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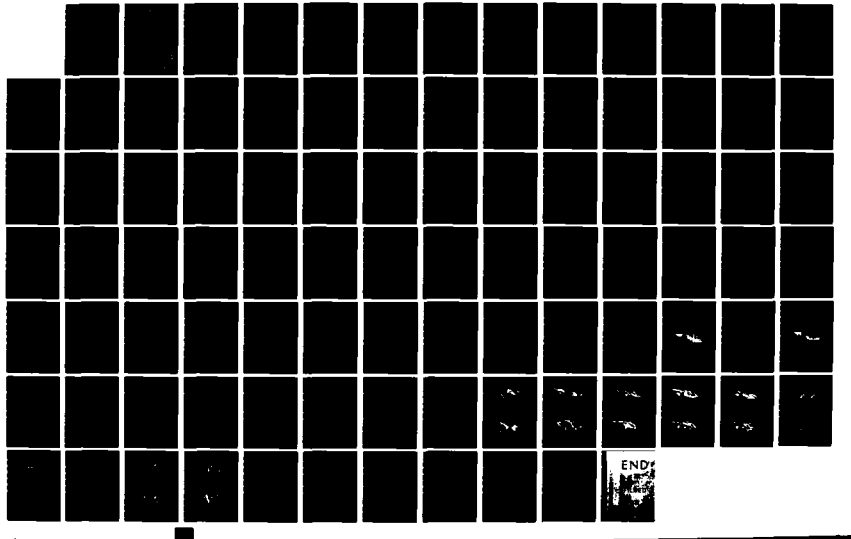

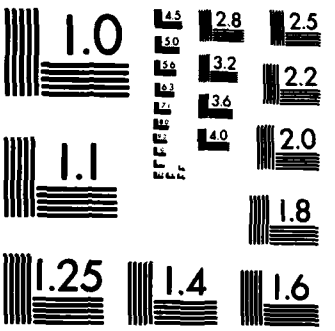

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LEAST SQUARE APPROXIMATION BY LINEAR COMBINATIONS OF (MULTI)POLES

WILLI FREEDEN

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THE OHIO STATE UNIVERSITY
COLUMBUS, OHIO 43210

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The purpose of the paper is to transcribe the standard procedure of expanding an external gravitational potential by trial functions (e.g. mass points, multipoles, spherical harmonics) from the classical spherical case to non-spherical models.

The external gravitational potential of the earth is shown to be expandable into a series being convergent outside and on any closed surface completely surrounding the earth's surface in the

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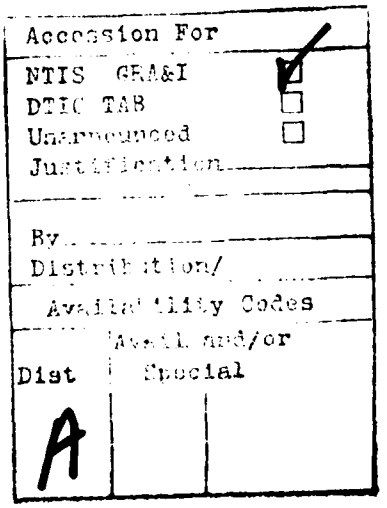
outer space. In addition, for each compact subset of the outer space with positive distance to the earth's surface, the convergence is uniform.

A numerical method of computing the external gravitational potential by truncated series expansions concludes the paper.

FOREWORD

This report was prepared by Dr. Willi Freeden, Associate Professor, Institute for Pure and applied Mathematics, West Germany and Post Doctoral Researcher, Department of Geodetic Science and Surveying, The Ohio State University, under Air Force Contract No. F19628-82-K-0017, Project Supervisor, Urho A. Uotila, Professor, Department of Geodetic Science and Surveying. The contract covering this research is administered by the Air Force Geophysics Laboratory, Hanscom Air Force Base, Massachusetts, with Dr. Christopher Jekeli, Contract Manager.



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1. Introduction

From classical potential theory we know that the (Newtonian) potential of the earth's body E

$$V(x) = \kappa \int_E \rho(y) \frac{1}{|x-y|} dV(y) \qquad (1.1)$$

(κ : gravitational constant)

is a harmonic function in free space. That is, V is arbitrarily often differentiable in the earth's exterior E_e and it satisfies Laplace's equation

$$\Delta V(x) = 0 \qquad (1.2)$$

for all points $x \in E_e$. Moreover, V is regular at infinity, i.e. V and ∇V tend to zero

$$|V(x)| = O(\frac{1}{|x|})$$
$$|\nabla V(x)| = O(\frac{1}{|x|^2}) \ , \qquad (1.3)$$

if $|x|$ tends to infinity. Since the density function ρ is better known only for parts of the upper crust of the earth, the closed representation (1.1) of the potential V, however, cannot be used in numerical computations. Instead, we have to look for suitable approximations. In this connection a result first formulated by C. Runge (1885) and later generalized by J.L. Walsh (1929) and T. Krarup (1969) is of particular importance. It states:

Any function V harmonic outside the earth's surface S and regular at infinity, may be approximated by a function U, harmonic outside an arbitrarily given sphere (Bjerhammar sphere) inside the earth and regular at infinity in the sense that for any given $\varepsilon > 0$, the relations

$$|V(x) - U(x)| < \varepsilon$$

resp. (1.4)

$$|\nabla V(x) - \nabla U(x)| < \varepsilon$$

hold for all points x of the Euclidean space R^3 outside and on any closed surface completely surrounding the earth's surface in the outer space.

The value ε may be arbitrarily small, and the surrounding surface may be arbitrarily close to the earth's surface (cf. Moritz (1980)).

In this formulation the Runge-Walsh-Krarup theorem is a pure existence theorem, i.e. it guarentees only the existence of an approximating function U and does not provide a method to find it. Nothing is said about the structure of the approximation U. The theorem describes merely the theoretical background for practical purposes of determining an approximation to the external gravitational potential of the earth.

In a spherically symmetric model, however, the situation is completely different. For a spherical earth, constructive approximations of V are available by use of trial functions with a larger harmonicity domain. As trial functions the following (base) systems $\{\phi_n\}$ can be used, for example:

(i) " solid spherical harmonics "

$$|x|^{-(n+1)} S_n(\xi) \quad (x=r\xi \ , \ r=|x|) \ ,$$

$\{S_n\}$ maximal linearly independent system of surface spherical harmonics (origin at earth's center of gravity)

(ii) " mass points "

$$\frac{1}{|x-x_n|}$$

$\{x_n\}$ "fundamental" sequence of points in the complement of $E_e \cup S$ (cf. Chapter 8)

(iii) " multipoles "

$$\left(\frac{\partial}{\partial x_o}\right)^j \frac{1}{|x-x_o|}$$

$$[j]=n$$

x_o is a fixed point in the complement of $E_e \cup S$, $j = (j_1,j_2,j_3)$ is a triple of non-negative integers,

$$[j] = j_1+j_2+j_3 \ , \ \left(\frac{\partial}{\partial x_o}\right)^j = \left.\frac{\partial^{[j]}}{\partial x_1^{j_1} \partial x_2^{j_2} \partial x_3^{j_3}}\right|_{x_o}$$

We recapitulate the standard method of approximating an external potential V in a spherical model.

Spherical approximation

Let us suppose that S is a sphere with radius R about the origin and E_e is the outer space of S:

$$S = \{x \in R^3 \mid |x| = R\}$$
$$E_e = \{x \in R^3 \mid |x| > R\}.$$

Let V be continuous in $E_e \cup S$ and twice continuously differentiable in E_e. Moreover, let V satisfy $\Delta V(x) = 0$ for all $x \in E_e$.

Let us denote by $\{\phi_n\}$ one of the linearly independent systems of trial functions listed above. Corresponding to the countably infinite sequence $\{\phi_n\}$ there exists a system $\{\phi_n^*\}$ of trial functions ϕ_n^* orthonormalized with respect to the inner product of the space $\mathcal{L}^2(S)$ of all square integrable functions on S (Gram-Schmidt orthonormalizing process)

$$\int_S \phi_n^*(y)\ \phi_\ell^*(y)dS(y) = \delta_{n\ell}\ . \qquad (1.5)$$

(dS : surface element)

Then the potential V can be represented by the expansion

$$V(x) = \sum_{n=0}^{\infty} (V,\phi_n^*)\ \phi_n^*(x), \qquad (1.6)$$

where the numbers

$$(V,\phi_n^*) = \int_S V(y)\ \phi_n^*(y)dS(y) \qquad (1.7)$$

are the Fourier (or orthogonal) coefficients of V on S with respect to the system $\{\phi_n^*\}$.

More explicitly this reads: given an error bound $\varepsilon > 0$, then there exists an integer $N = N(\varepsilon)$ such that

$$|V(x) - \sum_{n=0}^{N} (V,\phi_n^*)\ \phi_n^*(x)| \leq \varepsilon \qquad (1.8)$$

for all points $x \in G$ with $G \subset E_e$ and positive distance to S ($dist(G,S) \geq \lambda > 0$). In **each** compact subset $G \subset E_e$ the convergence is uniform.

Therefore we have a constructive version of Runge's theorem by means of the truncations

$$V^{(N)}(x) = \sum_{n=0}^{N} (V,\phi_n^*)\ \phi_n^*(x) \qquad (1.9)$$

in a spherical (earth's) model.

The approximation technique is formulated so as to have the permanence property: the transition from $V^{(N)}$ to $V^{(N+1)}$ necessitates merely the addition of one more term, all the terms obtained formerly remaining unchanged.

Furthermore, $V^{(N)}$ has the following minimum property

$$\int_S |V(y) - \sum_{n=0}^{N} (V,\phi_n^*)\ \phi_n^*(y)|^2\ dS \qquad (1.10)$$

$$= \min_{a_n} \int_S |V(y) - \sum_{n=0}^{N} a_n\ \phi_n^*(y)|^2\ dS$$

for any selection of coefficients a_n.

The considerations given above lead to the impression that a least square approximation by truncated expansions $V^{(N)}$ of the form (1.9) is intimately related to spherical boundaries. Our purpose, however, is to prove that the essential steps leading to a least square approximation of an external gravitational potential by truncations of series expansions into (multi)poles can be transcribed in the same way to general (non-spherical) models.

We summarize the main results.

Not-necessarily spherical models:

Let us suppose that S is the boundary-surface of a not-necessarily spherical earth's model (e.g. ellipsoid, telluroid, geoid, real(regular)earth's surface). Let E_e be the outer space.

Let V be continuous in $E_e \cup S$ and twice continuously differentiable in E_e. Moreover, let V satisfy $\Delta V(x) = 0$ for all $x \in E_e$.

Let us denote by $\{\phi_n\}$ one of the linearly independent systems listed above. Corresponding to the sequence $\{\phi_n\}$ there exists a system $\{\phi_n^*\}$ of trial functions ϕ_n^* orthonormalized with respect to the inner product of the space $\mathcal{L}^2(S)$ of all square integrable functions on S. Then the potential V can be respresented by the expansion

$$V(x) = \sum_{n=0}^{\infty} (V, \phi_n^*)\ \phi_n^*(x) \ , \qquad (1.11)$$

where the numbers

$$(V,\phi_n^*) = \int_S V(y)\ \phi_n^*(y)\ dS(y) \qquad (1.12)$$

are the Fourier (or orthogonal) coefficients of V on S with respect to the system $\{\phi_n^*\}$.

More explicitly: given an error bound $\varepsilon > 0$, then there exists an integer $N=N(\varepsilon)$ such that

$$\left|V(x) - \sum_{n=0}^{N} (V,\phi_n^*)\ \phi_n^*(x)\right| \leq \varepsilon \qquad (1.13)$$

for all $x \in G$ with $G \subset E_e$ and $dist(G,S) \geq \lambda > 0$. In each compact subset $G \subset E_e$, the convergence is uniform.

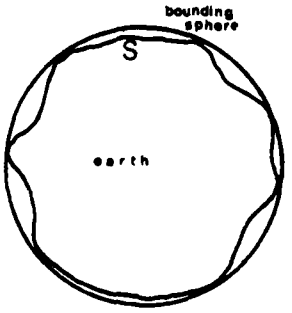

Figure 1

It should be pointed out that the series (1.11) guarentees ordinary point-wise convergence not only outside the smallest bounding sphere, but indeed in the (whole) outer space E_e. Our approach therefore avoids inadequate phenomena of divergence in regions between a bounding sphere and the surface S. The price to be paid is the orthonormalization process with respect to the not-necessarily spherical surface S.

As in the spherical case, the advantages are the permanence property and the minimum property.

At the surface S, the approximation by trial functions ϕ_n^* depends closely on the smoothness presupposed for V.

Provided that V is only assumed to be continuous on the surface S, then it can be proved that, to every $\varepsilon > 0$, there exist an integer $N = N(\varepsilon)$ and coefficients $b_o,\ldots,b_N$ such that

$$\sup_{x\in S} \left|V(x) - \sum_{n=o}^{N} b_n \phi_n^*(x)\right| \leq \varepsilon \tag{1.14}$$

and by maximum principle of potential theory

$$\sup_{x\in E_e \cup S} \left|V(x) - \sum_{n=o}^{N} b_n \phi_n^*(x)\right| \leq \varepsilon \tag{1.15}$$

(cf. Freeden(1980) Chapter 6). But the coefficients $b_o,\ldots,b_N$ do not coincide, in general, with the Fourier coefficients $(V,\phi_o^*),\ldots,(V,\phi_N^*)$.

Hence, "downward continuation" of the expansion

$$\sum_{n=o}^{\infty} (V,\phi_n^*)\phi_n^* \tag{1.16}$$

to the surface S cannot be assured generally under the assumption of continuity of V over the surface S.

The series expansion

$$V(x) = \sum_{n=o}^{\infty} (V,\phi_n^*)\phi_n^*(x), \tag{1.17}$$

however, remains true for all $x \in E_e \cup S$ in the ordinary pointwise sense if stronger conditions of smoothness are imposed on V in addition . In the spherical case, for example, the continuous differentiability of V on S is sufficient to verify the validity of (1.17) on S for spherical harmonics. (A detailed discussion of "downward continuation" of series expansions into trial functions for non-spherical cases seems to be open).

In detail we are concerned with the following investigations:

In Chapter 2 and 3 notations are introduced. Essential tools of the presentation are the "jump relations" and the "limit formulas" of potential theory (Chapt.4) and a regularization theorem (Chapt.5). The adventages of an approximation by harmonic trial functions are discussed in Chapt.6 and 7. Then non-constructive approximation theorems (Chapt.8) and constructive approximation theorems (Chapt.9) are developed. Chapt.10 gives an application to the external gravitational potential of the earth. In Chapt.11 a numerical method based on the Cholesky factorization is presented. The efficiency of the numerical method is analyzed. Finally (Chapt.12), some simple examples give an impression of the accuracy.

Acknowledgements:

The author is indebted to Dr. R. Reuter, Aachen, for the numerical computations.

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2. Basic Settings of Functional Analysis

The following definitions and notations are standard in functional analysis.

A linear space is a set of elements such that to any finite subset $\varphi_o,\ldots,\varphi_n$ the linear combination $\alpha_o\varphi_o + \ldots + \alpha_n\varphi_n$ (α_n : scalars) is also a member of the linear space. We shall only consider real linear spaces. Hence $\alpha_o,\ldots,\alpha_n$ are assumed to be real. Any linear space possesses a unique zero element. A collection $\varphi_o,\ldots,\varphi_n$ of elements out of a linear space is called linearly dependent if there exist numbers $\alpha_o,\ldots,\alpha_n$, not all of them being zero, such that

$$\alpha_o\varphi_o + \ldots + \alpha_n\varphi_n = 0. \tag{2.1}$$

Otherwise the elements $\varphi_o,\ldots,\varphi_n$ are called linearly independent.

For each subset of a linear space we consider the space generated by this subset, i.e. linear combinations of the form

$$\alpha_o\varphi_o + \ldots + \alpha_n\varphi_n .$$

The union of all such spaces is called the span of $\varphi_o,\ldots,\varphi_n$:

$$\text{span}(\varphi_o,\ldots,\varphi_n).$$

A <u>normed space</u> X is a linear space which assigns to any of its elements φ a non-negative number $\|\varphi\|$, called the norm of φ, with the following properties:

(i) $\|\varphi\| \geq 0$; $\|\varphi\| = 0 \Leftrightarrow \varphi = 0$

(ii) $\|\varphi + \psi\| \leq \|\varphi\| + \|\psi\|$

(iii) $\|\alpha\varphi\| = |\alpha| \, \|\varphi\|$. ($\alpha$: real).

The norm describes distances between pairs of elements $\varphi, \psi \in X$. The <u>distance</u> between φ and ψ is defined by

$$\text{dist}(\varphi,\psi) = \|\varphi - \psi\| \, . \tag{2.2}$$

The distance between two subsets A,B of X is defined by

$$\text{dist}(A,B) = \inf_{\substack{\varphi\in A \\ \psi\in B}} \|\varphi - \psi\| \, . \tag{2.3}$$

Distances can be used to define <u>convergence</u>. Two cases are of particular interest.

(i) <u>Convergence to a limit element</u>.
Let $\{\varphi_n\}_{n=0,1,\ldots}$ be a sequence of elements out of a normed linear space X. Let φ be another element of X. We say that the sequence $\{\varphi_n\}_{n=0,1,\ldots}$ converges to the limit element φ, in symbols,

$$\lim_{n\to\infty} \varphi_n = \varphi \quad \text{or} \quad \varphi_n \to \varphi \quad \text{in } X \, ,$$

if, given $\varepsilon > 0$, there exists an integer $N = N(\varepsilon)$ such that

$$\|\varphi_n - \varphi\| \leq \varepsilon \tag{2.4}$$

for all $n \geq N(\varepsilon)$.

(ii) Convergence in the sense of Cauchy

A sequence $\{\varphi_n\}_{n=o,1,\ldots}$ is called convergent in the sense of Cauchy, if given $\varepsilon > 0$, there exists an integer $N = N(\varepsilon)$ such that

$$\|\varphi_n - \varphi_m\| \leq \varepsilon \tag{2.5}$$

for all $n,m \geq N(\varepsilon)$.

We see that a Cauchy sequence is defined without using a limit element. It merely requires that the elements of the sequence cluster in a certain way.

A normed space X is called complete, if, to every Cauchy convergent sequence $\{\varphi_n\}_{n=o,1,\ldots}$ in X, there exists an element $\varphi \in X$, such that $\{\varphi_n\}_{n=o,1,\ldots}$ converges to the limit element φ.

As soon as the elements cluster in the sense of Cauchy in a complete normed space, a limit element must exist.

Complete normed spaces are called Banach spaces.

For a normed space X, a system $\{\varphi_n\}_{n=o,1,\ldots}$ is called a basis, if for any $\varepsilon > 0$, $\varphi \in X$ there exist an integer $N=N(\varepsilon)$ and coefficients $\alpha_o,\ldots,\alpha_N$ such that

$$\|\varphi - \sum_{k=o}^{N} \alpha_k \varphi_k\| \leq \varepsilon.$$

A linear space X is called an inner product space, if to any pair φ,ψ of its members a real number (φ,ψ), called the inner product, is assigned which has the following properties:

(i) $(\varphi,\psi) = (\psi,\varphi)$

(ii) $(\alpha\varphi,\psi) = \alpha(\varphi,\psi)$ (α: real)

(iii) $(\varphi_1+\varphi_2,\psi) = (\varphi_1,\psi) + (\varphi_2,\psi)$

(iv) $(\varphi,\varphi) \geq 0$, $(\varphi,\varphi) = 0 \Leftrightarrow \varphi = 0$.

A consequence of these properties is the Schwarz inequality

$$(\varphi,\psi)^2 \leq (\varphi,\varphi)(\psi,\psi). \tag{2.6}$$

We say that a system $\{\varphi_k\}_{k=0,1,\ldots}$ of functions of an inner product space X is an orthogonal system if its elements are mutually perpendicular, that is

$$(\varphi_k,\varphi_i) = 0 \tag{2.7}$$

if $i \neq k$, and in addition

$$(\varphi_k,\varphi_k) \neq 0$$

for all k.

An orthogonal system $\{\varphi_n\}_{n=0,1,\ldots}$ is called orthonormal system if $(\varphi_k,\varphi_k) = 1$ for all k. One can always obtain an orthonormal system by dividing each element by its norm.

A further property of the inner product which may be deduced from the defining properties (i) - (iv) is the continuity of the inner product. This means that if $\varphi_n \to \varphi$, $\varphi_m \to \psi$, then $(\varphi_n, \psi_m) \to (\varphi, \psi)$.

The inner product induces a norm by defining

$$\|\varphi\| = \sqrt{(\varphi,\varphi)}. \tag{2.8}$$

Hence, an inner product space X is automatically a normed space. Schwarz's inequality may be reformulated by taking the square root

$$|(\varphi,\psi)| \leq \|\varphi\| \cdot \|\psi\|. \tag{2.9}$$

A Hilbert space is an inner product space which, in addition, is complete.

If a basis $\{\varphi_n\}_{n=0,1,\ldots}$ in an Hilbert space X is orthonormalized, each element φ of the space X is representable by the generalized Fourier series

$$\varphi = \sum_{n=0}^{\infty} (\varphi,\varphi_n)\varphi_n \tag{2.10}$$

in the sense of X.

An orthonormal basis in a Hilbert space is often called Hilbert basis.

3. The Banach Space $C^{(o)}(S)$ and the Hilbert Space $\mathcal{L}^2(S)$

If G is a set of points in R^3, ∂G will represent its boundary. The set $\overline{G} = G \cup \partial G$ will be called the closure of G.

A set $G \subset R^3$ is called a region if it is open and connected.

The restriction of a function f to a subset M of its domain is denoted by $f|M$, for a set I of functions we define $I|M$ as the set of restrictions $f|M$:

$$I|M = \{f|M | f \in I\}$$

A function f possessing k continuous derivatives is said to be of class $C^{(k)}$.

A surface $S \subset R^3$ will be called regular, if it satisfies the following properties:

(i) S divides three-dimensional Euclidean space R^3 into the bounded region $E = E_i$ (inner space) and the unbounded region E_e (outer space) defined by $E_e = R^3 - \overline{E}$

(ii) S is a closed and compact surface with no double points

(iii) S is a $C^{(2)}$-surface

From the definition it is clear that all (geodetically relevant) earth's models are included. Regular surfaces are for example sphere, ellipsoid, spheroid, telluroid, geoid or real (regular) earth's surface.

By $C^{(o)}(S)$ we denote the set of all continuous functions on S, endowed with the norm

$$\|f\| = \sup_{x \in S} |f(x)| . \tag{3.1}$$

$C^{(o)}(S)$ is a complete normed space, i.e. a Banach space.

In $C^{(o)}(S)$ we introduce the inner product

$$(f,g) = \int_S f(x)g(x)dS, \tag{3.2}$$

the inner product $(\cdot,\cdot)$ implies the norm

$$|f| = \sqrt{(f,f)} . \tag{3.3}$$

In connection with (3.2), $C^{(o)}(S)$ is an inner product space.

For each $f \in C^{(o)}(S)$ we have the inequality

$$|f| \leq C \|f\| \tag{3.4}$$

with

$$C = C(S) = \sqrt{\int_S dS} . \tag{3.5}$$

We denote by $\mathscr{L}^2(S)$ the space of real functions defined on S with second power absolutely integrable over S. $\mathscr{L}^2(S)$ is a Banach space with the norm

$$|f| = \sqrt{\int_S |f(x)|^2 \, dS} . \tag{3.6}$$

Furthermore, $\mathscr{L}^2(S)$ is a Hilbert space with inner product $(\cdot,\cdot)$ analogously defined by (3.2).

$\mathcal{L}^2(S)$ is the completion of $C^{(o)}(S)$ with respect to the norm $|\cdot|$.

4. Limit Relations and Jump Relations in Potential Theory

At each point y of the boundary $S = \partial E$ we can construct a normal $n(y)$ pointing into the outer space E_e. The set

$$S_\tau = \{x \in R^3 \mid x = y + \tau n(y),\ y \in S\} \tag{4.1}$$

generates a parallel surface which is exterior to E for $\tau > 0$ and interior for $\tau < 0$. It is well-known from differential geometry (cf. Cl.Müller(1969),§ 13) that if $|\tau|$ is sufficiently small, then the surface S_τ is regular, and the normal to one parallel surface is a normal to the other.

Let g be a continuous function on S ($g \in C^{(0)}(S)$). Then the functions U_N defined by

$$U_N(x) = \int_S g(y) \left(\frac{\partial}{\partial n(y)}\right)^{N-1} \frac{1}{|x-y|}\, dS(y) \tag{4.2}$$

are infinitely often differentiable in E_i and E_e and satisfy the Laplace equation. In addition, they are regular at infinity.

The function U_1 defined by

$$U_1(x) = \int_S g(y) \frac{1}{|x-y|}\, dS(y)$$

is called the potential of the single layer on S, while U_2 given by

$$U_2(x) = \int_S g(y)\, \frac{\partial}{\partial n(y)}\, \frac{1}{|x-y|}\, dS(y)$$

is called the potential of the double layer on S.

For $g \in C^{(o)}(S)$ the functions U_N can be continued continuously to the surface S, but the limits depend from which parallel surface (inner or outer) the points x tend to S.

On the other hand, the functions U_N also are defined on the surface S, i.e. the integrals

$$U_N(x) = \int_S g(y) \left(\frac{\partial}{\partial n(y)}\right)^{N-1} \frac{1}{|x-y|} \, dS(y)$$

exist for $x \in S$.

Furthermore, the integral

$$U_1'(x) = \int_S g(y) \left(\frac{\partial}{\partial n(x)}\right) \frac{1}{|x-y|} \, dS(y) \tag{4.3}$$

exists for $x \in S$.

However, the integrals do not coincide in general with the inner or outer limits of the potentials.

From classical potential theory (cf. e.g. O.D.Kelogg (1929), J. Schauder (1931)) it is known that for $x \in S$

$$\lim_{\substack{\tau \to o \\ \tau > o}} U_1(x \pm \tau n(x)) = U_1(x) \tag{4.4a}$$

$$\lim_{\substack{\tau \to o \\ \tau > o}} \frac{\partial U_1}{\partial n}(x \pm \tau n(x)) = \mp 2\pi g(x) + U_1'(x) \tag{4.4b}$$

$$\lim_{\substack{\tau \to o \\ \tau > o}} U_2(x \pm \tau n(x)) = \pm 2\pi g(x) + U_2(x) \tag{4.4c}$$

(" limit relations ")

and

$$\lim_{\substack{\tau\to o\\ \tau>o}}(U_1(x+\tau n(x)) - U_1(x-\tau n(x)) = 0 \qquad (4.5a)$$

$$\lim_{\substack{\tau\to o\\ \tau>o}}(\frac{\partial U_1}{\partial n}(x+\tau n(x)) - \frac{\partial U_1}{\partial n}(x-\tau n(x)) = -4\pi g(x) \qquad (4.5b)$$

$$\lim_{\substack{\tau\to o\\ \tau>o}}(U_2(x+\tau n(x)) - U_2(x-\tau n(x))) = 4\pi g(x) \qquad (4.5c)$$

$$\lim_{\substack{\tau\to o\\ \tau>o}}(\frac{\partial U_2}{\partial n}(x+\tau n(x)) - \frac{\partial U_2}{\partial n}(x-\tau n(x))) = 0. \qquad (4.5d)$$

(" jump relations ")

In addition , O.D. Kellogg has proved that the above relations hold uniformly with respect to all $x \in S$.

This means that

$$\lim_{\substack{\tau\to o\\ \tau>o}} \sup_{x\in S} |U_1(x \pm \tau n(x)) - U_1(x)| = 0 \qquad (4.6a)$$

$$\lim_{\substack{\tau\to o\\ \tau>o}} \sup_{x\in S} |\frac{\partial U_1}{\partial n}(x \pm \tau n(x)) \pm 2\pi g(x) - U_1'(x)| = 0 \qquad (4.6b)$$

and

$$\lim_{\substack{\tau\to o\\ \tau>o}} \sup_{x\in S} |U_1(x + \tau n(x) - U_1(x - \tau n(x))| = 0 \qquad (4.7a)$$

$$\lim_{\substack{\tau\to o\\ \tau>o}} \sup_{x\in S} \left| \frac{\partial U_1}{\partial n}(x + \tau n(x)) - \frac{\partial U_1}{\partial n}(x - \tau n(x)) + 4\pi g(x) \right| = C \qquad (4.7b)$$

$$\lim_{\substack{\tau\to o\\ \tau>o}} \sup_{x\in S} |U_2(x + \tau n(x)) - U_2(x - \tau n(x)) - 4\pi g(x)| = 0 \qquad (4.7c)$$

$$\lim_{\substack{\tau\to o\\ \tau>o}} \sup_{x\in S} \left| \frac{\partial U_2}{\partial n}(x + \tau n(x)) - \frac{\partial U_2}{\partial n}(x - \tau n(x)) \right| = 0. \qquad (4.7d)$$

Herein

$$\frac{\partial U}{\partial n}(x \pm \tau n(x)) = n(x) \cdot (\nabla U)(x \pm \tau n(x)).$$

These relations can be generalized to the Hilbert space $\mathcal{L}^2(S)$ using exclusively functional analytic means (cf. Freeden(198o), Kersten (198o)).

For square-integrable functions g, i.e. $g \in \mathcal{L}^2(S)$, we then have

$$\lim_{\substack{\tau\to o\\ \tau>o}} \int_S |U_1(x \pm \tau n(x)) - U_1(x)|^2 \, dS(x) = 0 \qquad (4.8a)$$

$$\lim_{\substack{\tau\to o\\ \tau>o}} \int_S \left| \frac{\partial U_1}{\partial n}(x \pm \tau n(x)) - U_1'(x) \pm 2\pi g(x) \right|^2 dS(x) = 0 \qquad (4.8b)$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \int_S |U_2(x \pm \tau n(x)) - U_2(x) \mp 2\pi g(x)|^2 dS(x) = 0 \qquad (4.8c)$$

and

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \int_S |U_1(x + \tau n(x)) - U_1(x - \tau n(x))|^2 dS(x) = 0 \qquad (4.9a)$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \int_S \left|\frac{\partial U_1}{\partial n}(x + \tau n(x)) - \frac{\partial U_1}{\partial n}(x - \tau n(x)) + 4\pi g(x)\right|^2 dS(x) = 0 \qquad (4.9b)$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \int_S |U_2(x + \tau n(x)) - U_2(x - \tau n(x)) - 4\pi g(x)|^2 dS(x) = 0 \qquad (4.9c)$$

$$\lim_{\substack{\tau\to 0\\ \tau>0}} \int_S \left|\frac{\partial U_2}{\partial n}(x + \tau n(x)) - \frac{\partial U_2}{\partial n}(x - \tau n(x))\right|^2 dS(x) = 0. \qquad (4.9d)$$

5. Regularization Theorem

Let $\mathcal{P}(E_e)$ be the space of functions V with the following properties:

(i) V is twice continuously differentiable in E_e and continuous in $\bar{E}_e = E_e \cup S$, i.e. $V \in C^{(2)}(E_e) \cap C^{(0)}(\bar{E}_e)$

(ii) V satisfies Laplace's equation in E_e :

$$\Delta V(x) = 0 \quad \text{for} \quad x \in E_e .$$

(iii) V is regular at infinity:

$$|V(x)| = O\left(\frac{1}{|x|}\right)$$

$$|\nabla V(x)| = O\left(\frac{1}{|x|^2}\right)$$

if $|x| \to \infty$.

Suppose now that $\{\phi_n\}_{n=0,1,\ldots}$ is a sequence of functions ϕ_n belonging to the class $\mathcal{P}(E_e)$. We denote by $\{\varphi_n\}_{n=0,1,\ldots}$ the system of restrictions $\varphi_n = \phi_n|S, n=0,1,\ldots$.

Assume that the system $\{\varphi_n\}_{n=0,1,\ldots}$ converges uniformly to the function φ on S, i.e. for given $\varepsilon > 0$, there exists an integer $N = N(\varepsilon)$ such that

$$\sup_{x \in S} |\varphi_n(x) - \varphi(x)| < \varepsilon \tag{5.1}$$

for all $n > N$. Then the maximum principle of potential theory assures that the sequence $\{\phi_n\}_{n=0,1,\ldots}$ converges uniformly to the (unique) solution ϕ of the exterior (Dirichlet) problem $\phi \in \mathcal{P}(E_e)$, $\phi|S = \varphi$.

In other words, this means that

$$\sup_{x \in \bar{E}} | \phi_n(x) - \phi(x) | \leq \varepsilon \tag{5.2}$$

for all $n \geq N$.

If the system $\{\varphi_n\}_{n=0,1,\ldots}$ is only convergent to φ in quadratic sense, i.e. for given $\varepsilon > 0$, there exists an integer $N = N(\varepsilon)$ such that

$$\sqrt{\int_S |\varphi_n(x) - \varphi(x)|^2 dS} \leq \varepsilon \ , \tag{5.3}$$

then we cannot expect in general that the sequence $\{\phi_n\}_{n=0,1,\ldots}$ is uniformly convergent in the <u>whole</u> space $\bar{E}_e = E_e \cup S$. Our intend, however, is to the show that the quadratic convergence of a sequence $\{\varphi_n\}_{n=0,1,\ldots}$ to a function φ on the surface S implies ordinary pointwise convergence of the sequence $\{\phi_n\}_{n=0,1,\ldots}$ to the solution ϕ of the exterior (Dirichlet) problem for each point x of the outer space E_e. In addition, for each compact subset $B \subset E_e$ with positive distance to the surface S (dist $(B,S) \geq \lambda > 0$) the convergence of $\{\phi_n\}_{n=0,1,\ldots}$ is uniform.

For that purpose we need the following <u>regularization theorem</u>:

> Let B be a compact subset of the outer space E_e with dist$(B,S) \geq \lambda > 0$.
>
> Suppose that Φ, Ψ are functions of the class $\mathcal{P}(E_e)$ with $\Phi|S = \varphi$, $\Psi|S = \psi$. Then there exists a positive constant $C = C(S,B)$ such that
>
> $$\sup_{x \in B} | \Phi(x) - \Psi(x) | \leq C \sqrt{\int_S |\varphi(y) - \psi(y)|^2 dS}. \tag{5.4}$$

Proof. By the smoothness condition imposed on $S = \partial E$, there exists the classical Green function $\mathcal{G}$ for the scalar Dirichlet problem related to E_e, i.e.:

$$\Phi(x) = \int_S \varphi(y) \frac{\partial}{\partial n(y)} \mathcal{G}(x,y) dS(y)$$

$$\Psi(x) = \int_S \psi(y) \frac{\partial}{\partial n(y)} \mathcal{G}(x,y) dS(y)$$

for all $x \in E_e$.

Consequently, we have

$$\Phi(x) - \Psi(x) = \int_S [\varphi(y) - \psi(y)] \frac{\partial}{\partial n(y)} \mathcal{G}(x,y) dS(y)$$

for all $x \in E_e$.

Put

$$C = C(S,B) = \sup_{x \in B} \left[\int_S \left| \frac{\partial}{\partial n(y)} \mathcal{G}(x,y) \right|^2 dS(y) \right]^{1/2}.$$

Then, for each $x \in B$, we have by the Cauchy-Schwarz inequality

$$\begin{aligned} |\Phi(x) - \Psi(x)|^2 &= \left| \int_S [\varphi(y) - \psi(y)] \frac{\partial}{\partial n(y)} \mathcal{G}(x,y) dS(y) \right|^2 \\ &\leq \int_S |\varphi(y) - \psi(y)|^2 dS(y) \int_S \left| \frac{\partial}{\partial n(y)} \mathcal{G}(x,y) \right|^2 dS(y) \\ &\leq C^2 \int_S |\varphi(y) - \psi(y)|^2 dS . \end{aligned}$$

Thus

$$\sup_{x\in B} |\phi(x) - \psi(x)|^2 \leq C^2 \int_S |\varphi(y) - \psi(y)|^2 dS(y). \qquad \square$$

As consequence of the regularity theorem we obtain the following Corollaries:

Corollary 1: Let $\{\phi_n\}_{n=0,1,\ldots}$ denote a sequence in $\mathcal{P}(E_e)$. Denote by $\{\varphi_n\}_{n=0,1,\ldots}$ the restrictions $\varphi_n = \phi_n|S$ of the functions ϕ_n to the surface S.

Suppose that ϕ is the (unique) solution of the Dirichlet problem

$$\phi \in \mathcal{P}(E_e) \ , \ \phi|S = \varphi \ .$$

If now $\varphi_n \to \varphi$ in $\mathcal{L}^2(S)$:

$$\lim_{n\to\infty} \sqrt{\int_S |\varphi_n(x) - \varphi(x)|^2 dS(x)} = 0,$$

then $\phi_n \to \phi$ uniformly on each compact subset $B \subset E_e$ with $\text{dist}(B,S) \geq \lambda > 0$,

$$\lim_{n\to\infty} \sup_{x\in B} |\phi_n(x) - \phi(x)| = 0.$$

Proof. According to the regularization theorem we have

$$\sup_{x\in B} |\phi(x) - \phi_n(x)| \leq C \sqrt{\int_S |\varphi(y) - \varphi_n(y)|^2 dS(y)}$$

for $n = 0,1,\ldots$.

Thus the $\mathcal{L}^2$-convergence

$$\sqrt{\int_S |\varphi(y) - \varphi_n(y)|^2 \, dS(y)} \to 0$$

implies

$$\sup_{x \in B} |\phi(x) - \phi_n(x)| \to 0 \,,$$

as n tends to infinity. □

Corollary 2: Let $\{\varphi_n^*\}_{n=0,1,\ldots}$ be an orthonormal system in $\mathcal{L}^2(S), \varphi_n^* \in C^{(0)}(S)$. Denote by $\{\phi_n^*\}_{n=0,1,\ldots}$ the sequence of functions $\phi_n^* \in \mathcal{P}(E_e)$ with $\phi_n^* | S = \varphi_n^*$ for $n=0,1,\ldots$.

Suppose that ϕ is the (unique) solution of the (Dirichlet) problem

$$\phi \in \mathcal{P}(E_e) \quad , \quad \phi | S = \varphi \,.$$

Then the sum

$$\phi^{(N)}(x) = \sum_{n=0}^{N} (\varphi, \varphi_n^*) \phi_n^*$$

$$= \sum_{n=0}^{N} \left[\int_S \varphi(y) \varphi_n^*(y) dS(y) \right] \phi_n^*(x)$$

converges uniformly to $\phi(x)$ as $N \to \infty$ on each compact subset $B \subset E_e$ with $\operatorname{dist}(B,S) \geq \lambda > 0$,

i.e.

$$\lim_{N\to\infty} \sup_{x\in B} |\phi^{(N)}(x) - \phi(x)|$$

$$= \lim_{N\to\infty} \sup_{x\in B} \left| \sum_{n=o}^{N} \left[\int_S \varphi(y)\varphi_n^*(y)dS(y) \right] \phi_n^*(x) - \phi(x) \right|$$

$$= 0.$$

Proof. According to the regularization theorem it follows that

$$\sup_{x\in B} \left| \sum_{n=o}^{N} \left[\int_S \varphi(y)\varphi_n^*(y)dS(y) \right] \phi_n^*(x) - \phi(x) \right|$$

$$\leq C \sqrt{\int_S \left| \sum_{n=o}^{N} \left[\int_S \varphi(y)\varphi_n^*(y)dS(y) \right] \varphi_n^*(x) - \varphi(x) \right|^2 dS(x)} \ .$$

As element of the Hilbert space $\mathcal{L}^2(S)$, the function φ can be represented by its Fourier series with respect to the system $\{\varphi_n^*\}_{n=o,1,\ldots}$:

$$\lim_{N\to\infty} \sqrt{\int_S \left| \sum_{n=o}^{N} \left[\int_S \varphi(y)\varphi_n^*(y)dS(y) \right] \varphi_n^*(x) - \varphi(x) \right|^2 dS(x)}$$

$$= 0.$$

This proves Corollary 2. □

Before giving a list of the most important systems of trial functions satisfying the assumptions of the Corollaries stated above, we recapitulate some functional analytic aspects concerned with the closure and completeness of sequences in (general) inner product spaces.

6. Closed and Complete Systems

A sequence $\{\varphi_n\}_{n=o,1,\ldots}$ is called complete in an inner product space X with inner product $(\cdot,\cdot)$, if, for every $\varphi \in X$, the condition

$$(\varphi,\varphi_n) = 0$$

for all $n = o,1,\ldots$ implies

$$\varphi = 0.$$

A sequence $\{\varphi_n\}_{n=o,1,\ldots}$ is called closed in a normed space X with norm $\|\cdot\|$, if, given $\varphi \in X$ and $\varepsilon > 0$, there exist an integer $N = N(\varepsilon)$ and constants $a_o,\ldots, a_N$ such that

$$\|\varphi - \sum_{n=o}^{N} a_n\varphi_n\| < \varepsilon .$$

In an inner product space X, which can be regarded as normed space with norm

$$\|\cdot\| = \sqrt{(\cdot,\cdot)},$$

a closed sequence is complete, but the reverse is not true in general.

In a Hilbert space, however, the properties of closure and completeness are equivalent.

In the special case that $\{\varphi_n^*\}_{n=o,1,\ldots}$ is an orthonormal system in a Hilbert space X the following statements are equivalent:

(i) $\{\varphi_n^*\}_{n=0,1,\ldots}$ is an orthonormal Hilbert-basis

(ii) $\{\varphi_n^*\}$ is closed in X

(iii) $\{\varphi_n^*\}$ is complete in X

(iv) For each $\varphi \in X$,

$$\|\varphi\|^2 = \sum_{n=0}^{\infty} |(\varphi,\varphi_n^*)|^2$$

is valid.

(v) For each $\varphi \in X$ the Fourier expansion

$$\varphi = \sum_{n=0}^{\infty} (\varphi,\varphi_n^*)\varphi_n^*$$

holds (in sense of the topology of X).

(vi) For $\varphi,\Psi \in X$,

$$(\varphi,\Psi) = \sum_{n=0}^{\infty} (\varphi,\varphi_n^*)(\Psi,\varphi_n^*)$$

(" Parseval's identity ")

Proofs of these equivalences can be found e.g. in Davis (1963).

7. Trial Functions / Base Systems

Some base systems particularly important in geodetic applications should be mentioned now:

(i) solid spherical harmonics

Let $\{S_{n,j}\}_{j=1,\ldots,2n+1}$ denote a maximal linearly independent system of (Laplace's) surface spherical harmonics of oder m. Let $\{H_{n,j}\}_{\substack{n=o,1,\ldots \\ j=1,\ldots,2n+1}}$ be the system of solid spherical harmonics defined by

$$H_{n,j}(x) = |x|^{-(n+1)} S_{n,j}(\xi)$$

$$(x = |x|\ \xi, \xi^2 = 1).$$

Then the system

$$\{H_{n,j}|S\}_{\substack{n=o,1,\ldots \\ j=1,\ldots,2n+1}} \tag{7.1}$$

is linearly independent and complete in the Hilbert space $\mathcal{L}^2(S)$.

Proof.

The linear independence is a consequence of the linear independence of the system $\{S_{n,j}\}$ of surface spherical harmonics $S_{n,j}$.

Therefore it remains to prove that $\{H_{n,j}|S\}_{\substack{n=o,1,\ldots \\ j=1,\ldots,2n+1}}$ is complete (and consequently closed) in the Hilbert space $\mathcal{L}^2(S)$.

Suppose that φ is a square-integrable function ($\varphi \in \mathcal{L}^2(S)$) satisfying

$$(\varphi, H_{n,j}) = \int_S \varphi(x)\, H_{n,j}(x)\, dS = 0$$

for all $n = 0,1,\ldots$; $j = 1,\ldots, 2n+1$.

We remember that the series expansion

$$\frac{1}{|x-y|} = 4\pi \sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{|x|^n}{|y|^{n+1}} \sum_{j=1}^{2n+1} S_{n,j}(\xi)\, S_{n,j}(\eta)$$

$$(x = |x|\xi \ , \ y = |y|\eta, \ \xi^2 = \eta^2 = 1)$$

is analytic in the domain given by

$$\left\{ \begin{array}{c} 0 \leq |x| \leq r_o \ , \ r_o < \rho_o, \rho_o = \inf\limits_{y \in S} |y| \\ \\ x \in E_i \ , \ y \in S \end{array} \right\} .$$

For all $x \in E_i$ with $|x| \leq r_o$ we thus find by virtue of our assumption

$$\begin{aligned} U_1(x) &= \int_S \varphi(y)\, \frac{1}{|x-y|}\, dS(y) \\ &= 4\pi \sum_{n=0}^{\infty} \frac{1}{2n+1} |x|^n \sum_{j=1}^{2n+1} S_{n,j}(\xi) \int_S \varphi(y) H_{n,j}(y)\, dS(y) \\ &= 0. \end{aligned}$$

Analytic continuation shows that the single-layer potential $U_1(x)$ vanishes at each point $x \in E_i$.

In other words, the equations

$$U_1(x - \tau n(x)) = 0$$

$$\frac{\partial U}{\partial n(x)}\,(x - \tau n(x)) = 0$$

hold for all $x \in S$ and all sufficiently small $\tau > 0$.

But this yields using the relations (4.8) and (4.9)

$$\lim_{\substack{\tau \to 0 \\ \tau \to 0}} \int_S |U_1(x + \tau n(x))|^2 dS(x) = 0$$

$$\lim_{\substack{\tau \to 0 \\ \tau \to 0}} \int_S \left| \frac{\partial U_1}{\partial n(x)}\,(x + \tau n(x)) + 4\pi\varphi(x) \right|^2 dS(x) = 0$$

and

$$\lim_{\substack{\tau \to 0 \\ \tau \to 0}} \int_S \left| U_1'(x) + 2\pi\varphi(x) \right|^2 dS(x) = 0 .$$

The last equation can be rewritten in the explicit form

$$\int_S \left| \int_S \varphi(y)\, \frac{\partial}{\partial n(x)}\, \frac{1}{|x-y|}\, dS(y) + 2\pi\varphi(x) \right|^2 dS(x) = 0.$$

In other words we have

$$-\frac{1}{2\pi} \int_S \varphi(y)\, \frac{\partial}{\partial n(x)}\, \frac{1}{|x-y|}\, dS(y) = \varphi(x)$$

in the sense of $\mathcal{L}^2(S)$.

Iteration yields

$$\left(-\frac{1}{2\pi}\right)^2 \int_S \int_S \varphi(y_2) \frac{\partial}{\partial n(y_1)} \frac{1}{|y_1-y_2|} \frac{\partial}{\partial n(x)} \frac{1}{|x-y_1|} dS(y_1) dS(y_2)$$

$$= \varphi(x)$$

and

$$\left(-\frac{1}{2\pi}\right)^3 \iiint_{SSS} \left\{\varphi(y_3) \frac{\partial}{\partial n(y_2)} \frac{1}{|y_3-y_2|} \frac{\partial}{\partial n(y_1)} \frac{1}{|y_2-y_1|} \frac{\partial}{\partial n(x)} \frac{1}{|y_1-x|}\right\}$$

$$dS(y_1) dS(y_2) dS(y_3)$$

$$= \varphi(x)$$

Induction states that

$$\left(-\frac{1}{2\pi}\right)^n \int_S \cdots \int_S \left\{\varphi(y_n) \frac{\partial}{\partial n(y_{n-1})} \frac{1}{|y_n-y_{n-1}|} \cdots \frac{\partial}{\partial n(x)} \frac{1}{|y_1-x|}\right\} dS(y_1)$$

$$\cdots dS(y_n)$$

$$= \varphi(x)$$

for all integers n (in the sense of $\mathcal{L}^2(S)$). Since iteration reduces the order of singularity (cf. Kellogg (1929)) the left hand side is a continuous function of variable x at S ($n > n_0, n_0 \in \mathbb{N}$). Thus the function φ can be replaced by a function $\tilde{\varphi} \in C^{(0)}(S)$ satisfying

$$\int_S |\varphi(x) - \tilde{\varphi}(x)|^2 \, dS = 0 ,$$

i.e. $\varphi = \tilde{\varphi}$ in the sense of $\mathcal{L}^2(S)$.

For the continuous function $\tilde{\varphi}$, however, the classical limit and jump formulas are valid:

$$\lim_{\substack{\tau \to 0 \\ \tau \to 0}} U_1(x + \tau n(x)) = 0$$

$$\lim_{\substack{\tau \to 0 \\ \tau \to 0}} \frac{\partial U_1}{\partial n(x)}(x + \tau n(x)) = 4\pi \tilde{\varphi}(x).$$

The uniqueness theorem of the exterior Dirichlet problem then yields $U_1(x) = 0$ identically for all $x \in E_e$. But this means that $\tilde{\varphi} = 0$. Consequently, $\varphi = 0$ on the surface S. □

(ii) singularity functions (mass points)

Let F be a (regular) surface in the inner region E_i with $\text{dist}(F,S) \geq \lambda > 0$.

Suppose that $\{x_n\}_{n=0,1,\ldots}$ is a dense point-sequence in F (i.e.: to every $\varepsilon > 0$ and every point $x \in F$ there exists a point x_n such that $|x-x_n| \leq \varepsilon$).

Denote by

$$\phi_n(x) = \frac{1}{|x-x_n|} \quad (x_n \in F \subset E_i) \tag{7.2}$$

the singularity function (fundamental solution) at $x_n \in F$.

Then the system

$$\{\phi_n|S\}_{n=0,1,\ldots}$$

is linearly independent and complete in the Hilbert space $\mathcal{L}^2(S)$.

Proof.

Provided that $x_n \neq x_m$ if $n \neq m$ we are immediately able to prove the linear independence of the system $\{\phi_n|S\}_{n=0,1,\ldots}$ in $\mathcal{L}^2(S)$.

Our purpose is to verify the completeness of $\{\phi_n|S\}$ in $\mathcal{L}^2(S)$.

To this end we again consider a function $\varphi \in \mathcal{L}^2(S)$. We require that

$$\int_S \varphi(x)\phi_n(x)dS = 0$$

for $n = 0,1,\ldots$.

The single-layer potential

$$U_1(x) = \int_S \varphi(y) \frac{1}{|x-y|} dS(y)$$

then vanishes at all points $x_n \in F$.

As U_1 is continuous on the surface F, the density of the system $\{x_n\}_{n=o,1,\ldots}$ in F implies

$$U_1(x) = 0$$

for all $x \in F$. The same arguments as given before then assure that U_1 vanishes identically in R^3. But this yields $\varphi = 0$ in $\mathcal{L}^2(S)$. □

(iii) "multipoles"

Let x_o be a fixed point in $R^3 - \bar{E}_e$. Denote by $\{\phi_n\}_{n=o,1,\ldots}$ the sequence of "multipoles" defined by

$$\left(\frac{\partial}{\partial x_o}\right)^{\alpha} \frac{1}{|x-x_o|}\Bigg|_{[\alpha]=n} , \tag{7.3}$$

where

$$[\alpha] = \alpha_1+\alpha_2+\alpha_3 \quad , \quad \left(\frac{\partial}{\partial x_o}\right)^{\alpha} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \left(\frac{\partial}{\partial x_3}\right)^{\alpha_3}\Bigg|_{x_o} .$$

Then the system $\{\phi_n|S\}_{n=o,1,\ldots}$ is linearly independent and complete in the Hilbert space $\mathcal{L}^2(S)$.

Proof.

It is clear that the potential U_1 with all its derivatives of arbitrary order vanishes in x_o. For an analytic function this implies the vanishing in a neighbourhood of the point x_o, hence in $R^3 - \bar{E}_e$. The next steps are exactly those presented above. □

Other types of trial functions have been listed in Freeden - Kersten (1982).

Remark: From theory of spherical harmonics (cf. e.g. Freeden(1979)) it is well-known that every(surface) spherical harmonic S_n may be expressed in the form

$$S_n(\xi) = \sum_{j=1}^{2n+1} a_j \, P_n(\xi \eta_j) \quad , \quad (|\xi|=1) \ ,$$

where P_n is the Legendre-polynomial of degree n and $\eta_1, \ldots, \eta_{2n+1}$ is a system of points η_j with $|\eta_j|=1$ for which

$$\det\left(P_n(\eta_i \eta_\ell)\right)_{\substack{i=1,\ldots,2n+1 \\ \ell=1,\ldots,2n+1}} \neq 0.$$

According to Maxwell's representation theorem

$$(n\nabla)^n \frac{1}{|x|} = (-1)^n n! \frac{P_n(\xi n)}{|x|^{n+1}}$$

the Legendre polynomials may be obtained by repeated differentiations of the fundamental solution of the Laplace equation in the direction of the unit vector n. But this yields (cf. Cl. Müller (1966))

$$\frac{S_n(\xi)}{|x|^{n+1}} = \frac{(-1)^n}{n!} \sum_{j=1}^{2n+1} a_j (\eta_j \nabla)^n \frac{1}{|x|} .$$

In other words, spherical harmonics may be described always by combinations of multipoles. Therefore any approximation by spherical harmonics may be interpreted as approximation by combinations of multipoles.

8. Non - constructive Approximation Theorem

In the following we summarize our results developed in Chapter 7 and give some first applications.

Let us denote by

$$\{\phi_n\}_{n=o,1,\ldots}$$

one of the following linearly independent systems of functions $\phi_n \in \mathcal{P}(E_e)$:

(i) solid spherical harmonics

$$|x|^{-(n+1)} S_{n,j}(\xi) \quad , \quad (x=|x|\xi,\ \xi^2=1),$$

where $S_{n,j}$ represents a maximal linearly independent system of (Laplace's) surface spherical harmonics.

(ii) "mass points" (reciprocal distances)

$$\frac{1}{|x-x_n|} \quad , \quad x_n \in E \ , \ E = E_i \ ,$$

where $\{x_n\}_{n=o,1,\ldots}$ is fundamental in E,

i.e.: (α) $\operatorname{dist}(x_n,S) > 0$ for all n

(β) for each $\phi \in C^{(o)}(E \cup S) \cap C^{(2)}(E)$ with $\Delta\phi = 0$ in E the assumption $\phi(x_n) = 0, n=o,1,\ldots,$ implies $\phi|E = 0$.

(iii) "multipoles"

$$\left(\frac{\partial}{\partial x_o}\right)^{\alpha} \frac{1}{|x - x_o|}\Bigg|_{|\alpha|=n} \quad ,$$

where x_o is a point of the complement of $\bar{E}_e = E_e \cup S$.

Then the sequence

$$\{\varphi_n\}_{n=o,1,\ldots}$$

of restrictions

$$\varphi_n = \phi_n|S \quad , \quad n=o,1,\ldots ,$$

is a closed system in the Hilbert space $\mathcal{L}^2(S)$, i.e. to every $\varphi \in \mathcal{L}^2(S)$ and every $\varepsilon > 0$ there exist an integer $N=N(\varepsilon)$ and coefficients $a_o,\ldots,a_N$ such that

$$\sqrt{\int_S |\varphi(x) - \sum_{n=o}^{N} a_n\varphi_n(x)|^2 dS} < \varepsilon.$$

In particular, to every $\varphi \in C^{(o)}(S)$ and every $\varepsilon > 0$, there exist an integer $N = N(\varepsilon)$ and coefficients $a_o,\ldots,a_N$ such that

$$\sqrt{\int_S |\varphi(x) - \sum_{n=o}^{N} a_n\varphi_n(x)|^2 dS} < \varepsilon$$

(Observe the space $C^{(o)}(S)$ is subset of the space $\mathcal{L}^2(S)$).

By application of the regularization theorem we obtain:

Let ϕ be the (unique) function of class $\mathcal{P}(E_e)$ with $\phi|S = \varphi, \varphi \in C^{(o)}(S)$. Then, to every $\varepsilon > 0$, there exist an integer $N = N(\varepsilon)$ and coefficients $a_o,\ldots,a_N$ such that

$$\sup_{x \in B} \left| \phi(x) - \sum_{n=o}^{N} a_n \phi_n(x) \right|$$

$$\leq C \sqrt{\int_S \left| \varphi(y) - \sum_{n=o}^{N} a_n \varphi_n(y) \right|^2 dS(y)}$$

$$\leq C \cdot \varepsilon$$

for each compact subset $B \subset E_e$ with dist $(B,S) \geq \lambda > 0$.

Unfortunately, the theorem is non - constructive, because we have no further information about the coefficients $a_o,\ldots,a_N$. In order to derive constructive approximation theorems we have to orthonormalize the system $\{\varphi_n\}_{n=o,1,\ldots}$ by virtue of the well-known Gram-Schmidt (orthonormalization) process.

9. Constructive Approximation Theorem

Let $\{\phi_n\}_{n=o,1,\ldots}$ be one of the systems of harmonic functions $\phi_n \in \mathcal{P}(E_e)$ listed in Chapter 8 (i.e. solid spherical harmonics, mass points, multipoles). Let us denote by $\{\varphi_n\}_{n=o,1,\ldots}$ as usual the sequence of restrictions $\varphi_n = \phi_n|S, n=o,1,\ldots$. Then, corresponding to the countably infinite sequence $\{\varphi_n\}_{n=o,1,\ldots}$ a system $\{\varphi_n^*\}_{n=o,1,\ldots}$ of functions φ_n^* satisfying

$$(\varphi_n^*, \varphi_\ell^*) = \int_S \varphi_n^*(y)\varphi_\ell^*(y)dS(y) = \delta_{n\ell} \qquad (9.1)$$

can be determined recursively in the following way (cf. Davis (1961) Sec.8.3)

$$\varphi_o^* = \frac{\psi_o}{\|\psi_o\|}, \quad \psi_o = \varphi_o, \quad \|\psi_o\|^2 = (\psi_o, \psi_o)$$

$$\varphi_1^* = \frac{\psi_1}{\|\psi_1\|}, \quad \psi_1 = \varphi_1 - (\varphi_1, \varphi_o^*)\varphi_o^*$$

$$\varphi_2^* = \frac{\psi_2}{\|\psi_2\|}, \quad \psi_2 = \varphi_2 - \sum_{k=o}^{1} (\varphi_2, \varphi_k^*)\varphi_k^*$$

$$\vdots \qquad\qquad \vdots$$

$$\varphi_{n+1}^* = \frac{\psi_{n+1}}{\|\psi_{n+1}\|}, \quad \psi_{n+1} = \varphi_{n+1} - \sum_{k=o}^{n} (\varphi_{n+1}, \varphi_k^*)\varphi_k^*$$

$$\vdots \qquad\qquad \vdots$$

("Gram - Schmidt orthonormalization")

According to the exterior Dirichlet problem of potential theory, there exists one and only one function $\phi_n^* \in \mathcal{P}(E_e)$ satisfying $\phi_n^* | S = \varphi_n^*$, $n = o,1,\ldots$.

Therefore, as application of Corollary 2 (Chapt. 5), we obtain:

Let ϕ be the (unique) function of class $\mathcal{P}(E_e)$ with $\phi | S = \varphi, \varphi \in C^{(o)}(S)$. Then the $\mathcal{L}^2$-convergence of the sum

$$\sum_{n=o}^{N} (\varphi,\varphi_n^*)\varphi_n^* = \sum_{n=o}^{N} \left[\int_S \varphi(y)\varphi_n^*(y)\,dS\right]\varphi_n^*$$

to the function φ, i.e.

$$\lim_{N\to\infty} \sqrt{\int_S \left| \sum_{n=o}^{N} (\varphi,\varphi_n^*)\varphi_n^*(y) - \varphi(y) \right|^2 dS(y)} = o$$

implies ordinary pointwise convergence of the sum

$$\sum_{n=o}^{N} (\varphi,\varphi_n^*)\phi_n^*(x) = \sum_{n=o}^{N} \left[\int_S \varphi(y)\phi_n^*(y)\,dS\right]\phi_n^*(x)$$

to $\phi(x)$ as $N \to \infty$ for each $x \in E_e$. On each compact subset B with $B \subset E_e$ and $\mathrm{dist}(B,S) \geq \lambda > o$ the convergence is uniform:

$$\lim_{N\to\infty} \sup_{x\in B} \left| \sum_{n=o}^{N} (\varphi,\varphi_n^*)\phi_n^*(x) - \phi(x) \right| = o$$

The approximation technique proposed here, indeed, is constructed so as to have the so - called permanence property:

> The transition from the N-th truncation
>
> $$\sum_{n=o}^{N} (\varphi,\varphi_n^*)\phi_n^*(x)$$
>
> to (N+1)-th sum
>
> $$\sum_{n=o}^{N+1} (\varphi,\varphi_n^*)\phi_n^*(x)$$
>
> necessitates merely the addition of one more term, all the terms obtained formerly remaining unchanged.

The price to be paid for the convenience of the permanence property is the process of orthonormalization.

The theorem is constructive, since the numbers

$$(\varphi,\varphi_n^*) = \int_S \varphi(y)\varphi_n^*(y)dS(y) \qquad (9.2)$$

are the Fourier coefficients of φ with respect to the system $\{\varphi_n^*\}_{n=o,1,\ldots}$.

It is clear from the structure of the recursion process that ψ_{n+1}, and hence, φ_{n+1}^*, is a linear combination of $\varphi_o,\ldots,\varphi_{n+1}$ (cf. (9.1)). In order to determine the linear combination φ_{n+1}^* from $\varphi_o,\ldots,\varphi_{n+1}$ we have to evaluate scalar products and norms, i.e. surface integrals over S. Assumed the surface S is available in explicit form (e.g. in the case of sphere, ellipsoid, spheroid) a direct attack

to the integrals occuring in the orthonormalization procedure seems possible using efficient methods of numerical quadrature or cubature.

Furthermore, we have

$$\int_S \left| \phi(y) - \sum_{n=o}^{N} a_n \phi_n^*(y) \right|^2 dS \tag{9.3}$$

$$= \int_S |\phi(y)|^2 dS - \sum_{n=o}^{N} \left| \int_S \phi(y)\phi_n^*(y) dS \right|^2$$

$$+ \sum_{n=o}^{N} \left| a_n - \int_S \phi(y)\phi_n^*(y) dS \right|^2$$

for any selection of constants $a_o, \ldots, a_n$.

Since the first two terms on the right hand side are independent of the coefficients $a_o, \ldots, a_n$, the problem of finding the best approximation by linear combination of $\phi_o, \ldots, \phi_n^*$:

$$\min_{a_n} \int_S \left| \phi(y) - \sum_{n=o}^{N} a_n \phi(y) \right|^2 dS \tag{9.4}$$

is solved if and only if

$$a_n = \int_S \phi(y)\phi_n^*(y) dS. \tag{9.5}$$

In other words,

$$\int_S \left| \phi(y) - \sum_{n=o}^{N} (\phi, \phi_n^*)\phi_n^*(y) \right|^2 dS \tag{9.6}$$

$$< \int_S \left| \phi(y) - \sum_{n=o}^{N} a_n \phi_n^*(y) \right|^2 dS$$

for any selection of constants $a_o,\ldots,a_n$.

This is characteristic for (Fourier) expansions.

10. Approximation Scheme

The constructive character of our approximation theorem now gives rise to the following scheme for the computation of the earth's external gravitational potential V:

1. Choose a system $\{\phi_n\}_{n=0,1,\ldots}$ of the following type:

(i) solid spherical harmonics

$$|x|^{-(n+1)} S_{n,j}(\xi) \quad , \quad (x=|x|\xi, \xi^2=1),$$

where $S_{n,j}$ represents a maximal linearly independent system of (Laplace's) surface spherical harmonics

(ii) singularity functions

$$\frac{1}{|x-x_n|} \quad ,$$

where $\{x_n\}_{n=0,1,\ldots}$ is fundamental in $E = E_i$ (cf. Chapter 8).

(iii) "multipoles"

$$\left(\frac{\partial}{\partial x_0}\right)^{\alpha} \frac{1}{|x-x_0|} \Big|_{[\alpha]=n} \quad ,$$

where x_0 is a point of the complement of $\bar{E}_e = E_e \cup S$.

(iv) unions of the base systems mentioned under (i) - (iii)

2. Choose an appropriate "approximation" to the earth's surface S (e.g. sphere, ellipsoid, spheroid, ...)

3. Orthonormalize the system

$$\{\phi_n\}_{n=0,1,\ldots} \quad , \quad \phi_n|_S = \varphi_n \ ,$$

with respect to the boundary S. This yields a closed and complete system

$$\{\varphi_n^*\}_{n=0,1\ldots}$$

in the Hilbert space $\mathcal{L}^2(S)$.
Define the system

$$\{\phi_n^*\}_{n=0,1,\ldots}$$

by $\phi_n^* \in \mathcal{P}(E_e)$, $\phi_n^*|_S = \varphi_n^*$.

4. Calculate the coefficients

$$(V,\varphi_n^*) = \int_S V(y)\ \varphi_n^*(y)\,dS(y)$$

by numerical integratior methods.

5. the sum

$$V^{(N)}(x) = \sum_{n=0}^{N} \left[\int_S V(y) \varphi_n^*(y) \, dS(y) \right] \phi_n^*(x)$$

serves as (globally valid) approximation for the gravitational potential

$$V \in \mathcal{P}(\Gamma_e) \quad .$$

Therefore the whole approximation process essentially amounts to computing various types of surface integrals over S.

Remark: For a spherical earth's model

$$\bar{E} = \{x \in R^3 \mid |x| \leq R\}$$

(R: mean radius of the earth) and the base system of spherical harmonics our (numerical) procedure coincides with the conventional computation of the N-th truncated series expansion of a harmonic function into solid spherical harmonics (cf. e.g. Heiskanen-Moritz, Chapter 1.16, (1967)).

From a geodetic point of view it should be noted that the external gravitational potential V of the earth may be uniformly approximated by the sum

$$V^{(N)}(x) = \sum_{n=0}^{N} (V, \varphi_n^*) \, \phi_n^*(x) = \sum_{n=0}^{N} \left[\int_S V(y) \varphi_n^*(y) \, dS(y) \right] \phi_n^*(x)$$

in the sense that for any given $\varepsilon > 0$ there exists an integer $N = N(\varepsilon)$ such that the relation

$$\left| \sum_{n=0}^{N} \left[\int_S V(y)\varphi_n^*(y)dS(y) \right] \phi_n^*(x) - V(x) \right| < \varepsilon$$

holds uniformly everywhere outside and on any closed surface completely surrounding the (earth's) surface S in the outer space E_e. The surrounding surface may be arbitrarily close to the earth's surface.

Consequently the external gravitational potential V of the earth can be expanded into the series

$$\sum_{n=0}^{\infty} \left[\int_S V(y)\varphi_n^*(y)dS(y) \right] \phi_n^*(y)$$

actually outside and on <u>any</u> closed surface completely surrounding the earth's surface S in the outer space E_e. In other words, we have guaranteed the possibility of an approximation of the external gravitational potential in the <u>whole</u> outer space E_e by a <u>constructive version of Runge's theorem.</u>

11. Numerical Approximation

From the structure of the Gram-Schmidt orthonormalizing process we know that the functions ϕ_n^* are related to the functions ϕ_n by the following linear equations

$$\phi_n^* = \sum_{j=0}^{N} D_{n,j}\, \phi_j \quad , \qquad (n=0,\ldots,N) \tag{11.1}$$

where the matrix

$$D = (D_{n,j})_{\substack{n=0,\ldots,N \\ j=0,\ldots,N}} \tag{11.2}$$

is a lower triangular matrix (cf. Chapter 9). Thus it is clear that there are several ways to express the best approximation $\phi^{(N)}$ to an element $\phi \in \mathcal{P}(E_e)$ by a combination of given (independent) trial functions:

(i) as a linear combination

$$\phi^{(N)}(x) = \sum_{n=0}^{N} a_n\, \phi_n(x) \tag{11.3}$$

of the given elements $\phi_0,\ldots,\phi_N$

(ii) as a linear combination

$$\phi^{(N)}(x) = \sum_{n=0}^{N} c_n\, \phi_n^*(x) \quad , \quad c_n = (\phi, \phi_n^*) , \tag{11.4}$$

of the orthonormalized elements $\phi_0^*, \ldots, \phi_N^*$.

In order to exploit the permanence property orthonormalization is indispensable. Orthonormalization, however, is equivalent to the determination of the matrix D, and computation of the matrix D is nothing else than the inversion of a well-defined (triangular) matrix C (cf. (11.18)). From a numerical point of view, however, inversion is often not very economical.

The situation is different if we are interested only in one best approximation $\phi^{(N)}$ of special (prescribed) degree N. In this case, a numerical technique can be used without any need of inverting. This technique is based on Cholesky's factorization theorem as will be described now.

To this end we begin with the proof of the following theorem:

Let the sum given by

$$\phi^{(N)}(x) = \sum_{n=0}^{N} a_n \phi_n(x) \tag{11.5}$$

be the best approximation to ϕ from among the linear combination of $\phi_0, \ldots, \phi_N$. Then, the coefficients $a_0, \ldots, a_n$ are the solutions of the following linear system

$$\begin{aligned} (\phi_0,\phi_0)\, a_0 \;+\; \ldots \;+\; (\phi_N,\phi_0)\, a_N &= (\phi,\phi_0) \\ \vdots \qquad\qquad\quad \vdots \qquad\qquad\quad \vdots & \qquad\qquad (11.6) \\ (\phi_0,\phi_N)\, a_0 \;+\; \ldots \;+\; (\phi_N,\phi_N)\, a_N &= (\phi,\phi_N). \end{aligned}$$

("normal equations")

Proof: The coefficients $a_0,\ldots,a_N$ are determined such that

$$\sum_{n=0}^{N} a_n \phi_n(x) = \sum_{n=0}^{N} (\phi,\phi_n^*)\phi_n^*. \tag{11.7}$$

Now, it is easy to see that

$$(\phi - \sum_{j=0}^{N} (\phi,\phi_j^*)\ \phi_j^*,\phi_n^*) \tag{11.8}$$

$$= (\phi,\phi_n^*) - (\phi,\phi_n^*)$$

$$= 0$$

for n=0,...,N.

But this means that

$$(\phi - \sum_{k=0}^{N} a_k \phi_k,\phi_n) = 0 \tag{11.9}$$

for n=0,...,N.

In other words, ϕ minus its best approximation by linear combinations of $\phi_0,\ldots,\phi_N$ is orthogonal to each ϕ_n. □

The normal equations (11.6) can be rewritten in the vectorial form

$$Ga = b\ ,\ b^T = (b_0,\ldots,b_N) \tag{11.10}$$

where $b \in R^{N+1}$, is the (N+1)-tupel

$$b = \begin{pmatrix} (\phi, \phi_0) \\ \vdots \\ (\phi, \phi_N) \end{pmatrix} \tag{11.11}$$

and G is the Gram matrix

$$G = \begin{pmatrix} (\phi_0, \phi_0) & \cdots & (\phi_0, \phi_N) \\ \vdots & & \vdots \\ (\phi_N, \phi_0) & \cdots & (\phi_N, \phi_N) \end{pmatrix} . \tag{11.12}$$

As symmetric and positive definite matrix, G can be decomposed uniquely in the form

$$G = C \cdot C^T , \tag{11.13}$$

where C is a lower triangular matrix with positive diagonal elements. This splitting of G is known as the Cholesky factorization.

Hence, the solution $a \in R^{N+1}$, $a^T = (a_0, \ldots, a_N)$, can be found by the following direct method:

(i) factorization of the Gram matrix:

$$G = C \cdot C^T$$

(ii) computation of the vector

$$d \in R^{N+1} , \quad d^T = (d_0, \ldots, d_N) ,$$

of the linear system

$$C\,d = b$$

by forward substitution

(iii) computation of the vector

$$a \in R^{N+1}\ ,\ a^T = (a_0,\ldots,a_N),$$

of the linear system

$$C^T a = d$$

by backward substitution.

Remark: As is well-known from numerical analysis, the procedure is very economical (as regards to the algorithmic operations) and very stable. The solution $a \in R^{N+1}$ is obtainable without any need of inverting.

Observing the representation (11.1) we get

$$(\phi_k^*, \phi_\ell^*) = \sum_{i=0}^{N} \sum_{j=0}^{N} D_{k,i} D_{\ell,j} (\phi_i, \phi_j) = \delta_{k,\ell} \ . \tag{11.14}$$

This is equivalent to

$$DGD^T = I \ . \tag{11.15}$$

(I: unit matrix)

Consequently,

$$G = D^{-1} \cdot (D^{-1})^T \ . \tag{11.16}$$

On the other hand, we have the (unique) decomposition

$$G = C \cdot C^T \ . \tag{11.17}$$

Thus a comparison between the Gram-Schmidt orthonormalizing and the Cholesky factorizing process shows that

$$C = D^{-1} \ . \tag{11.18}$$

Moreover, the vector $c \in R^{N+1}$, $c^T = (c_0, \ldots, c_N)$, is equal to the vector $d \in R^{N+1}$, $d^T = (d_0, \ldots, d_N)$:

$$c_j = (\phi, \phi_j) = d_j \ . \tag{11.19}$$

This is easily seen from (11.1) by the following implications:

$$\begin{aligned} Db = c \quad &\Leftrightarrow \quad b = D^{-1}c \\ &\Leftrightarrow \quad b = Cc \\ &\Leftrightarrow \quad b = Cd \qquad \text{(cf. (ii))} \end{aligned}$$

i.e.:

$$c = d \ . \tag{11.20}$$

Hence, by Cholesky's method, the best approximation $\phi^{(N)}$ to an element ϕ with respect to the (orthonormalized) system $\phi_0^*, \ldots, \phi_N^*$ can be determined without explicit computation of the system $\{\phi_n^*\}$ itself.

12. Simple Examples

In order to get an impression of the least square method proposed here we discuss some simple numerical examples:

Our aim is to approximate ϕ by best approximations

$$\phi^{(N)}(x) = \sum_{k=1}^{N} a_k \phi_k(x) \qquad (12.1)$$

with respect to the system of (buried) mass points

$$\phi_k(x) = \frac{1}{|x-x_k|} \quad , \quad x_k \in E_i \; . \qquad (12.2)$$

(i) Boundaries

The surface S we choose for our computations are given by

$$S = \{x \mid (\frac{x_1}{a})^2 + (\frac{x_2}{b})^2 + (\frac{x_3}{c})^2 = 1\}$$

with

(α) $a = b = c = 1$ (Example 1)

(β) $a = 1{,}25$; $b = 1$; $c = 0{,}75$.
(Examples 2,3)

By the equations

$$x_1 = a \sin\vartheta \cos\varphi \qquad (0 \leq \varphi < 2\pi)$$
$$x_2 = b \sin\vartheta \sin\varphi \qquad (0 \leq \vartheta \leq \pi)$$
$$x_3 = c \cos\vartheta$$

the rectangular coordinates x_1, x_2, x_3 are related to the ellipsoidal coordinates: a,b,c (axes), ϑ (polar distance), φ (geocentric longitude).

(ii) Boundary values

We discuss two types of boundary values defined on the surface S under consideration

(α) potential of a mass point

$$\varphi(x) = \frac{1}{|x-y|} \quad , \quad x \in S \ , \ y^T = \frac{\sqrt{3}}{4}\,(1,1,1)$$

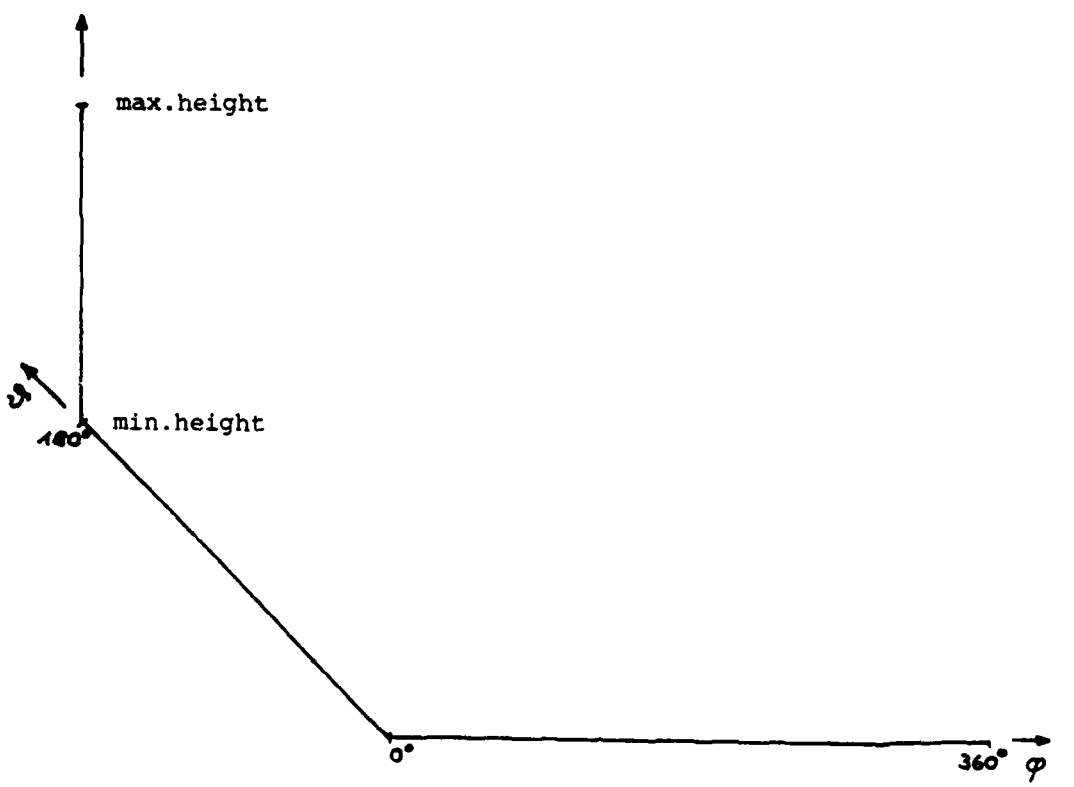
(Examples 1,2)

(β) polynomial

$$\varphi(x) = 3 + x_1^2 + x_2^2 - 2x_3^2 \quad , \quad x \in S, \ x^T = (x_1, x_2, x_3).$$

(Example 3)

The boundary values are shown pictorially by use of ellipsoidal coordinates in the following figures:



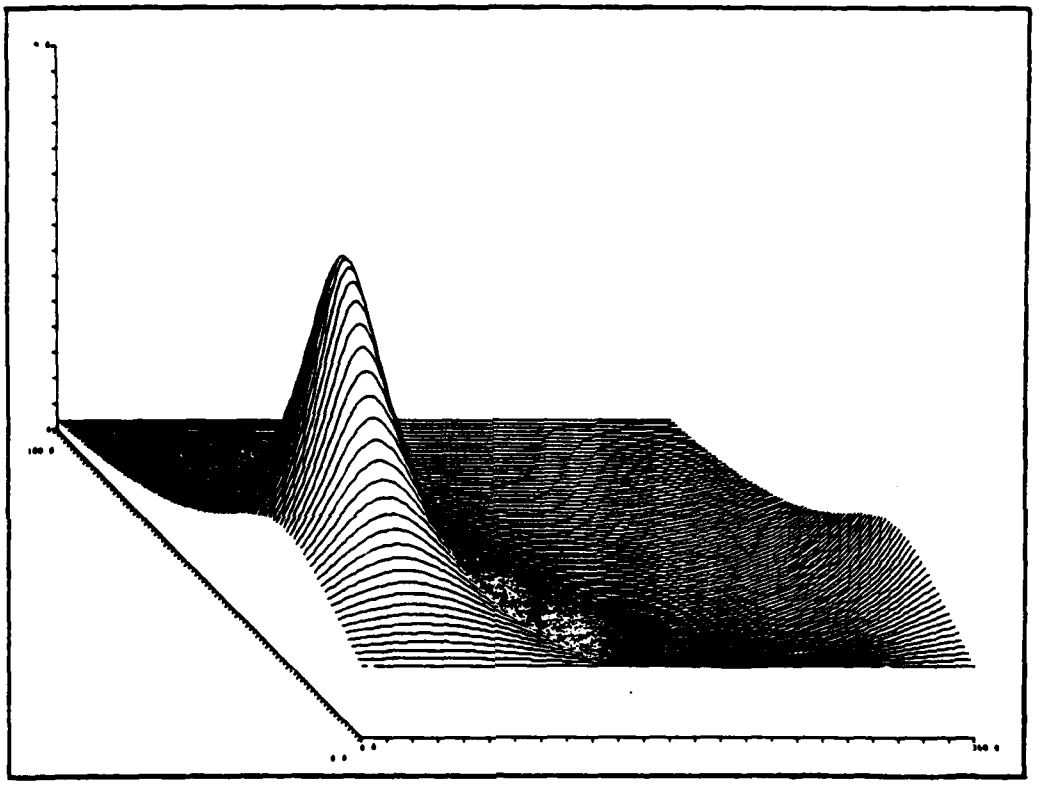


Figure 2: $\varphi(x) = |x-y|^{-1}$, $y = \frac{\sqrt{3}}{4}(1,1,1)$

$S = \{x | x_1^2 + x_2^2 + x_3^2 = 1\}$

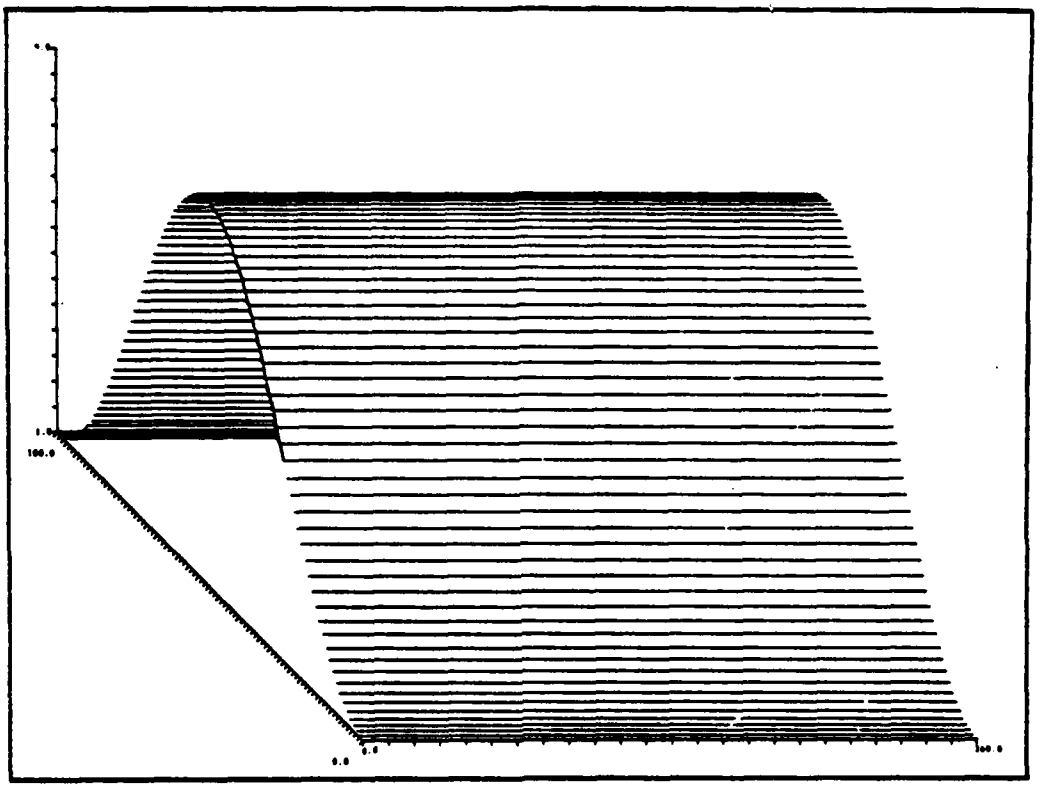


Figure 3: $\varphi(x) = 3 + x_1^2 + x_2^2 - 2x_3^2$

$S = \{x | x_1^2 + x_2^2 + x_3^2 = 1\}$

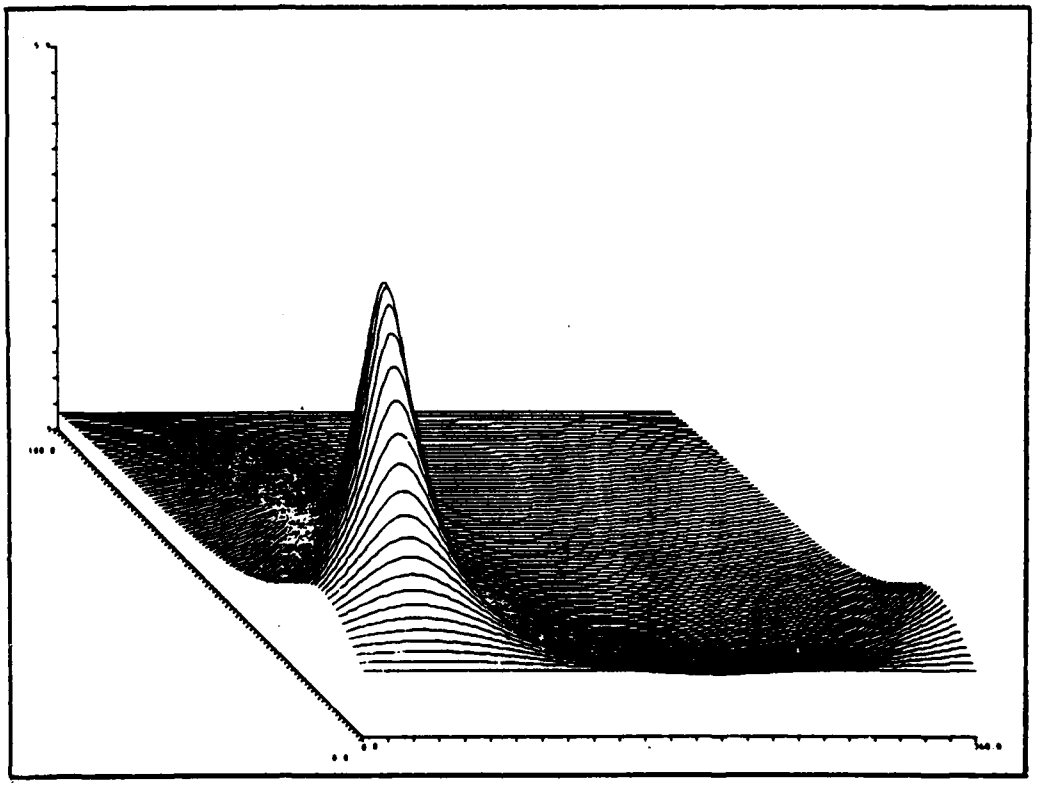


Figure 4: $\varphi(x) = |x-y|^{-1}$, $y^T = \frac{\sqrt{3}}{4}(1,1,1)$

$$S = \{x \mid (\frac{x_1}{1,25})^2 + x_2^2 + (\frac{x_3}{0,75})^2 = 1\}$$

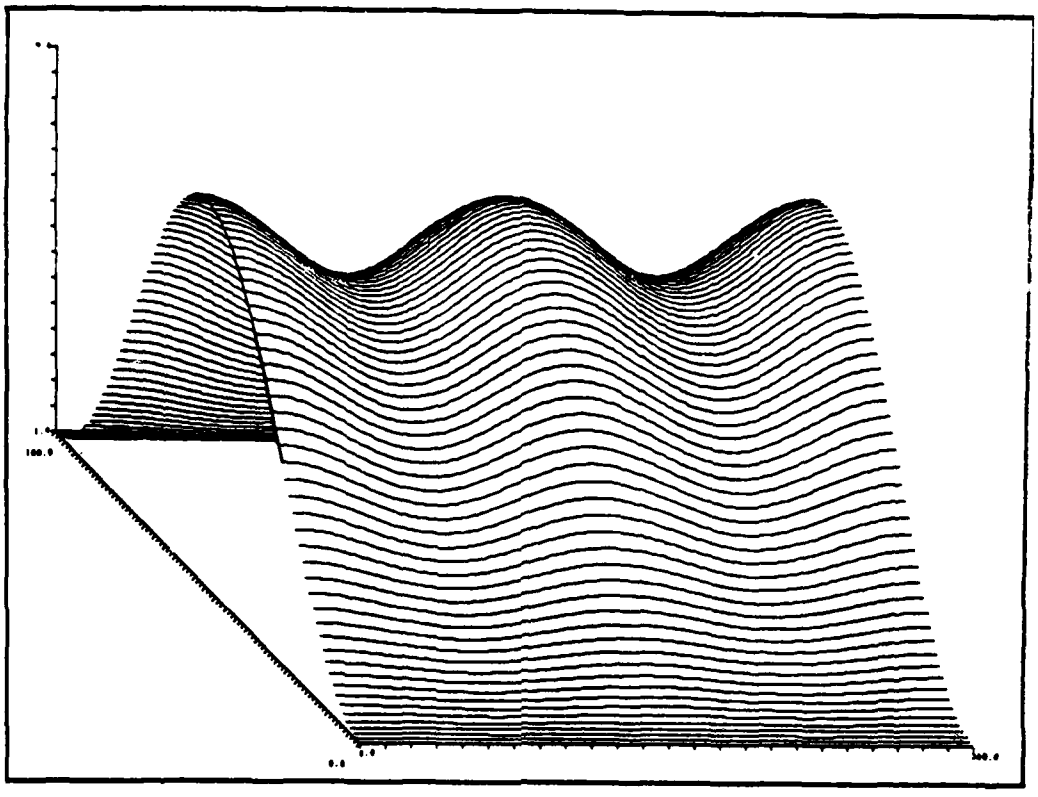


Figure 5: $\varphi(x) = 3 + x_1^2 + x_2^2 - 2x_3^2$

$$S = \{x \mid (\frac{x_1}{1,25})^2 + x_2^2 + (\frac{x_3}{0,75})^2 = 1\}$$

(iii) Trial functions

As trial functions we use the singularity functions

$$\phi_k(x) = \frac{1}{|x-x_k|} \quad , \quad x_k \in E_i .$$

The set $\tilde{\mathcal{F}}_N$ of mass points $x_k \in E_i$

$$\tilde{\mathcal{F}}_N = \{x_1, \ldots, x_N\} \tag{12.3}$$

is generated by radial projection of the points ξ_k of the system $\mathcal{S}_N$

$$\mathcal{S}_N = \{\xi_1, \ldots, \xi_N\} \tag{12.4}$$

to a surface $\tilde{S}$. The systems $\mathcal{S}_N$ are unions of regular polyhedra:

$\mathcal{S}_4 = \{PS1\}$ (Tetrahedron)

$\mathcal{S}_8 = \{PS1\} \cup \{PS2\}$ (Cube)

$\mathcal{S}_{20} = \{PS1\} \cup \{PS2\} \cup \{PS3\}$ (Dodecahedron)

$\mathcal{S}_{32} = \mathcal{S}_{20} \cup \{PS4\}$ (Icosahedron/Dodecahedron)

$\mathcal{S}_{56} = \mathcal{S}_{32} \cup \{PS5\} \cup \{PS6\}$ (Icosahedra/Dodecahedra)

where the point systems (PSi), i=1,...,6, are given as follows:

point	systems
PS1	$\alpha(-1,-1,-1)$, $\alpha(-1,1,1)$ $\alpha(1,-1,1)$, $\alpha(1,1,-1)$ $(\alpha = \sqrt{1/3})$
PS2	$\alpha(+1,+1,+1)$, $\alpha(-1,-1,+1)$ $\alpha(-1,+1,-1)$, $\alpha(+1,-1,-1)$ $(\alpha = \sqrt{1/3})$
PS3	$\alpha(0,\pm\tau,\pm 1/\tau)$ $\alpha(\pm 1/\tau, 0,\pm\tau)$ $\alpha(\pm\tau,\pm 1/\tau,0)$ $(\alpha = \sqrt{1/3}, \tau=(1+\sqrt{5})/2)$
PS4	$\beta(0,\pm 1,\pm\tau)$ $\beta(\pm\tau,0,\pm 1)$ $\beta(\pm 1,\pm\tau,0)$ $(\tau = (1+\sqrt{5})/2 , \beta = 1/\sqrt{(1+\tau^2)})$
PS5	$\alpha(0,\pm 1/\tau,\pm\tau)$ $\alpha(\pm\tau, 0,\pm 1/\tau)$ $\alpha(\pm 1/\tau,\pm\tau, 0)$ $(\alpha = \sqrt{1/3}, \tau = (1+\sqrt{5})/2)$
PS6	$\beta(0,\pm\tau,\pm 1)$ $\beta(\pm 1, 0,\pm\tau)$ $\beta(\pm\tau,\pm 1, 0)$ $(\tau = (1+\sqrt{5})/2 , \beta = 1/\sqrt{(1+\tau^2)})$

The point system $\mathcal{S}_{62}$ is chosen as the union of $\mathcal{S}_{56}$ and the points

$$(\pm 1,0,0) \quad , \quad (0,\pm 1,0) \quad , \quad (0,0,\pm 1).$$

(iv) Integration method

The integrals over S occuring in the Gram matrix G were computed in the following way:

The integrals over S are transformed using polar coordinates. Then all the computations of the (transformed) integrals are performed by (iterated) application of Romberg's quadrature formula (Romberg extrapolation). Using the subroutine DCADRE (cf. de Boor (1977)) we are able to compute both the approximation of the integral and the estimation of the error between integral and quadrature formula. In all the cases, the actual resulting error was about 10^{-8}.

Remark:

For the Cholesky factorization, the forward and backward substitution powerful routines are available (cf. Dongarra et.al.(1979)).

(v) error estimates

The accuracy of the best approximation $\phi^{(N)}$ to ϕ will be investigated by discussing the errors

$$\text{LS}: \quad \int_S [\phi(y) - \phi^{(N)}(y)]^2 \, dS$$

(least square error)

$$\text{MAR}: \quad \max_{x \in S} \left| \frac{\phi(x) - \phi^{(N)}(x)}{\phi(x)} \right|$$

(maximal absolute relative error)

$$\text{M}: \quad \int_S \left| \frac{\phi(y) - \phi^{(N)}(y)}{\phi(y)} \right| dS \; .$$

(mean absolute relative error)

The error LS is determined by the Romberg integration mentioned above.

The errors MAR,M are evaluated for a discrete lattice of 181 × 91 points in the (φ, ϑ)-plane (φ, ϑ: polar coordinates).

$$M = \frac{1}{181 \cdot 91} \sum_{i=1}^{181} \sum_{j=1}^{91} \left| \frac{\phi(x_{ij}) - \phi^{(N)}(x_{ij})}{\phi(x_{ij})} \right|$$

Example 1:

$$S = \{x | x_1^2 + x_2^2 + x_3^2 = 1\}$$

(unit sphere)

$$\tilde{S} = \{x | x_1^2 + x_2^2 + x_3^2 = 7/10\}$$

Boundary values:

$\varphi(x) = |x-y|^{-1}$, $y = \frac{\sqrt{3}}{4}(1,1,1)$, $x \in S$.

N	LS	MAR	M
4	0,25	0,80	0,24
8	$0,12 \cdot 10^{-2}$	$0,93 \cdot 10^{-1}$	$0,16 \cdot 10^{-1}$
20	$0,50 \cdot 10^{-3}$	$0,62 \cdot 10^{-1}$	$0,77 \cdot 10^{-2}$
32	$0,23 \cdot 10^{-3}$	$0,48 \cdot 10^{-1}$	$0,50 \cdot 10^{-2}$
56	$0,21 \cdot 10^{-3}$	$0,46 \cdot 10^{-1}$	$0,40 \cdot 10^{-2}$

Example 2:

$$S = \{x \mid (\frac{x_1}{1,25})^2 + x_2^2 + (\frac{x_3}{0,75})^2 = 1\}$$

(ellipsoid)

$$\tilde{S} = \{x \mid x = rw \ , \ w \in S \ , \ r = 0,75\}$$

Boundary values:

$$\varphi(x) = |x-y|^{-1} \ , \ y = \frac{\sqrt{3}}{4}(1,1,1) \ , \ x \in S$$

N	LS	MAR	M
4	0,32	1,12	0,25
8	$0,33 \cdot 10^{-2}$	0,13	$0,21 \cdot 10^{-1}$
20	$0,17 \cdot 10^{-2}$	0,10	$0,12 \cdot 10^{-1}$
32	$0,11 \cdot 10^{-2}$	$0,82 \cdot 10^{-1}$	$0,78 \cdot 10^{-2}$
56	$0,99 \cdot 10^{-3}$	$0,817 \cdot 10^{-1}$	$0,59 \cdot 10^{-2}$

Example 3:

$$S = \{x \mid \left(\frac{x_1}{1,25}\right)^2 + x_2^2 + \left(\frac{x_3}{0,75}\right)^2 = 1\}$$

(ellipsoid)

$$\tilde{S} = \{x = rw \ , \ w \in S \ , \ r = 0,75\}$$

Boundary values:

$$\varphi(x) = 3 + x_1^2 + x_2^2 - 2x_3^2$$

N	LS	MAR	M
4	1,37	1,01	0,42
8	1,25	0,85	0,42
2o	0,29	0,42	0,13
32	$0,53 \cdot 10^{-1}$	0,24	$0,38 \cdot 10^{-1}$
5o	$0,10 \cdot 10^{-1}$	0,10	$0,19 \cdot 10^{-1}$
62	$0,98 \cdot 10^{-3}$	$0,35 \cdot 10^{-1}$	$0,88 \cdot 10^{-2}$

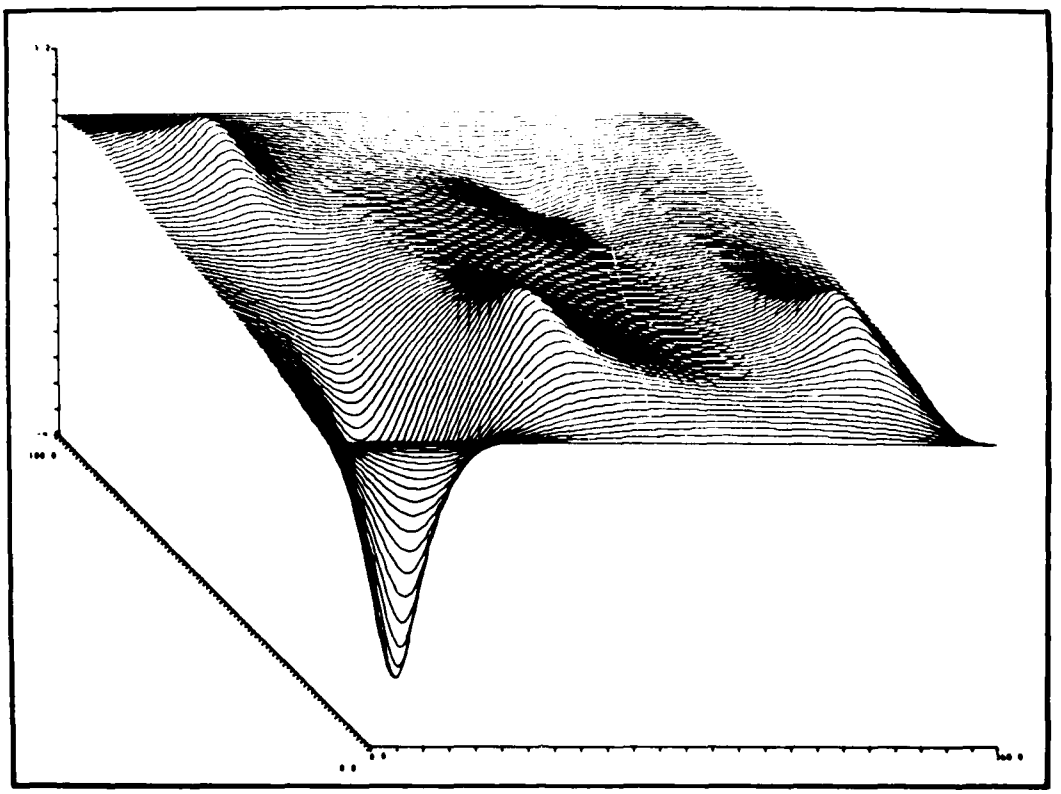


Figure 6a: error $\phi(x) - \phi^{(N)}(x)$, $x \in S$ (N=4, Example 2)

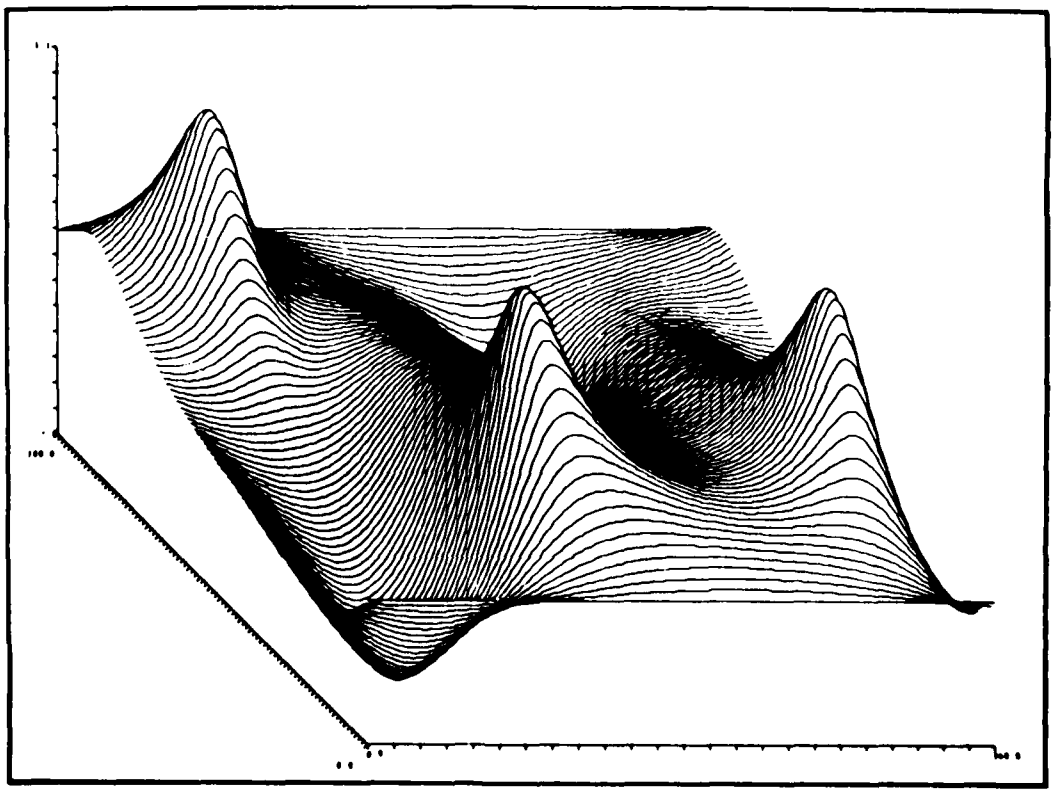


Figure 6b: relative error $\frac{\phi(x) - \phi^{(N)}(x)}{\phi(x)}$, $x \in S$ (N=4, Example 2)

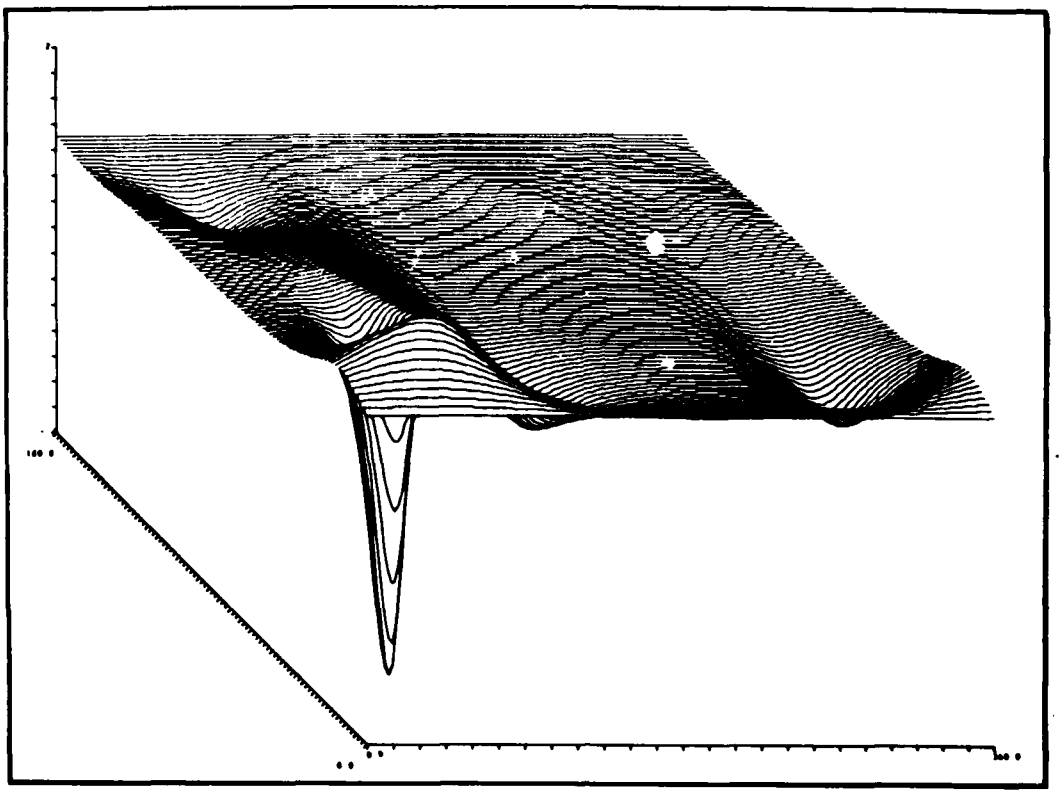


Figure 7a: error $\phi(x) - \phi^{(N)}(x)$, $x \varepsilon S$ (N=8, Example 2)

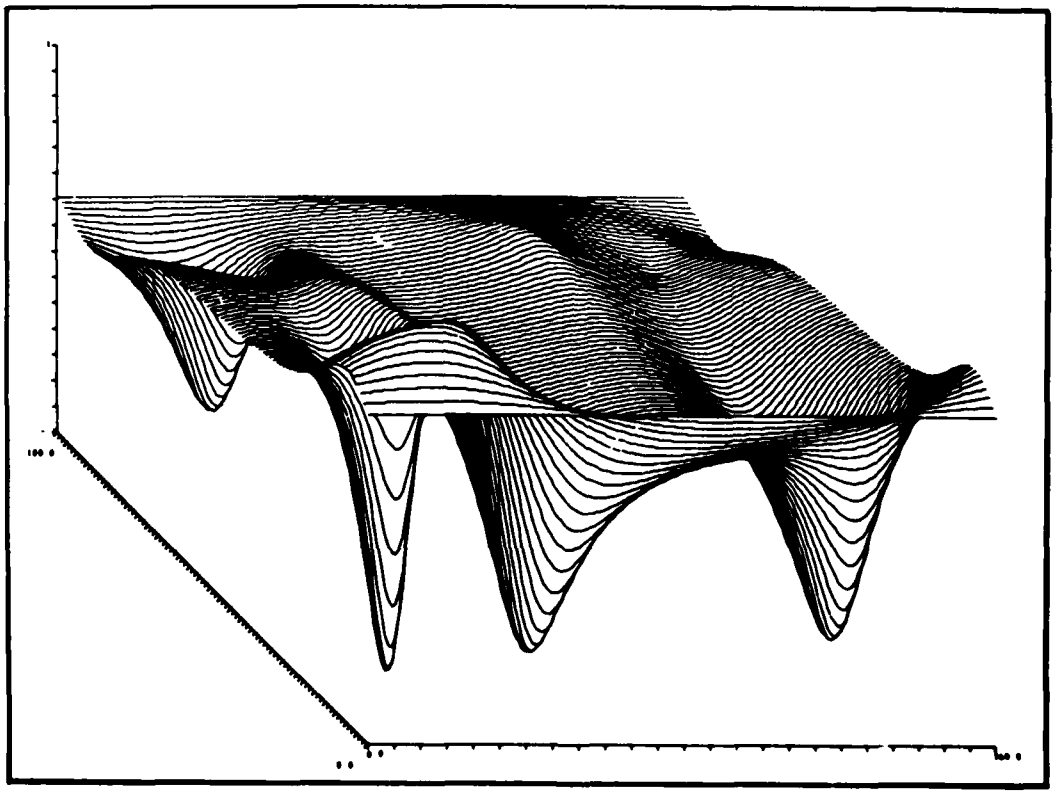


Figure 7b: relative error $\frac{\phi(x) - \phi^{(N)}(x)}{\phi(x)}$, $x \varepsilon S$ (N=8, Example 2)

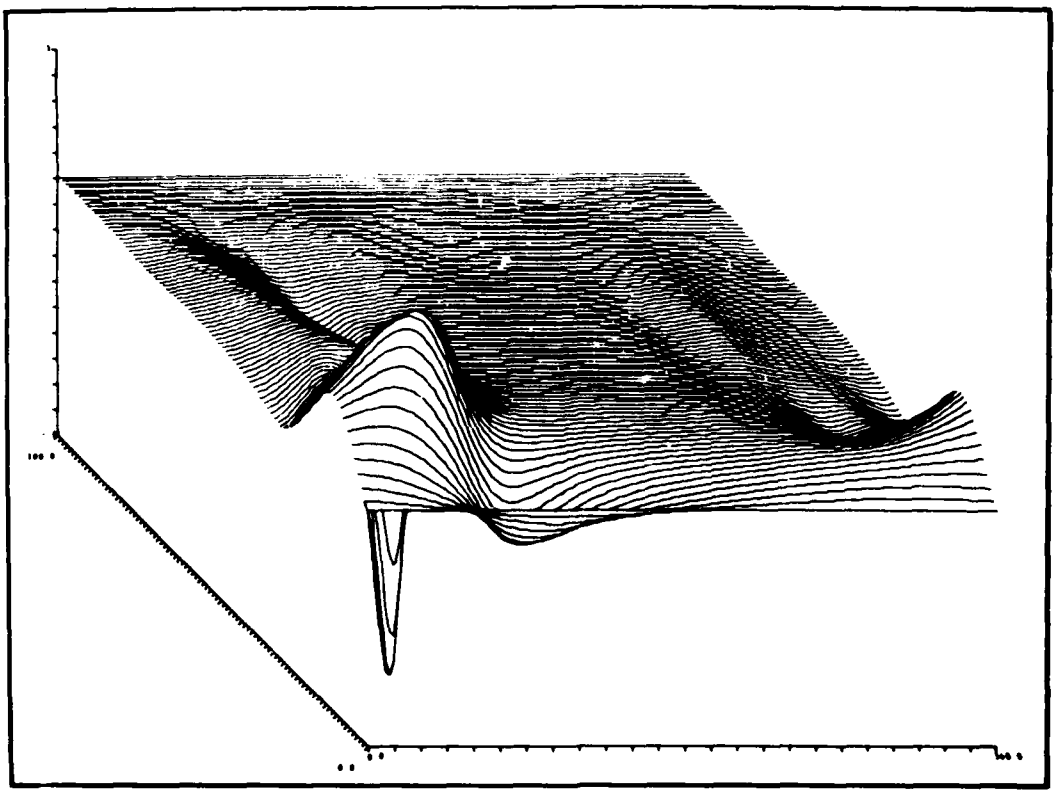


Figure 8a: error $\phi(x) - \phi^{(N)}(x)$, $x \in S$ (N=20, Example 2)

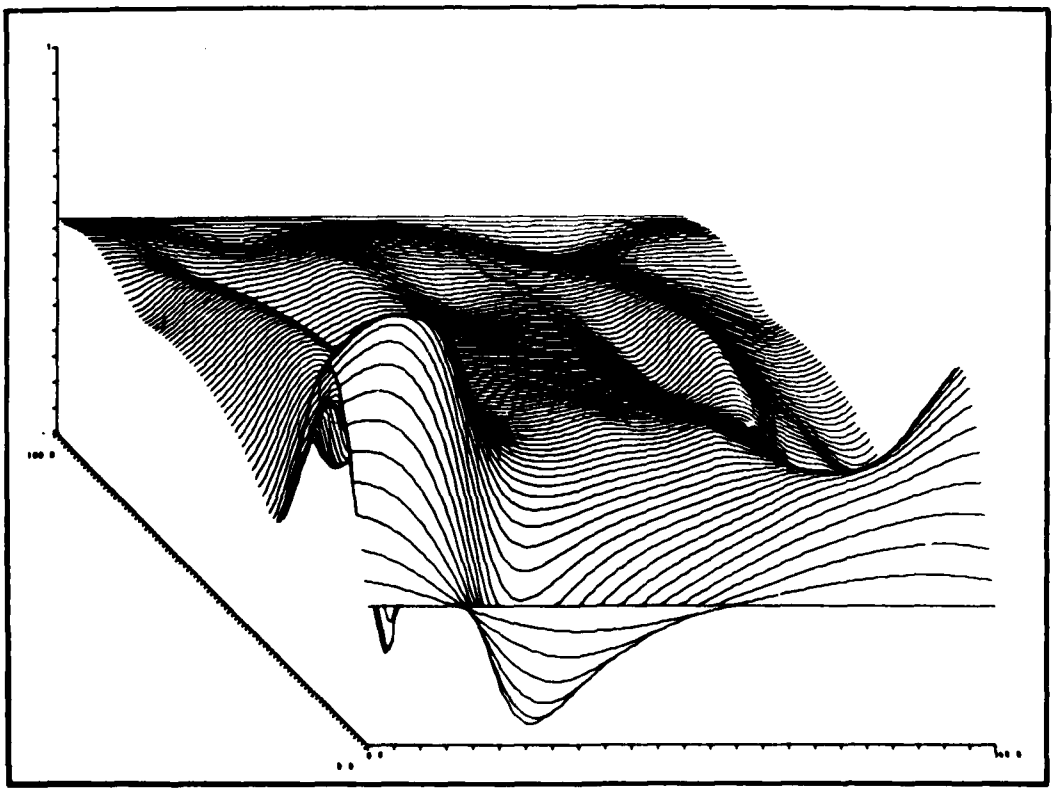


Figure 8b: relative error $\frac{\phi(x) - \phi^{(N)}(x)}{\phi(x)}$, $x \in S$ (N=20, Example 2)

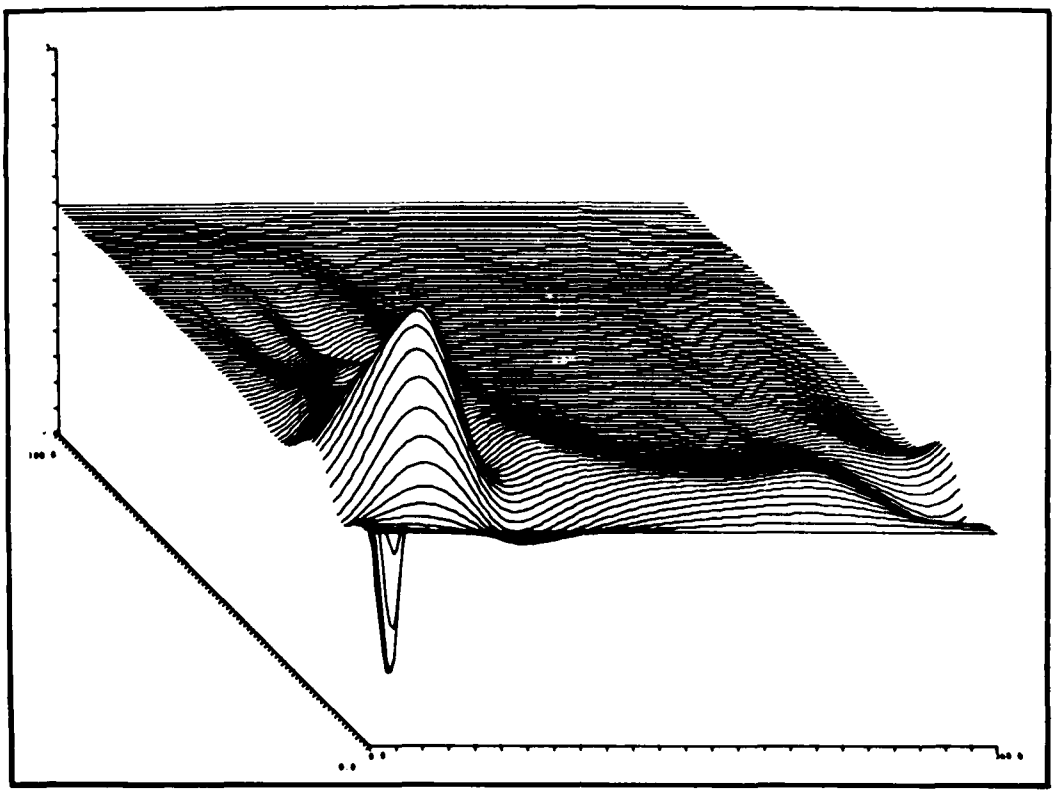


Figure 9a: error $\phi(x) - \phi^{(N)}(x)$, $x \in S$ (N=32, Example 2)

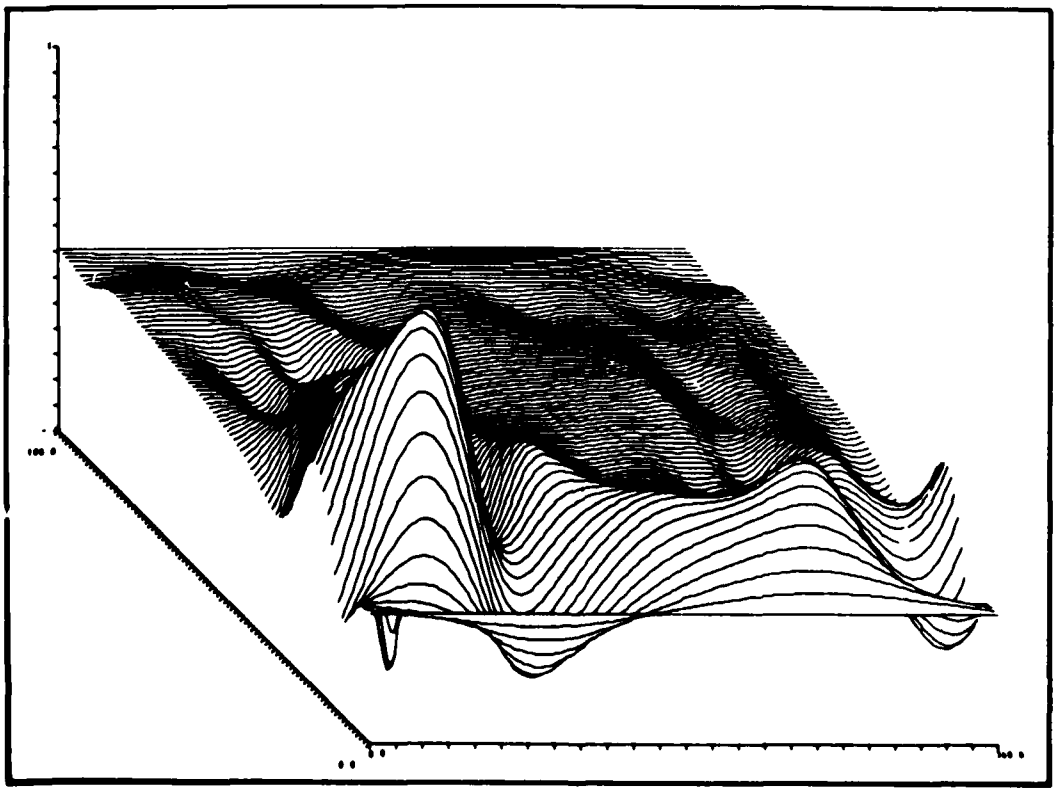


Figure 9b: relative error $\frac{\phi(x) - \phi^{(N)}(x)}{\phi(x)}$, $x \in S$ (N=32, Example 2)

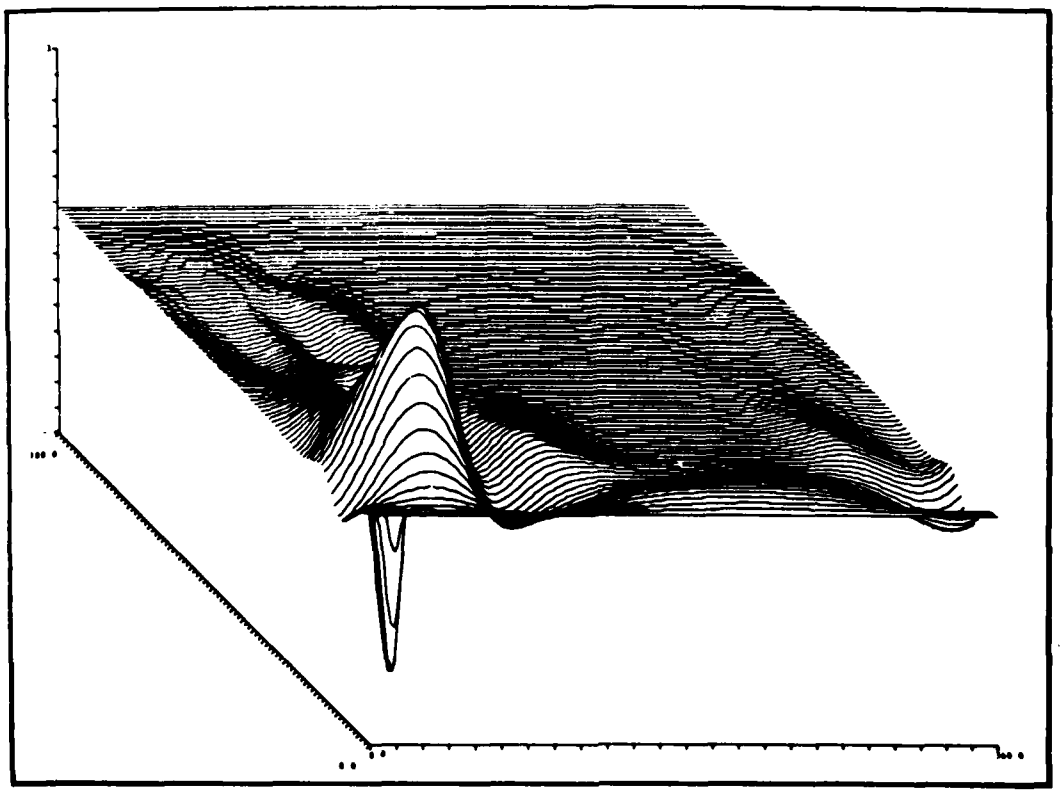


Figure 10a: error $\phi(x) - \phi^{(N)}(x)$, $x \in S$ (N=56, Example 2)

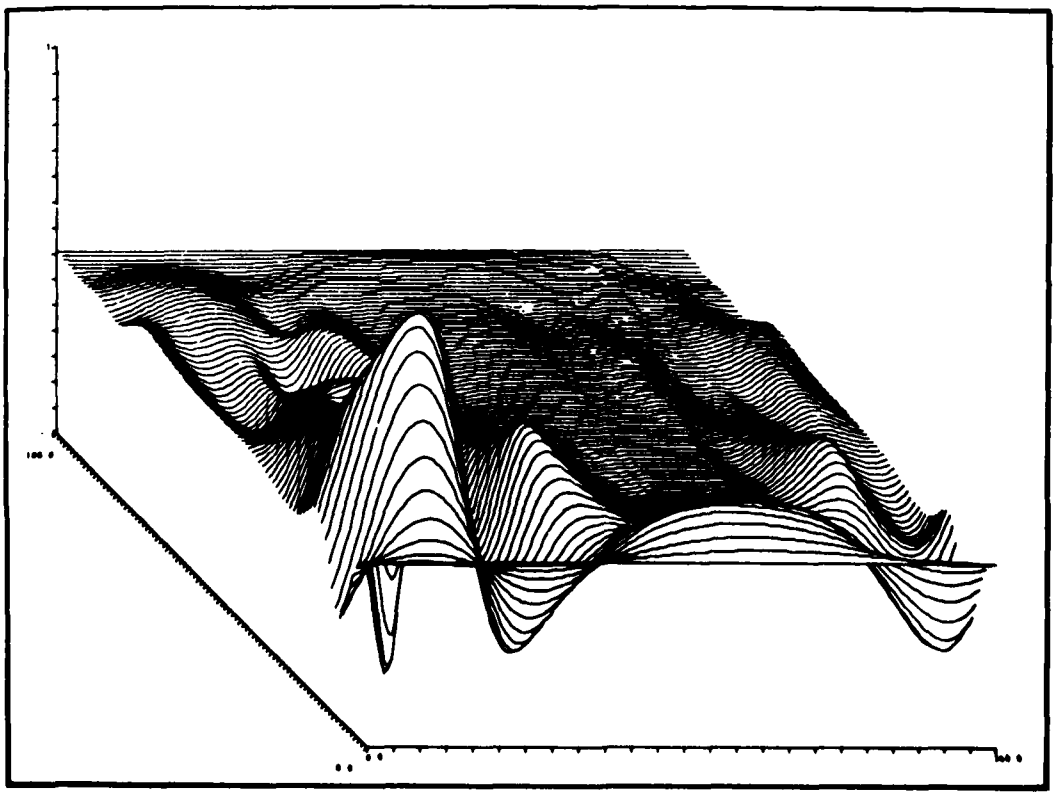


Figure 10b: relative error $\frac{\phi(x) - \phi^{(N)}(x)}{\phi(x)}$, $x \in S$ (N=56, Example 2)

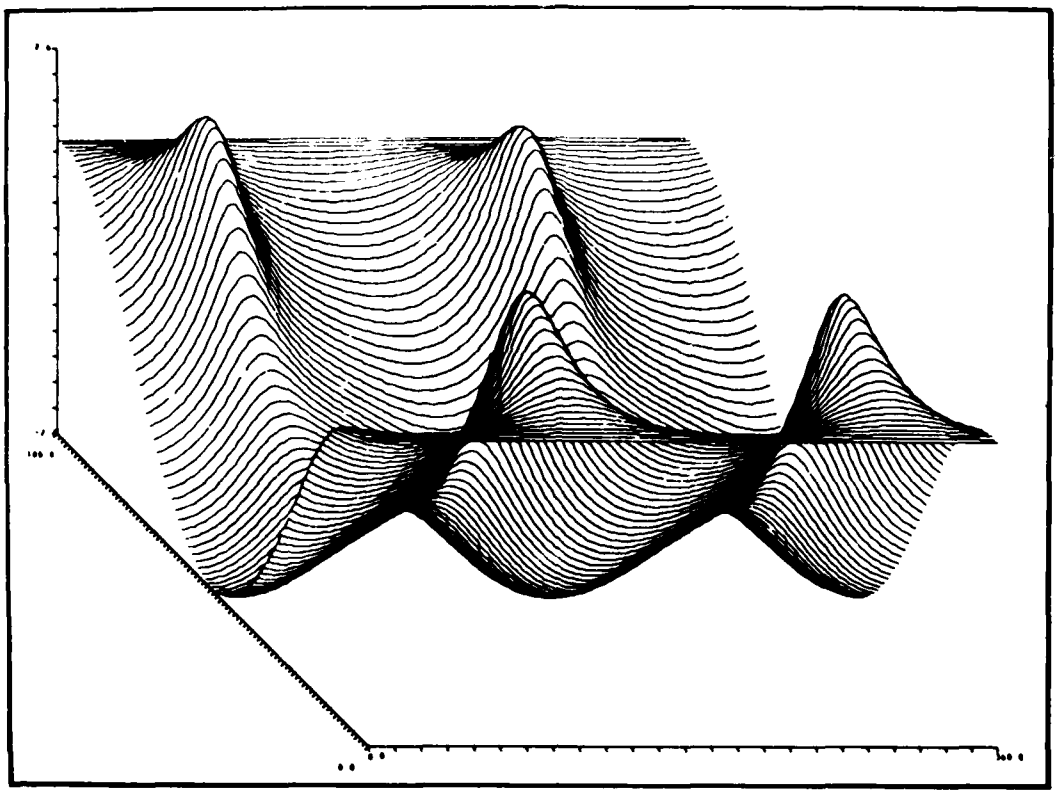


Figure 11a: error $\phi(x) - \phi^{(N)}(x)$, $x \in S$ (N=4, Example 3)

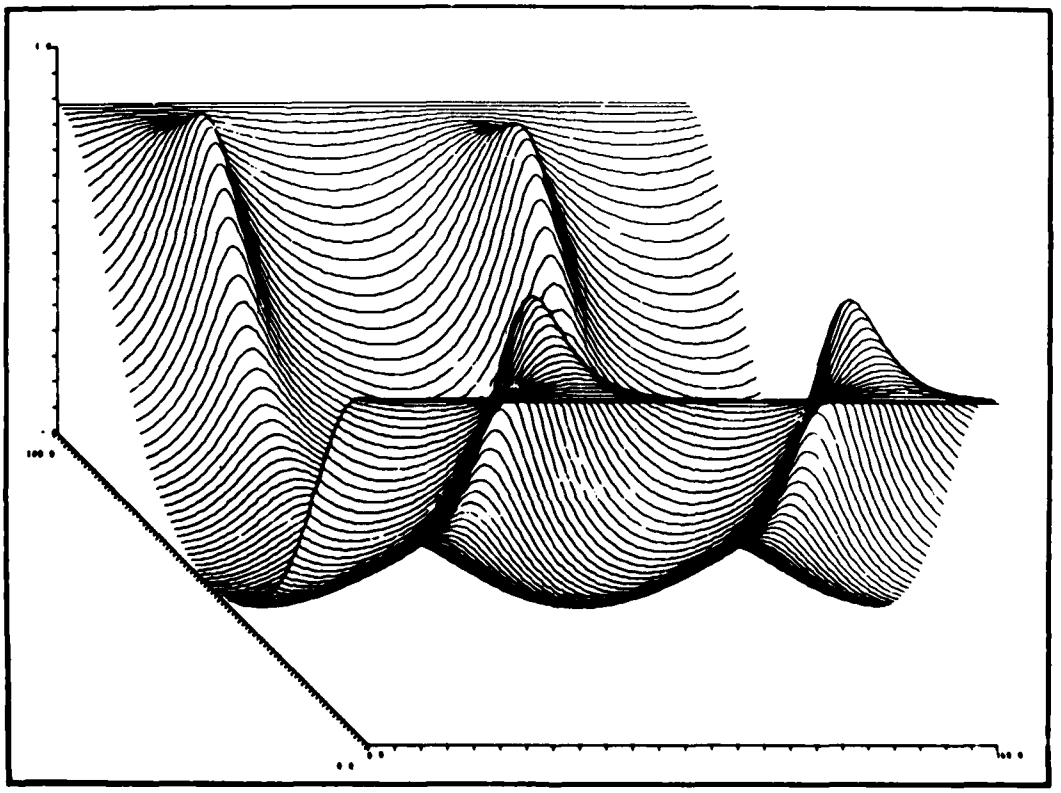


Figure 11b: relative error $\frac{\phi(x) - \phi^{(N)}(x)}{\phi(x)}$, $x \in S$ (N=4, Example 3)

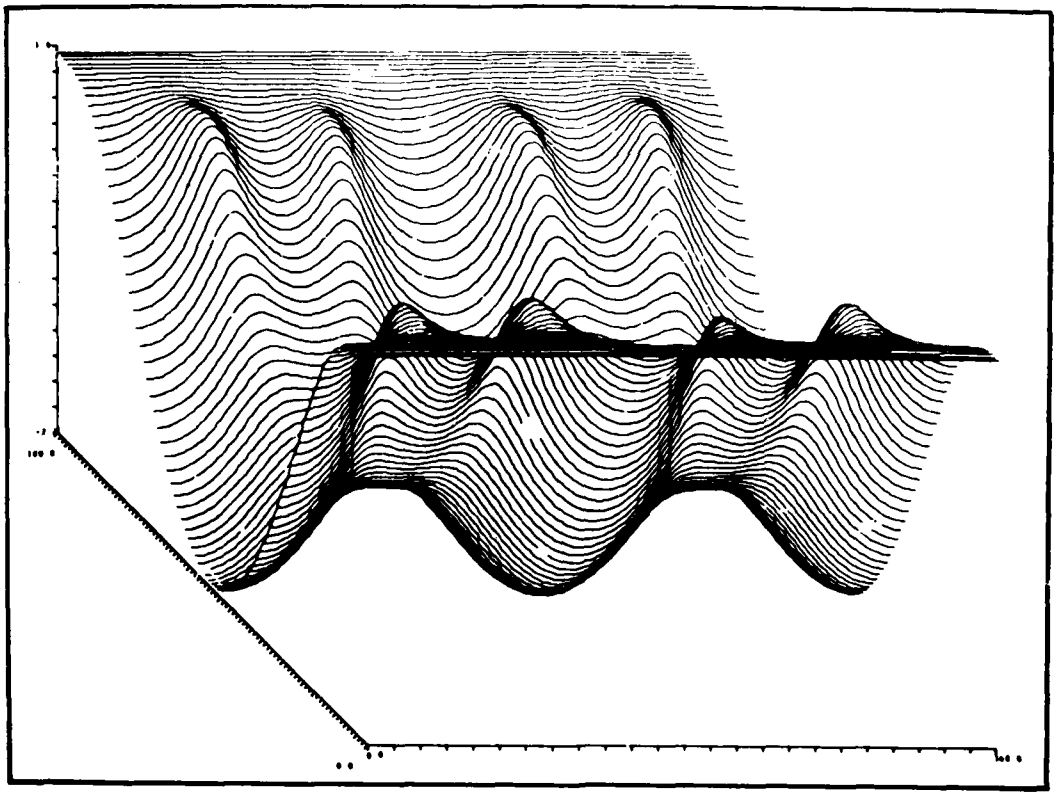


Figure 12a: error $\phi(x) - \phi^{(N)}(x)$, $x \in S$ (N=8, Example 3)

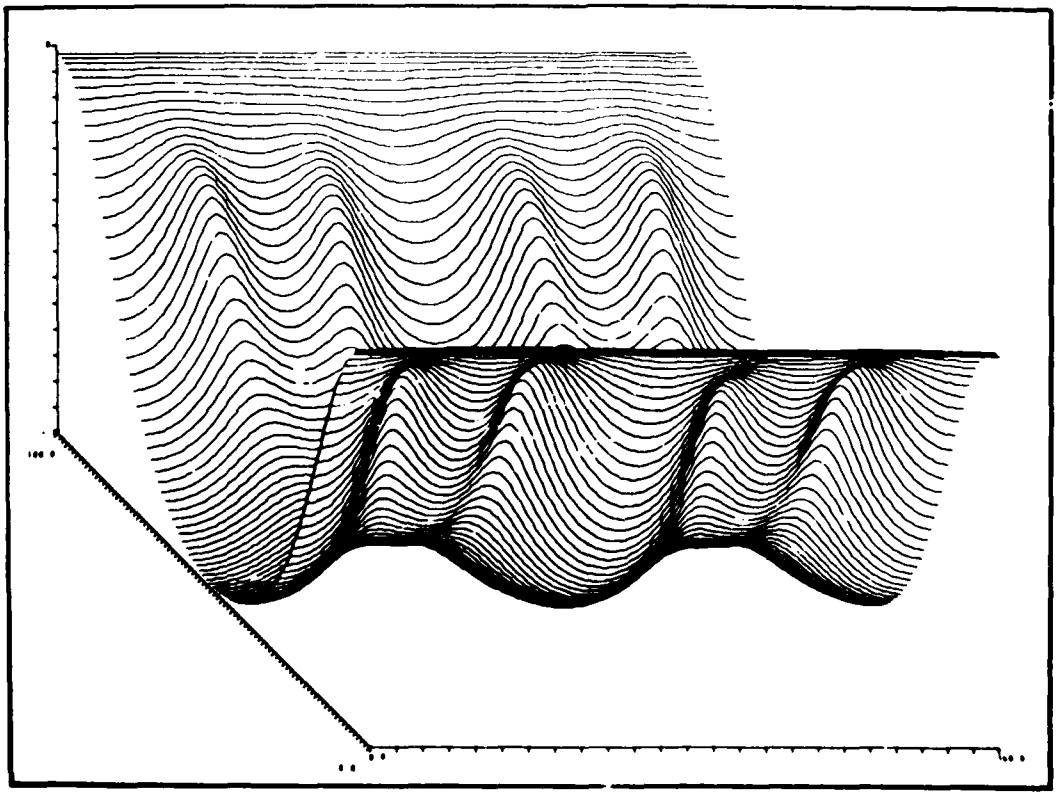


Figure 12b: relative error $\frac{\phi(x) - \phi^{(N)}(x)}{\phi(x)}$, $x \in S$ (N=8, Example 3)

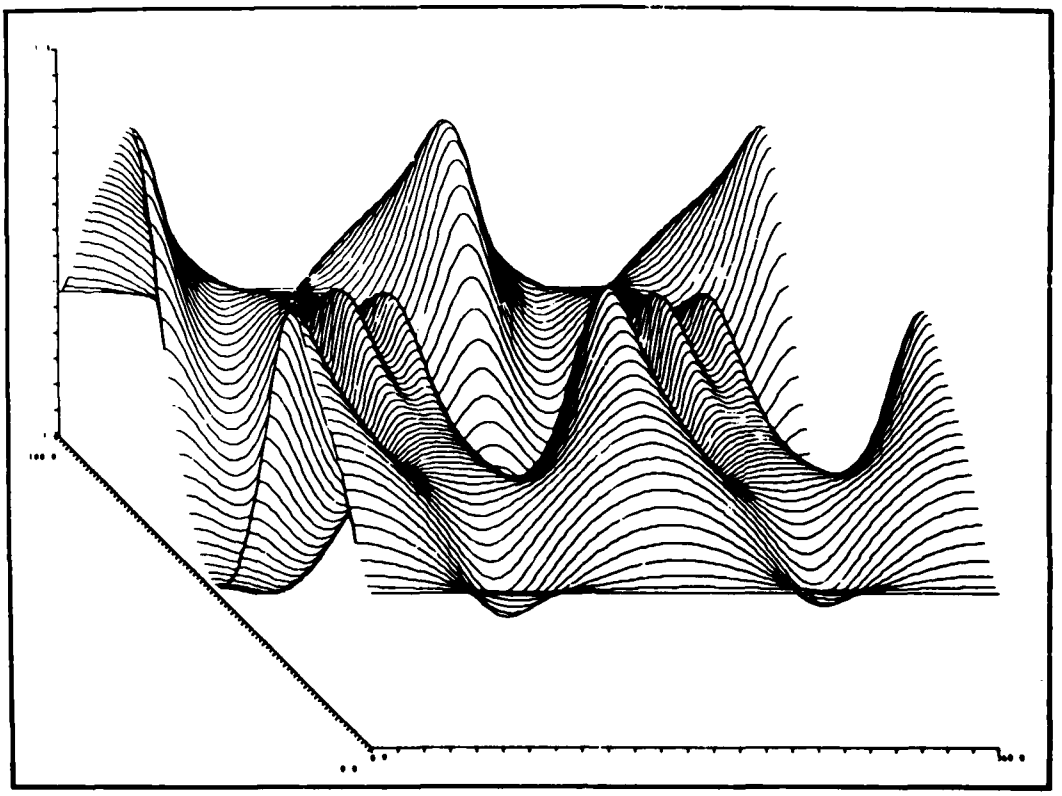


Figure 13a: error $\phi(x) - \phi^{(N)}(x)$, $x \in S$ (N=20, Example 3)

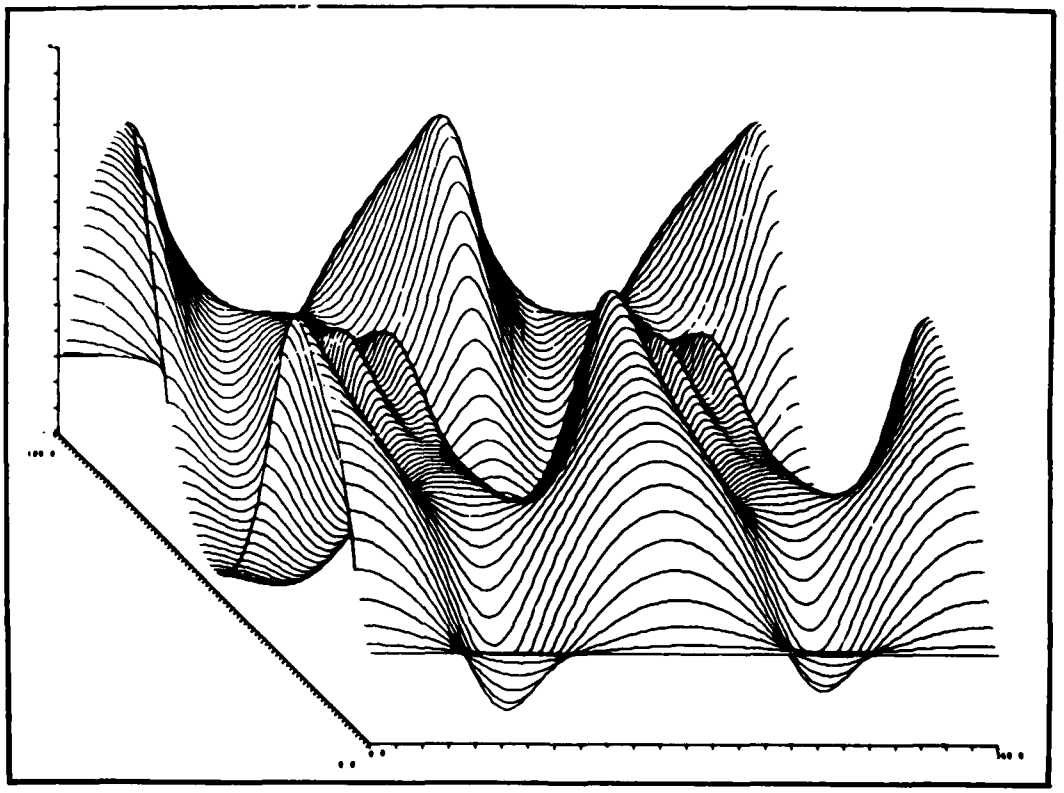


Figure 13b: relative error $\frac{\phi(x) - \phi^{(N)}(x)}{\phi(x)}$, $x \in S$ (N=20, Example 3)

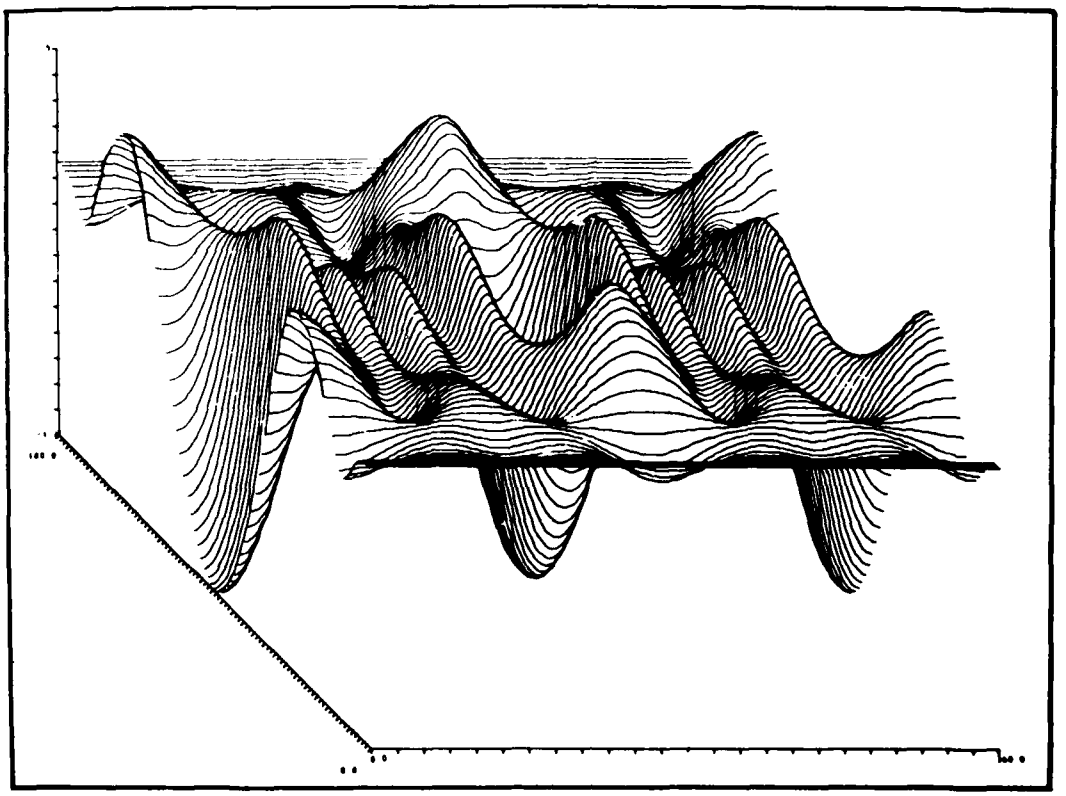


Figure 14a: error $\phi(x) - \phi^{(N)}(x)$, $x \in S$ (N=32, Example 3)

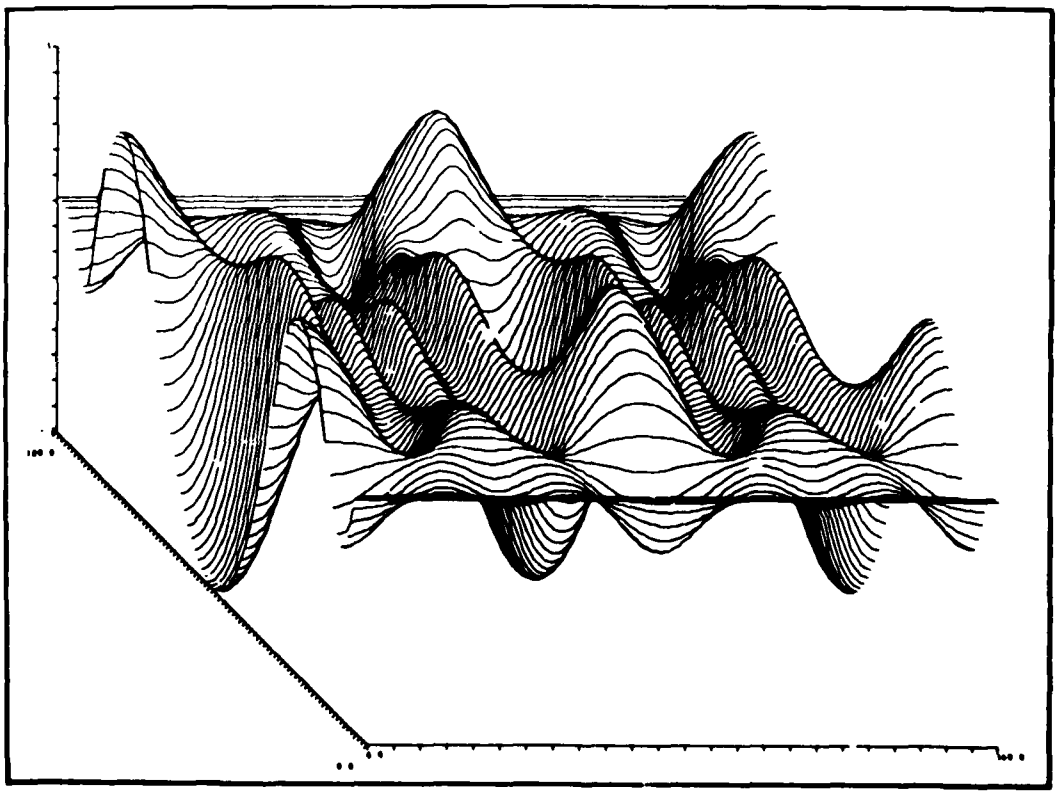


Figure 14b: relative error $\frac{\phi(x) - \phi^{(N)}(x)}{\phi(x)}$, $x \in S$ (N=32, Example 3)

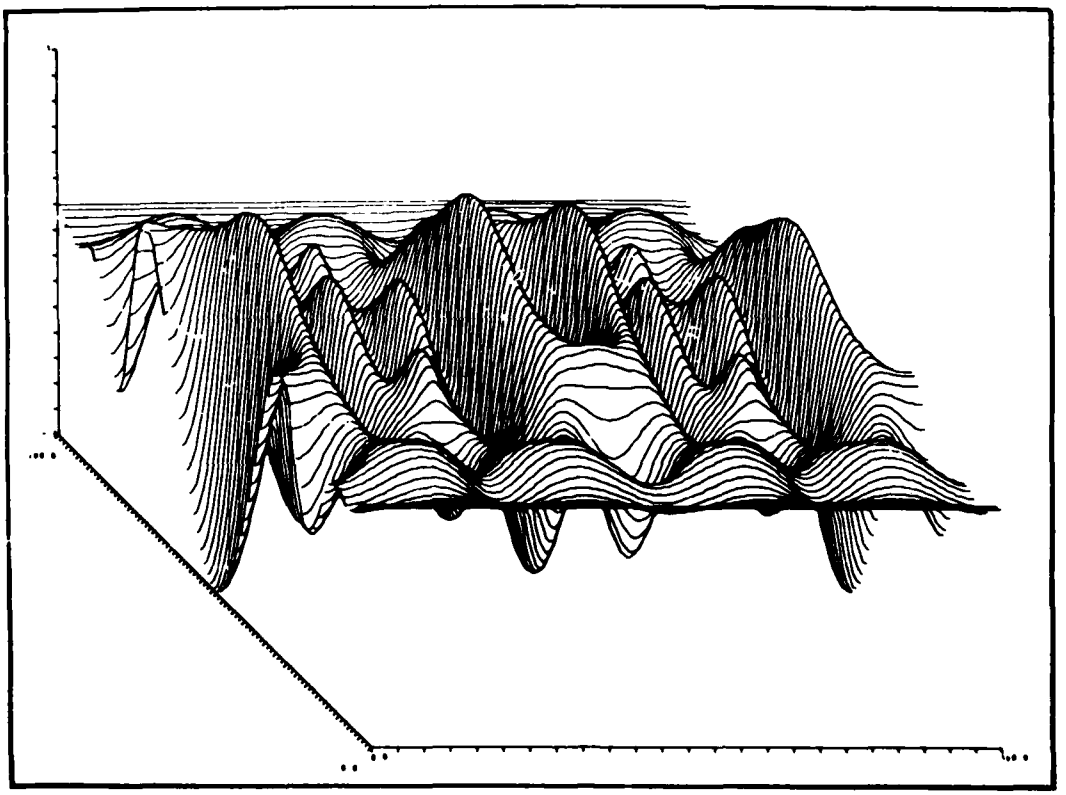


Figure 15a: error $\phi(x) - \phi^{(N)}(x)$, $x \in S$ (N=56, Example 3)

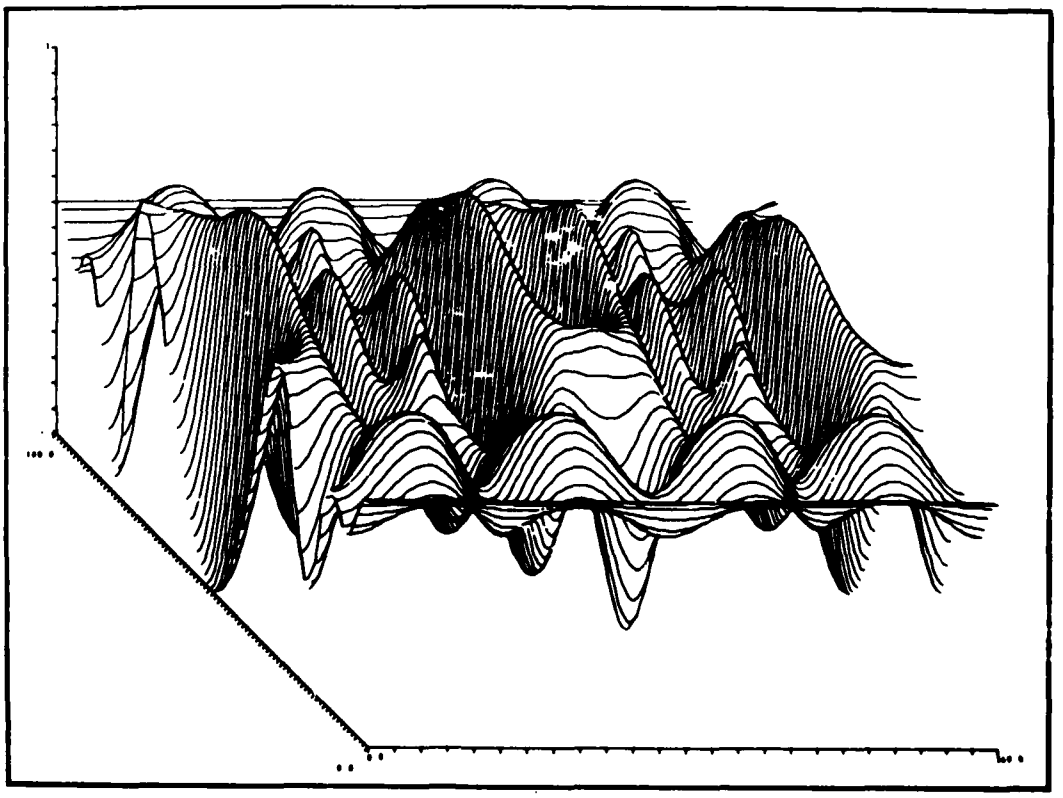


Figure 15b: relative error $\frac{\phi(x) - \phi^{(N)}(x)}{\phi(x)}$, $x \in S$ (N=56, Example 3)

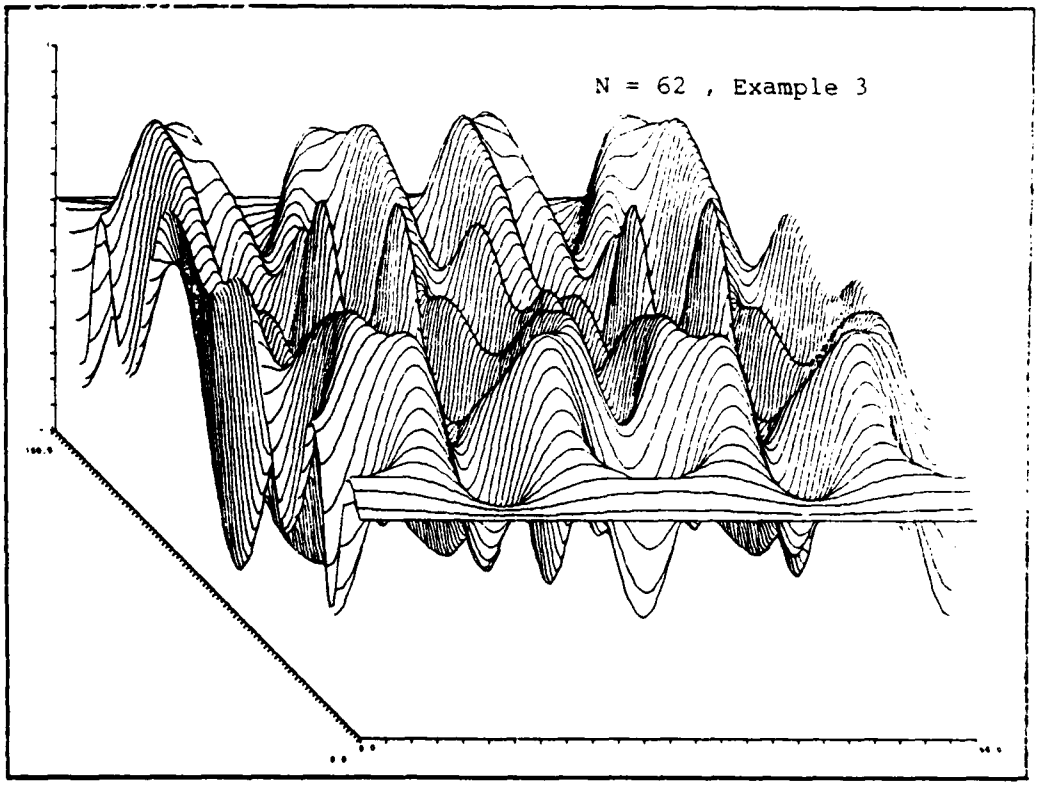


Figure 16a: error $\phi(x) - \phi^{(N)}(x)$, $x \in S$

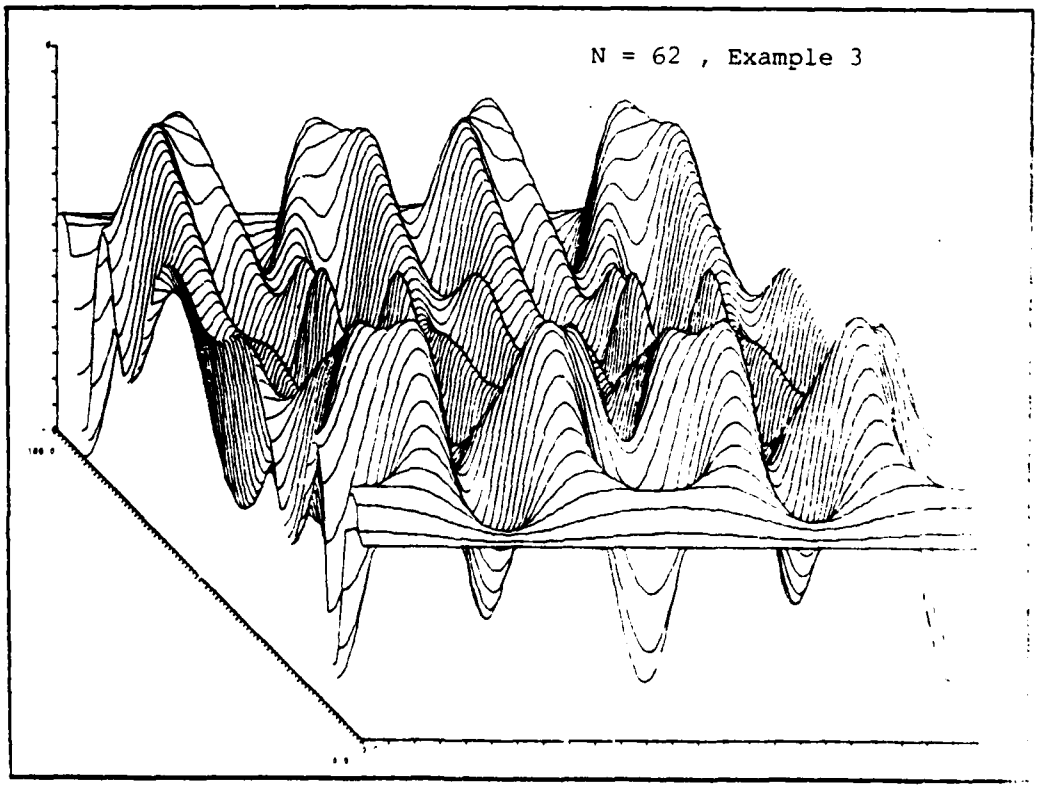


Figure 16b: relative error $\frac{\phi(x) - \phi^{(N)}(x)}{\phi(x)}$, $x \in S$

The numerical computations were done on the CYBER 175 at the "Rechenzentrum Aachen"

Final Remarks:

Mass model representations have been shown to be adequate tools for the purpose of representing the external gravitational potential. The relations of the models to the physical realitiy are transparent: the harmonicity of the approximation by mass model representations is guaranteed. Point masses (reciprocal distances) are easy to handle, reciprocal distances and their derivatives provide simple expressions of any desired gravitational field quantity. The relation between point masses and spherical harmonic coefficients of the external potential is straightforward (cf. Chapt. 7). By combination of the base functions (e.g. low degree spherical harmonics and mass points) physical meanings (center of mass, moments of inertia etc.) can be easily implemented into the model. Least square approximations by mass model representations can be deduced for not-necessarily spherical earth's models, the approximations are best in the sense of the root - mean-square error.

There are of course some drawbacks in the approximation of this technique: theoretically there exist infinite numbers of mass distributions compatible physically with the earth's gravitational field; in practical applications, however, we have to select a finite number of (multi)poles which are both economical computationally and relevant geophysically. Approximation of boundary values φ and external gravitational potential ϕ is achieved by sums $\phi^{(N)}$ of oscillating character. As illustrated in our Example 3 (Chapter 12), the oscillations often grow in number, but they decrease in size with increasing N.

Least square approximation as described in this report provides a suitable procedure for macro modeling by successive oscillations. It is not a technique of osculating character (as, for example, the Taylor series).

R e m a r k : A detailed discussion of oscillating and osculating approximation in classical Fourier theory has been given by A. Sommerfeld (1947): PARTIELLE DIFFERENTIALGLEICHUNGEN DER PHYSIK, Vorlesungen über Theoretische Physik, Bd. 6, Akademische Verlagsgesellschaft, Leipzig (6.Auflage (1966)).

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