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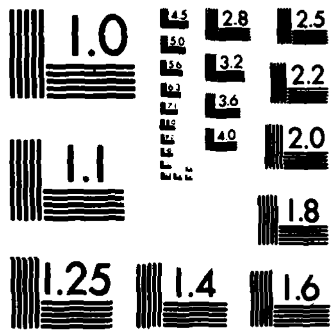

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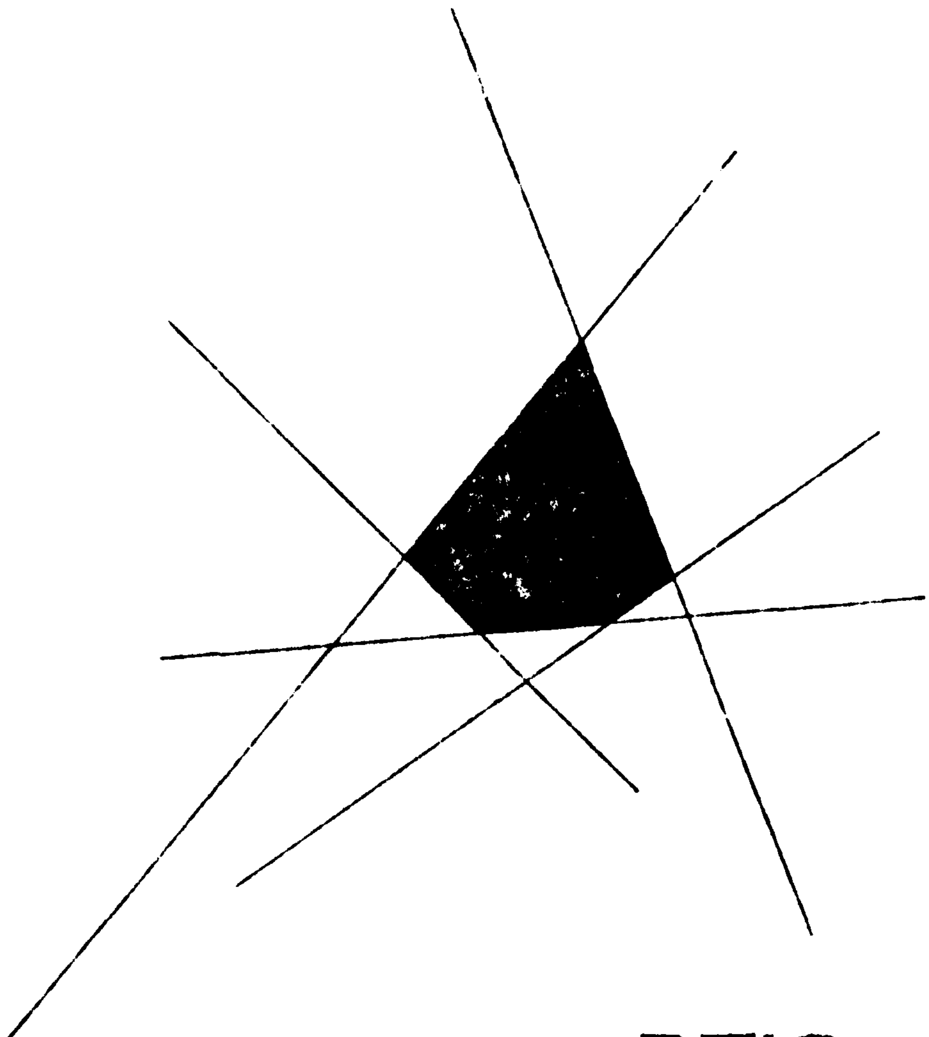

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ABSTRACT

A common model for the changes over time of the price (or sometimes the logarithm of the price) of a commodity is the random walk model. This is a Markov model which supposes that the change in price in any time period is a random variable, independent of the past, and having a given distribution F. In this note, we propose a generalized model in which the distribution of price change at any time depends upon the "environmental state" at that time. That is, we suppose that if S_n and Y_n represent the price and the environmental state at time n then, given $Y_n = i$, $S_{n+1} - S_n$ is a random variable with distribution F_i. We also suppose that the environmental state changes in a Markovian fashion. An application of this model to a stock option example is presented.

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0. INTRODUCTION

A common model for the changes over time of the price (or sometimes the logarithm of the price) of a commodity is the random walk model. This is a Markov model which supposes that the change in price in any time period is a random variable, independent of the past, and having a given distribution F. Whereas such a model might be appropriate over a short time span, it does not seem realistic over a larger time frame for it supposes that the future will behave as does the present in the sense that the change distribution is fixed. More sophisticated models suppose that, for each n, the price change from the nth to the $(n + 1)$st time period has its own distribution F_n, and they sometimes even allow this distribution to depend upon the price during the nth time period. However, these models seem even less applicable than the straight random walk model for they assume that at time $t = 0$ an individual would be knowledgeable about the price change distribution at time n.

In this note, we propose a model in which the distribution of price change at any time depends upon the "evnironmental state" at that time. That is, we suppose that if S_n and Y_n represent the price and the environmental state at time n then, given $Y_n = i$, $S_{n+1} - S_n$ is a random variable with distribution F_i. We also suppose that the environmental state changes in a Markovian fashion.

In Section 1, we present the specifics of the above model and in Section 2 we compute, conditional on initial conditions, the mean and variance of the price at time n . As, in most applications, the "environmental state" will be unobserved, we show, in Section 3, how to derive its posterior distribution at any time. In Section 4, a stock option example is considered.

1. THE MODEL

Let S_n denote the price and Y_n the environmental state at time n, where the set of possible values of Y_n are taken to be the nonnegative integers. We suppose that for a given environmental state i there is a joint distribution for the next environmental state and the change in price. That is, we suppose there are transition probabilities P_{ij} and a family of distributions F_{ij} such that

$$P\{S_{n+1} - S_n \le x \,,\, Y_{n+1} = j \mid Y_n = i \,,\, Y_{n-1}, \ldots, Y_0 \,,\, S_n, \ldots, S_0\}$$

$$= P_{ij}F_{ij}(x) \; .$$

In words, if the present price is s and the environmental state is i, then the environmental state will change to j with probability P_{ij} and if this occurs the new price will be s plus a random variable having distribution F_{ij}. Of course, it is not necessary for us to think in terms of the environment changing first. We can also imagine that, given $S_n = s$, $Y_n = i$, the price change will have distribution F_i defined by

$$F_i = \sum_j P_{ij}F_{ij} \; ,$$

and given that the price change is x the next environmental state will be j with probability $P_{ij}(x)$ given by

$$P_{ij}(x) = \frac{P_{ij}dF_{ij}(x)}{\sum_j P_{ij}dF_{ij}(x)} \; . \tag{1.1}$$

Let

$$\mu(i) = \int x dF_i(x)$$

$$\sigma^2(i) = \int (x - \mu(i))^2 dF_i(x)$$

denote the mean and variance of the price change for environmental state i . We first note that, in the long run, the commodity's price grows as a weighted average of the $\mu(i)$.

<u>Proposition 1</u>:

Assume that the Markov chain with transition probabilities P_{ij} , $i,j \geq 0$ is ergodic and let $\{\pi_i , i \geq 0\}$ denote the stationary probabilities. Then, with probability 1,

$$\frac{S_n}{n} \to \sum_i \pi_i \mu(i) \quad \text{as} \quad n \to \infty .$$

<u>Proof</u>:

Let $N(j,n)$ denote the number of time points k , $1 \leq k \leq n$, for which $Y_{k-1} = j$. Letting $X_n = S_n - S_{n-1}$, $n \geq 1$, we have that

$$\begin{aligned} S_n &= S_0 + \sum_{k=1}^{n} X_k \\ &= S_0 + \sum_j \sum_{\substack{k:Y_{k-1}=j \\ 1 \leq k \leq n}} X_k . \end{aligned}$$

Hence,

$$\frac{S_n}{n} = \frac{S_0}{n} + \sum_j \left(\sum_{\substack{k:Y_{k-1}=j \\ 1 \leq k \leq n}} X_k / N(j,n) \right) \frac{N(j,n)}{n} .$$

Now given $Y_{k-1} = j$, X_k has, independent of the past, distribution F_j and so by the strong law of large numbers

$$\sum_{\substack{k:Y_{k-1}=j \\ 1\leq k\leq n}} X_k/N(j,n) \to u_j \quad \text{as} \quad n \to \infty .$$

Also, by the strong law for renewal processes,

$$\frac{N(j,n)}{n} \to \frac{1}{E[\text{time between visits to } j]} = \pi_j \ . ||$$

2. MEAN AND VARIANCE OF S_n

Let us suppose that $S_0 = s$ and $Y_0 = i$ are given and let $X_n = S_n - S_{n-1}$, $n \geq 1$. Hence,

$$S_n = \sum_{j=1}^{n} X_j + s \, .$$

As $E[X_j \mid Y_{j-1}] = \mu(Y_{j-1})$, we see that

$$\begin{aligned} E[S_n] &= s + \sum_{j=1}^{n} E[\mu(Y_{j-1})] \\ &= s + \sum_{j=1}^{n} \sum_{k} P_{ik}^{j-1} \mu(k) \end{aligned}$$

where the P_{ik}^{j} are the j stage transition probabilities of the Markov chain $[P_{ik}]$.

To compute $\mathrm{Var}\,(S_n)$, we use

$$\begin{aligned} \mathrm{Var}\,(S_n) &= \mathrm{Var}\left[\sum_{j=1}^{n} X_j\right] \\ &= \sum_{j=1}^{n} \mathrm{Var}\,(X_j) + 2 \sum_{j<\ell}\sum \mathrm{Cov}\,(X_j, X_\ell) \, . \end{aligned} \tag{2.1}$$

By the conditional variance formula,

$$\text{(2.2)}\qquad \begin{aligned} \mathrm{Var}\,(X_j) &= E\left[\sigma^2(Y_{j-1})\right] + \mathrm{Var}\,[\mu(Y(j-1))] \\ &= \sum_k P_{ik}^{j-1}\,\sigma^2(k) + \sum_k P_{ik}^{j-1}(\mu(k))^2 - E^2(X_j)\ . \end{aligned}$$

Also, we have for $j < \ell$

$$\begin{aligned} E[X_j X_\ell] &= \sum_k E[X_j X_\ell \mid Y_{j-1} = k] P_{ik}^{j-1} \\ &= \sum_k \int x E[X_\ell \mid X_j = x\ ,\ Y_{j-1} = k] dF_k(x) P_{ik}^{j-1}\ . \end{aligned}$$

To compute $E[X_\ell \mid X_j\ ,\ Y_{j-1}]$, we condition on Y_j thusly:

$$E[X_\ell \mid X_j = x\ ,\ Y_{j-1} = k] = \sum_m P_{km}(x) \sum_r P_{mr}^{\ell-j-1}\mu(r)\ .$$

Hence, for $j < \ell$

$$\text{(2.3)}\qquad E[X_j X_\ell] = \sum_k \int x \sum_m P_{km}(x) \sum_r P_{mr}^{\ell-j-1}\mu(r) dF_k(x) P_{ik}^{j-1}$$

where $P_{km}(x)$ is given by (1.1). $\mathrm{Var}\,(S_n)$ can now be computed from (2.1), (2.2), and (2.3) by using

$$E[X_j] = \sum_k P_{ik}^{j-1}\mu(k)\ .$$

<u>Remark</u>:

As can be seen by the above, the formula for $\mathrm{Var}\,(S_n)$ is quite involved when the number of environmental states is not quite small.

3. DISTRIBUTION OF THE STATE OF NATURE

In most applications, the state of nature will never be explicitly observed. In such a case, we will suppose that a prior distribution is given for the initial state of nature. For instance, suppose

$$P\{Y_0 = i\} = P_i^o \quad , \quad P_i^o \geq 0 \quad , \quad \sum_i P_i^o = 1 \ .$$

If at any time the probability distribution of the current state of nature is $\underline{P} = (P_1, P_2, \ldots)$ and the change in the commodity's price is observed to equal x , then the next state of nature is $\underline{P}(x) = (P_1(x), \ldots)$ where

$$P_j(x) = \frac{\sum_i P_i dF_i(x) P_{ij}(x)}{\sum_i P_i dF_i(x)}$$

and where $P_{ij}(x)$ is given by (1.1).

4. A STOCK OPTION EXAMPLE

Consider a stock whose price changes in accordance with the model of Section 1 and suppose that we own an option to buy one share of this stock at any time within N days for a fixed price c. We need never exercise the option but if we do at a time when the stock's price is s then our profit is $s - c$. Under the assumption that the environmental state is observable, we are interested in the strategy that maximizes the expected profit.

If we let $V_n(s,i)$ denote the maximal expected profit when there are n days to go for the exercising of the option, the present price is s and the environmental state is i, then V_n satisfies the optimality equation

$$V_n(s,i) = \max \left\{ s - c \, , \, \sum_j P_{ij} \int V_{n-1}(s + x, j) dF_{ij}(x) \right\}$$

with the boundary condition

$$V_0(s,i) = \max \{0, s - c\} .$$

Lemma 2:

(i) $V_n(s,i) - s$ is decreasing in s, for fixed i.

(ii) For fixed i, $V_n(s,i)$ is increasing, continuous, and convex in s and is increasing in n.

Proof:

Part (i) follows by induction. It is immediate for $n = 0$ and so assume if for $n - 1$. Now

$$V_n(s,i) - s = \max \left\{ -c \ , \ \sum_j P_{ij} \int [V_{n-1}(s + x,j) - (s + x)] dF_{ij}(x) + \mu(i) \right\} .$$

By the induction hypothesis, $V_{n-1}(s + x,j) - (s + x)$ is, for each x, decreasing in s and so (i) follows. Part (ii) also follows by induction. For instance, by the induction hypothesis, and the optimality equation, $V_n(s,i)$ is the maximum of two functions, both convex in s, and so is itself convex in s.||

Proposition 3:

The optimal policy is as follows: There are numbers $s(n,i)$, increasing in n, $n \geq 1$, such that the option should be exercised when there are n days to go, the price is s, and the environmental state is i if and only if $s \geq s(n,i)$.

Proof:

In the situation given, it follows from the optimality equation that it is optimal to exercise the option if

$$V_n(s,i) - s = -c .$$

Let

$$s(n,i) = \min \{s : V_n(s,i) - s = -c\}$$

where we take $s(n,i)$ to equal ∞ if the above set is vacuous. By Lemma 2, we have for all $s \geq s(n,i)$

$$V_n(s,i) - s \leq V_n(s(n,i)) - s(n,i) = -c$$

implying that one should exercise the option in state (s,i) when n days remain. That $s(n,i)$ increases in n follows from $V_n(s,i)$ being increasing in n which is obvious.||

Remark:

It is easy to verify that if $\mu(i) \geq 0$, then $s(n,i) = \infty$.

To obtain conditions under which $s(n,i)$ increases in i , we need the following definition.

Definition:

The distribution F is said to be *more variable* than the distribution G if $\int f(x)dF(x) \geq \int f(x)dG(x)$ for all increasing convex functions f .

Theorem 4:

Suppose that the price change when in environmental state i is independent of the next environmental state. That is, suppose that $F_i(t) \equiv F_{ij}(t)$ does not depend on j . Then if

(i) F_i increases in variability in i

and

(ii) $\sum_{j=k}^{\infty} P_{ij}$ increases in i for all k

then $V_n(s,i)$, and thus $s(n,i)$, is increasing in i .

Proof:

By induction. It is immediate for $n = 0$ and so assume for $n - 1$.

Now let

$$h(i,j) = \int V_{n-1}(s + x,j)dF_i(x)$$

and note that, for fixed j, it follows from the monotonicity and convexity in x of $V_{n-1}(s + x,j)$ (from Lemma 2) and condition (i) of the hypothesis that $h(i,j)$ increases in i. Therefore,

$$\begin{aligned}\sum_j P_{i+1,j}h(i + 1,j) &\geq \sum_j P_{i+1,j}h(i,j) \\ &\geq \sum_j P_{i,j}h(i,j)\end{aligned}$$

where the second inequality follows from the induction hypothesis, which implies that $h(i,j)$ increases in j, and hypothesis (ii) which is equivalent to the condition that $\sum_j P_{ij}k(j)$ increases in i whenever $k(j)$ increases in j. Hence, $\sum_j P_{ij}h(i,j)$ increases in i and the result follows as

$$V_n(s,i) = \max\left\{s - c , \sum_j P_{ij}h(i,j)\right\} . ||$$

<u>Remark</u>:

The above example was considered by Taylor for the case of a single environmental state.

REFERENCE

[1] Taylor, H., "Evaluating a Call Option and Optimal Timing Strategy in the Stock Market," Management Science, Vol. 12, pp. 111-120, (1967).

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