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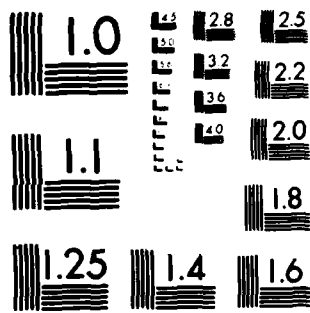
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Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables taking values in R^m and having a common distribution function F. Let $S_n = X_1 + \ldots + X_n$, where addition is coordinatewise. Call F recurrent if the random walk $\{S_n\}$ is recurrent. Shepp (1962) has used certain definitions of unimodality and peakedness to show that if F and G are symmetric unimodal and F is less peaked than G, then the recurrence of F implies the recurrence of G. This paper extends Shepp's result to a wider class of "symmetric" and "unimodal" distributions.

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RECURRENCE OF SYMMETRIC RANDOM WALKS

S.W. Dharmadhikari
Southern Illinois University
Carbondale, Il. 62901

Kumar Joag-dev
University of Illinois
Urbana, Il. 61801

SYNOPTIC ABSTRACT

Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables taking values in R^m and having a common distribution function F. Let $S_n = X_1 + \ldots + X_n$, where addition is coordinatewise. Call F recurrent if the random walk $\{S_n\}$ is recurrent. Shepp (1962) has used certain definitions of unimodality and peakedness to show that if F and G are symmetric unimodal and F is less peaked than G, then the recurrence of F implies the recurrence of G. This paper extends Shepp's result to a wider class of "symmetric" and "unimodal" distributions.

1. INTRODUCTION

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically
distributed random vectors taking values in R^m. Denote the common
distribution function (d.f.) by F. Let $S_n = X_1 + \ldots + X_n$,
where addition is by coordinates. The d.f. F is called
recurrent if, for every open set N containing the origin, the
random walk S_n visits N infinitely often with probability 1.
Otherwise F is called _transient_.

Let P_F denote the probability defined on R^m by the d.f. F.
The dimension of F can be defined in a natural way as follows.
Degenerate distributions have dimension zero. The dimension is
1 if P_F is not degenerate and concentrates all probability on a
line. The dimension is ≥ 3 if P_F does not concentrate its mass
on any plane. Chung and Fuchs (1951) proved that every d.f. F of
dimension ≥ 3 is transient. Shepp (1962) has considered the
recurrence of symmetric unimodal distributions when F has dimen-
sion 1 or 2. For distributions on the line, there are natural
definitions of symmetry and unimodality. Using these and also a
concept of peakedness given by Birnbaum (1948), Shepp showed
that if F is symmetric, unimodal and less peaked than a symmetric
d.f. G, then the recurrence of F implies the recurrence of G. He
also extended this result to distributions of dimension 2. But
here his definitions of "symmetry" and "unimodality" are somewhat
restrictive. In this paper, we weaken his requirements and extend
his results to a wider class of "symmetric" and "unimodal"
distributions on R^2.

2. PRELIMINARIES

A d.f. F on R is called _unimodal about a mode_ ν if F is
convex on $(-\infty, \nu)$ and concave on (ν, ∞). Further F is called
symmetric (about 0) if $P_F[(a,b)] = P_F[(-b,-a)]$, for all $a < b$.
It is easy to show that, if F is unimodal and symmetric, then 0
is a mode of F. It is also known (see Olshen and Savage (1970))
that the class of all symmetric unimodal distributions on R

coincides with the closed convex hull of the set of all uniform distributions on symmetric intervals (-a,a).

For distributions in higher dimensions, unimodality and symmetry can be defined in several different ways. Shepp (1962) has used mirror symmetry about the coordinate axes and defined unimodality as follows.

Definition 1. A distribution is called _symmetric unimodal_ (SSUM, for short) if it is in the closed convex hull of the set of all uniform distributions on symmetric rectangles with sides parallel to the coordinate axes.

We note, however, that Shepp's definitions of symmetry and unimodality are quite restrictive. These can be weakened as follows. If $A \subset R^m$, write $-A$ for $\{-x: x \in A\}$. We call A _centrally symmetric_ if $A = -A$. A d.f. F on R^m is called _centrally symmetric_ if $P_F(A) = P_F(-A)$ for all (Borel) sets $A \subset R^m$.

Definition 2. A distribution is called _central convex unimodal_ (CCUM) if it is in the closed convex hull of the set of all uniform distributions on centrally symmetric convex sets.

Lemma 1. _Every SSUM distribution is CCUM. The converse holds only in the one-dimensional case._

Proof. The first assertion is immediate because (a) a set which has mirror symmetry about the coordinate axes is also centrally symmetric and (b) a rectangle is also a convex set. On the real line, a symmetric convex set is also a symmetric rectangle (interval) and therefore the notions SSUM and CCUM coincide. In higher dimensions the CCUM class is strictly wider than the SSUM class. To see this, let P denote the uniform distribution on the unit ball in R^m, $m \geq 2$. Then P is CCUM but not SSUM. The lemma is thus proved.

While the CCUM class is fairly wide, it does not include all centrally symmetric distributions whose densities are "unimodal" along rays.

Example 1. In R^2, let A denote the triangle defined by $x \geq 0$, $y \geq 0$, $x + y \leq 1$. Let P be the uniform distribution on $A \cup (-A)$.

Then P is centrally symmetric but it is not CCUM and, a fortiori, cannot be SSUM.

The SSUM class is quite narrow compared with the CCUM class. Observe that: a) If F is a bivariate d.f. with density $f(x,y)$ then the condition $f(x,y) = f(\pm x, \pm y)$ is <u>necessary</u> for F to be SSUM.

b) The bivariate normal distribution with zero mean vector is always CCUM, whereas it is SSUM if, and only if, the correlation coefficient equals zero.

c) More generally, if the density of F decreases along rays and has elliptic contours, then F is CCUM. For F to be SSUM it is <u>necessary</u> that the axes of the ellipses lie along the coordinate axes.

d) As noted above, the bivariate normal distribution with a <u>nonzero</u> correlation coefficient is not SSUM. But the distribution becomes SSUM if the axes are suitably rotated. However, there are CCUM distributions which cannot be so placed in SSUM even if the axes are rotated. An example of this type is the uniform distribution on an ellipse.

The above remarks provide a motivation for generalizing Shepp's results to a wider class of distributions.

Birnbaum (1948) defined peakedness for distributions on the line. Sherman (1955) gave a generalization to R^m as follows. Given two d.f.'s F and G, <u>F is said to be less peaked than G</u>, and we write $F \leq G$, if $P_F(C) \leq P_G(C)$ for every centrally symmetric convex set C.

The final concept we need is that of a unimodal correspondent. Let F be the d.f. of a random vector X. The <u>unimodal correspondent</u> (U,F) of F is defined to be the d.f. of UX, where U is a real random variable independent of X and uniformly distributed on (0,1).

<u>Proof</u>. Let F_c assign probability $\frac{1}{2}$ to each of the points $\pm c$. Then (U,F_c) corresponds to the uniform distribution on $(-c,c)$. The latter distribution is clearly CCUM. If F concentrates all

probability on the finite set $\{\pm c_1, \ldots, \pm c_n\}$, then (U,F) is CCUM, because it is a mixture of the CCUM distributions (U, F_{c_i}).

Suppose now that F is a general centrally symmetric distribution. Then F is the limit of a sequence $\{F^{(n)}\}$ of centrally symmetric distributions, where each $F^{(n)}$ puts all probability on a finite set. Since each $(U, F^{(n)})$ is CCUM, the limit (U,F) is also CCUM.

Let X have d.f. F. If C is a centrally symmetric convex set, then, for every $u \in (0,1)$, $P_F(uX \in C) \geq P_F(X \in C)$. Integrating over $u \in (0,1)$, we see that $P_{(U,F)}(C) \geq P_F(C)$. The lemma is thus proved.

Finally, recall the following result of Sherman (1955). The symbol F*H denotes the convolution of F and H.

Lemma 3. If $F \leq G$ and H is CCUM, then $F*H \leq G*H$.

3. RECURRENCE AND PEAKEDNESS

Shepp (1962) proved that <u>if F, G are symmetric d.f.'s on R, F is unimodal and $F \leq G$, then the recurrence of F implies the recurrence of G.</u> He also extended this result to SSUM distributions on R^2. In this section we show that his result is valid for the wider class of CCUM distributions in R^2. In view of the result of Chung and Fuchs (1951) mentioned earlier, the problem is trivial for distributions of dimension ≥ 3, because all d.f.'s are then transient.

For the remainder of the paper F is a centrally symmetric d.f. on R^2 with characteristic function φ and ψ denotes the characteristic function of (U,F). The definition of (U,F) shows that

$$\psi(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\sin(ux+vy)}{(ux+vy)} \, d\,F(x,y) \tag{1}$$

The symbol F^{n*} denotes the n-fold convolution of F with itself and P_F^{n*} denotes the probability determined by F^{n*}. According to Chung and Fuchs (1951), either of the following two conditions is necessary and sufficient for the recurrence of F.

a) $\sum_{n=1}^{\infty} P_F^{n*}(C) = \infty$, for every centrally symmetric open convex set C.

b) $\int_{-1}^{1}\int_{-1}^{1}[1-\varphi(u,v)]^{-1}dudv = \infty$

Lemma 4. If (U,F) is recurrent then F is recurrent.

Proof. As observed by Shepp (1962), for some $c > 0$,

$$1 - \cos(ux + vy) \leq c[1 - \frac{\sin(ux + vy)}{(ux + vy)}].$$

Integrating w.r.t. F and using (1), we get

$$1 - \varphi(u,v) \leq c[1 - \psi(u,v)]$$

Therefore

$$c\int_0^1\int_0^1 [1 - \varphi(u,v)]^{-1} du\, dv \geq \int_0^1\int_0^1 [1 - \psi(u,v)]^{-1} du\, dv. \tag{2}$$

If the right side of (2) is infinite, then so is the left side. The lemma now follows from the condition (b) above.

Lemma 5. If F and G are CCUM and $F \leq G$, then the recurrence of F implies the recurrence of G.

Proof. By Lemma 3, $F^{n*} \leq G^{n*}$, for all n. Therefore

$$\sum_{n=1}^{\infty} P_F^{n*}(C) \leq \sum_{n=1}^{\infty} P_G^{n*}(C) \tag{3}$$

for all centrally symmetric open convex sets C. If the left side of (3) is infinite, then so is the right side. The lemma now follows from the condition (a) above.

Corollary. A CCUM d.f. F is recurrent if, and only if, (U,F) is recurrent.

Proof. Let F be CCUM. By Lemma 2, $F \leq (U,F)$. Therefore, Lemma 5 shows that the recurrence of F implies the recurrence of (U,F). The converse also holds by Lemma 4. The corollary follows.

Theorem 1. Let F, G be centrally symmetric and let F be CCUM. If $F \leq G$, then the recurrence of F implies the recurrence of G.

Proof. By Lemma 2, $F \leq G \leq (U,G)$. Now F and (U,G) are both CCUM and F is recurrent. So by Lemma 5, (U,G) is recurrent. Hence G is recurrent by Lemma 4.

4. CRITERION FOR RECURRENCE

In this section we obtain a criterion for the recurrence of bivariate CCUM distributions. For nonzero u and v in R, let $A(u,v)$ be the set of points (x,y) in R^2 such that $0 < (x(u) + (y/v) \leq 1$. The set $B(u,v)$ will correspond to $(x/u) + (y/v) > 1$. Write

$$D(u,v) = \int\int_{A(u,v)} (vx + uy)^2 dF(x,y) + u^2 v^2 \int\int_{B(u,v)} dF(x,y) \tag{4}$$

It is possible to write $D(u,v)$ in a more convenient form. If F

is the d.f. of (X,Y), let $G_{v,u}$ denote the d.f. of $vX+uY$. Then

$$D(u,v) = \int_0^{u,v} t^2 dG_{v,u}(t) + u^2 v^2 [1 - G_{v,u}(uv)].$$

Now, given any univariate d.f. H, we can use integration by parts to show that

$$\int_0^k t^2 dH(t) + k^2[1 - H(k)] = 2\int_0^k t[1 - H(t)]dt.$$

Therefore

$$D(u,v) = 2\int_0^{uv} t[1 - G_{v,u}(t)] \, dt. \tag{5}$$

Theorem 2. <u>Let F be a CCUM d.f. on R^2. Then F is recurrent if, and only if, the integral</u>

$$\int_1^\infty \int_1^\infty \{[D(u,v)]^{-1} + [D(u,-v)]^{-1}\} \, du \, dv \tag{6}$$

<u>diverges</u>.

<u>Proof</u>. Since F is CCUM, the recurrence of F is equivalent to the recurrence of (U,F). From (1) we have

$$1 - \psi(u,v) = \int_{-\infty}^\infty \int_{-\infty}^\infty [1 - \frac{\sin(ux+vy)}{(ux+vy)}]dF(x,y). \tag{7}$$

However, we can find positive constants c and d such that

$$c(ux+vy)^2 \le 1 - \frac{\sin(ux+vy)}{(ux+vy)} \le d(ux+vy)^2 \text{ if } |ux+vy| \le 1$$

and $c \le 1 - \frac{\sin(ux+vy)}{(ux+vy)} \le d$ if $|ux+vy| > 1$.

Therefore, if we define

$$I(u,v) = \int\int_{0<ux+vy\le1} (ux+vy)^2 \, dF(x,y) + \int\int_{ux+vy \, 1} dF(x,y),$$

then (7) shows that

$2c \, I(u,v) \le 1 - \psi(u,v) \le 2d \, I(u,v)$.

It follows, from the criterion (b) stated in Section 3, that (U,F) is recurrent if, and only if, the integral

$$\int_{-1}^1 \int_{-1}^1 [I(u,v)]^{-1} \, du \, dv \tag{8}$$

diverges. By the central symmetry of F, we have $I(u,v) = I(-u,-v)$. Therefore the divergence of (8) is equivalent to the divergence of

$$\int_{0}^{1}\int_{0}^{1} \{ [I(u,v)]^{-1} + [I(u,-v)^{-1}\} \ du \ dv \qquad (9)$$

If we now note that $D(u,v) = u^2 v^2 I(u^{-1},v^{-1})$, we see that Theorem 2 follows from (9) via the substitutions $u' = u^{-1}$ and $v' = v^{-1}$.

<u>Remark</u>. The criterion for recurrence given by Theorem 2 is, in reality, a criterion for the recurrence of (U,F). We therefore see from Lemma 4 that the divergence of (8) is <u>sufficient</u> for the recurrence of F, even if F is just centrally symmetric and not CCUM.

The importance of Theorem 2 is that the criterion for recurrence is given in terms of the d.f. In contrast, the Chung and Fuchs criterion (condition (b) of Section 3) is given in terms of the characteristic function. Therefore, Theorem 2 can be used in some situations when the characteristic function is not available in a convenient form.

<u>Example 2</u>. The application of Theorem 2 is particularly simple if the distribution has circular symmetry. Under this condition the distribution of $X \cos \theta + Y \sin \theta$ is the same for all values of θ. Therefore, if F_1 denotes the x-marginal of F, then

$$G_{u,v}(t) = P[uX + vY \leq t] = F_1(t/\sqrt{(u^2 + v^2)}) \qquad (10)$$

If we write

$$\xi(t) = \int_{0}^{t} x[1 - F_1(x)]dx,$$

then (5) and (10) show that

$$D(u,v) = 2(u^2 + v^2)\xi[uv/\sqrt{u^2 + v^2}].$$

Since $D(u,v) = D(u,-v)$ in the present case, we see that the divergence of (6) is equivalent to the divergence of

$$\int_{1}^{\infty}\int_{1}^{\infty} [(u^2 + v^2)\xi(uv/\sqrt{u^2 + v^2})]^{-1} \ du \ dv \qquad (11)$$

If we change to polar coordinates, we easily see that (11) diverges if, and only if,

$$\int_{1}^{\infty} [r\xi(r)]^{-1} \ dr \qquad (12)$$

diverges. Thus, if (12) diverges, then F is recurrent. Conversely, if (12) converges and F is CCUM, then F is transient.

Suppose now that F has a density $f(x,y) = g(x^2+y^2)$. Then F is CCUM as soon as g is nonincreasing on $(0,\infty)$. For F to be SSUM, however, g has to be convex. Thus Theorem 2 enables us to decide the transience of a wide class of "unimodal" and "symmetric" distributions. As an example, suppose $g(t) = c_1 \log t/t^2$ for large t. Then it is easy to check that $1 - F_1(t) \sim c_2 \log t/t^2$. Therefore

$$\xi(t) = \int_0^t x[1-F_1(x)]dx \sim c_3 \int_0^t [\log x/x]\, dx = c_3 (\log t)^2 .$$

Therefore (12) converges and F is transient.

<u>Remark</u>. If F does not have circular symmetry, results of the type considered in Example 2 can still be proved. One needs to put conditions on the asymptotic behavior of

$\int_0^t x[1 - F_1(x)]dx$ and $\int_0^t y[1 - F_2(y)]dy$, where F_1 and F_2 are the x- and y-marginals of F. The calculations are tedious but straightforward. A catalog of the values of the integral (5) for various types of CCUM distributions would be quite useful.

Tweedie (1975) has considered ergodicity and recurrence of Markov chains on general state spaces. For the present paper, the relevant condition for recurrence is given by his Corollary 5.4. Consider, for simplicity, the one-dimensional case. Suppose F has mean zero. Then F is well known to be recurrent. But this result does not follow from Tweedie's conditions. His sufficient condition requires that

$$\int |y|\, dF(y - x) \le |x| \tag{13}$$

for all large $|x|$. Since F has mean zero,

$$\int y\, dF(y - x) = x.$$

Therefore, by Jensen's inequality,

$$\int |y|\, dF(y - x) \ge |x|.$$

Thus the inequality sign goes the wrong way and (13) cannot hold. So, in the context of our paper, Tweedie's conditions are too strong. It should be noted, however, that Tweedie's results

concern Markov chains on general spaces. In his set-up, chains on three or higher dimensional spaces can be recurrent. Consequently, the thrust of his results is away from space-homogeneous, renewal-type chains considered in this paper.

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