to guarantee a positive random variable, and
- M_f, size of FFT and storage required.

Selection of L and Δ is largely a matter of trial and error and is amply documented in the examples in [1].

A PARTICULAR SECOND-ORDER PROCESSOR

Before we embark on the analysis of the particular second-order processor of interest in this section, we solve the following simpler statistical problem. Let s and t be real jointly-Gaussian random variables with means m_s, m_t, standard deviations σ_s, σ_t, and correlation coefficient ρ; thus s and t are statistically dependent. Consider the random variable

$$x = as^2 + bt^2 + cst + ds + et, \tag{1}$$

where weightings a, b, c, d, e are arbitrary real constants. The characteristic function of random variable x is defined by*

$$f_x(\xi) = \overline{\exp(i\xi x)} = \overline{\exp(i\xi(as^2 + bt^2 + cst + ds + et))} =$$

$$= \iint du\, dv \exp(i\xi(au^2 + bv^2 + cuv + du + ev))\, p_{st}(u,v), \tag{2}$$

where the joint probability density function of s and t is

$$p_{st}(u,v) = \left(2\pi\, \sigma_s \sigma_t \sqrt{1-\rho^2}\right)^{-1} \exp\left[-\frac{\left(\frac{u-m_s}{\sigma_s}\right)^2 + \left(\frac{v-m_t}{\sigma_t}\right)^2 - 2\rho\left(\frac{u-m_s}{\sigma_s}\right)\left(\frac{v-m_t}{\sigma_t}\right)}{2(1-\rho^2)}\right]. \tag{3}$$

Substitution of (3) in (2) and use of the double integral

$$\iint dx\, dy \exp[-\alpha x^2 - \beta y^2 + 2\gamma xy + \mu x + \nu y] = \frac{\pi}{\sqrt{\alpha\beta-\gamma^2}} \exp\left[\frac{\beta\mu^2 + \alpha\nu^2 + 2\gamma\mu\nu}{4(\alpha\beta-\gamma^2)}\right]$$

$$\text{for} \quad \alpha_r > 0, \quad \beta_r > 0, \quad \alpha_r\beta_r > \gamma_r^2 \quad , \tag{4}$$

---

* Integrals without limits are over $(-\infty,+\infty)$.

(where sub r denotes the real part of complex constants α, β, γ, μ, ν) yields, after an extensive amount of manipulations, the characteristic function of random variable x as the compact closed form expression

$$f_x(\xi) = \left(1 - i\xi D_1 - \xi^2 D_2\right)^{-1/2} \exp\left[i\xi \frac{N_0 - i\xi N_1 - \xi^2 N_2}{1 - i\xi D_1 - \xi^2 D_2}\right] . \qquad (5)$$

The required real constants in (5) are given directly in terms of the processor weights and statistical parameters as

$$D_1 = 2(a\sigma_s^2 + b\sigma_t^2 + c\rho\sigma_s\sigma_t) \quad ,$$

$$D_2 = (4ab - c^2)(1 - \rho^2)\sigma_s^2\sigma_t^2 \quad ,$$

$$N_0 = am_s^2 + bm_t^2 + cm_s m_t + dm_s + em_t \quad ,$$

$$N_1 = (4ab - c^2)(\tfrac{1}{2} m_s^2\sigma_t^2 + \tfrac{1}{2} m_t^2\sigma_s^2 - \rho m_s m_t \sigma_s\sigma_t) +$$

$$+ (2ae - cd)\sigma_s(m_t\sigma_s - \rho m_s\sigma_t) +$$

$$+ (2bd - ce)\sigma_t(m_s\sigma_t - \rho m_t\sigma_s) -$$

$$- (\tfrac{1}{2} d^2\sigma_s^2 + \tfrac{1}{2} e^2\sigma_t^2 + de\rho\sigma_s\sigma_t) \quad ,$$

$$N_2 = -(ae^2 + bd^2 - cde)(1-\rho^2)\sigma_s^2\sigma_t^2 \quad . \qquad (6)$$

For later reference, the mean and variance of x follow from (5), upon expansion of $\ln f_x(\xi)$ in a power series in ξ, as

$$\mu_x = N_0 + \frac{1}{2} D_1,$$

$$\sigma_x^2 = \frac{1}{2} D_1^2 + 2N_0 D_1 - D_2 - 2N_1. \qquad (7)$$

(When $D_1 = 0$ in (6), it can be shown that $D_2 < 0$; thus characteristic function (5) never possesses any singularities along the real ξ axis.)

Second-Order Processor

Now let x be the sum of K independent terms of the form of (1):

$$x = \sum_{k=1}^{K} \left(a_k s_k^2 + b_k t_k^2 + c_k s_k t_k + d_k s_k + e_k t_k\right) \quad , \tag{8}$$

where real constants a_k, b_k, c_k, d_k, e_k can depend arbitrarily on k, and where means* m_{sk}, m_{tk}, standard deviations σ_{sk}, σ_{tk}, and correlation coefficients ρ_k are unrestricted (except that $\sigma_{sk} \geq 0$, $\sigma_{tk} \geq 0$, $|\rho_k| \leq 1$). The pair of random variables s_k, t_k is statistically independent of the pair s_n, t_n for all $k \neq n$. Thus random variable x is composed of a sum of K groups of random variables, where each group is statistically independent of every other group, but each group itself contains two statistically dependent random variables.

This processor in (8) is the general form of interest in this section. It can be time-varying when the weights $\{a_k, b_k, c_k, d_k, e_k\}$ vary with k, and nonstationary when the statistical parameters $\{m_{sk}, m_{tk}, \sigma_{sk}, \sigma_{tk}, \rho_k\}$ vary with k.

The characteristic function of (8) follows from (5) as

$$f_x(\xi) = \left[\prod_{k=1}^{K} \left\{1 - i\xi D_1(k) - \xi^2 D_2(k)\right\}\right]^{-1/2} *$$

$$* \exp\left[i\xi \sum_{k=1}^{K} \frac{N_0(k) - i\xi N_1(k) - \xi^2 N_2(k)}{1 - i\xi D_1(k) - \xi^2 D_2(k)}\right] \quad , \tag{9}$$

---

* These means can be interpreted as the deterministic signal components of the channels s and t, if desired.

where the identification of $D_1(k)$, etc., is obvious from (6). Only one (continuous) square root and one exponential per ξ value is required in (9), regardless of the number of terms added, K. The mean and variance of random variable x in (8) follows from (9) as

$$\mu_x = \sum_{k=1}^{K} \left[N_0(k) + \frac{1}{2} D_1(k)\right],$$

$$\sigma_x^2 = \sum_{k=1}^{K} \left[\frac{1}{2} D_1^2(k) + 2N_0(k)\, D_1(k) - D_2(k) - 2N_1(k)\right]. \quad (10)$$

Any analytical attempt at determining the probability density function or cumulative distribution function corresponding to characteristic function (9) would be a formidable task indeed. However, it is a very simple task via the method of [1] to get accurate numerical values for the cumulative and exceedance distribution functions. The program listing in appendix A accomplishes this task, based upon characteristic function (9) and the constants listed in (6). All the weights $\{a_k, b_k, c_k, d_k, e_k\}_1^K$ and statistical parameters $\{m_{sk}, m_{tk}, \sigma_{sk}, \sigma_{tk}, \rho_k\}_1^K$ are arbitrary. Observe that (9) is far more general than the characteristic function considered in [3, eq. 5], which itself required a very lengthy analytic treatment to get the probability density function and cumulative distribution function. In fact, there is little hope of getting any tractable analytic results for (9) when K is greater than 2.

Special Case 1

Suppose weightings a, b, c, d, e in (8) are independent of k and that statistics σ_s, σ_t, ρ are also independent of k. The decision variable x in (8) then simplifies to

$$x = \sum_{k=1}^{K} (as_k^2 + bt_k^2 + cs_k t_k + ds_k + et_k) \quad . \quad (11)$$

Then D_1, D_2, N_2 are independent of k. If we define mean parameters

$$M_{20} = \sum_{k=1}^{K} m_{sk}^2 , \quad M_{02} = \sum_{k=1}^{K} m_{tk}^2 , \quad M_{11} = \sum_{k=1}^{K} m_{sk}\, m_{tk} \quad ,$$

$$M_{10} = \sum_{k=1}^{K} m_{sk} , \quad M_{01} = \sum_{k=1}^{K} m_{tk} \quad , \tag{12}$$

the characteristic function of x in (9) then takes the simpler form

$$f_x(\xi) = \left(1 - i\xi D_1 - \xi^2 D_2\right)^{-K/2} \exp\left[i\xi \frac{N_0' - i\xi N_1' - \xi^2 N_2'}{1 - i\xi D_1 - \xi^2 D_2}\right] , \tag{13}$$

where D_1 and D_2 are still given by (6), and

$$N_0' = aM_{20} + bM_{02} + cM_{11} + dM_{10} + eM_{01} \quad ,$$

$$N_1' = (4ab - c^2)(\tfrac{1}{2}\,\sigma_t^2 M_{20} + \tfrac{1}{2}\,\sigma_s^2 M_{02} - \rho\sigma_s\sigma_t M_{11}) +$$

$$+ (2ae - cd)\sigma_s(\sigma_s M_{01} - \rho\sigma_t M_{10}) +$$

$$+ (2bd - ce)\sigma_t(\sigma_t M_{10} - \rho\sigma_s M_{01}) -$$

$$- K\,(\tfrac{1}{2}\, d^2\sigma_s^2 + \tfrac{1}{2}\, e^2\sigma_t^2 + de\rho\sigma_s\sigma_t) \quad ,$$

$$N_2' = -\, K(ae^2 + bd^2 - cde)(1 - \rho^2)\sigma_s^2\sigma_t^2 \quad . \tag{14}$$

(The choice of K = 2 and $N_2' = 0$ in (13) corresponds to the form given in [3, eq. 5].) Observe that the characteristic function in (13) (and therefore the performance) of the processor in (11) depends on the means $\{m_{sk}\}$ and $\{m_{tk}\}$ only through the parameters $\{M_{ij}\}$ defined in (12). The mean and variance of random variable x in (11) follow from characteristic function (13) as

$$\mu_x = N_0' + \frac{1}{2} KD_1,$$

$$\sigma_x^2 = \frac{1}{2} KD_1^2 + 2N_0'D_1 - KD_2 - 2N_1' \quad . \tag{15}$$

Special Case 2

Let us also assume d = 0, e = 0 in (11) above; then the pertinent decision variable is given by

$$x = \sum_{k=1}^{K} (as_k^2 + bt_k^2 + cs_kt_k) \quad . \tag{16}$$

D_1 and D_2 are still given by (6), and there follows from (14),

$$N_0' = aM_{20} + bM_{02} + cM_{11} \quad ,$$

$$N_1' = (4ab - c^2)(\frac{1}{2}\sigma_t^2M_{20} + \frac{1}{2}\sigma_s^2M_{02} - \rho\sigma_s\sigma_tM_{11}) \quad ,$$

$$N_2' = 0 \quad . \tag{17}$$

The characteristic function of x is given by (13), with $N_2' = 0$. The mean and variance of x in (16) are given by (15).

Fading for Special Case 2

Let the mean parameters $\{M_{ij}\}$ in (12) be subject to slow fading; i.e., replace

$$M_{20} \text{ by } rM_{20},\ M_{02} \text{ by } rM_{02},\ M_{11} \text{ by } rM_{11}, \tag{18}$$

where power scale factor r has probability density function

$$p_r(u) = \frac{\nu^\nu}{\Gamma(\nu)}\, u^{\nu-1}\, e^{-\nu u} \quad \text{for } u > 0,\ \nu > 0;$$

$$\bar{r} = 1,\ \sigma_r^2 = \frac{1}{\nu},\ \chi_r(n) = \frac{(n-1)!}{\nu^{n-1}} \quad \text{for } n \geq 1. \tag{19}$$

This form of fading is encountered in diversity combination receivers; see, for example, [4, eq. 9 et seq.] and [5, eq. 24 et seq.]. Then (13), (17), and (18) yield the conditional characteristic function, for a specified r, as

$$f_x(\xi|r) = \left(1 - i\xi D_1 - \xi^2 D_2\right)^{-K/2} \exp\left[i\xi r\, \frac{N_0' - i\xi N_1'}{1 - i\xi D_1 - \xi^2 D_2}\right] . \tag{20}$$

Weighting (20) according to the probability density function in (19), and performing the integral, there follows, for the characteristic function of the decision variable x in (16), the result

$$f_x(\xi) = \frac{\left(1 - i\xi D_1 - \xi^2 D_2\right)^{\nu - \frac{K}{2}}}{\left(1 - i\xi(D_1 + N_0'/\nu) - \xi^2(D_2 + N_1'/\nu)\right)^{\nu}} . \tag{21}$$

(The limit of (21) as $\nu \to +\infty$ is again (13) with $N_2' = 0$, as in (17); this agrees with the fact that the corresponding limit of the probability density function in (19) is $p_r(u) = \delta(u-1)$.) The mean and variance of x in (16) follow from characteristic function (21) as

$$\mu_x = N_0' + \frac{1}{2} K D_1 ,$$

$$\sigma_x^2 = \frac{1}{2} K D_1^2 + 2N_0' D_1 - K D_2 - 2N_1' + N_0'^2/\nu . \tag{22}$$

Observe that mean μ_x is independent of ν, the power law in fading (19). A program for the cumulative and exceedance distribution functions corresponding to characteristic function (21) is given in appendix B.

SPECIAL FORMS OF SECOND-ORDER PROCESSOR (8)

Before embarking on the analysis of the other two classes of processors, we will explicitly detail some of the special forms that processor (8) reduces to, under particular selections of the weightings and statistical parameters. A rather broad collection of typical processors will be seen to be included. In the following, any unspecified weights $\{a_k, b_k, c_k, d_k, e_k\}$ are zero, and any unspecified statistical parameters that do not appear in the final characteristic function are irrelevant.

I. Gaussian

$$d_k = 1$$

$$x = \sum_{k=1}^{K} s_k$$

$$f_x(\xi) = \exp\left[i\xi \sum_{k=1}^{K} m_{sk} - \frac{1}{2}\xi^2 \sum_{k=1}^{K} \sigma_{sk}^2\right]$$

II. Chi-square of K Degrees of Freedom

$$a_k = 1, \quad m_{sk} = 0, \quad \sigma_{sk} = 1$$

$$x = \sum_{k=1}^{K} s_k^2$$

$$f_x(\xi) = \left(1 - i\xi 2\right)^{-K/2}$$

III. Non-Central Chi-Square (Q_M Distribution if K = 2M)

$$a_k = 1, \quad \sigma_{sk} = \sigma_s$$

$$x = \sum_{k=1}^{K} s_k^2$$

$$f_x(\xi) = \left(1 - i\xi 2\sigma_s^2\right)^{-K/2} \exp\left[\frac{i\xi \sum_{k=1}^{K} m_{sk}^2}{1 - i\xi 2\sigma_s^2}\right]$$

IV. Weighted Energy Detector

$$a_k \neq 0, \quad d_k \neq 0$$

$$x = \sum_{k=1}^{K} (a_k s_k^2 + d_k s_k)$$

$$f_x(\xi) = \left[\prod_{k=1}^{K} \left\{1 - i\xi 2a_k \sigma_{sk}^2\right\}\right]^{-\frac{1}{2}} \exp\left[i\xi \sum_{k=1}^{K} \frac{a_k m_{sk}^2 + d_k m_{sk} + i\xi \frac{1}{2} d_k^2 \sigma_{sk}^2}{1 - i\xi 2a_k \sigma_{sk}^2}\right]$$

V. Weighted Cross-Correlator

$$c_k \neq 0$$

$$x = \sum_{k=1}^{K} c_k s_k t_k$$

$$D_1(k) = 2c_k \rho_k \sigma_{sk} \sigma_{tk},$$

$$D_2(k) = -c_k^2 (1 - \rho_k^2) \sigma_{sk}^2 \sigma_{tk}^2,$$

$$N_0(k) = c_k m_{sk} m_{tk},$$

$$N_1(k) = -c_k^2 \left(\frac{1}{2} m_{sk}^2 \sigma_{tk}^2 + \frac{1}{2} m_{tk}^2 \sigma_{sk}^2 - \rho_k m_{sk} m_{tk} \sigma_{sk} \sigma_{tk}\right),$$

$$N_2(k) = 0 \quad .$$

Characteristic function $f_x(\xi)$ is given by (9).

VI. Two-Channel Energy Detector

$$a_k \neq 0, \quad b_k \neq 0$$

$$x = \sum_{k=1}^{K} (a_k s_k^2 + b_k t_k^2)$$

$$D_1(k) = 2(a_k \sigma_{sk}^2 + b_k \sigma_{tk}^2) \ ,$$

$$D_2(k) = 4 a_k b_k (1 - \rho_k^2) \sigma_{sk}^2 \sigma_{tk}^2 \ ,$$

$$N_0(k) = a_k m_{sk}^2 + b_k m_{tk}^2,$$

$$N_1(k) = 4 a_k b_k \left(\frac{1}{2} m_{sk}^2 \sigma_{tk}^2 + \frac{1}{2} m_{tk}^2 \sigma_{sk}^2 - \rho_k m_{sk} m_{tk} \sigma_{sk} \sigma_{tk}\right),$$

$$N_2(k) = 0.$$

Characteristic function $f_x(\xi)$ is given by (9). A simple application of this particular processor was encountered in [6, eqs. 25-26].

VII. Two-Channel Energy Detector and Cross-Correlator

$$a_k \neq 0, \quad b_k \neq 0, \quad c_k \neq 0$$

$$x = \sum_{k=1}^{K} (a_k s_k^2 + b_k t_k^2 + c_k s_k t_k)$$

$$D_1(k) = 2(a_k \sigma_{sk}^2 + b_k \sigma_{tk}^2 + c_k \rho_k \sigma_{sk} \sigma_{tk}) \quad ,$$

$$D_2(k) = (4a_k b_k - c_k^2)(1 - \rho_k^2)\sigma_{sk}^2 \sigma_{tk}^2 \quad ,$$

$$N_0(k) = a_k m_{sk}^2 + b_k m_{tk}^2 + c_k m_{sk} m_{tk} \quad ,$$

$$N_1(k) = (4a_k b_k - c_k^2)(\frac{1}{2} m_{sk}^2 \sigma_{tk}^2 + \frac{1}{2} m_{tk}^2 \sigma_{sk}^2 - \rho_k m_{sk} m_{tk} \sigma_{sk} \sigma_{tk}) \quad ,$$

$$N_2(k) = 0 \quad .$$

Characteristic function $f_x(\xi)$ is given by (9).

NARROWBAND CROSS-CORRELATOR

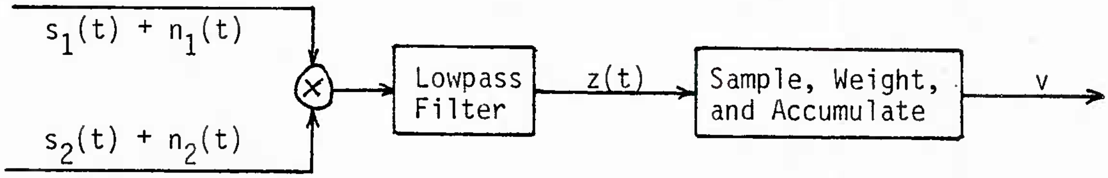

Figure 1. Narrowband Cross-Correlator

The processor of interest in this section is depicted in figure 1. Input signals $s_1(t)$ and $s_2(t)$ are arbitrary deterministic narrowband real waveforms:

$$s_j(t) = \mathrm{Re}\left\{\underline{s}_j(t)\exp(i2\pi f_o t)\right\} = A_j(t)\cos(2\pi f_o t + P_j(t)) =$$

$$= a_j(t)\cos(2\pi f_o t) - b_j(t)\sin(2\pi f_o t) \quad \text{for } j = 1,2, \tag{23}$$

where input signal complex envelope

$$\underline{s}_j(t) = A_j(t)\exp(iP_j(t)) = a_j(t) + ib_j(t) \tag{24}$$

in terms of polar or rectangular low-frequency components, respectively.

Input noises $n_1(t)$ and $n_2(t)$ are zero-mean correlated narrowband jointly-Gaussian processes which may be nonstationary:

$$n_j(t) = \mathrm{Re}\left\{\underline{n}_j(t)\exp(i2\pi f_o t)\right\} = x_j(t)\cos(2\pi f_o t) - y_j(t)\sin(2\pi f_o t), \tag{25}$$

where noise complex envelope

$$\underline{n}_j(t) = x_j(t) + iy_j(t) \quad \text{for } j = 1,2. \tag{26}$$

The statistics of the input noise complex envelopes are arbitrary:

$$\overline{|\underline{n}_1(t)|^2} = 2\sigma_1^2 ,$$

$$\overline{|\underline{n}_2(t)|^2} = 2\sigma_2^2 ,$$

$$\overline{\underline{n}_1(t)\ \underline{n}_2^*(t)} = 2\sigma_1\sigma_2\gamma, \quad \text{where } \gamma = \rho + i\lambda = |\gamma|\exp(i\phi),$$

$$\overline{\underline{n}_j(t)\ \underline{n}_m(t)} = 0 \text{ for all } j,\ m. \tag{27}$$

The quantities σ_1, σ_2, γ can all vary with time t, for nonstationary noise processes. There follows, for the statistics of the in-phase and quadrature components defined in (25),

$$\overline{x_1^2} = \overline{y_1^2} = \sigma_1^2, \qquad \overline{x_1 y_1} = 0 ,$$

$$\overline{x_2^2} = \overline{y_2^2} = \sigma_2^2, \qquad \overline{x_2 y_2} = 0 ,$$

$$\overline{x_1 x_2} = \overline{y_1 y_2} = \sigma_1\sigma_2\rho ,$$

$$\overline{x_2 y_1} = -\overline{x_1 y_2} = \sigma_1\sigma_2\lambda . \tag{28}$$

The reason for breaking out this narrowband cross-correlator as a separate problem is now apparent from (28). Namely, at each time instant, a group of <u>four</u> random variables are statistically dependent on each other. This case does not fall into the framework of (8) above, since only two random variables were dependent there.

Using the narrowband character of all the waveforms in (24) and (26), the lowpass filter output in figure 1 may be expressed as

$$z(t) = \frac{1}{2}[x_1(t) + a_1(t)][x_2(t) + a_2(t)] + \frac{1}{2}[y_1(t) + b_1(t)][y_2(t) + b_2(t)]. \tag{29}$$

The final system output in figure 1 is the weighted sum of K terms,

$$v = \sum_{k=1}^{K} w(k)\; z(t_k) \;, \qquad (30)$$

where it is assumed that the time separations between samples at instants $\{t_k\}$ lead to statistically independent random variables $\{z(t_k)\}$. The weights and statistics can change with sample time t_k, in an arbitrary fashion.

Based upon the method in [7], we find the characteristic function of z(t) in (29) to be given by

$$f_z(\xi,t) = \frac{1}{1-i\xi D_1 + \xi^2 D_2} \; \exp\left[i\xi \, \frac{N_0 + i\xi N_1}{1-i\xi D_1 + \xi^2 D_2}\right], \qquad (31)$$

where the constants (in their most compact form) are given by

$$D_1 = \sigma_1 \sigma_2 \rho \;,$$

$$D_2 = \frac{1}{4}\, \sigma_1^2 \sigma_2^2 (1-\rho^2-\lambda^2) \;,$$

$$N_0 = \frac{1}{2}(a_1 a_2 + b_1 b_2) \;,$$

$$N_1 = \frac{1}{8}\left[\sigma_2^2(a_1^2 + b_1^2) + \sigma_1^2(a_2^2 + b_2^2) - 2\sigma_1\sigma_2\rho(a_1 a_2 + b_1 b_2) - 2\sigma_1\sigma_2\lambda(a_2 b_1 - a_1 b_2)\right]. \qquad (32)$$

(The characteristic function and constants in (31) and (32) are not to be interchanged or confused with any earlier results in previous sections. In fact, observe there is no square root involved in (31).) All of the parameters in (32) can vary with time t.

In terms of the signal polar definitions in (24) and the complex noise correlation coefficient γ in (27), alternative expressions to (32) (where we have emphasized the t-dependence) are

$$D_1 = \sigma_1(t)\ \sigma_2(t)\ \mathrm{Re}\{\gamma(t)\} = \sigma_1(t)\ \sigma_2(t)\ |\gamma(t)|\ \cos\phi(t),$$

$$D_2 = \frac{1}{4}\ \sigma_1^2(t)\ \sigma_2^2(t)\ \left(1 - |\gamma(t)|^2\right),$$

$$N_0 = \frac{1}{2}\ \mathrm{Re}\left\{\underline{s}_1^*(t)\ \underline{s}_2(t)\right\} = \frac{1}{2}\ A_1(t)\ A_2(t)\ \cos[P_1(t)-P_2(t)],$$

$$N_1 = \frac{1}{8}\left[\sigma_2^2(t)\left|\underline{s}_1(t)\right|^2 + \sigma_1^2(t)\left|\underline{s}_2(t)\right|^2 - 2\ \sigma_1(t)\ \sigma_2(t)\ \mathrm{Re}\left\{\underline{s}_1^*(t)\ \underline{s}_2(t)\ \gamma(t)\right\}\right] =$$

$$= \frac{1}{8}\left[\sigma_2^2(t)\ A_1^2(t) + \sigma_1^2(t)\ A_2^2(t) - 2\ \sigma_1(t)\ \sigma_2(t)\ A_1(t)\ A_2(t)\ |\gamma(t)|\ \cos[P_1(t)-P_2(t)-\phi(t)]\right]. \tag{33}$$

The mean and variance of z(t) in (29) follow from (31) as

$$\mu_z = D_1 + N_0,$$

$$\sigma_z^2 = D_1^2 + 2D_2 + 2D_1N_0 + 2N_1. \tag{34}$$

Finally, the characteristic function of the narrowband cross-correlator output v in (30) follows from (31) as

$$f_v(\xi) = \prod_{k=1}^{K} f_z\left(\xi w(k), t_k\right) =$$

$$= \left[\prod_{k=1}^{K}\left\{1 - i\xi w(k)\ D_1(k) + \xi^2 w^2(k)\ D_2(k)\right\}\right]^{-1} \exp\left[i\,\xi \sum_{k=1}^{K} \frac{w(k)\ N_0(k) + i\xi w^2(k)\ N_1(k)}{1 - i\xi w(k)\ D_1(k) + \xi^2 w^2(k)\ D_2(k)}\right], \tag{35}$$

where we have allowed all the parameters in (32) and (33) to vary with time t_k. The mean and variance of output v follow from (35) as

$$\mu_v = \sum_{k=1}^{K} w(k)\ [D_1(k) + N_0(k)],$$

$$\sigma_v^2 = \sum_{k=1}^{K} w^2(k)\left[D_1^2(k) + 2D_2(k) + 2D_1(k)N_0(k) + 2N_1(k)\right]. \tag{36}$$

A program for the evaluation of the cumulative and exceedance distribution functions via (35) is given in appendix C.

In comparison with earlier results in [7], we have obtained the following extensions here:

1. The input signals are arbitrary narrowband waveforms; they are not limited to two sine waves at the same frequency;
2. The Gaussian input noises can be nonstationary;
3. The number of terms summed to yield the narrowband cross-correlator output can be greater than 1;
4. The characteristic function is in its most compact form, and the constants are expressed directly in terms of given quantities, having eliminated all auxiliary variables.

Output Signal-to-Noise Ratio

It is sometimes desirable to have simple expressions for the output signal-to-noise ratio of the narrowband cross-correlator in figure 1. In terms of the lowpass filter output z(t), we observe first from (32)-(34) that

$$\mu_z(s) = \mu_z(s+n) - \mu_z(n) = N_0 = \frac{1}{2} A_1 A_2 \cos(P_1-P_2) . \tag{37}$$

We then have two alternative definitions of the signal-to-noise ratio at the lowpass filter output:

$$R_z(n) = \frac{\mu_z^2(s)}{\sigma_z^2(n)} = \frac{A_1^2 A_2^2 \cos^2(P_1-P_2)}{2 \sigma_1^2 \sigma_2^2(1+\rho^2-\lambda^2)} ,$$

$$R_z(s+n) = \frac{\mu_z^2(s)}{\sigma_z^2(s+n)} = \frac{A_1^2 A_2^2 \cos^2(P_1-P_2)}{4(D_1^2 + 2D_2 + 2D_1N_0 + 2N_1)} . \tag{38}$$

These closed form expressions allow for arbitrary noise correlations and are considerably simpler than [7, eqs. 41-43]. The signal-to-noise ratios of system output v in figure 1 are K times greater than either form in (38).

Specialization to Narrowband Energy Detector

If the signal and noise parameters in (24) and (27) are chosen as

$$a_1(t) = a_2(t) = a(t) ,$$

$$b_1(t) = b_2(t) = b(t) ,$$

$$\sigma_1(t) = \sigma_2(t) = \sigma(t) ,$$

$$\rho(t) = 1, \lambda(t) = 0 , \tag{39}$$

then figure 1 reduces to identical input channels, that is, a narrowband energy detector. There follows from (32),

$$D_1 = \sigma^2(t), \quad D_2 = 0 ,$$

$$N_0 = \frac{1}{2}\left(a^2(t) + b^2(t)\right) = \frac{1}{2} A^2(t), \quad N_1 = 0 , \tag{40}$$

and (31) becomes

$$f_z(\xi,t) = \frac{1}{1 - i\xi\sigma^2(t)} \exp\left[i\xi \frac{A^2(t)/2}{1 - i\xi\sigma^2(t)}\right]. \tag{41}$$

Corresponding results for the system output v are easily obtained from this.

REDUCTION OF HERMITIAN AND LINEAR FORM

The most general case of interest in this section is as follows: random complex matrix

$$X = [x_1\ x_2\ \dots\ x_M]^T \tag{42}$$

is Mx1; constant complex matrix

$$A = [a_1\ a_2\ \dots\ a_M]^T \tag{43}$$

is Mx1; and constant complex matrix

$$B = \begin{bmatrix} b_{11} & \dots & b_{1M} \\ & & \\ b_{M1} & \dots & b_{MM} \end{bmatrix} \tag{44}$$

is MxM and Hermitian. The Hermitian and linear form we consider is

$$q = X^H B X + \frac{1}{2}(X^H A + A^H X) =$$

$$= \sum_{m,n=1}^{M} x_m^* b_{mn} x_n + \frac{1}{2} \sum_{m=1}^{M} (x_m^* a_m + a_m^* x_m), \tag{45}$$

which is real. Random variable q is a weighted sum of all possible products of $\{x_m^*\}$ and $\{x_n\}$, plus linear combinations.* A and B are called the weighting matrices.

---

* For M=2 or 4, and real variables and weights, (45) reduces to the earlier forms given in (1) and (29).

We will concentrate in this section on reducing form (45) to a weighted sum of squares of uncorrelated random variables. This stepping stone does not require any Gaussian assumptions on X and is therefore useful as a separate item.

The relevant statistics pertaining to random vector X are

$$\overline{X} = E \quad \text{(mean matrix)},$$

$$\tilde{X} = X - \overline{X} = X - E,$$

$$\text{Cov}\{X\} = \overline{\tilde{X}\tilde{X}^H} = K \quad \text{(covariance matrix)}, \tag{46}$$

where statistics matrices E and K are given. MxM matrix K is always Hermitian and non-negative definite. We assume K is positive definite; otherwise eliminate the linearly dependent components of $\tilde{X}$. We allow x_m and x_n to be correlated with each other for any m and n; this situation is much more general than the investigations above.

Let C be a constant MxM matrix and form the linearly transformed variables

$$W = C^H X = [w_1 \; w_2 \; \dots \; w_M]^T. \tag{47}$$

Then the statistics of W are given by

$$\overline{W} = C^H E,$$

$$\tilde{W} = W - \overline{W} = C^H \tilde{X},$$

$$\text{Cov}\{W\} = \overline{\tilde{W}\tilde{W}^H} = C^H \overline{\tilde{X}\tilde{X}^H} C = C^H K C. \tag{48}$$

Also, from (47), since

$$X = C^{-H} W, \tag{49}$$

then we can express (45) as

$$q = W^H C^{-1} B C^{-H} W + \frac{1}{2}(W^H D + D^H W), \tag{50}$$

where we define constant Mx1 matrix

$$D = C^{-1} A = [d_1 \; d_2 \; \ldots \; d_M]^T. \tag{51}$$

We want to have, from (48) and (50),

$$C^H K C = I \tag{52}$$

and

$$C^{-1} B C^{-H} = \Lambda = \mathrm{diag}(\lambda_1 \; \lambda_2 \; \ldots \; \lambda_M); \quad \text{i.e. } C^H B^{-1} C = \Lambda^{-1} , \tag{53}$$

for then, in addition to the relation between the means,

$$\overline{W} = C^H E, \tag{54}$$

we have the desirable properties

$$\mathrm{Cov}\{W\} = I, \tag{55}$$

and

$$q = W^H \Lambda W + \frac{1}{2}(W^H D + D^H W) = \sum_{m=1}^{M} \lambda_m |w_m|^2 + \mathrm{Re} \sum_{m=1}^{M} d_m^* w_m \quad . \tag{56}$$

That is, the random vector W given by (47) is composed of uncorrelated unit-variance components, and q is a weighted sum of magnitude-squares of these components, in addition to a linear sum.

We now have to address the problem of determining the MxM matrices C and Λ in (52) and (53). From [8, p. 106, Theorem 2], we identify

$$M \rightarrow K, \quad K \rightarrow B^{-1}, \Lambda \rightarrow \Lambda^{-1}; \tag{57}$$

then according to [8, p. 107, eq. 29], we must solve for C and Λ in the equation

$$B^{-1}C = KC\Lambda^{-1}, \quad \text{i.e.} \quad BKC = C\Lambda. \tag{58}$$

So the only matrix that need be considered is the MxM product BK. C is the modal matrix, and Λ the eigenvalue matrix, of BK. Also, from (51),

$$D = C^H A, \text{ since } C^{-1} = C^H. \tag{59}$$

Letting $C = [C^{(1)} \ \dots \ C^{(M)}]$, where eigenvector $C^{(m)}$ is a Mx1 matrix, (58) can be expressed as

$$BKC^{(m)} = \lambda_m C^{(m)} \quad \text{for} \quad 1 \leq m \leq M. \tag{60}$$

Several important properties hold for Λ and C:

The $\{\lambda_m\}_1^M$ are all real, but can be positive, zero, or negative.
If K and B are real, then C is real. (61)
If B is positive definite, then $\lambda_m > 0$ for $1 \leq m \leq M$.
If A = 0 and E = 0, there is no need to solve (58) for C,
because D = 0 and $\overline{W} = 0$.

QUADRATIC AND LINEAR FORM

If random vector X is real Gaussian, if A is real, and if B is real symmetric, then mean E and covariance K are real, and it follows that modal matrix C is also real. Also from (47) and (59), W and D are real. Equation (45) reduces to

$$q = X^T B X + X^T A = \sum_{m,n=1}^{M} x_m b_{mn} x_n + \sum_{m=1}^{M} a_m x_m \,, \tag{62}$$

which is a quadratic form and linear form.

Letting mean $\overline{W}$ in (54) be expressed as

$$\overline{W} = [\nu_1 \; \nu_2 \; \cdots \; \nu_M]^T, \tag{63}$$

the Gaussian character of X and the linear transformation (47) allow us to write the probability density function of W as a product:

$$p(W) = \prod_{m=1}^{M} \left\{ (2\pi)^{-1/2} \exp\left(-\frac{1}{2} (w_m - \nu_m)^2 \right) \right\}. \tag{64}$$

Here we used property (55). Since we now have, from (56),

$$q = \sum_{m=1}^{M} \left(\lambda_m w_m^2 + d_m w_m \right), \tag{65}$$

the characteristic function of q is

$$f_q(\xi) = \overline{\exp(i\xi q)} = \overline{\exp\left(i\xi \sum_{m=1}^{M} \left(\lambda_m w_m^2 + d_m w_m \right) \right)} =$$

$$= \left[\prod_{m=1}^{M} \left\{ 1 - i2\lambda_m \xi \right\} \right]^{-1/2} \exp\left[i\xi \sum_{m=1}^{M} \frac{\lambda_m \nu_m^2 + d_m \nu_m + i\xi d_m^2/2}{1 - i2\lambda_m \xi} \right], \tag{66}$$

where the square root must be a continuous function of ξ, <u>not</u> a principal value square root.* Notice that only one square root and one exponential is required per ξ value. Observe that the characteristic function depends on the separate values $\{\nu_m\}_1^M$ and $\{d_m\}_1^M$, not merely on their sums. If A = E = 0, the exponential is unity, by virtue of (54) and (59). And if M=2, (66) reduces to (5), while M=4 leads to form (31).

To summarize, the characteristic function $f_q(\xi)$ in (66) for random variable q in (62) requires the constants $\{\lambda_m\}$, $\{d_m\}$, and $\{\nu_m\}$ for $1 \leq m \leq M$. The initially given quantities are weighting matrices A, B and statistics matrices E, K. We first solve the equation (58),

$$BKC = C\Lambda , \tag{67}$$

for eigenvalue matrix Λ and modal matrix C corresponding to BK. Then

$$\Lambda = \operatorname{diag}(\lambda_1 \ \lambda_2 \ \ldots \ \lambda_M) \ ,$$

$$D = C^T A = [d_1 \ d_2 \ \ldots \ d_M]^T \ ,$$

$$\overline{W} = C^T E = [\nu_1 \ \nu_2 \ \ldots \ \nu_M]^T \ . \tag{68}$$

If the mean of input X is zero, E = 0, and if the linear weighting form is zero, A = 0, then there is no need to solve for modal matrix C of BK in (67). Then $D = \overline{W} = 0$ and the exponential term in (66) is unity. One only need compute eigenvalue matrix Λ of BK in this case.

A program for the evaluation of the cumulative and exceedance distribution functions corresponding to characteristic function (66) is listed in appendix D. The inputs to the program are considered to be M, $\{\lambda_m\}$,

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* That is, the square root is the analytic continuation of the function defined as 1 at ξ =0.

$\{d_m\}$, $\{\nu_m\}$; that is, it is presumed that (67) and (68) have already been solved prior to use of the program.

The cumulants of q are obtained from (66) as

$$\chi_q(n) = \begin{cases} \sum_{m=1}^{M} \left(\lambda_m + \lambda_m \nu_m^2 + d_m \nu_m\right) = \mu_q & \text{for } n = 1 \\ 2^{n-1}(n-1)! \sum_{m=1}^{M} \lambda_m^{n-2} \left[\lambda_m^2 + n\left(\lambda_m \nu_m + \frac{1}{2} d_m\right)^2\right] & \text{for } n \geq 2 \end{cases}. \tag{69}$$

In particular, the variance of q is

$$\chi_q(2) = 2\sum_{m=1}^{M} \left[\lambda_m^2 + 2\left(\lambda_m \nu_m + \frac{1}{2} d_m\right)^2\right] = \sigma_q^2 \quad . \tag{70}$$

If another random variable is formed by the sum of several independent random variables q_j with the form (62), but with different sizes M_j, the new characteristic function is the product of terms like (66).

Breakdown of X into Two Components

It is useful to investigate a particular version of the general results above, because the resultant forms correspond to some often-realized practical energy detectors and correlators. We let M = 2N, and

$$X = \begin{bmatrix} U \\ V \end{bmatrix} \quad , \quad B = \begin{bmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{bmatrix} \quad , \quad A = \begin{bmatrix} A^{(1)} \\ A^{(2)} \end{bmatrix} \quad , \tag{71}$$

where U, V, $A^{(1)}$, $A^{(2)}$ are Nx1 real matrices, and $\{B^{ij}\}$ are NxN real matrices. Also B^{11} and B^{22} are symmetric, while $B^{21} = B^{12^T}$. Then (62) can be expressed as

$$q = X^T B X + X^T A = \begin{bmatrix} U^T & V^T \end{bmatrix} \begin{bmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} + \begin{bmatrix} U^T & V^T \end{bmatrix} \begin{bmatrix} A^{(1)} \\ A^{(2)} \end{bmatrix} =$$

$$= U^T B^{11} U + U^T B^{12} V + V^T B^{21} U + V^T B^{22} V + U^T A^{(1)} + V^T A^{(2)} =$$

$$= U^T B^{11} U + 2U^T B^{12} V + V^T B^{22} V + U^T A^{(1)} + V^T A^{(2)} =$$

$$= \sum_{m,n=1}^{N} \left(u_m\, b_{mn}^{11}\, u_n + 2u_m\, b_{mn}^{12}\, v_n + v_m\, b_{mn}^{22}\, v_n \right) + \sum_{n=1}^{N} \left(u_n a_n^{(1)} + v_n a_n^{(2)} \right) =$$

= all possible auto and cross combinations of random

variables $\{u_n\}_1^N$ and $\{v_n\}_1^N$, plus linear combinations. (72)

We also have, from (47) and (68),

$$W = C^T X = C^T \begin{bmatrix} U \\ V \end{bmatrix}, \quad \overline{W} = C^T \begin{bmatrix} \overline{U} \\ \overline{V} \end{bmatrix} = C^T \begin{bmatrix} E_u \\ E_v \end{bmatrix} \quad , \quad D = C^T \begin{bmatrix} A^{(1)} \\ A^{(2)} \end{bmatrix} \quad ,$$

$$K = \mathrm{Cov}\{X\} = \overline{\tilde{X}\tilde{X}^T} = \overline{\begin{bmatrix} \tilde{U} \\ \tilde{V} \end{bmatrix} \begin{bmatrix} \tilde{U}^T & \tilde{V}^T \end{bmatrix}} = \begin{bmatrix} K_{uu} & K_{uv} \\ K_{vu} & K_{vv} \end{bmatrix} \quad . \tag{73}$$

Then the fundamental matrix required in (67) is expressable as

$$BK = \begin{bmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{bmatrix} \begin{bmatrix} K_{uu} & K_{uv} \\ K_{vu} & K_{vv} \end{bmatrix}, \tag{74}$$

which is a 2Nx2N matrix. Also random variable (65) is now

$$q = \sum_{m=1}^{2N} \left(\lambda_m w_m^2 + d_m w_m \right) \quad , \tag{75}$$

which has 2N terms. The characteristic function of q, in general, follows from (66) and (68) as

$$f_q(\xi) = \left[\prod_{m=1}^{2N} \left\{1-i2\lambda_m\xi\right\}\right]^{-1/2} \exp\left[i\xi\sum_{m=1}^{2N} \frac{\lambda_m\nu_m^2 + d_m\nu_m + i\xi d_m^2/2}{1-i2\lambda_m\xi}\right], \quad (76)$$

where

$$\overline{W} = [\nu_1 \ \ldots \ \nu_{2N}]^T = C^T\begin{bmatrix}E_u \\ E_v\end{bmatrix}, \quad D = [d_1 \ d_2 \ \ldots \ d_{2N}]^T = C^T\begin{bmatrix}A^{(1)} \\ A^{(2)}\end{bmatrix}. \quad (77)$$

(If $A^{(1)} = A^{(2)} = E_u = E_v = 0$, then $D = 0$ and $\overline{W} = 0$, and there is no need to solve for modal matrix C; the exponential in $f_q(\xi)$ in (76) is then unity.)

As a special case, if $A = 0$, $B^{11} = 0$, $B^{22} = 0$, then (71) and (73) yield

$$B = \begin{bmatrix}0 & B^{12} \\ B^{21} & 0\end{bmatrix}, \quad D = 0 , \quad (78)$$

and (72) gives

$$q = 2U^TB^{12}V = 2\sum_{m,n=1}^{N} u_m \, b_{mn}^{12} \, v_n =$$

$$= \text{all possible cross combinations of } \{u_n\}_1^N \text{ and } \{v_n\}_1^N. \quad (79)$$

Then (74) specializes to

$$BK = \begin{bmatrix}0 & B^{12} \\ B^{21} & 0\end{bmatrix}\begin{bmatrix}K_{uu} & K_{uv} \\ K_{vu} & K_{vv}\end{bmatrix} = \begin{bmatrix}B^{12}K_{vu} & B^{12}K_{vv} \\ B^{21}K_{uu} & B^{21}K_{uv}\end{bmatrix} \quad (80)$$

and (75) reduces to

$$q = \sum_{m=1}^{2N} \lambda_m w_m^2 \quad , \tag{81}$$

with characteristic function

$$f_q(\xi) = \left[\prod_{m=1}^{2N} \left\{1 - i2\lambda_m \xi\right\}\right]^{-1/2} \cdot \exp\left[i\xi \sum_{m=1}^{2N} \frac{\lambda_m \nu_m^2}{1 - i2\lambda_m \xi}\right] \tag{82}$$

following directly from (76) and (78).

For the particular example of

$$B^{12} = \frac{1}{2}\,\mathrm{diag}(\ell_1\ \ell_2\ \ldots \ell_N) + \frac{1}{2}\,[g_1\ g_2\ \ldots\ g_N]^T\,[h_1\ h_2\ \ldots\ h_N], \tag{83}$$

then

$$q = \sum_{n=1}^{N} \ell_n u_n v_n + \left(\sum_{n=1}^{N} g_n u_n\right)\left(\sum_{n=1}^{N} h_n v_n\right) , \tag{84}$$

with the same characteristic function (82).

As a still more-special case, let $B^{12} = \frac{1}{2}\, I$; then (79) and (81) give the simple cross-correlator (but with correlated inputs for all time separations)

$$q = \sum_{n=1}^{N} u_n v_n = \sum_{m=1}^{2N} \lambda_m w_m^2 \quad , \tag{85}$$

and (80) and (77) become

$$BK = \frac{1}{2}\begin{bmatrix} K_{vu} & K_{vv} \\ K_{uu} & K_{uv} \end{bmatrix}, \quad \overline{W} = C^T \begin{bmatrix} E_u \\ E_v \end{bmatrix} = [\nu_1\ \ldots\ \nu_{2N}]^T \; . \tag{86}$$

The important equations that must be solved are always

$$BKC = C\Lambda \tag{87}$$

or

$$BKC^{(m)} = \lambda_m C^{(m)} \quad \text{for} \quad 1 \le m \le M = 2N, \tag{88}$$

where all matrices B, K, C, Λ are 2Nx2N. The characteristic function of (85) is again (82).

Special Case of Correlator (85)

Here we let components U and V have the same covariance and a scaled cross-correlation; that is, let

$$K_{uu} = K_o, \quad K_{vv} = K_o, \quad K_{uv} = \rho K_o \quad , \tag{89}$$

where ρ is a scale factor. This case corresponds, for example, to a common signal in two independent components:

$$\begin{aligned} u(t) &= s(t) + n_1(t), \\ v(t) &= s(t) + n_2(t), \end{aligned} \tag{90}$$

where $s(t)$, $n_1(t)$, $n_2(t)$ are all independent and have a common covariance. Then (86) becomes

$$BK = \frac{1}{2}\begin{bmatrix} \rho K_o & K_o \\ K_o & \rho K_o \end{bmatrix} . \tag{91}$$

Now suppose that we can determine the NxN eigenvalue matrix Γ and modal matrix Q of K_o, that is

$$K_oQ = Q\Gamma \; ; \quad \Gamma = \text{diag}(\gamma_1, \gamma_2 \cdots \gamma_N). \tag{92}$$

Then we have the standard relations [8]

$$K_o = Q\Gamma Q^T \quad \text{where} \quad QQ^T = I. \tag{93}$$

We can now express the 2Nx2N matrix in (91) as

$$BK = \frac{1}{2}\begin{bmatrix} Q\rho\Gamma Q^T & Q\Gamma Q^T \\ Q\Gamma Q^T & Q\rho\Gamma Q^T \end{bmatrix} =$$

$$= \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix}\begin{bmatrix} \frac{1}{2}\rho\Gamma & \frac{1}{2}\Gamma \\ \frac{1}{2}\Gamma & \frac{1}{2}\rho\Gamma \end{bmatrix}\begin{bmatrix} Q^T & 0 \\ 0 & Q^T \end{bmatrix}, \tag{94}$$

and 2Nx2N identity matrix

$$I_{2N} = \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} Q^T & 0 \\ 0 & Q^T \end{bmatrix}. \tag{95}$$

There follows

$$BK - \lambda I_{2N} = \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix}\begin{bmatrix} \frac{1}{2}\rho\Gamma - \lambda I & \frac{1}{2}\Gamma \\ \frac{1}{2}\Gamma & \frac{1}{2}\rho\Gamma - \lambda I \end{bmatrix}\begin{bmatrix} Q^T & 0 \\ 0 & Q^T \end{bmatrix}. \tag{96}$$

But the middle matrix in (96) can be developed in detail in the partitioned form

$$\begin{bmatrix} \frac{1}{2}\rho\gamma_1 - \lambda & & & \frac{1}{2}\gamma_1 & & \\ & \ddots & & & \ddots & \\ & & \frac{1}{2}\rho\gamma_N - \lambda & & & \frac{1}{2}\gamma_N \\ \hline \frac{1}{2}\gamma_1 & & & \frac{1}{2}\rho\gamma_1 - \lambda & & \\ & \ddots & & & \ddots & \\ & & \frac{1}{2}\gamma_N & & & \frac{1}{2}\rho\gamma_N - \lambda \end{bmatrix} . \tag{97}$$

This matrix is singular when the k<u>th</u> row is equal to, or the negative of, the k+N <u>th</u> row. This leads to the eigenvalues $\{\lambda_n\}_1^{2N}$ of matrix BK:

$$\lambda_1 = \frac{1}{2}(\rho+1)\gamma_1, \ \ldots, \ \lambda_N = \frac{1}{2}(\rho+1)\gamma_N \ ,$$

$$\lambda_{N+1} = \frac{1}{2}(\rho-1)\gamma_1, \ \ldots, \ \lambda_{2N} = \frac{1}{2}(\rho-1)\gamma_N \ . \tag{98}$$

Thus we need only solve for the N eigenvalues $\{\gamma_n\}_1^N$ of matrik K_o, and then use them as above to determine all 2N eigenvalues of BK; this is a significant shortcut.

If also $E_u = E_v = 0$, then $\overline{W} = 0$ from (86), and the characteristic function of q in (85) follows from (82) and (98) as

$$f_q(\xi) = \left[\prod_{m=1}^{N} \left\{(1-i(\rho+1)\gamma_m\xi)(1-i(\rho-1)\gamma_m\xi)\right\}\right]^{-1/2} =$$

$$= \left[\prod_{m=1}^{N} \left\{1-i2\rho\gamma_m\xi + (1-\rho^2)\gamma_m^2\,\xi^2\right\}\right]^{-1/2} . \tag{99}$$

This is a generalization of [1, eq. 54], which held for a single pair of Gaussian random variables.

EXAMPLES

The program listed in appendix A for the second-order processor (8) and attendant characteristic function (9) has been employed to yield the result in figure 2. The particular values for the number of terms K, the weights, and the input statistics are listed in lines 20-120. There is no physical significance attached to this particular example; rather it has been run simply to illustrate the extreme generality that the technique is capable of. Some negative values for the weights, means, and correlation coefficients have been employed to emphasize this generality. This simple example (and others to follow) can be used as a check case on any user-written program to evaluate cumulative and exceedance distribution functions.

The selection of parameters L, Δ, b in lines 130-150 is discussed in detail in [1]; the reader is referred there for the deleterious effects that can occur for improper choices of L, Δ, b. The selection of M_f, the FFT size in line 160, is rather arbitrary; it controls the spacing at which the probability distributions are computed, but has no effect upon the accuracy of the results (except for round-off noise). Additional computational details on the particular program for characteristic function (9) are given in appendix A.

The ordinate scale for figure 2 is a logarithmic one. The lower right end of the exceedance distribution function curve decreases smoothly to the region 1E-11, where roundoff noise is encountered. The exceedance distribution function values continue to decrease with x until, finally, negative values (due to roundoff noise) are generated. For negative probability values, the logarithm of the absolute value is plotted, but mirrored below the 1E-12 level. These values have no physical significance, of course; they are plotted to illustrate the level of accuracy attainable by this procedure with appropriate choices of L and Δ.

The rates of decay of the cumulative and exceedance distribution functions in figure 2 are markedly different for this particular example. Additionally, since the decays are both linear on this logarithmic ordinate, it means that both tail distributions are exponential, not Gaussian. These attributes of the cumulative and exceedance distribution functions are easily

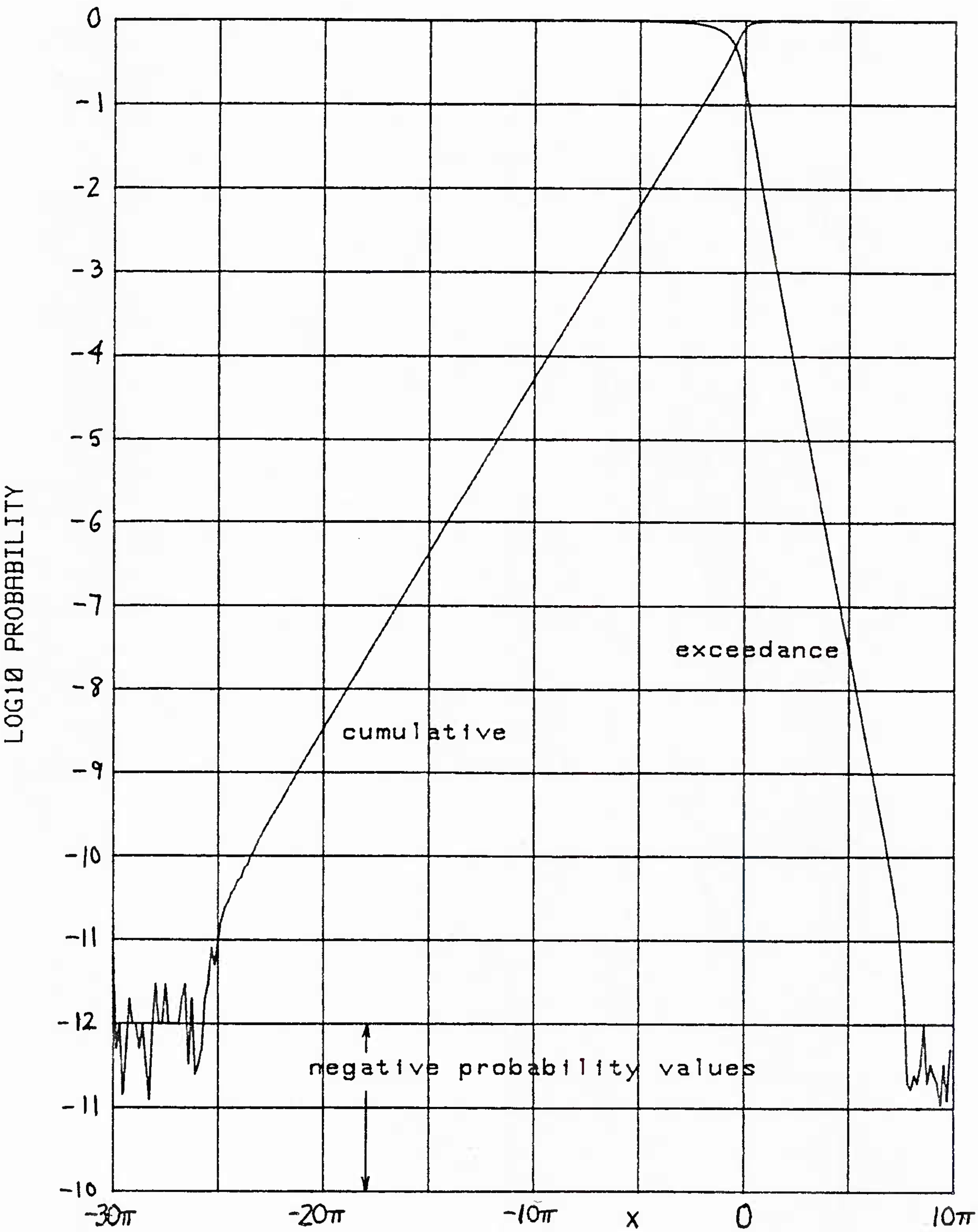

Figure 2. Distributions for Second-Order Processor

and quickly discernible by use of the numerical technique in [1], for a limitless variety of weights and input statistics, with a minimum of effort on the part of the user.

As a check on the program in appendix A, the second-order processor in (8) was simulated, and 10,000 independent trials were used to determine its performance for the exact same parameters as used for figure 2 above. The program is listed in appendix E and the results are given in figure 3. The corroboration is excellent, even near the 1E-4 probability level.

As the number of terms, K, in the second-order processor (8) is increased, and if the statistics are identical, the random variable x should approach Gaussian, at least near its mean. The example in figure 4 was run for K = 10, and all weights and statistics independent of k; the particular choices were

$$\begin{aligned} &a = .6,\ b = -.6,\ c = .3,\ d = -.2,\ e = .2, \\ &m_s = .5,\ m_t = -.5,\ \sigma_s = 1,\ \sigma_t = 1,\ \rho = .4, \\ &L = 4,\ \Delta = .05,\ b = 20\pi,\ M_f = 256. \end{aligned} \tag{100}$$

The cumulative and exceedance distribution functions in figure 4 both display a parabolic shape near the mean of x, which signifies Gaussian behavior of the random variable, as expected. However, on the tails, the distributions are tending to linear, which means an exponential decay there. This observation for this example confirms the comments of [7, p. 673].

The cumulative and exceedance distribution functions for an example of the second-order processor with fading are displayed in figure 5, as determined from characteristic function (21) and the corresponding program in appendix B. The power law, ν, for the fading probability density function (19) is 2.7 for this example, but can be easily changed. The particular constants employed are listed in lines 20-110 in appendix B.

An example of the distributions for the narrowband cross-correlator of figure 1 is presented in figure 6, as evaluated from characteristic function

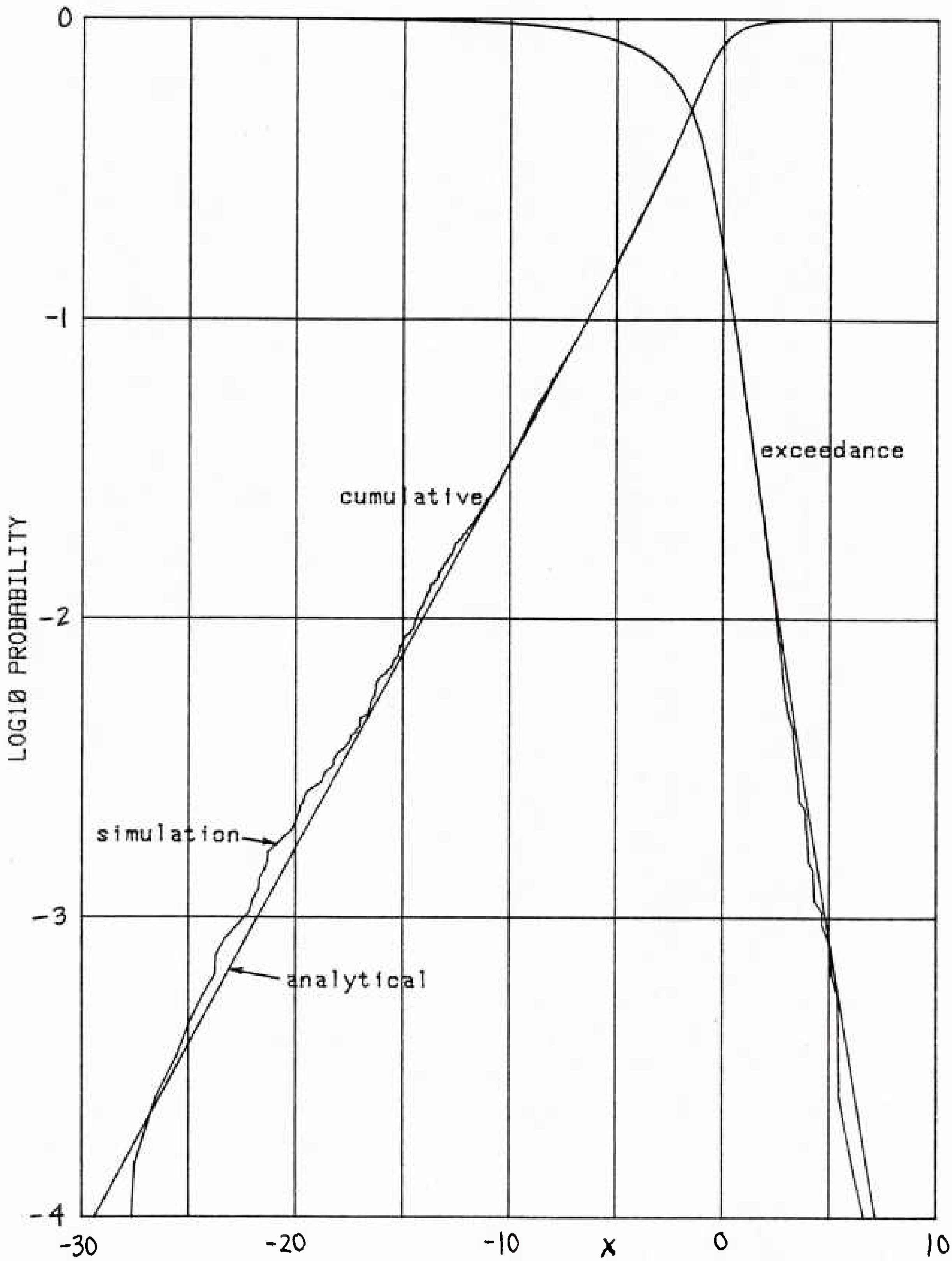

Figure 3. Simulation Comparison of Second-Order Processor

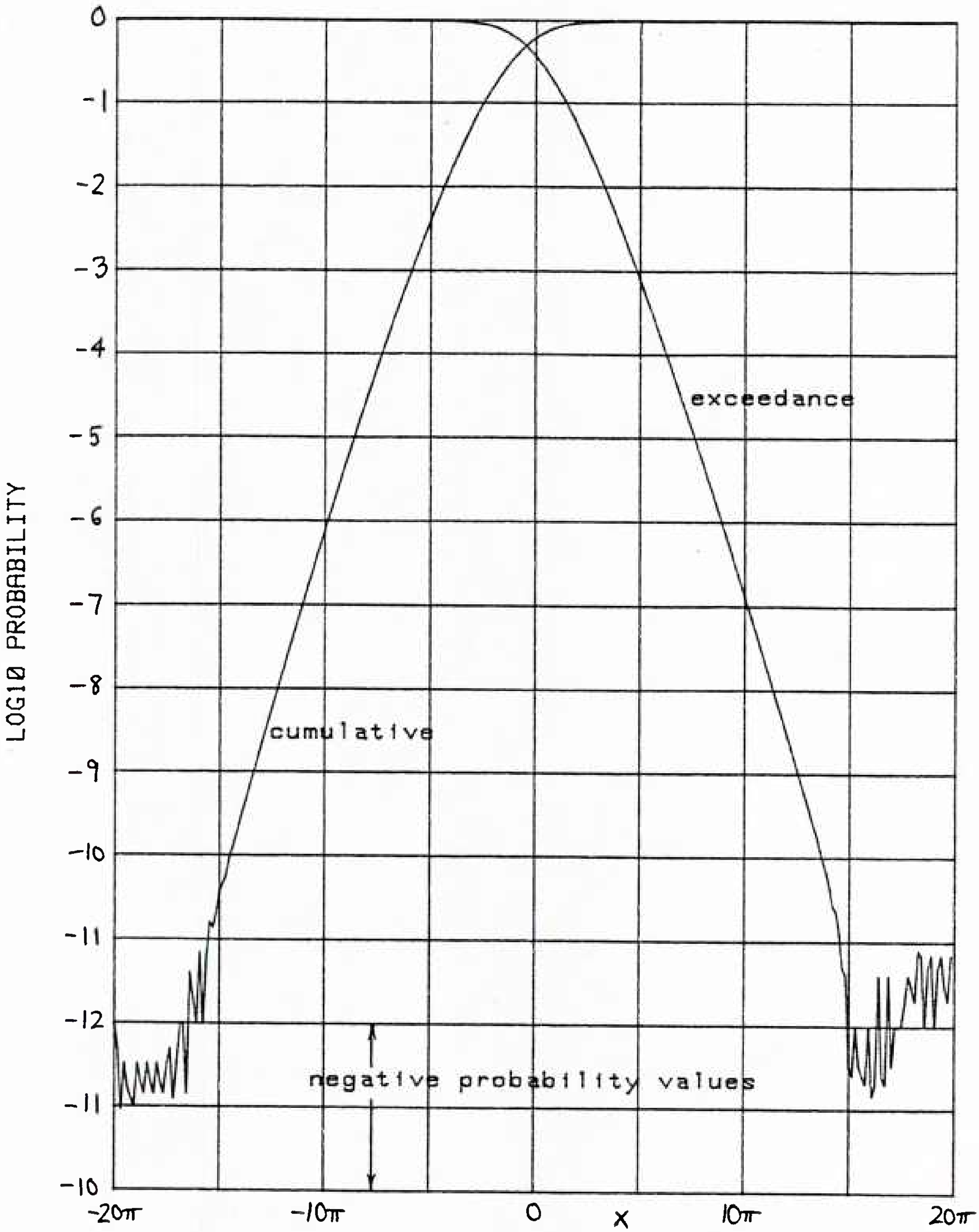

Figure 4. Second-Order Processor, K=10, Identical Statistics

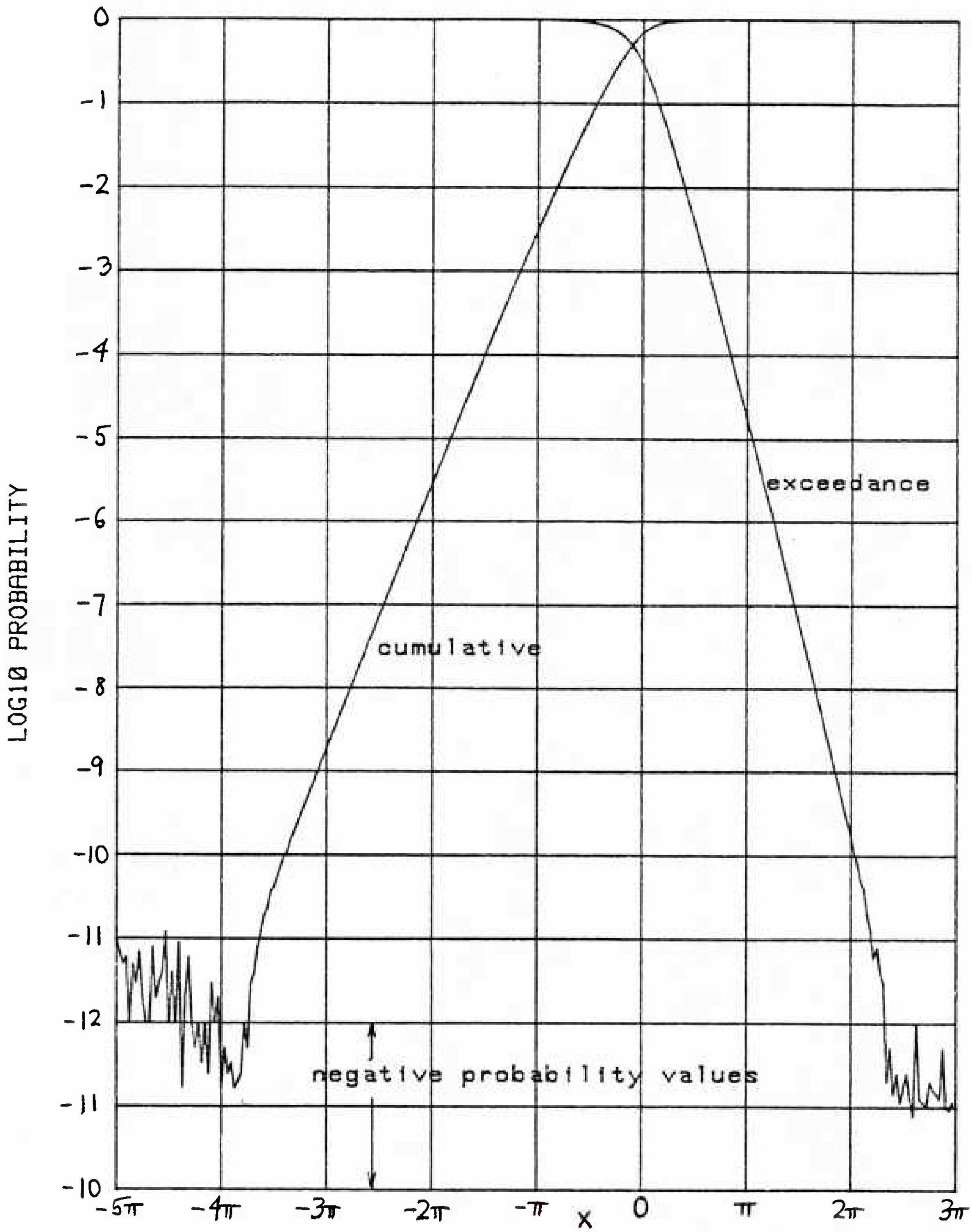

Figure 5. Second-Order Processor with Fading

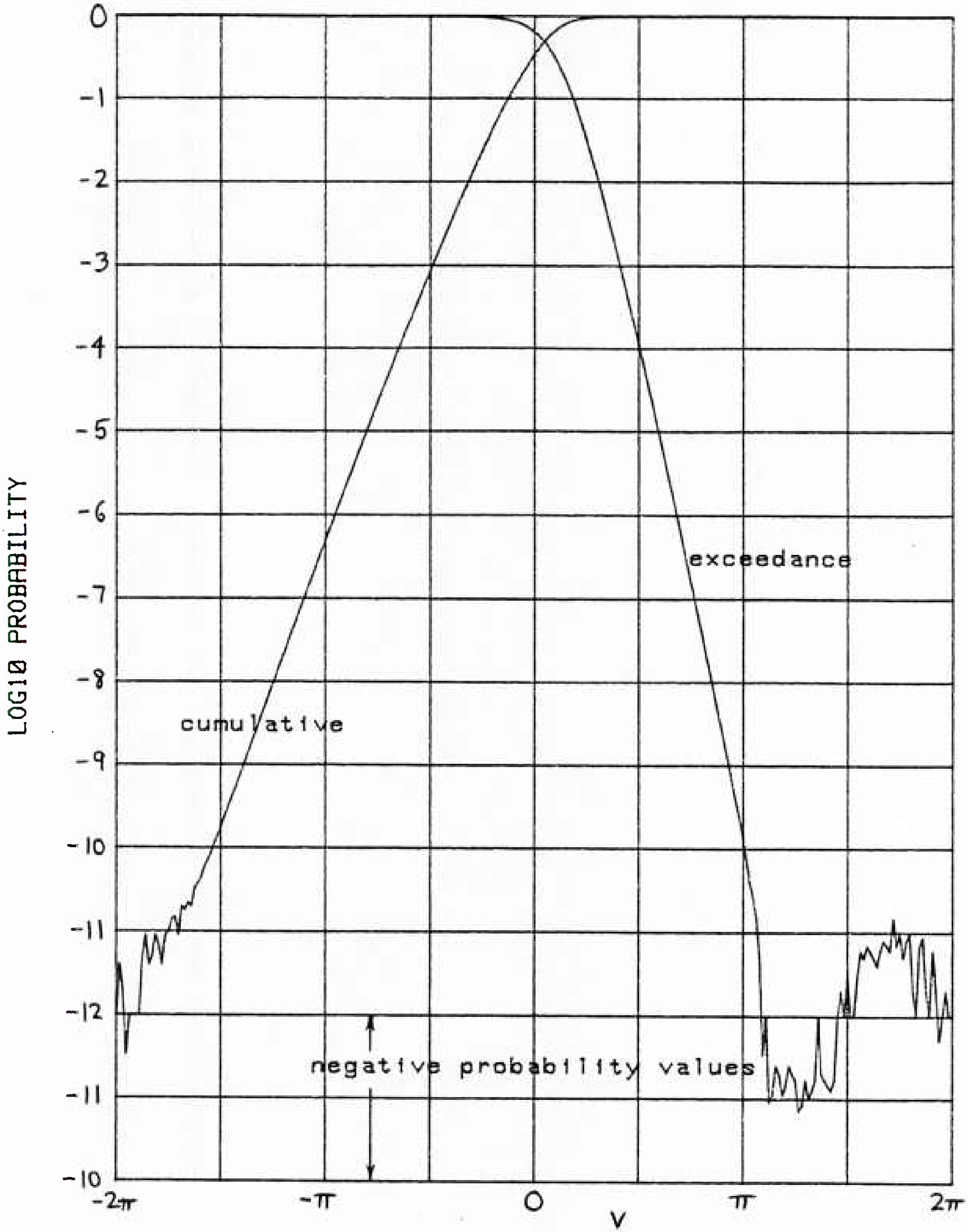

Figure 6. Distributions for Narrowband Cross-Correlator

(31) and the program in appendix C. The weightings, signal components, and noise statistics have no special values or interrelationships; the particular values used here are listed in lines 20-110.

The distributions for the reduced quadratic and linear form (65) and accompanying characteristic function (66) are presented in figure 7 for the numerical example employed in the program listing in appendix D. If the given form is instead that of (62), then (67)-(68) must first be solved before the program in appendix D can be employed; that is, one must augment these results with the capability for extracting the eigenvalues (and eigenvectors in some cases) of the MxM matrix BK. The size of the FFT, M_f, has been increased to 1024 in figure 7; this results in finer spacing of the distribution values and additional spikes in the round-off noise region centered about 1E-12.

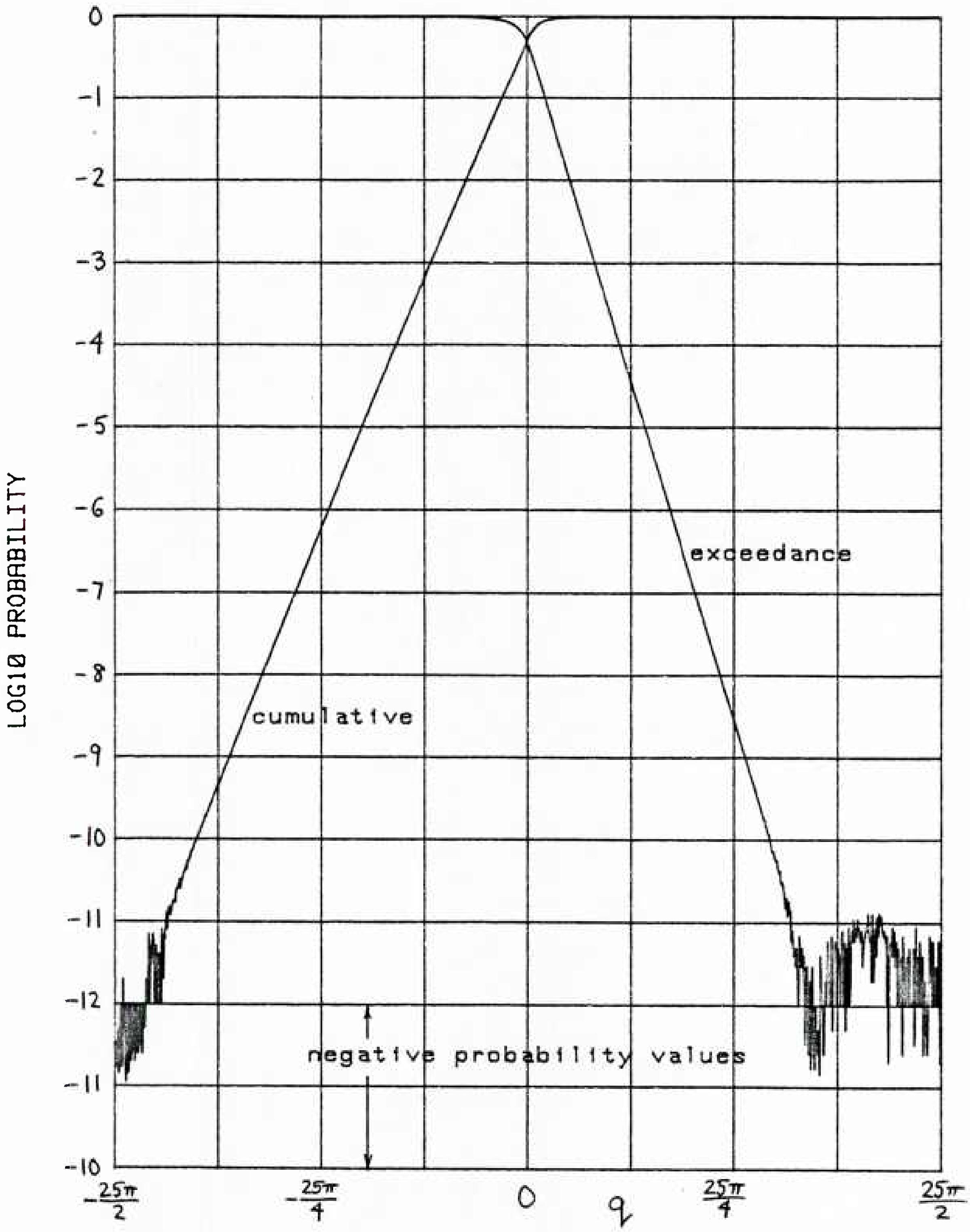

Figure 7. Distributions for Quadratic and Linear Form

SUMMARY AND DISCUSSION

Closed form expressions for the characteristic functions of the decision variables of three classes of second-order processors have been derived. The input noise to the processors must be Gaussian, but it can be nonstationary with arbitrary statistics. Programs for the direct evaluation of the exact cumulative and exceedance distribution functions have been generated and then exercised for completely general values of the weights, signal parameters, and noise statistics. There is no assumption needed about a large number of statistically independent contributors, nor need any signal-to-noise ratio be either small or large. The first two classes of processors are restricted in form, but include many of the practical devices often encountered in detection and estimation problems. The third class covers the most general second-order processor; it requires the solution for the eigenvalue and modal matrices of an MxM matrix (where M is the size of the general quadratic form) in addition to the program furnished here. The approach utilized here allows a user to quickly and easily obtain accurate quantitative information about the performance of a particular processor, and to investigate the effects of making changes in any of the input constants or parameters.

Approximations to the performance of continuous quadratic processors are possible by use of the above procedures. For example,

$$\iint dt_1\, dt_2\, x(t_1)\, \beta(t_1, t_2)\, x(t_2) \cong \Delta_1\, \Delta_2 \sum_{m,n} x(m\Delta_1)\, \beta(m\Delta_1, n\Delta_2)\, x(n\Delta_2), \qquad (101)$$

which is of the form $X^T BX$ encountered in (62). Also

$$\iint dt_1\, dt_2\, u(t_1)\, \beta(t_1, t_2)\, v(t_2) \cong \Delta_1\, \Delta_2 \sum_{m,n} u(m\Delta_1)\, \beta(m\Delta_1, n\Delta_2)\, v(n\Delta_2), \qquad (102)$$

which is of the form $U^T B^{12} V$ encountered in (79).

Receiver operating characteristics, that is, detection probability vs false alarm probability, can be easily determined from the above results. First store the exceedance distribution for zero signal strength in an array. Then plot the exceedance distributions for nonzero signal strengths vs this stored array of numbers, each point for a common threshold. The common thresholds are most easily realized by keeping sampling increment Δ and FFT size M_f the same throughout all the computations.

APPENDIX A. SECOND-ORDER PROCESSOR

This program computes the cumulative and exceedance distribution functions of random variable (8) via characteristic function (9). The required inputs are listed in lines 20-120 and are annotated consistently with (8). The parameters D_1, D_2, N_0, N_1, N_2 required in characteristic function (9) are pre-computed once in loop 290-510 for the sake of execution time. The mean of x is entered in line 520. When we enter loop 590-830 for the actual calculation of the characteristic function (9), the number of computations are minimized. For example, only one complex exponential and square root are required per ξ value, in lines 740-750. The square root in (9) is not a principal value square root, but in fact must yield a continuous function in ξ. In order to achieve this, the argument of the square root is traced continuously from $\xi = 0$ (line 530). If an abrupt change in phase is detected, a polarity indicator takes note of this fact (line 780) and corrects the final values of characteristic function $f_y(\xi)$ (line 790). More detail on the selection of L, Δ, b in lines 130-150 is available in [1].

```
10 ! SECOND-ORDER PROCESSOR
20    K=5                          !  Number of terms summed
30    DATA .6,-.5,.4,-.3,.2        !  a(k) weightings
40    DATA .9,.8,.7,-.6,-.5        !  b(k) weightings
50    DATA -.6,-.8,1,1.2,1.4       !  c(k) weightings
60    DATA .1,-.2,-.3,.4,.5        !  d(k) weightings
70    DATA -.7,.6,.5,.4,-.3        !  e(k) weightings
80    DATA .2,.3,.4,-.5,-.6        !  Means of random variables s(k)
90    DATA .8,-.7,-.6,.5,.4        !  Means of random variables t(k)
100   DATA .1,.3,.5,.7,.9          !  Standard deviations of s(k)
110   DATA .2,.4,.6,.8,1           !  Standard deviations of t(k)
120   DATA .4,-.5,.6,.7,-.8        !  Correlation coeffs. of s(k) and t(k)
130   L=25                         !  Limit on integral of char. function
140   Delta=.05                    !  Sampling increment on char. function
150   Bs=.75*(2*PI/Delta)          !  Shift b, as fraction of alias interval
160   Mf=2^8                       !  Size of FFT
170   PRINTER IS 0
180   PRINT "L =";L,"Delta =";Delta,"b =";Bs,"Mf =";Mf
190   REDIM A(1:K),B(1:K),C(1:K),D(1:K),E(1:K)
200   REDIM Ms(1:K),Mt(1:K),Ss(1:K),St(1:K),Rho(1:K)
210   REDIM D1(1:K),D2(1:K),N0(1:K),N1(1:K),N2(1:K)
220   REDIM X(0:Mf-1),Y(0:Mf-1)
230   DIM A(1:10),B(1:10),C(1:10),D(1:10),E(1:10)
240   DIM Ms(1:10),Mt(1:10),Ss(1:10),St(1:10),Rho(1:10)
250   DIM D1(1:10),D2(1:10),N0(1:10),N1(1:10),N2(1:10)
260   DIM X(0:1023),Y(0:1023)
270   READ A(*),B(*),C(*),D(*),E(*)       !  Enter
280   READ Ms(*),Mt(*),Ss(*),St(*),Rho(*) !  constants
```

```
290   FOR J=1 TO K                             ! Calculation
300   T1=Ms(J)^2                               ! of
310   T2=Mt(J)^2                               ! parameters
320   T3=Ss(J)^2
330   T4=St(J)^2
340   T5=Ms(J)*Mt(J)
350   T6=Rho(J)*Ss(J)*St(J)
360   T7=4*A(J)*B(J)-C(J)^2
370   T8=(1-Rho(J)^2)*T3*T4
380   T9=Mt(J)*Ss(J)
390   T10=Ms(J)*St(J)
400   T11=D(J)^2
410   T12=E(J)^2
420   T13=D(J)*E(J)
430   D1(J)=2*(A(J)*T3+B(J)*T4+C(J)*T6)
440   D2(J)=T7*T8
450   N0(J)=A(J)*T1+B(J)*T2+C(J)*T5+D(J)*Ms(J)+E(J)*Mt(J)
460   T=T7*(.5*(T1*T4+T2*T3)-T5*T6)
470   T=T+(2*A(J)*E(J)-C(J)*D(J))*Ss(J)*(T9-Rho(J)*T10)
480   T=T+(2*B(J)*D(J)-C(J)*E(J))*St(J)*(T10-Rho(J)*T9)
490   N1(J)=T-.5*(T11*T3+T12*T4)-T13*T6
500   N2(J)=-(A(J)*T12+B(J)*T11-C(J)*T13)*T8
510   NEXT J
520   Mux=SUM(N0)+.5*SUM(D1)                   ! Mean of random variable x
530   R=0                                      ! Argument of square root
540   P=1                                      ! Polarity indicator
550   Muy=Mux+Bs
560   X(0)=0
570   Y(0)=.5*Delta*Muy
580   N=INT(L/Delta)
590   FOR Ns=1 TO N
600   Xi=Delta*Ns                              ! Argument xi of char. fn.
610   X2=Xi*Xi                                 ! Calculation
620   Pr=1                                     ! of
630   Pi=Sr=Si=0                               ! characteristic
640   FOR J=1 TO K                             ! function
650   Dr=1-X2*D2(J)                            ! fy(xi)
660   Di=-Xi*D1(J)
670   CALL Mul(Pr,Pi,Dr,Di,A,B)
680   Pr=A
690   Pi=B
700   CALL Div(N0(J)-X2*N2(J),-Xi*N1(J),Dr,Di,A,B)
710   Sr=Sr+A
720   Si=Si+B
730   NEXT J
740   CALL Exp(-Xi*Si,Xi*(Sr+Bs),A,B)
750   CALL Sqr(Pr,Pi,C,D)
760   Ro=R
770   R=ATN(D/C)
780   IF ABS(R-Ro)>1.6 THEN P=-P
790   CALL Div(A,B,C*P,D*P,Fyr,Fyi)
800   Ms=Ns MOD Mf                             ! Collapsing
810   X(Ms)=X(Ms)+Fyr/Ns
820   Y(Ms)=Y(Ms)+Fyi/Ns
830   NEXT Ns
840   CALL Fft10z(Mf,X(*),Y(*))                ! 0 subscript FFT
```

```
850   PLOTTER IS "GRAPHICS"
860   GRAPHICS
870   SCALE 0,Mf,-14,0
880   LINE TYPE 3
890   GRID Mf/8,1
900   PENUP
910   LINE TYPE 1
920   B=Bs*Mf*Delta/(2*PI)              !  Origin for random variable x
930   MOVE B,0
940   DRAW B,-14
950   PENUP
960   FOR Ks=0 TO Mf-1
970   T=Y(Ks)/PI-Ks/Mf
980   X(Ks)=.5-T                        !  Cumulative probability in X(*)
990   Y(Ks)=Pr=.5+T                     !  Exceedance probability in Y(*)
1000  IF Pr>=1E-12 THEN Y=LGT(Pr)
1010  IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
1020  IF ABS(Pr)<1E-12 THEN Y=-12
1030  PLOT Ks,Y
1040  NEXT Ks
1050  PENUP
1060  PRINT Y(0);Y(1);Y(Mf-2);Y(Mf-1)
1070  FOR Ks=0 TO Mf-1
1080  Pr=X(Ks)
1090  IF Pr>=1E-12 THEN Y=LGT(Pr)
1100  IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
1110  IF ABS(Pr)<1E-12 THEN Y=-12
1120  PLOT Ks,Y
1130  NEXT Ks
1140  PENUP
1150  PAUSE
1160  DUMP GRAPHICS
1170  PRINT LIN(5)
1180  PRINTER IS 16
1190  END
1200  !
1210  SUB Mul(X1,Y1,X2,Y2,A,B)          !  Z1*Z2
1220  A=X1*X2-Y1*Y2
1230  B=X1*Y2+X2*Y1
1240  SUBEND
1250  !
1260  SUB Div(X1,Y1,X2,Y2,A,B)          !  Z1/Z2
1270  T=X2*X2+Y2*Y2
1280  A=(X1*X2+Y1*Y2)/T
1290  B=(Y1*X2-X1*Y2)/T
1300  SUBEND
1310  !
1320  SUB Exp(X,Y,A,B)                  !  EXP(Z)
1330  T=EXP(X)
1340  A=T*COS(Y)
1350  B=T*SIN(Y)
1360  SUBEND
1370  !
```

```
1380  SUB Sqr(X,Y,A,B)                    !  PRINCIPAL SQR(Z)
1390  IF X<>0 THEN 1430
1400  A=B=SQR(.5*ABS(Y))
1410  IF Y<0 THEN B=-B
1420  GOTO 1540
1430  F=SQR(SQR(X*X+Y*Y))
1440  T=.5*ATN(Y/X)
1450  A=F*COS(T)
1460  B=F*SIN(T)
1470  IF X>0 THEN 1540
1480  T=A
1490  A=-B
1500  B=T
1510  IF Y>=0 THEN 1540
1520  A=-A
1530  B=-B
1540  SUBEND
1550  !
1560  SUB Fft10z(N,X(*),Y(*))     ! N <= 2^10 = 1024, N=2^INTEGER     0 subscript
1570  DIM C(0:256)
1580  INTEGER I1,I2,I3,I4,I5,I6,I7,I8,I9,I10,J,K
1590  DATA 1,.999981175283,.999924701839,.999830581796,.999698818696,.9995294175
01,.999322384588,.999077727753,.998795456205,.998475580573,.998118112900
1600  DATA .997723066644,.997290456679,.996820299291,.996312612183,.995767414468
,.995184726672,.994564570734,.993906970002,.993211949235,.992479534599
1610  DATA .991709753669,.990902635428,.990058210262,.989176509965,.988257567731
,.987301418158,.986308097245,.985277642389,.984210092387,.983105487431
1620  DATA .981963869110,.980785280403,.979569765685,.978317370720,.977028142658
,.975702130039,.974339382786,.972939952206,.971503890986,.970031253195
1630  DATA .968522094274,.966976471045,.965394441698,.963776065795,.962121404269
,.960430519416,.958703474896,.956940335732,.955141168306,.953306040354
1640  DATA .951435020969,.949528180593,.947585591018,.945607325381,.943593458162
,.941544065183,.939459223602,.937339011913,.935183509939,.932992798835
1650  DATA .930766961079,.928506080473,.926210242138,.923879532511,.921514039342
,.919113851690,.916679059921,.914209755704,.911706032005,.909167983091
1660  DATA .906595704515,.903989293123,.901348847046,.898674465694,.895966249756
,.893224301196,.890448723245,.887639620403,.884797098431,.881921264348
1670  DATA .879012226429,.876070094195,.873094978418,.870086991109,.867046245516
,.863972856122,.860866938638,.857728610000,.854557988365,.851355193105
1680  DATA .848120344803,.844853565250,.841554977437,.838224705555,.834862874986
,.831469612303,.828045045258,.824589302785,.821102514991,.817584813152
1690  DATA .814036329706,.810457198253,.806847553544,.803207531481,.799537269108
,.795836904609,.792106577300,.788346427627,.784556597156,.780737228572
1700  DATA .776888465673,.773010453363,.769103337646,.765167265622,.761202385484
,.757208846506,.753186799044,.749136394523,.745057785441,.740951125355
1710  DATA .736816568877,.732654271672,.728464390448,.724247082951,.720002507961
,.715730825284,.711432195745,.707106781187,.702754744457,.698376249409
1720  DATA .693971460890,.689540544737,.685083667773,.680600997795,.676092703575
,.671558954847,.666999922304,.662415777590,.657806693297,.653172842954
1730  DATA .648514401022,.643831542890,.639124444864,.634393284164,.629638238915
,.624859488142,.620057211763,.615231590581,.610382806276,.605511041404
1740  DATA .600616479384,.595699304492,.590759701859,.585797857456,.580813958096
,.575808191418,.570780745887,.565731810784,.560661576197,.555570233020
1750  DATA .550457972937,.545324988422,.540171472730,.534997619887,.529803624686
,.524589682678,.519355990166,.514102744193,.508830142543,.503538383726
```

```
1760  DATA .498227666973,.492898192230,.487550160148,.482183772079,.476799230063
,.471396736826,.465976495768,.460538710958,.455083587126,.449611329655
1770  DATA .444122144570,.438616238539,.433093818853,.427555093430,.422000270800
,.416429560098,.410843171058,.405241314005,.399624199846,.393992040061
1780  DATA .388345046699,.382683432365,.377007410216,.371317193952,.365612997805
,.359895036535,.354163525420,.348418680249,.342660717312,.336889853392
1790  DATA .331106305760,.325310292162,.319502030816,.313681740399,.307849640042
,.302005949319,.296150888244,.290284677254,.284407537211,.278519689385
1800  DATA .272621355450,.266712757475,.260794117915,.254865659605,.248927605746
,.242980179903,.237023605994,.231058108281,.225083911360,.219101240157
1810  DATA .213110319916,.207111376192,.201104634842,.195090322016,.189068664150
,.183039887955,.177004220412,.170961888760,.164913120490,.158858143334
1820  DATA .152797185258,.146730474455,.140658239333,.134580708507,.128498110794
,.122410675199,.116318630912,.110222207294,.104121633872,.980171403296E-1
1830  DATA .919089564971E-1,.857973123444E-1,.796824379714E-1,.735645635997E-1,.
674439195637E-1,.613207363022E-1,.551952443497E-1,.490676743274E-1
1840  DATA .429382569349E-1,.368072229414E-1,.306748031766E-1,.245412285229E-1,.
184067299058E-1,.122715382857E-1,.613588464915E-2,0
1850  READ C(*)
1860  K=1024/N
1870  FOR J=0 TO N/4
1880  C(J)=C(K*J)
1890  NEXT J
1900  N1=N/4
1910  N2=N1+1
1920  N3=N2+1
1930  N4=N1+N3
1940  Log2n=INT(1.4427*LOG(N)+.5)
1950  FOR I1=1 TO Log2n
1960  I2=2^(Log2n-I1)
1970  I3=2*I2
1980  I4=N/I3
1990  FOR I5=1 TO I2
2000  I6=(I5-1)*I4+1
2010  IF I6<=N2 THEN 2050
2020  N6=-C(N4-I6-1)
2030  N7=-C(I6-N1-1)
2040  GOTO 2070
2050  N6=C(I6-1)
2060  N7=-C(N3-I6-1)
2070  FOR I7=0 TO N-I3 STEP I3
2080  I8=I7+I5
2090  I9=I8+I2
2100  N8=X(I8-1)-X(I9-1)
2110  N9=Y(I8-1)-Y(I9-1)
2120  X(I8-1)=X(I8-1)+X(I9-1)
2130  Y(I8-1)=Y(I8-1)+Y(I9-1)
2140  X(I9-1)=N6*N8-N7*N9
2150  Y(I9-1)=N6*N9+N7*N8
2160  NEXT I7
2170  NEXT I5
2180  NEXT I1
2190  I1=Log2n+1
2200  FOR I2=1 TO 10                ! 2^10=1024
2210  C(I2-1)=1
2220  IF I2>Log2n THEN 2240
2230  C(I2-1)=2^(I1-I2)
2240  NEXT I2
```

```
2250  K=1
2260  FOR I1=1 TO C(9)
2270  FOR I2=I1 TO C(8) STEP C(9)
2280  FOR I3=I2 TO C(7) STEP C(8)
2290  FOR I4=I3 TO C(6) STEP C(7)
2300  FOR I5=I4 TO C(5) STEP C(6)
2310  FOR I6=I5 TO C(4) STEP C(5)
2320  FOR I7=I6 TO C(3) STEP C(4)
2330  FOR I8=I7 TO C(2) STEP C(3)
2340  FOR I9=I8 TO C(1) STEP C(2)
2350  FOR I10=I9 TO C(0) STEP C(1)
2360  J=I10
2370  IF K>J THEN 2440
2380  A=X(K-1)
2390  X(K-1)=X(J-1)
2400  X(J-1)=A
2410  A=Y(K-1)
2420  Y(K-1)=Y(J-1)
2430  Y(J-1)=A
2440  K=K+1
2450  NEXT I10
2460  NEXT I9
2470  NEXT I8
2480  NEXT I7
2490  NEXT I6
2500  NEXT I5
2510  NEXT I4
2520  NEXT I3
2530  NEXT I2
2540  NEXT I1
2550  SUBEND
```

APPENDIX B. FADING FOR SECOND-ORDER PROCESSOR

This program computes the cumulative and exceedance distribution functions for characteristic function (21), when the power fading factor r in (18) has probability density function (19). The parameters D_1, D_2, N_0', N_1' are pre-computed once in lines 210-310. The logarithms in lines 430 and 440 have arguments that never cross the branch line along the negative real axis for the principal value logarithm; hence the calculated characteristic function is automatically continuous for all ξ.

```
10 ! FADING FOR SECOND-ORDER PROCESSOR
20     Nu=2.7                    !  Power law for fading
30     K=5                       !  Number of terms summed
40     Ak=.7                     !  a(k) weighting
50     Bk=-.9                    !  b(k) weighting
60     Ck=-.6                    !  c(k) weighting
70     DATA .2,.3,.4,-.5,-.6     !  Means of random variables s(k)
80     DATA .8,-.7,-.6,.5,.4     !  Means of random variables t(k)
90     Ss=.3                     !  Standard deviation of s(k)
100    St=.2                     !  Standard deviation of t(k)
110    Rho=-.4                   !  Correlation coeff. of s(k) and t(k)
120    L=150                     !  Limit on integral of char. function
130    Delta=.25                 !  Sampling increment on char. function
140    Bs=.625*(2*PI/Delta)      !  Shift b, as fraction of alias interval
150    Mf=2^8                    !  Size of FFT
160    PRINTER IS 0
170    PRINT "L =";L,"Delta =";Delta,"b =";Bs,"Mf =";Mf
180    REDIM Ms(1:K),Mt(1:K),X(0:Mf-1),Y(0:Mf-1)
190    DIM Ms(1:10),Mt(1:10),X(0:1023),Y(0:1023)
200    READ Ms(*),Mt(*)                        !  Enter constants
210    M20=DOT(Ms,Ms)                          !  Calculation
220    M02=DOT(Mt,Mt)                          !  of
230    M11=DOT(Ms,Mt)                          !  parameters
240    T1=Ss*Ss
250    T2=St*St
260    T3=Rho*Ss*St
270    T4=4*Ak*Bk-Ck*Ck
280    N0p=Ak*M20+Bk*M02+Ck*M11
290    N1p=T4*(.5*(T2*M20+T1*M02)-T3*M11)
300    D1=2*(Ak*T1+Bk*T2+Ck*T3)
310    D2=T4*(1-Rho*Rho)*T1*T2
320    D1p=D1+N0p/Nu
330    D2p=D2+N1p/Nu
340    Mux=N0p+.5*K*D1                         !  Mean of random variable x
350    Muy=Mux+Bs
360    T=Nu-.5*K
```

```
370   X(0)=0
380   Y(0)=.5*Delta*Muy
390   N=INT(L/Delta)
400   FOR Ns=1 TO N
410   Xi=Delta*Ns                          !  Argument xi of char. fn.
420   X2=Xi*Xi                             !  Calculation
430   CALL Log(1-X2*D2,-Xi*D1,A,B)         !  of
440   CALL Log(1-X2*D2p,-Xi*D1p,C,D)       !  characteristic
450   T1=T*A-Nu*C                          !  function
460   T2=T*B-Nu*D+Bs*Xi                    !  fy(xi)
470   CALL Exp(T1,T2,Fyr,Fyi)
480   Ms=Ns MOD Mf                         !  Collapsing
490   X(Ms)=X(Ms)+Fyr/Ns
500   Y(Ms)=Y(Ms)+Fyi/Ns
510   NEXT Ns
520   CALL Fft10z(Mf,X(*),Y(*))            !  0 subscript FFT
530   PLOTTER IS "GRAPHICS"
540   GRAPHICS
550   SCALE 0,Mf,-14,0
560   LINE TYPE 3
570   GRID Mf/8,1
580   PENUP
590   LINE TYPE 1
600   B=Bs*Mf*Delta/(2*PI)                 !  Origin for random variable x
610   MOVE B,0
620   DRAW B,-14
630   PENUP
640   FOR Ks=0 TO Mf-1
650   T=Y(Ks)/PI-Ks/Mf
660   X(Ks)=.5-T                           !  Cumulative probability in X(*)
670   Y(Ks)=Pr=.5+T                        !  Exceedance probability in Y(*)
680   IF Pr>=1E-12 THEN Y=LGT(Pr)
690   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
700   IF ABS(Pr)<1E-12 THEN Y=-12
710   PLOT Ks,Y
720   NEXT Ks
730   PENUP
740   PRINT Y(0);Y(1);Y(Mf-2);Y(Mf-1)
750   FOR Ks=0 TO Mf-1
760   Pr=X(Ks)
770   IF Pr>=1E-12 THEN Y=LGT(Pr)
780   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
790   IF ABS(Pr)<1E-12 THEN Y=-12
800   PLOT Ks,Y
810   NEXT Ks
820   PENUP
830   PAUSE
840   DUMP GRAPHICS
850   PRINT LIN(5)
860   PRINTER IS 16
870   END
880   !
```

```
890   SUB Exp(X,Y,A,B)                        !  EXP(Z)
900   T=EXP(X)
910   A=T*COS(Y)
920   B=T*SIN(Y)
930   SUBEND
940   !
950   SUB Log(X,Y,A,B)                        !  PRINCIPAL LOG(Z)
960   A=.5*LOG(X*X+Y*Y)
970   IF X<>0 THEN 1000
980   B=.5*PI*SGN(Y)
990   GOTO 1020
1000  B=ATN(Y/X)
1010  IF X<0 THEN B=B+PI*(1-2*(Y<0))
1020  SUBEND
1030  !
1040  SUB Fft10z(N,X(*),Y(*))     ! N <= 2^10 = 1024, N=2^INTEGER      0 subscript
```

APPENDIX C. NARROWBAND CROSS-CORRELATOR

This program computes the cumulative and exceedance distribution functions of random variable (30) via characteristic function (35). The parameters D_1, D_2, N_0, N_1 are pre-computed in lines 280-390 and weighted according to (35)-(36) in lines 400-440. All the functions employed are analytic.

```
10 ! NARROWBAND CROSS-CORRELATOR
20    K=5                          !  Number of terms summed
30    DATA .6,-.5,.4,-.3,.2        !  w(k)      weightings
40    DATA .9,.8,.7,-.6,-.5        !  a1(k)     signal 1 in-phase components
50    DATA -.6,-.8,1,1.2,1.4       !  b1(k)     signal 1 quadrature components
60    DATA .1,-.2,-.3,.4,.5        !  a2(k)     signal 2 in-phase components
70    DATA -.7,.6,.5,.4,-.3        !  b2(k)     signal 2 quadrature components
80    DATA .1,.3,.5,.7,.9          !  sigma1(k)  noise 1 standard deviations
90    DATA .2,.4,.6,.8,1           !  sigma2(k)  noise 2 standard deviations
100   DATA .4,-.5,.6,.7,-.8        !  rho(k)    noise in-phase corr. coeffs.
110   DATA .9,-.7,-.5,.3,-.1       !  lambda    noise quadrature corr. coeffs.
120   L=50                         !  Limit on integral of char. function
130   Delta=.5                     !  Sampling increment on char. function
140   Bs=.5*(2*PI/Delta)           !  Shift b, as fraction of alias interval
150   Mf=2^8                       !  Size of FFT
160   PRINTER IS 0
170   PRINT "L =";L,"Delta =";Delta,"b =";Bs,"Mf =";Mf
180   REDIM W(1:K),A1(1:K),B1(1:K),A2(1:K),B2(1:K)
190   REDIM S1(1:K),S2(1:K),Rho(1:K),Lambda(1:K)
200   REDIM D1(1:K),D2(1:K),N0(1:K),N1(1:K)
210   REDIM X(0:Mf-1),Y(0:Mf-1),W2(1:K)
220   DIM W(1:10),A1(1:10),B1(1:10),A2(1:10),B2(1:10)
230   DIM S1(1:10),S2(1:10),Rho(1:10),Lambda(1:10)
240   DIM D1(1:10),D2(1:10),N0(1:10),N1(1:10)
250   DIM X(0:1023),Y(0:1023),W2(1:10)
260   READ W(*),A1(*),B1(*),A2(*),B2(*)  !  Enter
270   READ S1(*),S2(*),Rho(*),Lambda(*)  !  constants
280   FOR J=1 TO K                       !  Calculation
290   S1s=S1(J)^2                        !  of
300   S2s=S2(J)^2                        !  parameters
310   T1=S1(J)*S2(J)
320   D1(J)=T2=T1*Rho(J)
330   D2(J)=.25*S1s*S2s*(1-Rho(J)^2-Lambda(J)^2)
340   T3=A1(J)*A2(J)+B1(J)*B2(J)
350   N0(J)=.5*T3
360   T4=A2(J)*B1(J)-A1(J)*B2(J)
370   T5=S2s*(A1(J)^2+B1(J)^2)+S1s*(A2(J)^2+B2(J)^2)
380   N1(J)=.125*(T5-2*T2*T3-2*T1*Lambda(J)*T4)
390   NEXT J
400   MAT W2=W.W
410   MAT D1=W.D1
420   MAT D2=W2.D2
430   MAT N0=W.N0
440   MAT N1=W2.N1
450   Mux=SUM(N0)+SUM(D1)                !  Mean of random variable v
460   Muy=Mux+Bs
```

```
470   X(0)=0
480   Y(0)=.5*Delta*Muy
490   N=INT(L/Delta)
500   FOR Ns=1 TO N
510   Xi=Delta*Ns                          ! Argument xi of char. fn.
520   X2=Xi*Xi                             ! Calculation
530   Pr=1                                 ! of
540   Pi=Sr=Si=0                           ! characteristic
550   FOR J=1 TO K                         ! function
560   Dr=1+X2*D2(J)                        ! fy(xi)
570   Di=-Xi*D1(J)
580   CALL Mul(Pr,Pi,Dr,Di,A,B)
590   Pr=A
600   Pi=B
610   CALL Div(N0(J),Xi*N1(J),Dr,Di,A,B)
620   Sr=Sr+A
630   Si=Si+B
640   NEXT J
650   CALL Exp(-Xi*Si,Xi*(Sr+Bs),A,B)
660   CALL Div(A,B,Pr,Pi,Fyr,Fyi)
670   Ms=Ns MOD Mf                         ! Collapsing
680   X(Ms)=X(Ms)+Fyr/Ns
690   Y(Ms)=Y(Ms)+Fyi/Ns
700   NEXT Ns
710   CALL Fft10z(Mf,X(*),Y(*))            ! 0 subscript FFT
720   PLOTTER IS "GRAPHICS"
730   GRAPHICS
740   SCALE 0,Mf,-14,0
750   LINE TYPE 3
760   GRID Mf/8,1
770   PENUP
780   LINE TYPE 1
790   B=Bs*Mf*Delta/(2*PI)                 ! Origin for random variable v
800   MOVE B,0
810   DRAW B,-14
820   PENUP
830   FOR Ks=0 TO Mf-1
840   T=Y(Ks)/PI-Ks/Mf
850   X(Ks)=.5-T                           ! Cumulative probability in X(*)
860   Y(Ks)=Pr=.5+T                        ! Exceedance probability in Y(*)
870   IF Pr>=1E-12 THEN Y=LGT(Pr)
880   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
890   IF ABS(Pr)<1E-12 THEN Y=-12
900   PLOT Ks,Y
910   NEXT Ks
920   PENUP
930   PRINT Y(0);Y(1);Y(Mf-2);Y(Mf-1)
940   FOR Ks=0 TO Mf-1
950   Pr=X(Ks)
960   IF Pr>=1E-12 THEN Y=LGT(Pr)
970   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
980   IF ABS(Pr)<1E-12 THEN Y=-12
990   PLOT Ks,Y
1000  NEXT Ks
1010  PENUP
1020  PAUSE
```

```
1030  DUMP GRAPHICS
1040  PRINT LIN(5)
1050  PRINTER IS 16
1060  END
1070  !
1080  SUB Mul(X1,Y1,X2,Y2,A,B)           !  Z1*Z2
1090  A=X1*X2-Y1*Y2
1100  B=X1*Y2+X2*Y1
1110  SUBEND
1120  !
1130  SUB Div(X1,Y1,X2,Y2,A,B)           !  Z1/Z2
1140  T=X2*X2+Y2*Y2
1150  A=(X1*X2+Y1*Y2)/T
1160  B=(Y1*X2-X1*Y2)/T
1170  SUBEND
1180  !
1190  SUB Exp(X,Y,A,B)                   !  EXP(Z)
1200  T=EXP(X)
1210  A=T*COS(Y)
1220  B=T*SIN(Y)
1230  SUBEND
1240  !
1250  SUB Fft10z(N,X(*),Y(*))     ! N <= 2^10 = 1024, N=2^INTEGER     0 subscript
```

APPENDIX D. REDUCED QUADRATIC AND LINEAR FORM

This program computes the cumulative and exceedance distribution functions of random variable (65) via characteristic function (66). The required inputs to the program are M and the $\{\lambda_m\}$, $\{d_m\}$, $\{\nu_m\}$ of (68). The square root in (66) must again be continuous and is handled exactly as in appendix A. The parameters required in the exponential of (66) are pre-computed in lines 170–210, and the mean of q is entered in line 220.

```
10 ! REDUCED QUADRATIC AND LINEAR FORM
20    M=5                          !  Number of terms summed
30    DATA .2,-.3,.4,.5,-.6        !  Lambda values
40    DATA -.1,.3,.5,.7,-.9        !  d values
50    DATA .6,.5,-.4,-.3,.2        !  Nu values
60    L=800                        !  Limit on integral of char. function
70    Delta=.08                    !  Sampling increment on char. function
80    Bs=.5*(2*PI/Delta)           !  Shift b, as fraction of alias interval
90    Mf=2^10                      !  Size of FFT
100   PRINTER IS 0
110   PRINT "L =";L,"Delta =";Delta,"b =";Bs,"Mf =";Mf
120   REDIM Lambda(1:M),D(1:M),Nu(1:M),A(1:M),B(1:M),C(1:M)
130   REDIM X(0:Mf-1),Y(0:Mf-1)
140   DIM Lambda(1:10),D(1:10),Nu(1:10),A(1:10),B(1:10),C(1:10)
150   DIM X(0:1023),Y(0:1023)
160   READ Lambda(*),D(*),Nu(*)              !  Enter constants
170   FOR Ms=1 TO M                          !  Calculation
180   A(Ms)=2*Lambda(Ms)                     !  of parameters
190   B(Ms)=(Lambda(Ms)*Nu(Ms)+D(Ms))*Nu(Ms)
200   C(Ms)=.5*D(Ms)^2
210   NEXT Ms
220   Muq=SUM(Lambda)+SUM(B)                 !  Mean of random variable q
230   R=0                                    !  Argument of square root
240   P=1                                    !  Polarity indicator
250   Muy=Muq+Bs
260   X(0)=0
270   Y(0)=.5*Delta*Muy
280   N=INT(L/Delta)
290   FOR Ns=1 TO N
300   Xi=Delta*Ns                            !  Argument xi of char. fn.
310   Pr=1                                   !  Calculation
320   Pi=Sr=Si=0                             !  of
330   FOR Ms=1 TO M                          !  characteristic
340   T=-A(Ms)*Xi                            !  function
350   CALL Mul(Pr,Pi,1,T,A,B)                !  fy(xi)
360   Pr=A
370   Pi=B
380   CALL Div(B(Ms),C(Ms)*Xi,1,T,A,B)
390   Sr=Sr+A
400   Si=Si+B
410   NEXT Ms
```

```
420   CALL Exp(-Si*Xi,(Sr+Bs)*Xi,A,B)
430   CALL Sqr(Pr,Pi,C,D)
440   Ro=R
450   R=ATN(D/C)
460   IF ABS(R-Ro)>1.6 THEN P=-P
470   CALL Div(A,B,C*P,D*P,Fyr,Fyi)
480   Ms=Ns MOD Mf                          !  Collapsing
490   X(Ms)=X(Ms)+Fyr/Ns
500   Y(Ms)=Y(Ms)+Fyi/Ns
510   NEXT Ns
520   CALL Fft10z(Mf,X(*),Y(*))             !  0 subscript FFT
530   PLOTTER IS "GRAPHICS"
540   GRAPHICS
550   SCALE 0,Mf,-14,0
560   LINE TYPE 3
570   GRID Mf/8,1
580   PENUP
590   LINE TYPE 1
600   B=Bs*Mf*Delta/(2*PI)                  !  Origin for random variable q
610   MOVE B,0
620   DRAW B,-14
630   PENUP
640   FOR Ks=0 TO Mf-1
650   T=Y(Ks)/PI-Ks/Mf
660   X(Ks)=.5-T                            !  Cumulative probability in X(*)
670   Y(Ks)=Pr=.5+T                         !  Exceedance probability in Y(*)
680   IF Pr>=1E-12 THEN Y=LGT(Pr)
690   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
700   IF ABS(Pr)<1E-12 THEN Y=-12
710   PLOT Ks,Y
720   NEXT Ks
730   PENUP
740   PRINT Y(0);Y(1);Y(Mf-2);Y(Mf-1)
750   FOR Ks=0 TO Mf-1
760   Pr=X(Ks)
770   IF Pr>=1E-12 THEN Y=LGT(Pr)
780   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
790   IF ABS(Pr)<1E-12 THEN Y=-12
800   PLOT Ks,Y
810   NEXT Ks
820   PENUP
830   PAUSE
840   DUMP GRAPHICS
850   PRINT LIN(5)
860   PRINTER IS 16
870   END
880   !
```

```
890   SUB Mul(X1,Y1,X2,Y2,A,B)             !  Z1*Z2
900   A=X1*X2-Y1*Y2
910   B=X1*Y2+X2*Y1
920   SUBEND
930   !
940   SUB Div(X1,Y1,X2,Y2,A,B)             !  Z1/Z2
950   T=X2*X2+Y2*Y2
960   A=(X1*X2+Y1*Y2)/T
970   B=(Y1*X2-X1*Y2)/T
980   SUBEND
990   !
1000  SUB Exp(X,Y,A,B)                     !  EXP(Z)
1010  T=EXP(X)
1020  A=T*COS(Y)
1030  B=T*SIN(Y)
1040  SUBEND
1050  !
1060  SUB Sqr(X,Y,A,B)                     !  PRINCIPAL SQR(Z)
1070  IF X<>0 THEN 1110
1080  A=B=SQR(.5*ABS(Y))
1090  IF Y<0 THEN B=-B
1100  GOTO 1220
1110  F=SQR(SQR(X*X+Y*Y))
1120  T=.5*ATN(Y/X)
1130  A=F*COS(T)
1140  B=F*SIN(T)
1150  IF X>0 THEN 1220
1160  T=A
1170  A=-B
1180  B=T
1190  IF Y>=0 THEN 1220
1200  A=-A
1210  B=-B
1220  SUBEND
1230  !
1240  SUB Fft10z(N,X(*),Y(*))    ! N <= 2^10 = 1024, N=2^INTEGER     0 subscript
```

```
280   FOR I=1 TO Tt
290   X=0
300   FOR J=1 TO K
310   V1=RND-.5                          ! GENERATE TWO
320   V2=RND-.5                          ! INDEPENDENT
330   S=V1*V1+V2*V2                      ! GAUSSIAN
340   IF S>.25 THEN 310                  ! RANDOM
350   Q=(L-LOG(S))/S                     ! VARIABLES VIA
360   Q=SQR(Q+Q)                         ! ACCEPTANCE
370   G1=V1*Q                            ! AND
380   G2=V2*Q                            ! REJECTION
390   S=Ms(J)+Al(J)*G1+Be(J)*G2
400   T=Mt(J)+St(J)*G1
410   X=X+A(J)*S*S+B(J)*T*T+C(J)*S*T+D(J)*S+E(J)*T
420   NEXT J
430   X(I)=X
440   NEXT I
450   MAT SORT X
460   PLOTTER IS "GRAPHICS"
470   GRAPHICS
480   SCALE -30,10,-4,0
490   GRID 5,1
500   PENUP
510   FOR I=1 TO Tt
520   Y=LGT((I-.5)/Tt)
530   PLOT X(I),Y
540   NEXT I
550   PENUP
560   FOR I=1 TO Tt
570   Y=LGT(1-(I-.5)/Tt)
580   PLOT X(I),Y
590   NEXT I
600   PENUP
610   END
```

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