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Accurate Efficient Evaluation of Cumulative or Exceedance Probability Distributions Directly From Characteristic Functions

Albert H. Nuttall
Surface Ship Sonar Department

Naval Underwater Systems Center
Newport, Rhode Island / New London, Connecticut

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Preface

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The technical reviewer for this report was Dr. John P. Ianniello (Code 3212).

Reviewed and Approved: 1 October 1983

W. A. Von Winkle

W. A. Von Winkle
Associate Technical Director for Technology

The author of this report is located at the
New London Laboratory, Naval Underwater Systems Center,
New London, Connecticut 06320.

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An accurate and efficient method of evaluating the entire cumulative or exceedance probability distribution, via one fast Fourier transform of the sampled characteristic function, is presented. The sampling rate applied to the characteristic function results in aliasing of the probability density function, while the limited extent of the sampling gives rise to a systematic disturbance in the calculated probability distribution. Both types of errors are easily recognizable and can be controlled by a trial and error procedure whereby the calculated distributions are plotted for observation and modification.

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The size of the fast Fourier transform determines the number of distribution values available, but has no effect upon the accuracy of the result. Regardless of the number of characteristic function evaluations required for accurate results, the storage required is just that corresponding to the size of the fast Fourier transform.

A program for the procedure is presented, and the inputs required of the user are indicated. Several representative examples and plots illustrate the utility of the approach.

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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

x	random variable of primary interest
$f_x(\xi)$	characteristic function of random variable x
$p_x(v)$	probability density function of random variable x
b	bias or shift added to x
y	random variable $y=x+b$
$f_y(\xi)$	characteristic function of random variable y
μ_x	mean of random variable x
$p_y(v)$	probability density function of random variable y
$P_x(v)$	cumulative distribution function of random variable x
$P_y(v)$	cumulative distribution function of random variable y
$g(\xi,v)$	auxiliary function (6)
Im	imaginary part
μ_y	mean of random variable y
$C(v)$	right-hand side of (8)
Δ	sampling increment in argument ξ of characteristic function
$\tilde{p}_y(v)$	aliased version of $p_y(v)$; (12)
$\delta_\Delta(\xi)$	impulse train (13)
M	size of FFT employed
z_n	sequence of characteristic function samples; (20)
$\hat{z}_n$	collapsed sequence; (21)
L	limit on integral of characteristic function; (28)
N	number of nonzero z_n; $N=L/\Delta$
overbar	ensemble average

ACCURATE EFFICIENT EVALUATION OF CUMULATIVE OR EXCEEDANCE PROBABILITY DISTRIBUTIONS DIRECTLY FROM CHARACTERISTIC FUNCTIONS

INTRODUCTION

The performance of a signal processor can often be evaluated in terms of the characteristic function of the decision variable, either numerically or in closed form; see for example, refs. 1 and 2. However, a closed form for the corresponding probability density function or cumulative distribution function is seldom available, and numerical procedures must be employed. Several such procedures have been published in the literature, refs. 3-8. However they have limited accuracy or they require extensive storage or analytical manipulations and calculations.

We present a technique which is limited in accuracy only by the round-off noise of the computer or by the errors of the special functions required in the characteristic function calculation. The amount of storage depends only on the number of cumulative or exceedance distribution function values requested and does not influence the accuracy of the final probability values. The entire cumulative and exceedance distribution function values result as the output of one fast Fourier transform (FFT). The size of the FFT dictates the storage required and the spacing of the calculated probability values, but not their accuracy.

The addition and subtraction of integrand functions given in ref. 7 can be entirely circumvented and yet enable use of an FFT, through proper manipulation of the origin contribution of the characteristic function. Specific connections with past results will be noted at appropriate points in the derivations.

DERIVATION OF PROCEDURE

Shifted Random Variable

The primary random variable of interest is the real quantity x with given characteristic function $f_x(\xi)$ which is related to the probability density function p_x of random variable x via Fourier transform *

$$f_x(\xi) = \int dv \exp(i\xi v)\, p_x(v). \tag{1}$$

We define secondary random variable y as

$$y = x+b, \tag{2}$$

where bias (shift) b is a constant, chosen such that random variable y has insignificant probability of being less than zero. However, we also pick b as small as possible, so that the characteristic function of y,

$$f_y(\xi) = f_x(\xi) \exp(ib\xi), \tag{3}$$

will vary slowly with ξ. In fact, b can be negative, as for example if x were limited to values larger than some positive threshold. The approach here is not limited to positive random variables x, as were some of the results in ref. 7, but is applicable to any random variable distribution.

By way of example, for an exponential probability density function for random variable x, we choose b=0; while for a Gaussian random variable, $b=\mu_x+8\sigma_x$ yields a probability less than 1E-15 of y being negative. The probability density function of random variable y therefore appears as depicted in figure 1.

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* Integrals and sums without limits are over $(-\infty,+\infty)$.

Figure 1. Probability Density Function of Secondary Random Variable y

The cumulative distribution functions of random variables y and x are related according to

$$\int_{-\infty}^{v} dt\ p_y(t) = P_y(v) = P_x(v-b); \quad P_x(v) = P_y(v+b). \tag{4}$$

Thus we can inspect $P_x(v)$ in the neighborhood of $v=-b$ (the lower edge of interest of x) by looking at cumulative distribution function $P_y(v)$ in the neighborhood of $v=0$. More precisely, we will investigate $P_y(v)$ for values of v greater than zero, since this is the region of significant variation of $P_y(v)$; this is called the positive neighborhood of $v=0$.

<u>Approximation to Cumulative Distribution Function</u>

From ref. 4, eq. 7, we have the cumulative distribution function of random variable y in terms of the characteristic function according to

$$P_y(v) = \frac{1}{2} - \int_{0+}^{+\infty} d\xi\ g(\xi,v), \tag{5}$$

where we have defined auxiliary function

$$g(\xi,v) = \operatorname{Im}\left\{\exp(-i\xi v)\ \frac{f_y(\xi)}{\pi\xi}\right\}. \tag{6}$$

Observe for later use that

$$g(0+,v) = \lim_{\xi\to 0+} \operatorname{Im}\left\{(1-i\xi v)\ \frac{1+i\xi\mu_y}{\pi\xi}\right\} = \frac{\mu_y - v}{\pi}, \tag{7}$$

where μ_y is the mean of random variable y.

For v in the neighborhood of zero, $\exp(-i\xi v)$ in (6) varies slowly with ξ, and we have the approximation, via the Trapezoidal rule, to (5) as

$$P_y(v) \cong \frac{1}{2} - \frac{\Delta}{2}\, g(0+,v) - \sum_{n=1}^{+\infty} \Delta\, g(n\Delta,v) \equiv C(v), \tag{8}$$

where the right-hand side of (8) has been defined as $C(v)$. Here, Δ is the sampling interval in ξ, and is small enough to track changes in $\exp(-i\xi v)*f_y(\xi)/\xi$. We choose the Trapezoidal rule in (8) over other integration rules, such as Simpson's rule, because it results in minimum aliasing for Fourier transforms relative to all other rules; see appendix A for elaboration and proof.

Observe from (8) that

$$P_y(0) \cong C(0) \quad \text{means} \quad C(0) \cong 0, \tag{9}$$

since $P_y(0)$ is insignificant by the choice of b in (2); this relation will be used later.

Relationship of Approximation

Although we want to evaluate the exact cumulative distribution function $P_y(v)$, we have instead arrived at an approximation $C(v)$ via (8). How are these two related? To determine the relationship, we manipulate (6)-(8) as follows:

$$C(v) = \frac{1}{2} + \frac{\Delta}{2}\,\frac{v-\mu_y}{\pi} - \sum_{n=1}^{+\infty} \operatorname{Im}\left\{\exp(-in\Delta v)\,\frac{f_y(n\Delta)}{\pi n}\right\} =$$

$$= \frac{1}{2} + \frac{\Delta}{2}\,\frac{v-\mu_y}{\pi} - \operatorname{Im}\left\{\sum_{n=1}^{+\infty}\exp(-in\Delta v)\,\frac{f_y(n\Delta)}{\pi n}\right\} = \tag{10}$$

$$= \frac{1}{2} + \frac{\Delta}{2}\,\frac{v-\mu_y}{\pi} - \frac{1}{i2\pi}\sum_{n\neq 0}\exp(-in\Delta v)\,\frac{f_y(n\Delta)}{n}. \tag{11}$$

The removal of the imaginary operation from within the summation in (10) is a crucial step; it does not create a problem in divergence since $n > 0$. This is in contrast with the integral of (5) and (6), where removal of the imaginary operation would create a divergent integral. This postponement of the removal of the imaginary operation, until after the approximation to the integral was developed in (8), is the major difference with the results in ref. 7.

Taking a derivative of (11), we obtain

$$C'(v) = \frac{\Delta}{2\pi} + \frac{\Delta}{2\pi} \sum_{n \neq 0} \exp(-in\Delta v)\, f_y(n\Delta) =$$

$$= \frac{\Delta}{2\pi} \sum_{n} \exp(-in\Delta v)\, f_y(n\Delta) =$$

$$= \frac{1}{2\pi} \int d\xi\, \exp(-i\xi v)\, f_y(\xi)\, \Delta \delta_\Delta(\xi) =$$

$$= p_y(v) \circledast \delta_{\frac{2\pi}{\Delta}}(v) = \sum_n p_y\left(v - n\,\frac{2\pi}{\Delta}\right) \equiv \tilde{p}_y(v), \tag{12}$$

where infinite impulse train

$$\delta_\Delta(\xi) = \sum_n \delta(\xi - n\Delta), \tag{13}$$

where $\circledast$ denotes convolution, and where we have used the relation

$$\frac{1}{2\pi} \int dt\, \exp(-i\omega t)\, \Delta \delta_\Delta(t) = \delta_{\frac{2\pi}{\Delta}}(\omega). \tag{14}$$

This last result follows from ref. 9, p. 28, rule 11, with $u(t) = \delta(t)$, $T=\Delta$, $F=1/T$, and $\omega=2\pi f$. Relation (12) indicates that $C'(v)$ is an infinitely aliased version of the probability density function $p_y(v)$, with resultant period $2\pi/\Delta$ in v. For small enough sampling increment Δ in (8), there will be very little overlap of the displaced versions of p_y in (12), thereby yielding the good approximation

$$\tilde{p}_y(v) = p_y(v) \quad \text{for} \quad 0 \leq v \leq 2\pi/\Delta. \tag{15}$$

The situation for relation (12) is depicted in figure 2.

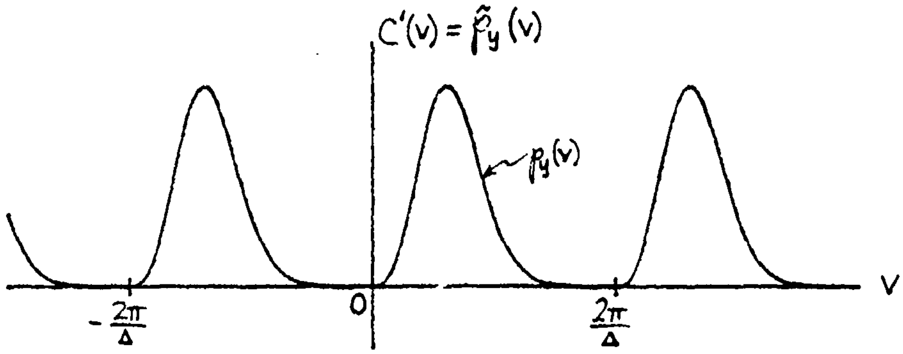

Figure 2. Infinitely Aliased Probability Density Function $\tilde{p}_y(v)$

There now follows from (12),

$$C(v) = C(0) + \int_0^v du\, \tilde{p}_y(u) \equiv C(0) + \tilde{P}_y(v), \tag{16}$$

where C(0) is given by (10) as

$$C(0) = \frac{1}{2} - \frac{\Delta\mu_y}{2\pi} - \mathrm{Im}\left\{\sum_{n=1}^{+\infty} \frac{f_y(n\Delta)}{\pi n}\right\}. \tag{17}$$

Relation (16) is an exact relation, showing that C(v) is the integral of the infinitely aliased version of $p_y(v)$, starting at v=0, plus an additive constant which is substantially zero; see (9).

So for v in the positive neighborhood of zero, (4), (8), (16), and (9) yield

$$P_x(v-b) = P_y(v) \cong C(v) = C(0) + \tilde{P}_y(v) \cong \tilde{P}_y(v). \tag{18}$$

Thus the quantity we want, the left-most term in (18), is well-approximated by calculated quantity C(v), which itself is approximately the integral of the infinitely aliased version of $p_y(v)$.

Calculation of C(v)

Let $v = \frac{2\pi k}{M\Delta}$ in (10), where M and k are arbitrary integers. Then

$$C\left(\frac{2\pi k}{M\Delta}\right) = \frac{1}{2} + \frac{k}{M} - \frac{\Delta\mu_y}{2\pi} - \mathrm{Im}\left\{\sum_{n=1}^{+\infty} \exp(-i2\pi nk/M) \frac{f_y(n\Delta)}{\pi n}\right\} =$$

$$= \frac{1}{2} + \frac{k}{M} - \frac{1}{\pi}\,\mathrm{Im}\left\{\sum_{n=0}^{+\infty} \exp(-i2\pi nk/M)\; z_n\right\}, \qquad (19)$$

where we define complex sequence

$$z_n = \left\{\begin{array}{lll} i\,\frac{1}{2}\Delta\mu_y & \text{for} & n=0 \\ f_y(n\Delta)/n & \text{for} & n\geq 1 \end{array}\right\}. \qquad (20)$$

Now define collapsed sequence (ref. 7, pp. 13-16) as

$$\hat{z}_n = \sum_{j=0}^{+\infty} z_{n+Mj} \quad \text{for} \quad 0 \leq n \leq M-1. \qquad (21)$$

Then since z_n receives the same weight as z_{n+Mj} in (19), regardless of the value of k, (19) can be expressed as

$$C\left(\frac{2\pi k}{M\Delta}\right) = \frac{1}{2} + \frac{k}{M} - \frac{1}{\pi}\,\mathrm{Im}\left\{\sum_{n=0}^{M-1} \exp(-i2\pi nk/M)\; \hat{z}_n\right\}. \qquad (22)$$

Relation (22) is exact and valid for all k. Since we are only interested in the positive neighborhood of v=0 in (18), we confine attention in (22) to $0 \leq k \leq M-1$.* Relation (22) can then be accomplished by an M-point FFT if M is chosen to be a power of 2. Notice that storage only for the M complex numbers $\{\hat{z}_n\}$ in (21) is required, even though the $\{z_n\}$ sequence in (20) is of infinite length.

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* Values for other k are available from (22) when we observe that $C\left(\frac{2\pi(M+k)}{M\Delta}\right) = 1 + C\left(\frac{2\pi k}{M\Delta}\right)$ for all k.

Observe that the size of M in no way affects the error of the calculation of $C(\frac{2\pi k}{M\Delta})$ or estimation of $P_y(v)$. Rather, M specifies the spacing at which $C(\frac{2\pi k}{M\Delta})$ is calculated, and can be coarse if desired. The accuracy of the estimate of $P_y(v)$ is governed thus far by Δ, through the aliasing depicted in figure 2.

Reference to (18) now yields

$$P_x(\frac{2\pi k}{M\Delta} - b) \cong C(\frac{2\pi k}{M\Delta}) \quad \text{for} \quad 0 \leq k \leq M-1, \tag{23}$$

where the latter quantity is given by (22). Thus the M-point FFT sweeps out the argument range $(-b,-b+2\pi/\Delta)$ for the cumulative distribution function P_x.

If we want the exceedance distribution function of y instead of the cumulative distribution function, we use (18) and (22) to get

$$1 - C(\frac{2\pi k}{M\Delta}) = \frac{1}{2} - \frac{k}{M} + \frac{1}{\pi}\,\mathrm{Im}\left\{\sum_{n=0}^{M-1} \exp(-i2\pi nk/M)\ \hat{z}_n\right\} \text{ for } 0 \leq k \leq M-1. \tag{24}$$

(By the footnote to (22), we have $1-C(2\pi/\Delta) = -C(0)$.)

Since μ_y must be known in (20) in order to use this approach, we need the mean μ_x of random variable x, since from (2)

$$\mu_y = \mu_x + b. \tag{25}$$

The quantity μ_x can be found analytically from characteristic function $f_x(\xi)$ according to

$$f_x'(0) = i\mu_x; \tag{26}$$

see (1).

In addition to the error caused by aliasing associated with nonzero sampling increment Δ, an additional error occurs because we cannot calculate all the coefficients $\{z_n\}$ in (20) and (21) out to $n=+\infty$. Rather, we terminate the calculation at integer $n=N$, such that $|z_n|$ is sufficiently small as to be negligible for $n \geq N$. Letting

$$L = N\Delta, \tag{27}$$

this is equivalent to ignoring the contribution to (5) of the tail error

$$-\int_L^{+\infty} d\xi \; g(\xi,v) = -\mathrm{Im} \int_L^{+\infty} d\xi \; \exp(-i\xi v) \, \frac{f_y(\xi)}{\pi\xi}. \tag{28}$$

If the asymptotic behavior of $f_y(\xi)$ for large ξ is known, this error can sometimes be evaluated in closed form and used to ascertain an adequate value of L. Instead, we have observed that tail error (28) causes a characteristic low-level sinusoidal variation in the calculated cumulative distribution function for small v near 0, and in the calculated exceedance distribution function for large v near $2\pi/\Delta$. When this sinusoidal variation is deemed excessive, L can be increased until the effect disappears or decreases to acceptable levels. This trial and error approach avoids the necessity of analytically upper bounding the magnitude of error (28), which is often very tedious and generally pessimistic.

So there are two errors to be concerned with: aliasing due to nonzero sampling interval Δ and tail error due to non infinite limit L. Later examples will demonstrate how these errors manifest themselves in the cumulative and exceedance distribution functions and how they can be controlled by a trial and error approach.

Relation to Requicha's Method, ref. 5

From ref. 5, eqs. 7, 9, 10, the cumulative distribution function is given by an expression that can be manipulated into the form (using current notation)

$$F_k = \frac{k}{M} - \frac{1}{\pi}\,\mathrm{Im}\left\{\sum_{n=1}^{M/2} \exp(-i2\pi kn/M)\,\frac{f_y(n\Delta)}{n}\right\} + \frac{1}{\pi}\,\mathrm{Im}\left\{\sum_{n=1}^{M/2} \frac{f_y(n\Delta)}{n}\right\}. \tag{29}$$

Although this is similar to the upper line of (19) here, it differs in several important respects:

1. F_k does not use mean μ_y at all; it is therefore not using a direct approximation to the specified integral in (5) and (6).

2. From (29), there follows $F_0 = 0$, $F_M = 1$; however, these results are not strictly true for the actual cumulative distribution function at these end points, thereby leading to poor estimates in the neighborhoods of these points. This is due to the arbitrary origin established in ref. 5, eq. 6.

3. The sums in (29) utilize characteristic function samples $f_y(n\Delta)$ only for $n \leq M/2$, where M is the size of the FFT. This is a very severe and unnecessary restriction; in fact, the sum on n in (29) ought to be conducted to the point where the tail contribution, (28), is negligible, regardless of the value of M.

4. In ref. 5, if eq. 4 is substituted into eq. 1, and the summation limits are extended to $\pm\infty$, we get exactly the second line of (12) here. When the probability density function is integrated to get the cumulative distribution function in ref. 5, eq. 6, the resultant cumulative distribution function is arbitrarily set to zero at v=0. We instead have from (9) and (17),

$$P_y(0) \cong C(0) = \frac{1}{2} - \frac{\Delta\mu_y}{2\pi} - \frac{1}{\pi}\,\mathrm{Im}\left\{\sum_{n=1}^{+\infty} \frac{f_y(n\Delta)}{n}\right\}, \tag{30}$$

which is small, but not necessarily zero. This consideration is very important on the tails of the cumulative and exceedance distribution functions.

Summary of Procedure

The cumulative distribution function of y is given by

$$P_y\left(\frac{2\pi k}{M\Delta}\right) \cong C\left(\frac{2\pi k}{M\Delta}\right) = \frac{1}{2} + \frac{k}{M} - \frac{1}{\pi}\,\mathrm{Im}\left\{\sum_{n=0}^{M-1} \exp(-i2\pi nk/M)\,\hat{z}_n\right\}$$

$$\text{for} \quad 0 \leq k \leq M-1, \tag{31}$$

where M is the size of the FFT and storage employed. Also

$$\hat{z}_n = \sum_{j=0}^{+\infty} z_{n+Mj} \quad \text{for} \quad 0 \leq n \leq M-1, \tag{32}$$

where

$$z_n = \begin{Bmatrix} i\frac{1}{2}\Delta\mu_y & \text{for} \quad n=0 \\ f_y(n\Delta)/n & \text{for } 1 \leq n \leq N \\ 0 & \text{for } n > N \end{Bmatrix}. \tag{33}$$

(The value for n=N should be scaled by 1/2 for the Trapezoidal rule). The zero values for z_n, when $n > N$, serve to terminate the collapsed sum in (32) at a finite upper limit. The value of N is given by the integer part of L/Δ, where Δ and L must be chosen so as to minimize aliasing and tail error, respectively. The characteristic function of random variable y needed in (33) is given by

$$f_y(\xi) = f_x(\xi)\exp(ib\xi), \tag{34}$$

in terms of the characteristic function of the primary random variable x, where shift b must be chosen such that $y = b+x$ is positive with probabilty virtually 1. The mean $\mu_y = b+\mu_x$ can be determined analytically from knowledge of characteristic function $f_x(\xi)$. Finally, the exceedance distribution function for random variable y is obtained by subtracting (31) from 1.

EXAMPLES

Programs for the following five examples are listed in appendix B.

1. Chi-Square

A chi-square variate of 2K degrees of freedom has probability density function (ref. 10)

$$p_x(v) = \frac{v^{K-1}\exp(-v/2)}{2^K (K-1)!} \quad \text{for } v > 0 \tag{35}$$

and characteristic function

$$f_x(\xi) = (1-i2\xi)^{-K}. \tag{36}$$

Since random variable x is obviously nonnegative by (35), we can choose shift b=0; i.e. y=x. A plot of the cumulative and exceedance distribution functions of random variable y obtained from characteristic function (36) with K=4 is given in figure 3 for $0 \leq v \leq 2\pi/\Delta$. The values of Δ and L have been chosen such that aliasing and tail error are insignificant.

The ordinate scale for figure 3 is a logarithmic one. The lower right end of the exceedance distribution function curve decreases smoothly to the region 1E-11, where round-off noise is encountered. The exceedance distribution function values continue to decrease with v until, finally, negative values (due to round-off noise) are generated. For negative probability values, the logarithm of the absolute value is plotted, but mirrored below the 1E-12 level. These values have no physical significance, of course; they are plotted to illustrate the level of accuracy attainable by this procedure with appropriate choices of Δ and L.

For this example, $N=L/\Delta=2666$, while M=256. Thus collapsing, according to (21) or (32), by over a factor of 10 has been employed and a small size FFT has been utilized. Nevertheless the error realized for the cumulative and exceedance distribution functions is in the 1E-12 range, the limit of accuracy of the Hewlett Packard 9845B Desk Calculator used here. Finer spacing in the distribution outputs is achievable by merely increasing M.

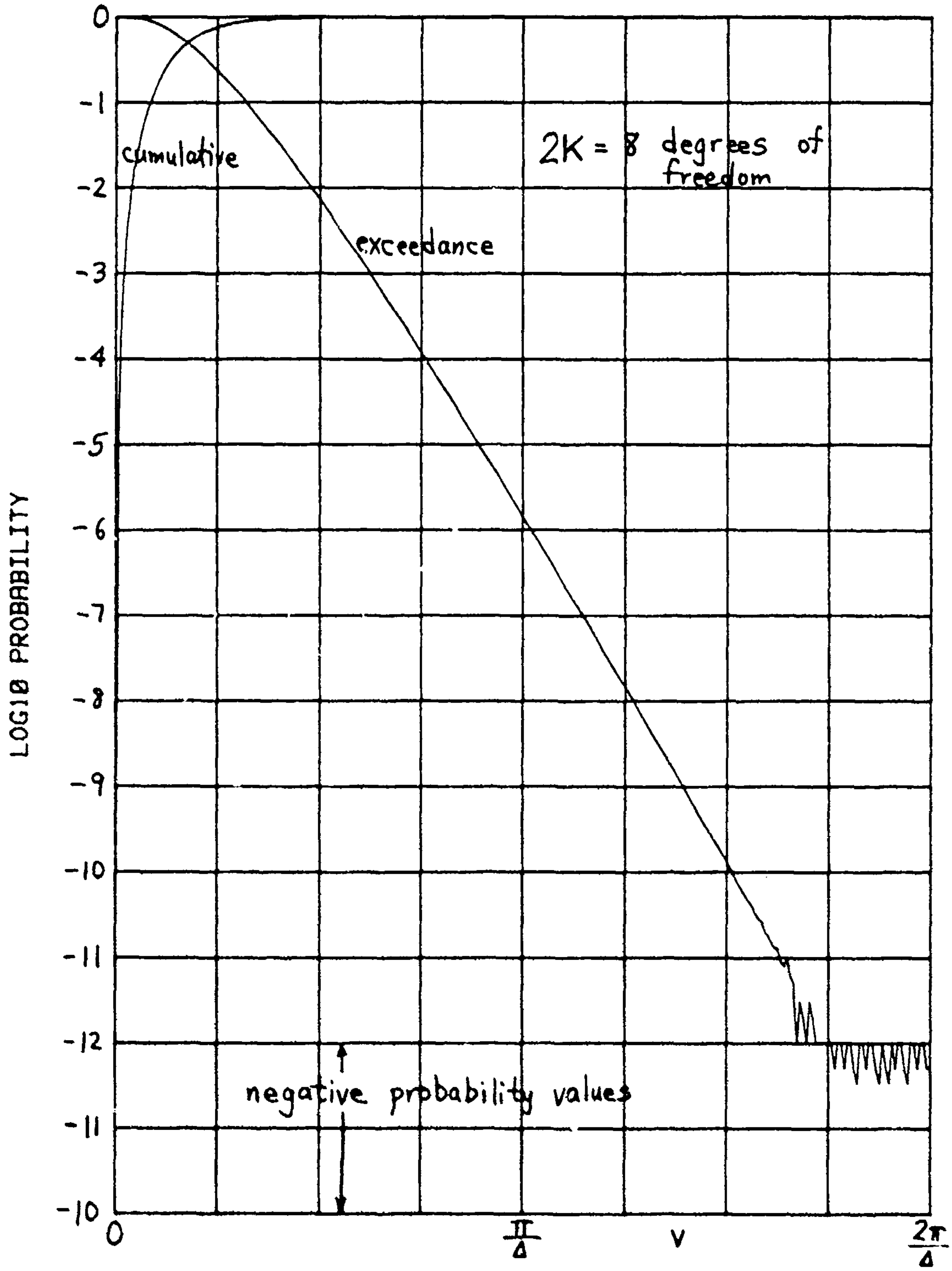

Figure 3. Chi-Square; L=200, Δ=.075, b=0, M=256

2. Gaussian

The characteristic function for a zero-mean unit-variance random variable is

$$f_x(\xi) = \exp(-\xi^2/2), \tag{37}$$

and the probability density function and cumulative distribution function are (ref. 11, eq. 10.5.3)

$$p_x(v) = (2\pi)^{-1/2} \exp(-v^2/2), \; P_x(v) = \Phi(v). \tag{38}$$

For $b = 5\pi/2$, using (4),

$$P_y(0) = P_x(-b) = \Phi(-b) = 2E{-}15. \tag{39}$$

which is negligible, as desired.

Plots of the cumulative and exceedance distribution functions for random variable y are given in figure 4 for L=7, Δ=.3. The logarithmic ordinate gives rise to the characteristic parabolic shape on the tails of the distributions. Once again, the probabilities decrease to the level of the round-off noise and fluctuate around 1E-12 near the edges of the fundamental aliased interval $(0,2\pi/\Delta)$. The fact that the cumulative distribution function of y starts in the round-off noise at v=0 indicates that $b=5\pi/2$ was large enough to guarantee $y > 0$ with probability virtually 1. Also indicated on the figure is the origin for random variable x. We have, from (4),

$$P_x(u) = P_y(u+b); \tag{40}$$

thus for example

$$\text{Prob}(x < 0) = P_x(0) = P_y(b) = .5. \tag{41}$$

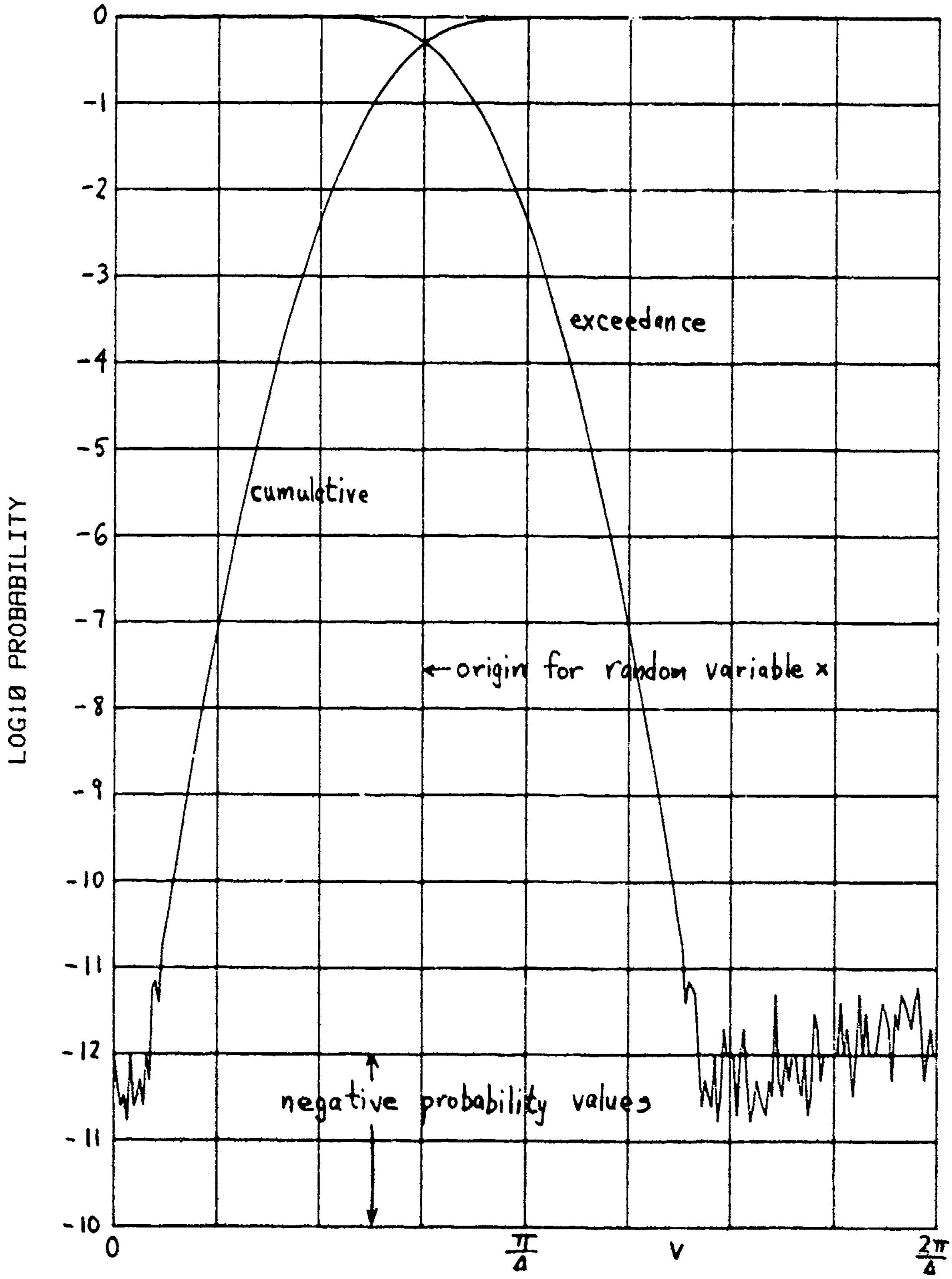

Figure 4. Gaussian; L=7, Δ=.3, b=2.5π, M=256

In figure 5, the only change is to decrease limit L from 7 to 6. The tail error mentioned in (28) et seq. then dominates the round-off noise and has a sinusoidal variation. Aliasing is not a problem, as witnessed by the fact that the cumulative and exceedance distribution functions of random variable y have decayed below 1E-12 well before the edges of the interval are reached.

When limit L is restored to 7, and sampling increment Δ is increased to .5, aliasing becomes significant, as shown in figure 6. The exceedance distribution function has not yet decayed to the round-off noise level at $v=2\pi/\Delta$, and the cumulative distribution function shows a large negative probability region near $v=0$. Shift b has been maintained at the value $5\pi/2$, corresponding to (39).

When L and Δ are restored to their values 7 and .3 as for figure 4, but b is decreased to $5\pi/3$, the probability of y becoming negative is, from (4) and (38), $\Phi(-5\pi/3) = .82E-7$. This is reflected in the cumulative distribution function for y in figure 7 at $v=0$, where the probability value is well above the round-off noise level. Also, the exceedance distribution function develops significantly negative values near $v = 2\pi/\Delta$.

Accurate evaluation of the cumulative and exceedance distribution functions can only be achieved when L, Δ, and b are properly chosen. Probably the optimum combination for the Gaussian variate is displayed in figure 8, where Δ has been increased to .4, the distributions are centered on the fundamental aliased interval $(0, 2\pi/\Delta)$ by choice of b, and L is taken at 7 to avoid tail error.

3. Smirnov

The limiting characteristic function of a measure of goodness of fit based on the sample distribution function was derived by Smirnov and is given by (ref. 12, eq. 30.104)

$$f_x(\xi) = \left(\frac{s}{\sin(s)}\right)^{1/2} \text{ where } s = (1+i)\sqrt{\xi} \text{ for } \xi \geq 0. \tag{42}$$

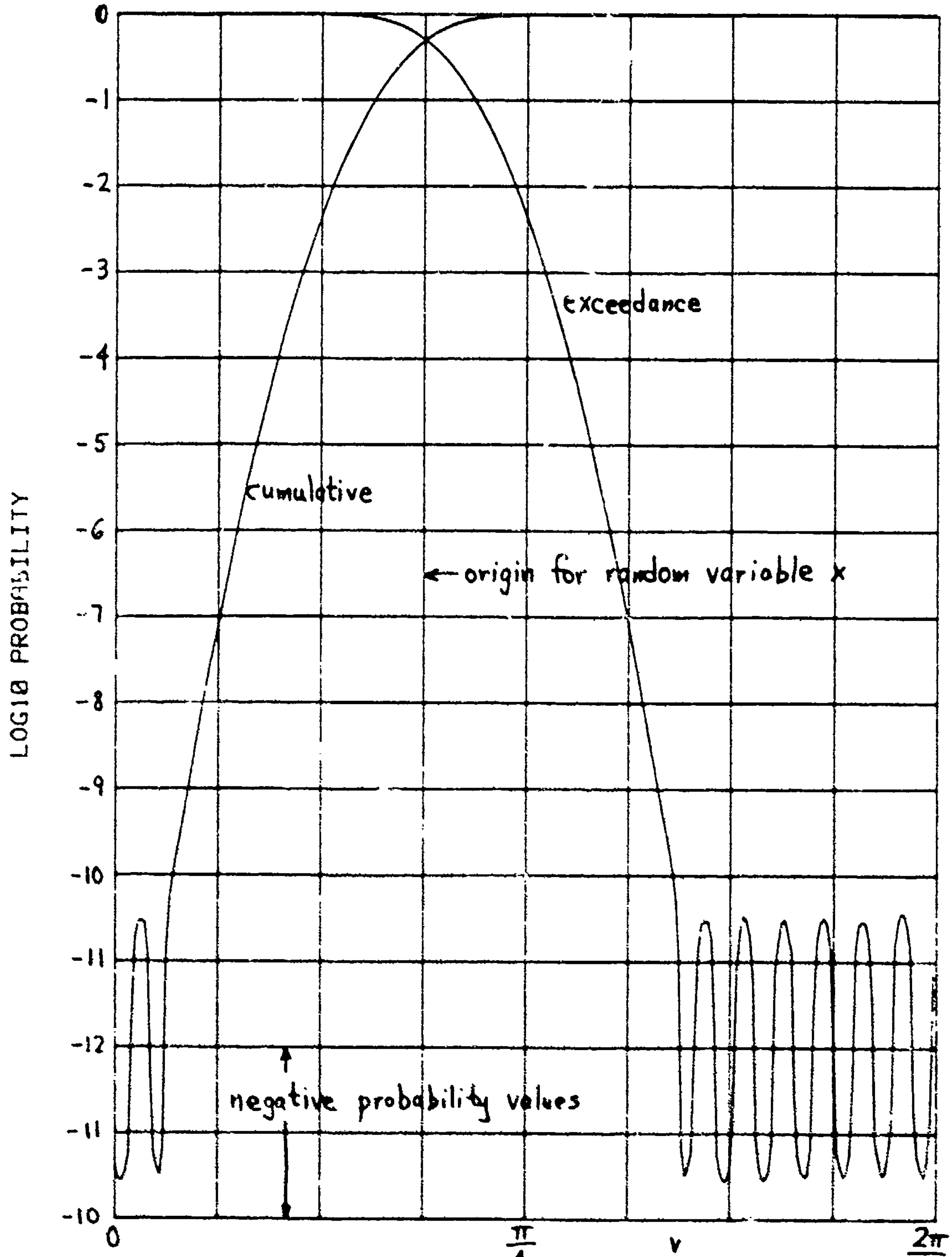

Figure 5. Gaussian; L=6, Δ=.3, b=2.5π, M=256

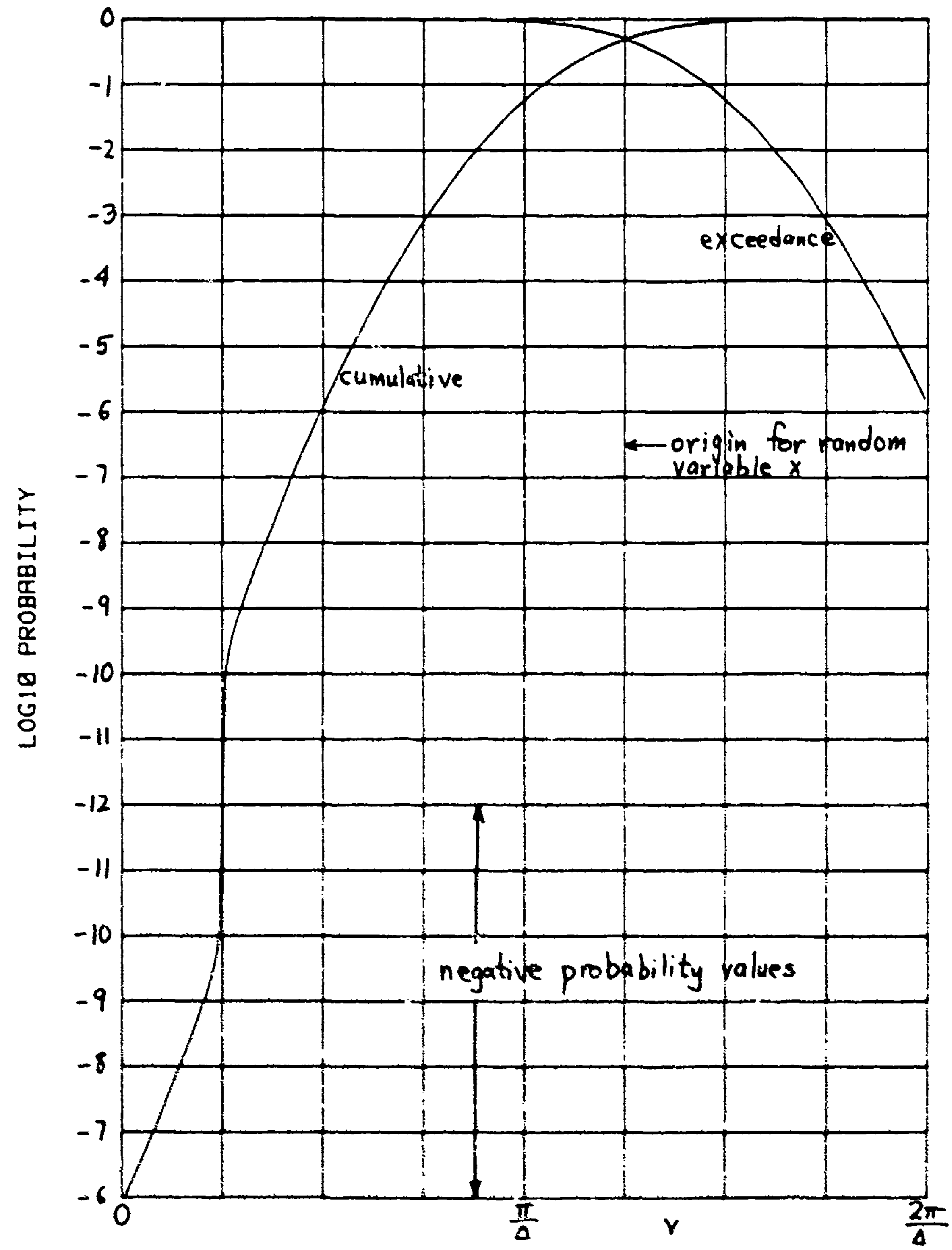

Figure 6. Gaussian; L=7, Δ=.5, b=2.5π, M=256

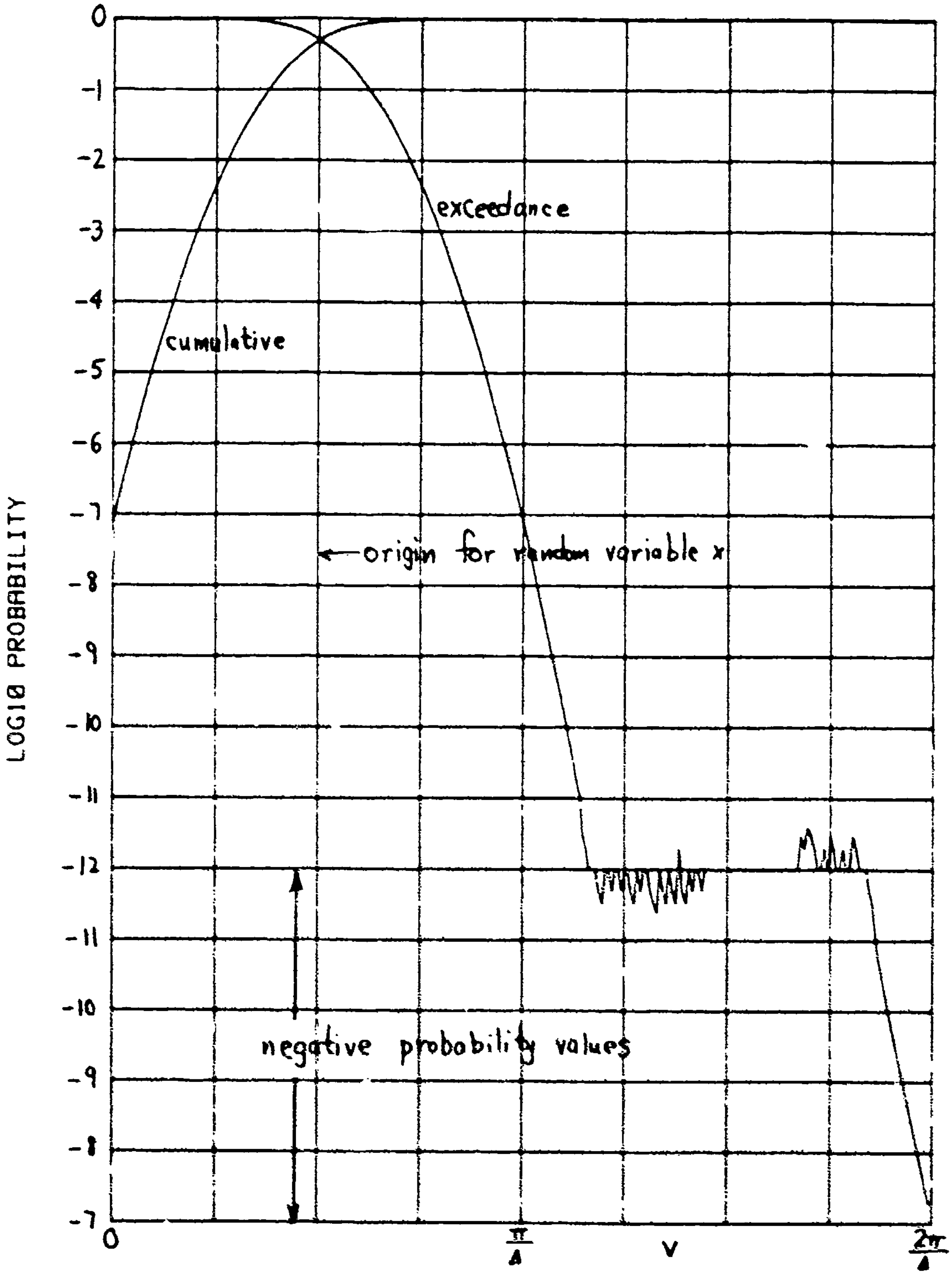

Figure 7. Gaussian; L=7, Δ=.3, b=5π/3, M=256

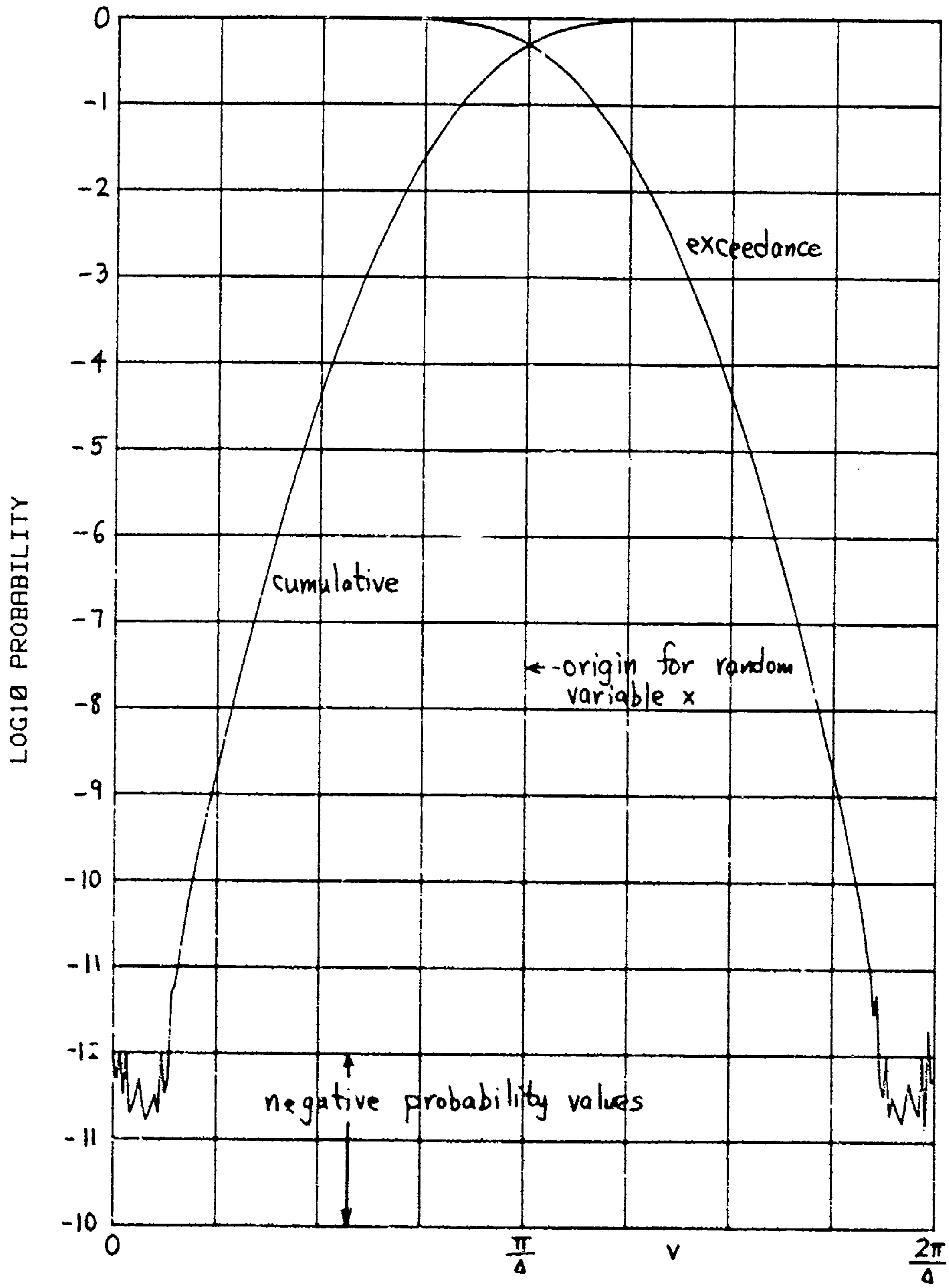

Figure 8. Gaussian; L=7, Δ=.4, b=2.5π, M=256

An expansion about $\xi=0$ yields

$$f_x(\xi) = 1+i\,\frac{1}{6}\xi - \frac{1}{40}\xi^2; \quad \text{i.e., } \mu_x = 1/6,\ \sigma_x^2 = 1/45. \tag{43}$$

And since the goodness of fit is always positive, random variable x is positive and we can choose

$$b=0. \tag{44}$$

Since

$$\sin((1+i)\sqrt{\xi}) \sim i\frac{1}{2}\exp(\sqrt{\xi}(1-i)) \quad \text{as } \xi\to+\infty, \tag{45}$$

it follows that

$$f_x(\xi) \sim 2^{3/4}\xi^{1/4}\exp\left(-\frac{1}{2}\sqrt{\xi} + i\left(\frac{1}{2}\sqrt{\xi} - \frac{\pi}{8}\right)\right) \text{ as } \xi\to+\infty. \tag{46}$$

The phase of this term rotates according to $\sqrt{\xi}/2$; if we were to choose $b\neq0$, $f_y(\xi)$ would rotate faster than $f_x(\xi)$ (linear with ξ rather than $\sqrt{\xi}$). This could necessitate a faster sampling rate, which is undesirable.

The cumulative and exceedance distribution functions are plotted in figure 9. L and Δ have been chosen so as to avoid tail error and aliasing. The exceedance distribution function is seen to decay exponentially until it reaches approximately 2E-11; the bump in the curve at this point is a manifestation of the limited accuracy of the trigonometric functions built into the calculator employed. Larger values of v lead to round-off noise around the 1E-12 level.

A comparison of results for this characteristic function, with Requicha's method described in (29) et seq., is given in figure 10 for FFT size M=1024. The plot labeled with N=L=512 is precisely Requicha's method. Aliasing is known to be insignificant for $\Delta=1$, as seen by reference to figure 9 and observing that extrapolation of the straight line section of the exceedance distribution function would result in probability values near 1E-13 at $v=2\pi/\Delta$. The dashed portion of the N=L=512 curve in figure 10 in fact

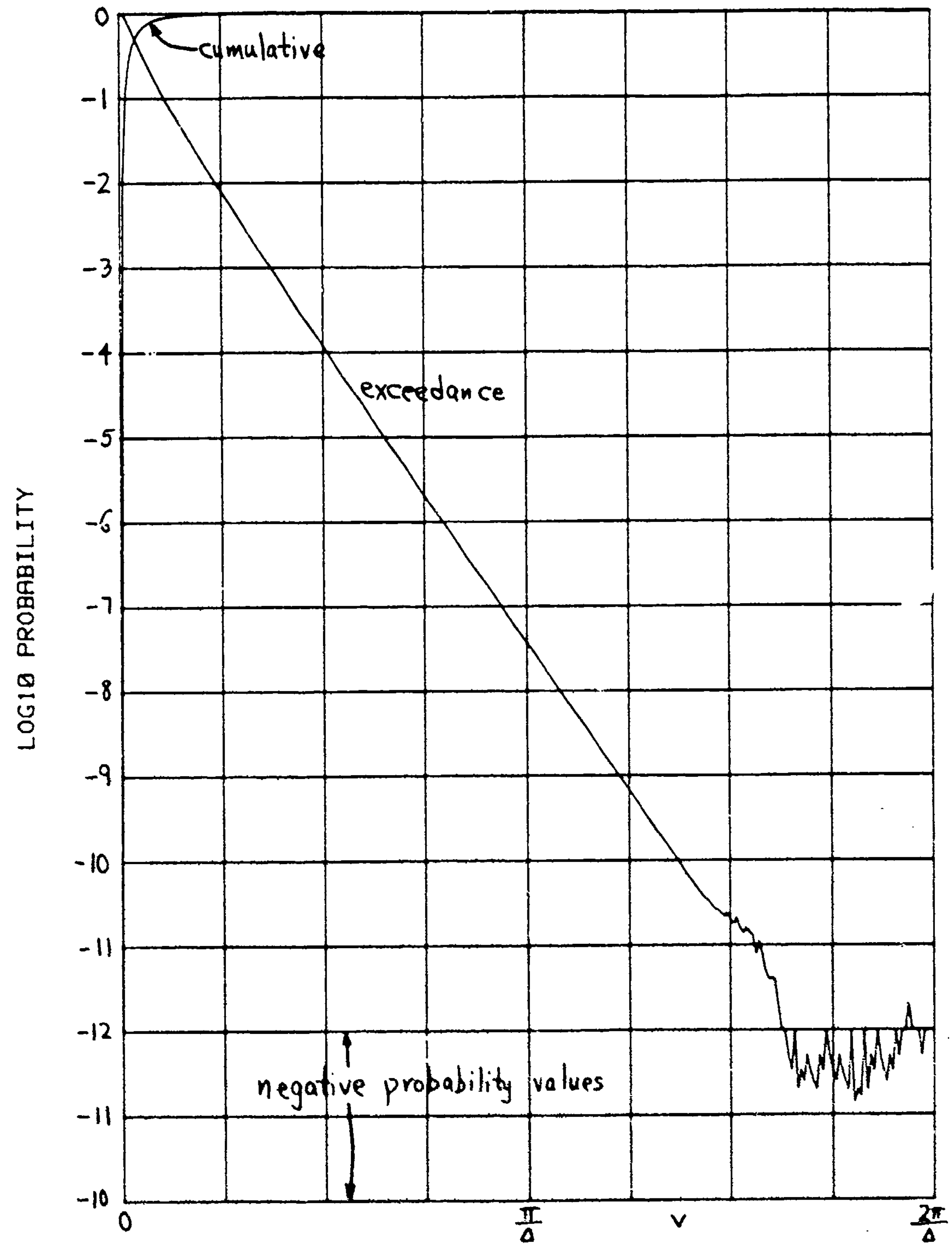

Figure 9. Smirnov; L=3000, Δ=1, b=0, M=256

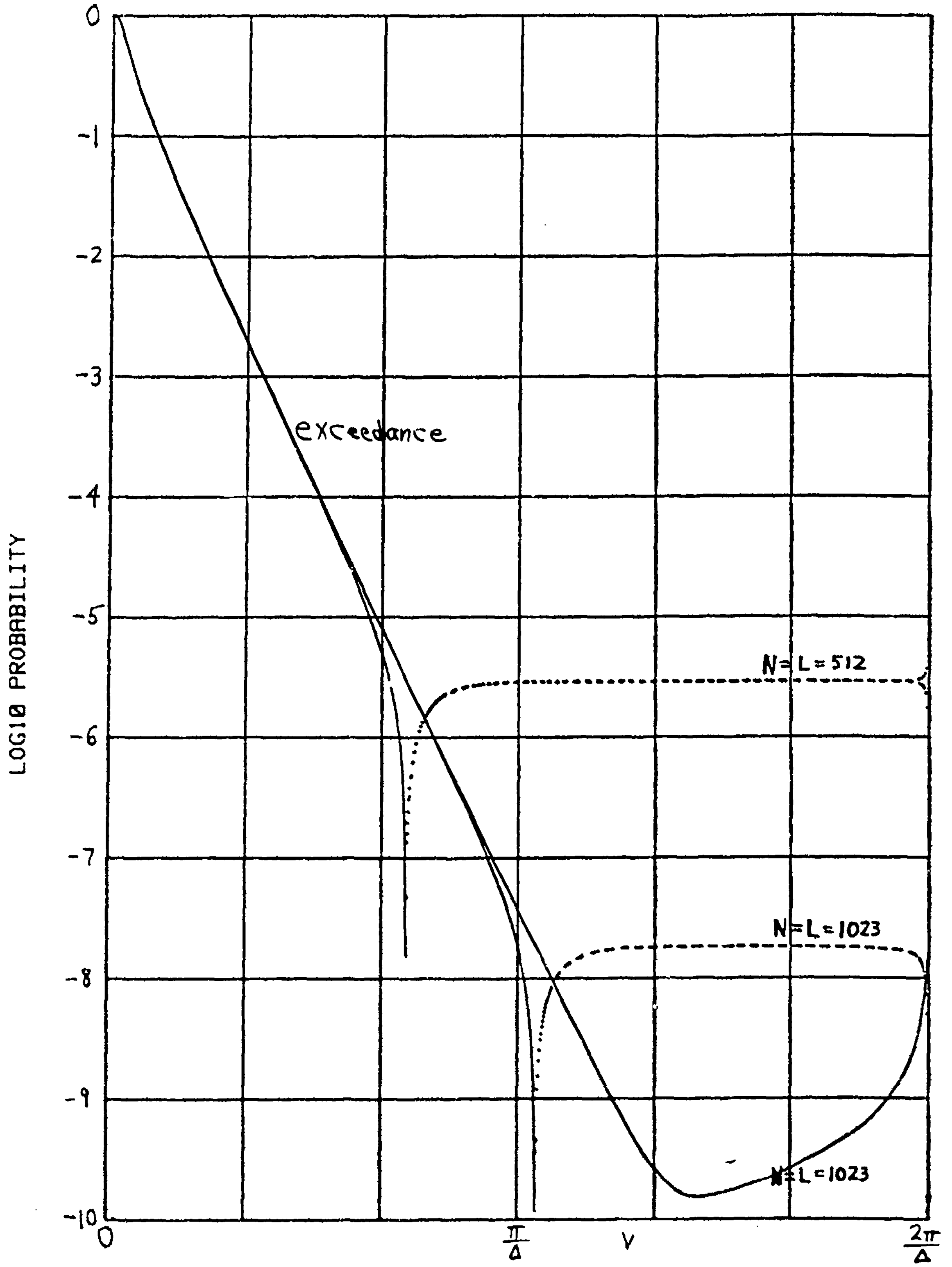

Figure 10. Comparison with Requicha's Method;
$\Delta=1$, $b=0$, $M=1024$

corresponds to negative probability estimates; these grossly inaccurate results are due to an inadequate value of limit L, leading to large tail error.

When N is simply increased to 1023, the middle curve in figure 10 results from Requicha's method. Again, negative estimates are indicated by the dashed portion of this curve, although two orders of magnitude smaller than above. The reasons for these errors have been delineated in (29) et seq.

The bottom-most curve in figure 10 (solid curve) is that obtained by the method proposed in this report for L = 1023. Exceedance distribution function estimates in the 1E-10 range are obtained, but the error returns to the 1E-8 range at $v=2\pi/\Delta$. No negative probability values occur. Also, by simply increasing limit L, while keeping FFT size M fixed, the error can be reduced significantly further, as already witnessed by figure 9.

4. Noncentral Chi-Square

Here the random variable x is given by

$$x = \sum_{k=1}^{K} (g_k + d_k)^2 \tag{47}$$

where $\{d_k\}$ are constants, and $\{g_k\}$ are independent Gaussian random variables with zero-mean and unit variance. The characteristic function of x is

$$f_x(\xi) = (1-i2\xi)^{-K/2} \exp\left(\frac{id^2\xi}{1-i2\xi}\right), \tag{48}$$

where deflection d is defined according to

$$d^2 = \sum_{k=1}^{K} d_k^2. \tag{49}$$

We actually consider a more general characteristic function than (48), namely

$$f_x(\xi) = (1-i2\xi)^{-\nu} \exp\left(\frac{id^2\xi}{1-i2\xi}\right) = \exp\left(\frac{id^2\xi}{1-i2\xi} - \nu \ln(1-i2\xi)\right), \tag{50}$$

where ν is an arbitrary positive real constant. Suppose that we use the principal value logarithm for $\ln(z)$, where the branch cut lies along the negative real axis of the complex z plane (ref. 13, sect. 4.1.1). Then since the argument of the logarithm in (50) never crosses the branch cut, form (50) gives the correct characteristic function values automatically for all real ξ, and any ν.

The probability density function and exceedance distribution function corresponding to (50) are (ref. 14, 6.631 4)

$$p_x(v) = \frac{1}{2} \exp\left(-\frac{d^2+v}{2}\right)\left(\frac{\sqrt{v}}{d}\right)^{\nu-1} I_{\nu-1}(d\sqrt{v}) \quad \text{for} \quad v > 0,$$

$$1 - P_x(v) = \int_{\sqrt{v}}^{+\infty} dt\ t \exp\left(-\frac{d^2+t^2}{2}\right)\left(\frac{t}{d}\right)^{\nu-1} I_{\nu-1}(dt) \equiv Q_\nu(d,\sqrt{v}) \quad \text{for} \quad v > 0. \tag{51}$$

Since the probability density function in (51) is never negative (ref. 13, sect. 9.6.1), (50) is a legal characteristic function. Also because random variable x is always positive according to (51), we choose shift b=0. Plots of the exceedance distribution function, as determined from characteristic function (50) are displayed for various values of d in figure 11. The values of L were chosen for each d value so as to control the tail error below the 1E-10 level plotted. Direct calculation of the exceedance distribution function directly from (51) would be a formidable task for arbitrary ν values.

5. Product of Correlated Gaussian Variates

Let

$$x = st \tag{52}$$

where s and t are zero-mean unit-variance Gaussian random variables with correlation coefficient ρ. The joint probability density function of s and t is

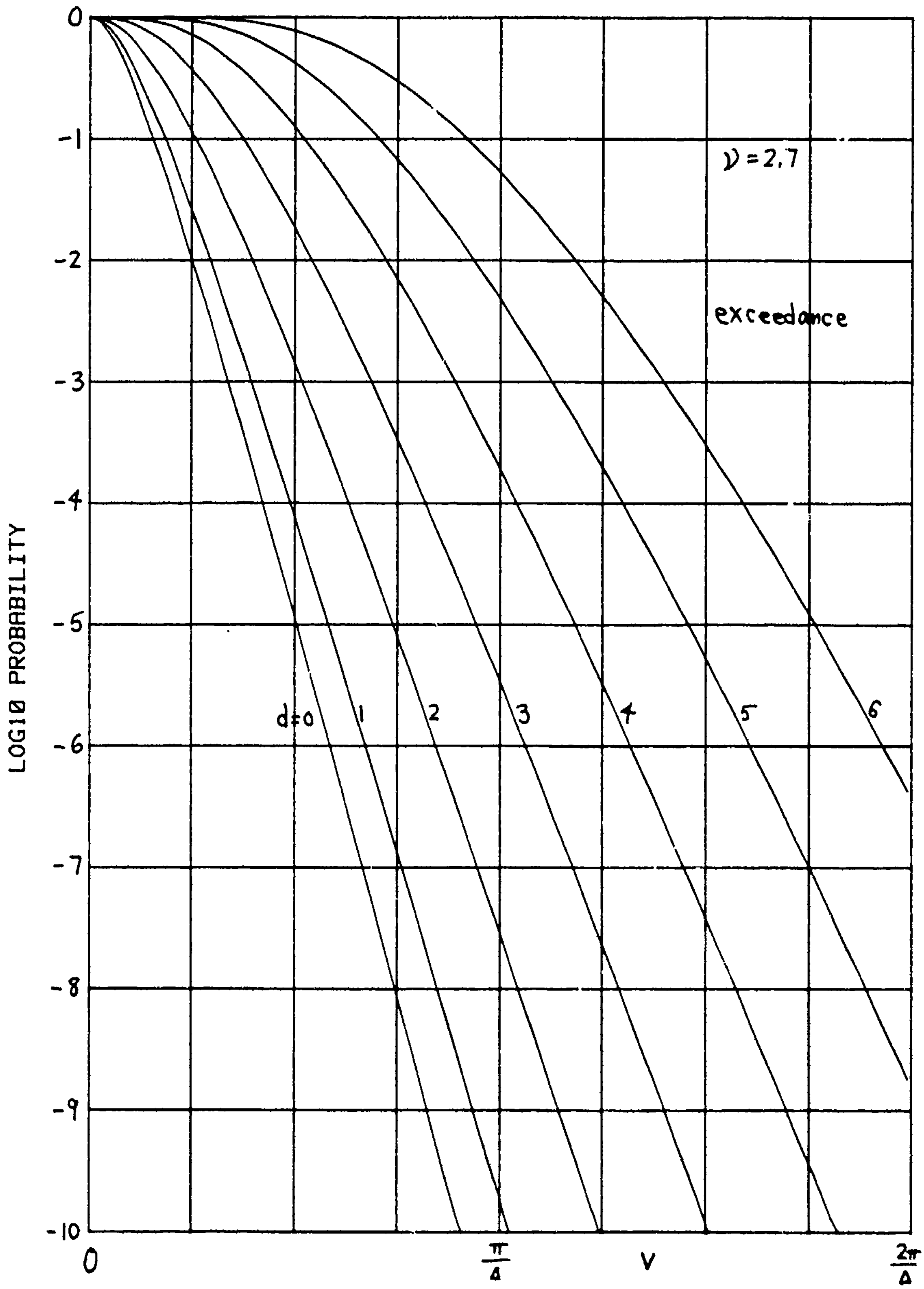

Figure 11. Non-Central Chi-Square; $\Delta = .05$, $b=0$, $M=256$

$$p_{st}(u,v) = \left(2\pi\sqrt{1-\rho^2}\right)^{-1} \exp\left[-\frac{u^2+v^2-2\rho uv}{2(1-\rho^2)}\right]. \tag{53}$$

The characteristic function of x is then

$$f_x(\xi) = \overline{\exp(i\xi st)} = \iint du\, dv \exp(i\xi uv)\, p_{st}(u,v) =$$

$$= \left[1-i2\rho\xi+(1-\rho^2)\xi^2\right]^{-1/2} = \left[1-i(1+\rho)\xi\right]^{-1/2} \left[1+i(1-\rho)\xi\right]^{-1/2}, \tag{54}$$

via repeated use of ref. 14, eq. 3.323 2. The corresponding probability density function of x is

$$p_x(v) = \int \frac{dy}{|y|}\, p_{st}\left(y,\frac{v}{y}\right) = \frac{1}{\pi\sqrt{1-\rho^2}} \exp\left(\frac{\rho v}{1-\rho^2}\right) K_0\left(\frac{|v|}{1-\rho^2}\right) \text{ for all } v, \tag{55}$$

via ref. 14, eq. 3.478 4.

(If we transform this probability density function according to (1) and use ref. 14, eq. 6.611 9 and ref. 13, eq. 4.4.15, we get precisely (54). Alternatively, if we transform (54) and modify the contour to wrap around the branch line along the imaginary axis and then use ref. 14, eq. 3.388 2, we get (55). Or we can use ref. 14, eq. 3.754 2.)

We actually consider a more general characteristic function than (54), namely

$$f_x(\xi) = \left[1-i2\rho\xi+(1-\rho^2)\xi^2\right]^{-\nu} = \exp\left(-\nu \ln\left[1-i2\rho\xi + (1-\rho^2)\xi^2\right]\right) =$$

$$= \left[\frac{1}{1-\rho^2} + \left\{\sqrt{1-\rho^2}\,\xi - i\frac{\rho}{\sqrt{1-\rho^2}}\right\}^2\right]^{-\nu}. \tag{56}$$

The mean of this random variable x is given by

$$\mu_x = 2\nu\rho. \tag{57}$$

The probability density function corresponding to (56) is

$$p_x(v) = \frac{1}{2\pi} \int d\xi \, \exp(-i\xi v) \, f_x(\xi) =$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty - i\rho/\sqrt{1-\rho^2}}^{+\infty - i\rho/\sqrt{1-\rho^2}} dy \, \exp\left(\frac{\rho v}{1-\rho^2} - i \frac{yv}{\sqrt{1-\rho^2}}\right) \left(\frac{1}{1-\rho^2} + y^2\right)^{-\nu}, \quad (58)$$

where we let

$$y = \sqrt{1-\rho^2}\,\xi - i\frac{\rho}{\sqrt{1-\rho^2}} \,. \quad (59)$$

We can move the contour in (58) to the real y-axis, because the branch points of the integrand are at $y = \pm\, i/\sqrt{1-\rho^2}$ which are outside the path of integration, since $|\rho| < 1$. Then using ref. 14, eq. 3.771 2 and ref. 13, eq. 6.1.17, we obtain

$$p_x(v) = \left(\sqrt{\pi}\,\Gamma(\nu)\sqrt{1-\rho^2}\right)^{-1} \left(\frac{|v|}{2}\right)^{\nu-\frac{1}{2}} \exp\left(\frac{\rho v}{1-\rho^2}\right) K_{\nu-\frac{1}{2}}\left(\frac{|v|}{1-\rho^2}\right) \quad \text{for all } v. \quad (60)$$

Since this probability density function is never negative (ref. 13, sect. 9.6.1), (56) is a legal characteristic function. If we Fourier transform (60) via ref. 14, 6.699 12, we get (56) directly.

There is no simple relation for the cumulative distribution function of this random variable. Nevertheless, it is a simple matter to evaluate directly from characteristic function (56). The $\ln$ in (56) causes no problems since its argument never crosses the branch cut. A plot for $\nu=7.7$ and $\rho=-.3$ is displayed in figure 12. The rate of decay of the distribution is different for each tail. The round-off noise is clearly visible at both ends of the range of v values.

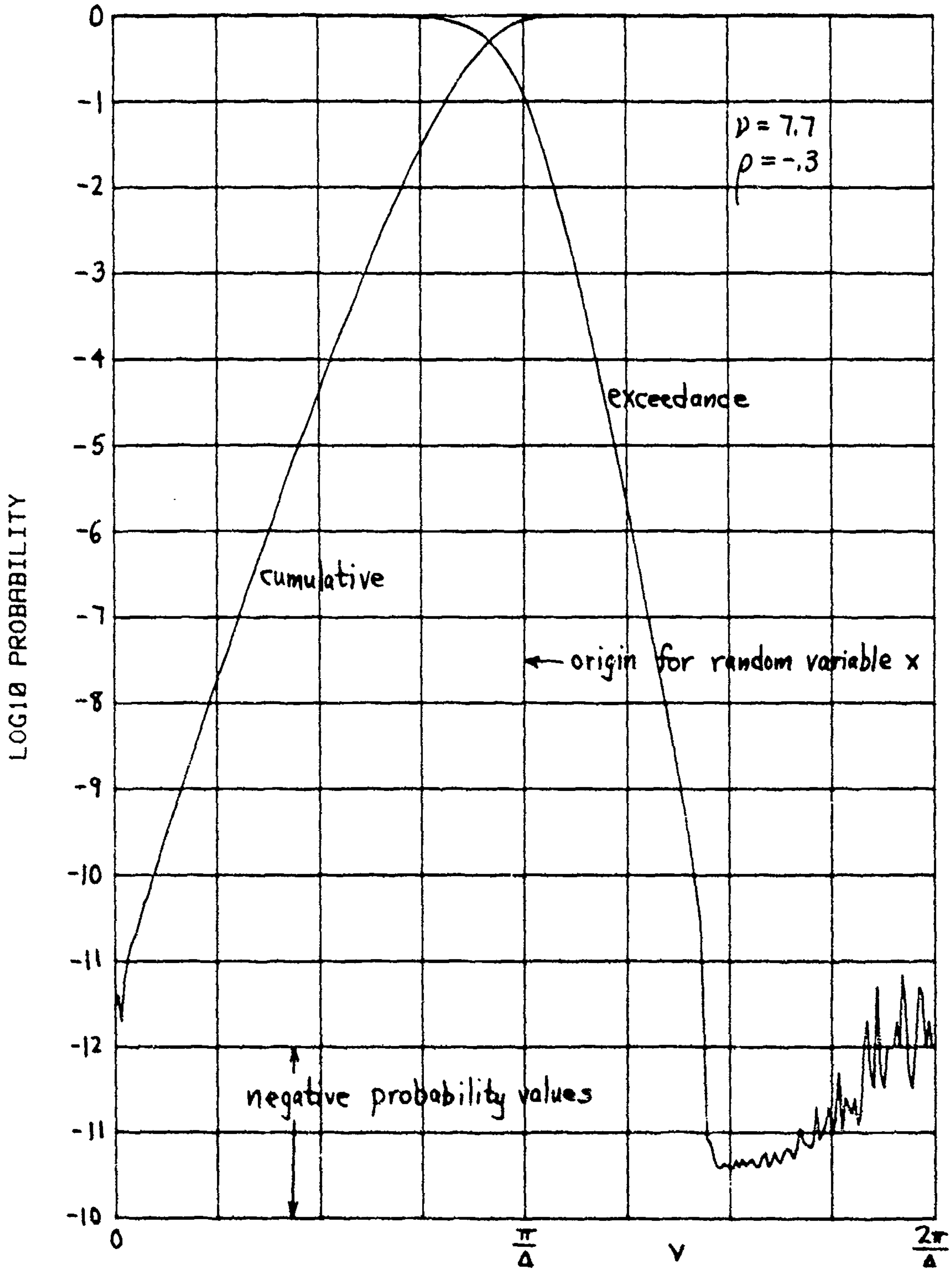

Figure 12. Gaussian Product; L=5, Δ=.06, b=50π/3, M=256

APPLICATIONS

We now have the capability to handle the following type of statistical problem in a fairly easy fashion. Consider random variable

$$x = \sum_{k=1}^{K} r_k^{\nu_k} , \tag{61}$$

where $\{r_k\}$ are arbitrary random variables, statistically independent of each other, and with different distributions. Power ν_k is arbitrary (except that ν_k must be a positive integer for those r_k that can become negative). Let the probability density function of random variable r_k be $p_k(v)$. Then the characteristic function of $r_k^{\nu_k}$ is

$$g_k(\xi) = \overline{\exp\left(i\xi r_k^{\nu_k}\right)} = \int dv \, \exp\left(i\xi v^{\nu_k}\right) p_k(v) =$$

$$= \frac{1}{\nu_k} \int \frac{dt}{t} \exp(i\xi t) \, t^{1/\nu_k} \quad p_k\left(t^{1/\nu_k}\right) . \tag{62}$$

If (62) is not integrable in closed form, it can be evaluated by means of an FFT (one for each k if the probability density functions or ν_k are all different). Then the characteristic function of random variable x in (61) is given by

$$f_x(\xi) = \prod_{k=1}^{K} \left\{g_k(\xi)\right\} . \tag{63}$$

Now the techniques of this report are directly applicable to (63).

An additional example is afforded by

$$x = \sum_{k=1}^{K} \alpha_k \, v_k^2 + \left(\sum_{k=1}^{K} \beta_k \, v_k \right)^2 + \sum_{k=1}^{K} \gamma_k \, v_k , \tag{64}$$

where $\{\alpha_k\}$, $\{\beta_k\}$, and $\{\gamma_k\}$ are constants, and $\{v_k\}$ are independent random variables with arbitrary probability density functions. The characteristic function of x is

$$f_x(\xi) = \overline{\exp(i\xi x)} = \int dV \; p_v(V) \, \exp\left(i\xi \left[\sum \alpha_k v_k^2 + \left(\sum \beta_k v_k \right)^2 + \sum \gamma_k v_k \right] \right), \tag{65}$$

where $V = (v_1, v_2, \ldots, v_K)$. Now since

$$\left(\frac{ia}{\pi} \right)^{1/2} \int dy \, \exp(-iay^2 + iby) = \exp\left(\frac{ib^2}{4a} \right) \text{ for } a \neq 0, \tag{66}$$

we identify $a = \xi/4$, $b = \xi \sum \beta_k v_k$, eliminate the square in the exponent, and express (65) as

$$f_x(\xi) = \int dV \; p_v(V) \, \exp\left(i\xi \sum \alpha_k v_k^2 + i\xi \sum \gamma_k v_k \right) *$$

$$* \left(\frac{i\xi}{4\pi} \right)^{1/2} \int dy \, \exp\left(-\frac{i\xi y^2}{4} + i\xi y \sum \beta_k v_k \right) =$$

$$= \left(\frac{i\xi}{4\pi} \right)^{1/2} \int dy \, \exp\left(-\frac{i\xi y^2}{4} \right) \prod_{k=1}^{K} \left\{ \int dv_k \; p_k(v_k) \, \exp(i\xi(\alpha_k v_k^2 + \gamma_k v_k + y\beta_k v_k)) \right\}, \tag{67}$$

where

$$p_v(V) = \prod_{k=1}^{K} \left\{ p_k(v_k) \right\} . \tag{68}$$

The inner integrals in (67) can either be done analytically or numerically. Then the remaining single integral on y must be numerically evaluated to find characteristic function $f_x(\xi)$. As an example, if v_k is exponentially distributed

$$p_k(v) = a_k \exp(-a_k v) \text{ for } v > 0, \tag{69}$$

then the inner integrals in (67) are w-functions; see ref. 13, ch. 7. A simpler method of handling general quadratic expressions like (64) with Gaussian $\mathbf{V}$ is presented in ref. 15.

SUMMARY

An accurate method for efficient evaluation of the cumulative and exceedance distribution functions has been derived and applied to several examples to illustrate its utility. Choice of the sampling increment Δ applied to the characteristic function controls the aliasing problem, and selection of the limit L minimizes the tail error; the effects of both of these parameters can be observed from sample plots of the distributions and can be modified if needed. Additionally, shift b must be chosen so as to yield a positive random variable with probability virtually 1. The number of distribution values yielded depends on the size of the FFT employed and can be independently selected to yield the desired spacing in distribution values.

APPENDIX A. SAMPLING FOR A FOURIER TRANSFORM

Suppose we are interested in evaluating Fourier transform

$$G(f) = \int dt \, \exp(-i2\pi ft) \, g(t). \tag{A-1}$$

If we sample at interval Δ in t in (A-1), and use integration weighting w(t), we have the approximation to G(f),

$$\begin{aligned} \tilde{G}(f) &\equiv \int dt \, \exp(-i2\pi ft) \, g(t) \, \delta_\Delta(t) \, w(t) \\ &= G(f) \circledast \frac{1}{\Delta} \delta_{\frac{1}{\Delta}}(f) \circledast W(f) \\ &= \frac{1}{\Delta} \sum_n G(f - \frac{n}{\Delta}) \circledast W(f), \end{aligned} \tag{A-2}$$

where infinite impulse train (sampling function)

$$\delta_\Delta(t) = \sum_n \delta(t - n\Delta), \tag{A-3}$$

and $\circledast$ denotes convolution.

The term

$$\frac{1}{\Delta} \sum_n G(f - \frac{n}{\Delta}) \tag{A-4}$$

in (A-2) is an infinitely aliased version of desired function G(f); this aliasing is an unavoidable effect due to sampling at increment Δ. However, to minimize any further aliasing in (A-2), we would like $W(f) = \delta(f)$, which requires $w(t) = 1$ for all t; strictly, all we need is

$$w(n\Delta) = 1 \quad \text{for all } n. \tag{A-5}$$

That is, the best weighting in (A-2) is uniform.

As an example, for Simpson's rule, we have weighting

$$w(n\Delta) = \ldots, \frac{2}{3}, \frac{4}{3}, \frac{2}{3}, \frac{4}{3}, \frac{2}{3}, \ldots = 1 + \frac{1}{3}(-1)^n \text{ or } 1 - \frac{1}{3}(-1)^n, \tag{A-6}$$

which can be represented as samples of time function

$$w(t) = 1 + \frac{1}{3}\exp(i\pi t/\Delta) \text{ or } 1 - \frac{1}{3}\exp(i\pi t/\Delta). \tag{A-7}$$

The corresponding transform is

$$W(f) = \int dt \exp(-i2\pi ft)\, w(t) =$$

$$= \delta(f) + \frac{1}{3}\,\delta\left(f - \frac{1}{2\Delta}\right) \quad \text{or} \quad \delta(f) - \frac{1}{3}\,\delta\left(f - \frac{1}{2\Delta}\right). \tag{A-8}$$

But this window function substituted in (A-2) results in an extra aliasing lobe in $\tilde{G}(f)$, halfway between the unavoidable major lobes of (A-4) at multiples of $1/\Delta$, of magnitude 1/3 as large. This effect very adversely affects the quality of $\tilde{G}(f)$ insofar as its approximation to the desired $G(f)$ is concerned. Thus the best sampling plan in (A-2) is the equal weight structure of (A-5) when one wants to approximate the Fourier transform of (A-1). For a bounded region, this is modified to the Trapezoidal rule, i.e., half-size weights at the boundaries.

APPENDIX B. LISTINGS OF PROGRAMS FOR FIVE EXAMPLES

The following listings are programs in BASIC for the Hewlett Packard 9845B Desktop Calculator. The FFT utilized is one with the capability of a zero subscript and is listed at the end of the appendix. Mathematically, the FFT programmed is

$$Z_m = \sum_{k=0}^{M-1} \exp(-i2\pi mk/M)\, z_k \quad \text{for } 0 \le m \le M-1,$$

where the arrays $\{z_k\}_0^{M-1}$ and $\{Z_m\}_0^{M-1}$ are handled directly, including the zero-subscript terms z_0 and Z_0.

A detailed explanation of the first program below for Chi-Squared random variables is as follows: line 20 specifies the parameter K, where 2K is the number of squared-Gaussian random variables summed to yield random variable x. Lines 30-60 require inputs L, Δ, b, M respectively, on the part of the user. Line 110 is the input of mean μ_x of random variable x, as evaluated analytically from characteristic function $f_x(\xi)$. Lines 180-210 specifically evaluate the characteristic function $f_y(\xi)$ at general point ξ. All of these lines mentioned thus far require inputs on the part of the user and are so noted in the listing by the presence of a single ! on each line; the comments after a double !! are for information purposes only and need not be modified. This convention is also adopted in the remaining listings.

Lines 220-240 accomplish the collapsing operation of (32)-(33). The cumulative and exceedance distribution functions are finally evaluated and stored in arrays X(*) and Y(*) in lines 400-410.

Some further elaboration is necessary for the listing of the Smirnov characteristic function as given by (42). Since a characteristic function is a continuous function of real ξ, the square root in (42) is <u>not</u> a principal value square root, but in fact must yield a continuous function in ξ. In order to achive this, the argument of the square root is traced continuously from $\xi=0$ (line 110). If an abrupt change in phase is detected, a polarity indicator takes note of this fact (line 250) and corrects the final values of characteristic function $f_y(\xi)$ (lines 260-270). No problems are encountered with complex sin(z) since it is analytic for all z.

```
10 ! CHI-SQUARE CHARACTERISTIC FUNCTION 1/(1-i 2 xi)^4
20    K=4                      !  2K=8 degrees of freedom
30    L=200                    !  Limit on integral of char. function
40    Delta=.075               !  Sampling increment on char. function
50    Bs=0                     !  Shift b
60    M=2^8                    !  Size of FFT
70    PRINTER IS 0
80    PRINT "L =";L,"Delta =";Delta,"b =";Bs,"M =";M
90    REDIM X(0:M-1),Y(0:M-1)
100   DIM X(0:1023),Y(0:1023)
110   Mux=2*K                            !  Mean of random variable x
120   Muy=Mux+Bs
130   X(0)=0
140   Y(0)=.5*Delta*Muy
150   N=INT(L/Delta)
160   FOR Ns=1 TO N
170   Xi=Delta*Ns                        !!  Argument xi of char. fn.
180   C=Xi+Xi                            !  Calculation of
190   CALL Mul(1,-C,1,-C,A,B)            !  characteristic
200   CALL Mul(A,B,A,B,C,D)              !  function fy(xi)
210   CALL Div(1,0,C,D,Fyr,Fyi)          !  for K=4
220   Ms=Ns MOD M                        !!  Collapsing
230   X(Ms)=X(Ms)+Fyr/Ns
240   Y(Ms)=Y(Ms)+Fyi/Ns
250   NEXT Ns
260   CALL Fft10z(M,X(*),Y(*))           !!  0 subscript FFT
```

```
270   PLOTTER IS "GRAPHICS"
280   GRAPHICS
290   SCALE 0,M,-14,0
300   LINE TYPE 3
310   GRID M/8,1
320   PENUP
330   LINE TYPE 1
340   B=Bs*M*Delta/(2*PI)              !!  Origin for random variable x
350   MOVE B,0
360   DRAW B,-14
370   PENUP
380   FOR Ks=0 TO M-1
390   T=Y(Ks)/PI-Ks/M
400   X(Ks)=.5-T                       !!  Cumulative probability in X(*)
410   Y(Ks)=Pr=.5+T                    !!  Exceedance probability in Y(*)
420   IF Pr>=1E-12 THEN Y=LGT(Pr)
430   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
440   IF ABS(Pr)<1E-12 THEN Y=-12
450   PLOT Ks,Y
460   NEXT Ks
470   PENUP
480   PRINT Y(0);Y(1);Y(M-2);Y(M-1)
490   FOR Ks=0 TO M-1
500   Pr=X(Ks)
510   IF Pr>=1E-12 THEN Y=LGT(Pr)
520   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
530   IF ABS(Pr)<1E-12 THEN Y=-12
540   PLOT Ks,Y
550   NEXT Ks
560   PENUP
570   PAUSE
580   DUMP GRAPHICS
590   PRINT LIN(5)
600   PRINTER IS 16
610   END
620   !
630   SUB Mul(X1,Y1,X2,Y2,A,B)          !  Z1*Z2
640   A=X1*X2-Y1*Y2
650   B=X1*Y2+X2*Y1
660   SUBEND
670   !
680   SUB Div(X1,Y1,X2,Y2,A,B)          !  Z1/Z2
690   T=X2*X2+Y2*Y2
700   A=(X1*X2+Y1*Y2)/T
710   B=(Y1*X2-X1*Y2)/T
720   SUBEND
730   !
740   SUB Fft10z(N,X(*),Y(*))     ! N <= 2^10 = 1024, N=2^INTEGER     0 subscript
```

```
10 ! GAUSSIAN CHARACTERISTIC FUNCTION exp(-.5 xi^2)
20    L=7                      !  Limit on integral of char. function
30    Delta=.3                 !  Sampling increment on char. function
40    Bs=.375*(2*PI/Delta) !  Shift b, as fraction of alias interval
50    M=2^8                    !  Size of FFT
60    PRINTER IS 0
70    PRINT "L =";L,"Delta =";Delta,"b =";Bs,"M =";M
80    REDIM X(0:M-1),Y(0:M-1)
90    DIM X(0:1023),Y(0:1023)
100   Mux=0                                  !  Mean of random variable x
110   Muy=Mux+Bs
120   X(0)=0
130   Y(0)=.5*Delta*Muy
140   N=INT(L/Delta)
150   FOR Ns=1 TO N
160   Xi=Delta*Ns                          !!  Argument xi of char. fn.
170   A=EXP(-.5*Xi*Xi)                      !  Calculation of
180   B=Bs*Xi                               !  characteristic
190   Fyr=A*COS(B)                          !  function
200   Fyi=A*SIN(B)                          !  fy(xi)
210   Ms=Ns MOD M                          !!  Collapsing
220   X(Ms)=X(Ms)+Fyr/Ns
230   Y(Ms)=Y(Ms)+Fyi/Ns
240   NEXT Ns
250   CALL Fft10z(M,X(*),Y(*))             !!  0 subscript FFT
260   PLOTTER IS "GRAPHICS"
270   GRAPHICS
280   SCALE 0,M,-14,0
290   LINE TYPE 3
300   GRID M/8,1
310   PENUP
320   LINE TYPE 1
330   B=Bs*M*Delta/(2*PI)                  !!  Origin for random variable x
340   MOVE B,0
350   DRAW B,-14
360   PENUP
370   FOR Ks=0 TO M-1
380   T=Y(Ks)/PI-Ks/M
390   X(Ks)=.5-T                           !!  Cumulative probability in X(*)
400   Y(Ks)=Pr=.5+T                        !!  Exceedance probability in Y(*)
410   IF Pr>=1E-12 THEN Y=LGT(Pr)
420   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
430   IF ABS(Pr)<1E-12 THEN Y=-12
440   PLOT Ks,Y
450   NEXT Ks
460   PENUP
470   PRINT Y(0);Y(1);Y(M-2);Y(M-1)
480   FOR Ks=0 TO M-1
490   Pr=X(Ks)
500   IF Pr>=1E-12 THEN Y=LGT(Pr)
510   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
520   IF ABS(Pr)<1E-12 THEN Y=-12
530   PLOT Ks,Y
540   NEXT Ks
550   PENUP
560   PAUSE
570   DUMP GRAPHICS
580   PRINT LIN(5)
590   PRINTER IS 16
600   END
610   !
620   SUB Fft10z(N,X(*),Y(*))    ! N <= 2^10 = 1024, N=2^INTEGER     0 subscript
```

```
10 ! SMIRNOV CHARACTERISTIC FUNCTION [s/sin(s)]^1/2 where s=(1+i)sqr(xi)
20    L=3000                 !  Limit on integral of char. function
30    Delta=1                !  Sampling increment on char. function
40    Bs=0                   !  Shift b
50    M=2^8                  !  Size of FFT
60    PRINTER IS 0
70    PRINT "L =";L,"Delta =";Delta,"b =",Bs,"M =";M
80    REDIM X(0:M-1),Y(0:M-1)
90    DIM X(0:1023),Y(0:1023)
100   Mux=1/6                                  !  Mean of random variable x
110   R=0                                      !  Argument of square root
120   P=1                                      !  Polarity indicator
130   Muy=Mux+Bs
140   X(0)=0
150   Y(0)=.5*Delta*Muy
160   N=INT(L/Delta)
170   FOR Ns=1 TO N
180   Xi=Delta*Ns                             !!  Argument xi of char. fn.
190   A=SQR(Xi)                                !  Calculation
200   CALL Sin(A,A,B,C)                        !  of
210   CALL Div(A,A,B,C,D,E)                    !  characteristic
220   CALL Sqr(D,E,A,B)                        !  function
230   Ro=R                                     !  fy(xi)
240   R=ATN(B/A)                               !
250   IF ABS(R-Ro)>1.6 THEN P=-P               !
260   Fyr=A*P                                  !
270   Fyi=B*P                                  !
280   Ms=Ns MOD M                             !!  Collapsing
290   X(Ms)=X(Ms)+Fyr/Ns
300   Y(Ms)=Y(Ms)+Fyi/Ns
310   NEXT Ns
320   CALL Fft10z(M,X(*),Y(*))                !!  0 subscript FFT
330   PLOTTER IS "GRAPHICS"
340   GRAPHICS
350   SCALE 0,M,-14,0
360   LINE TYPE 3
370   GRID M/8,1
380   PENUP
390   LINE TYPE 1
400   B=Bs*M*Delta/(2*PI)                     !!  Origin for random variable x
410   MOVE B,0
420   DRAW B,-14
430   PENUP
440   FOR Ks=0 TO M-1
450   T=Y(Ks)/PI-Ks/M
460   X(Ks)=.5-T                              !!  Cumulative probability in X(*)
470   Y(Ks)=Pr=.5+T                           !!  Exceedance probability in Y(*)
480   IF Pr>=1E-12 THEN Y=LGT(Pr)
490   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
500   IF ABS(Pr)<1E-12 THEN Y=-12
510   PLOT Ks,Y
520   NEXT Ks
```

```
530   PENUP
540   PRINT Y(0);Y(1);Y(M-2);Y(M-1)
550   FOR Ks=0 TO M-1
560   Pr=X(Ks)
570   IF Pr>=1E-12 THEN Y=LGT(Pr)
580   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
590   IF ABS(Pr)<1E-12 THEN Y=-12
600   PLOT Ks,Y
610   NEXT Ks
620   PENUP
630   PAUSE
640   DUMP GRAPHICS
650   PRINT LIN(5)
660   PRINTER IS 16
670   END
680   !
690   SUB Div(X1,Y1,X2,Y2,A,B)           !  Z1/Z2
700   T=X2*X2+Y2*Y2
710   A=(X1*X2+Y1*Y2)/T
720   B=(Y1*X2-X1*Y2)/T
730   SUBEND
740   !
750   SUB Sqr(X,Y,A,B)                   !  PRINCIPAL SQR(Z)
760   IF X<>0 THEN 800
770   A=B=SQR(.5*ABS(Y))
780   IF Y<0 THEN B=-B
790   GOTO 910
800   F=SQR(SQR(X*X+Y*Y))
810   T=.5*ATN(Y/X)
820   A=F*COS(T)
830   B=F*SIN(T)
840   IF X>0 THEN 910
850   T=A
860   A=-B
870   B=T
880   IF Y>=0 THEN 910
890   A=-A
900   B=-B
910   SUBEND
920   !
930   SUB Sin(X,Y,A,B)                   !  SIN(Z)
940   E=EXP(Y)
950   A=.5*SIN(X)*(E+1/E)
960   IF ABS(Y)<.1 THEN 990
970   S=.5*(E-1/E)
980   GOTO 1010
990   S=Y*Y
1000  S=Y*(120+S*(20+S))/120
1010  B=COS(X)*S
1020  SUBEND
1030  !
1040  SUB Fft10z(N,X(*),Y(*))     ! N <= 2^10 = 1024, N=2^INTEGER     0 subscript
```

```
10 ! NON-CENTRAL CHI-SQUARE CHARACTERISTIC FUNCTION
20 ! exp(i d^2 xi / s) / s^nu   where s = 1-i 2 xi
30    Nu=2.7                    !  Power law nu
40    Ds=3                      !  Deflection d
50    L=500                     !  Limit on integral of char. function
60    Delta=.05                 !  Sampling increment on char. function
70    Bs=0                      !  Shift b
80    M=2^8                     !  Size of FFT
90    PRINTER IS 0
100   PRINT "L =";L,"Delta =";Delta,"b =";Bs,"M =";M
110   REDIM X(0:M-1),Y(0:M-1)
120   DIM X(0:1023),Y(0:1023)
130   D2=Ds*Ds                              !  Calculate parameter
140   Mux=2*Nu+D2                           !  Mean of random variable x
150   Muy=Mux+Bs
160   X(0)=0
170   Y(0)=.5*Delta*Muy
180   N=INT(L/Delta)
190   FOR Ns=1 TO N
200   Xi=Delta*Ns                          !!  Argument xi of char. fn.
210   T=Xi+Xi                               !  Calculation of
220   CALL Div(0,D2*Xi,1,-T,A,B)            !  characteristic
230   CALL Log(1,-T,C,D)                    !  function
240   CALL Exp(A-Nu*C,B-Nu*D+Bs*Xi,Fyr,Fyi)   ! fy(xi)
250   Ms=Ns MOD M                          !!  Collapsing
260   X(Ms)=X(Ms)+Fyr/Ns
270   Y(Ms)=Y(Ms)+Fyi/Ns
280   NEXT Ns
290   CALL Fft10z(M,X(*),Y(*))             !!  0 subscript FFT
300   PLOTTER IS "GRAPHICS"
310   GRAPHICS
320   SCALE 0,M,-14,0
330   LINE TYPE 3
340   GRID M/8,1
350   PENUP
360   LINE TYPE 1
370   B=Bs*M*Delta/(2*PI)                  !!  Origin for random variable x
380   MOVE B,0
390   DRAW B,-14
400   PENUP
410   FOR Ks=0 TO M-1
420   T=Y(Ks)/PI-Ks/M
430   X(Ks)=.5-T                           !!  Cumulative probability in X(*)
440   Y(Ks)=Pr=.5+T                        !!  Exceedance probability in Y(*)
450   IF Pr>=1E-12 THEN Y=LGT(Pr)
460   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
470   IF ABS(Pr)<1E-12 THEN Y=-12
480   PLOT Ks,Y
490   NEXT Ks
500   PENUP
```

```
510    PRINT Y(0);Y(1);Y(M-2);Y(M-1)
520    FOR Ks=0 TO M-1
530    Pr=X(Ks)
540    IF Pr>=1E-12 THEN Y=LGT(Pr)
550    IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
560    IF ABS(Pr)<1E-12 THEN Y=-12
570    PLOT Ks,Y
580    NEXT Ks
590    PENUP
600    PAUSE
610    DUMP GRAPHICS
620    PRINT LIN(5)
630    PRINTER IS 16
640    END
650    !
660    SUB Div(X1,Y1,X2,Y2,A,B)            !  Z1/Z2
670    T=X2*X2+Y2*Y2
680    A=(X1*X2+Y1*Y2)/T
690    B=(Y1*X2-X1*Y2)/T
700    SUBEND
710    !
720    SUB Exp(X,Y,A,B)                    !  EXP(Z)
730    T=EXP(X)
740    A=T*COS(Y)
750    B=T*SIN(Y)
760    SUBEND
770    !
780    SUB Log(X,Y,A,B)                    !  PRINCIPAL LOG(Z)
790    A=.5*LOG(X*X+Y*Y)
800    IF X<>0 THEN 830
810    B=.5*PI*SGN(Y)
820    GOTO 850
830    B=ATN(Y/X)
840    IF X<0 THEN B=B+PI*(1-2*(Y<0))
850    SUBEND
860    !
870    SUB Fft10z(N,X(*),Y(*))    ! N <= 2^10 = 1024, N=2^INTEGER     0 subscript
```

```
10 ! GAUSSIAN PRODUCT CHARACTERISTIC FUNCTION (56)
20    Nu=7.7                      !  Power Nu
30    Rho=-.3                     !  Correlation coefficient
40    L=5                         !  Limit on integral of char. function
50    Delta=.06                   !  Sampling increment on char. function
60    Bs=.5*(2*PI/Delta)          !  Shift b, as fraction of alias interval
70    M=2^8                       !  Size of FFT
80    PRINTER IS 0
90    PRINT "L =";L,"Delta =";Delta,"b =";Bs,"M =";M
100   REDIM X(0:M-1),Y(0:M-1)
110   DIM X(0:1023),Y(0:1023)
120   T1=1-Rho*Rho                              !  Calculate
130   T2=2*Rho                                  !  parameters
140   Mux=2*Nu*Rho                              !  Mean of random variable x
150   Muy=Mux+Bs
160   X(0)=0
170   Y(0)=.5*Delta*Muy
180   N=INT(L/Delta)
190   FOR Ns=1 TO N
200   Xi=Delta*Ns                              !!  Argument xi of char. fn.
210   CALL Log(1+T1*Xi*Xi,-T2*Xi,A,B)           !  Calculation of
220   CALL Exp(-Nu*A,Bs*Xi-Nu*B,Fyr,Fyi)!  characteristic function fy(xi)
230   Ms=Ns MOD M                              !!  Collapsing
240   X(Ms)=X(Ms)+Fyr/Ns
250   Y(Ms)=Y(Ms)+Fyi/Ns
260   NEXT Ns
270   CALL Fft10z(M,X(*),Y(*))                 !!  0 subscript FFT
280   PLOTTER IS "GRAPHICS"
290   GRAPHICS
300   SCALE 0,M,-14,0
310   LINE TYPE 3
320   GRID M/8,1
330   PENUP
340   LINE TYPE 1
350   B=Bs*M*Delta/(2*PI)                      !!  Origin for random variable x
360   MOVE B,0
370   DRAW B,-14
380   PENUP
```

```
390   FOR Ks=0 TO M-1
400   T=Y(Ks)/PI-Ks/M
410   X(Ks)=.5-T                        !!  Cumulative probability in X(*)
420   Y(Ks)=Pr=.5+T                     !!  Exceedance probability in Y(*)
430   IF Pr>=1E-12 THEN Y=LGT(Pr)
440   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
450   IF ABS(Pr)<1E-12 THEN Y=-12
460   PLOT Ks,Y
470   NEXT Ks
480   PENUP
490   PRINT Y(0);Y(1);Y(M-2);Y(M-1)
500   FOR Ks=0 TO M-1
510   Pr=X(Ks)
520   IF Pr>=1E-12 THEN Y=LGT(Pr)
530   IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
540   IF ABS(Pr)<1E-12 THEN Y=-12
550   PLOT Ks,Y
560   NEXT Ks
570   PENUP
580   PAUSE
590   DUMP GRAPHICS
600   PRINT LIN(5)
610   PRINTER IS 16
620   END
630   !
640   SUB Exp(X,Y,A,B)                  !   EXP(Z)
650   T=EXP(X)
660   A=T*COS(Y)
670   B=T*SIN(Y)
680   SUBEND
690   !
700   SUB Log(X,Y,A,B)                  !   PRINCIPAL LOG(Z)
710   A=.5*LOG(X*X+Y*Y)
720   IF X<>0 THEN 750
730   B=.5*PI*SGN(Y)
740   GOTO 770
750   B=ATN(Y/X)
760   IF X<0 THEN B=B+PI*(1-2*(Y<0))
770   SUBEND
780   !
790   SUB Fft10z(N,X(*),Y(*))    ! N <= 2^10 = 1024, N=2^INTEGER     0 subscript
```

```
10    SUB Fft10z(N,X(*),Y(*))      ! N <= 2^10 = 1024, N=2^INTEGER     0 subscript
20    DIM C(0:256)
30    INTEGER I1,I2,I3,I4,I5,I6,I7,I8,I9,I10,J,K
40    DATA 1,.999981175283,.999924701839,.999830581796,.999698818696,.9995294175
01,.999322384588,.999077727753,.998795456205,.998475580573,.998118112900
50    DATA .997723066644,.997290456679,.996820299291,.996312612183,.995767414468
,.995184726672,.994564570734,.993906970002,.993211949235,.992479534599
60    DATA .991709753669,.990902635428,.990058210262,.989176509965,.988257567731
,.987301418158,.986308097245,.985277642389,.984210092387,.983105487431
70    DATA .981963869110,.980785280403,.979569765685,.978317370720,.977028142658
,.975702130039,.974339382786,.972939952206,.971503890986,.970031253195
80    DATA .968522094274,.966976471045,.965394441698,.963776065795,.962121404269
,.960430519416,.958703474896,.956940335732,.955141168306,.953306040354
90    DATA .951435020969,.949528180593,.947585591018,.945607325381,.943593458162
,.941544065183,.93[illegible]79223602,.937339011913,.935183509939,.932992798835
100   DATA .930766961079,.928506080473,.926210242138,.923879532511,.921514039342
,.919113851690,.916679059921,.914209755704,.911706032005,.909167983091
110   DATA .906595704515,.903989293123,.901348847046,.898674465694,.895966249756
,.893224301196,.890448723245,.887639620403,.884797098431,.881921264348
120   DATA .879012226429,.876070094195,.873094978418,.870086991109,.867046245516
,.863972856122,.860866938638,.857728610000,.854557988365,.851355193105
130   DATA .848120344803,.844853565250,.841554977437,.838224705555,.834862874986
,.831469612303,.828045045258,.824589302785,.821102514991,.817584813152
140   DATA .814036329706,.810457198253,.806847553544,.803207531481,.799537269108
,.795836904609,.792106577300,.788346427627,.784556597156,.780737228572
150   DATA .776888465673,.773010453363,.769103337645 .765167265622,.761202385484
,.757208846506,.753186799044,.749136394523,.745057785441,.740951125355
160   DATA .736816568877,.732654271672,.728464390448,.724247082951,.720002507961
,.715730825284,.711432195745,.707106781187,.702754744457,.698376249409
170   DATA .693971460890,.689540544737,.685083667773,.680600997795,.676092703575
,.671558954847,.666999922304,.662415777590,.657806693297,.653172842954
180   DATA .648514401022,.643831542890,.639124444864,.634393284164,.629638238915
,.624859488142,.620057211763, 615231590581,.610382806276,.605511041404
190   DATA .600616479384,.595699304492,.590759701859,.585797857456,.580813958096
,.575808191418,.570780745387,.565731810784,.560661576197,.555570233020
200   DATA .550457972937,.545324988422,.540171472730,.534997619887,.529803624686
,.524589682678,.519355990166,.514102744193,.508830142543,.503538383726
210   DATA .498227666973,.492898192230,.487550160148,.482183772079,.476799230063
,.471396736826,.465976495768,.460538710958,.455083587126,.449611329655
220   DATA .444122144570,.438616238539,.433093818853,.427555093430,.422000270800
,.416429560098,.410843171058,.405241314005,.399624199846,.393992040061
230   DATA .388345046699,.382683432365,.377007410216,.371317193952,.365612997805
,.359895036535,.354163525420,.348418680249,.342660717312,.336889853392
240   DATA .331106305760,.325310292162,.319502030816,.313681740399,.307849640042
,.302005949319,.296150888244,.290284677254,.284407537211,.278519689385
250   DATA .272621355450,.266712757475,.260794117915,.254865659605,.248927605746
,.242980179903,.237023605994,.231058108281,.225083911360,.219101240157
260   DATA .213110319916,.207111376192,.201104634842,.195090322016,.189068664150
,.183039887955,.177004220412,.170961888760,.164913120490,.158858143334
270   DATA .152797185258,.146730474455,.140658239333,.134580708507,.128498110794
,.122410675199,.116318630912,.110222207294,.104121633872,.980171403296E-1
280   DATA .919089564971E-1,.857973123444E-1,.796824379714E-1,.735645635997E-1,.
674439195637E-1,.613207363022E-1,.551952443497E-1,.490676743274E-1
290   DATA .429382569349E-1,.368072229414E-1,.306748031766E-1,.245412285229E-1,.
184067299058E-1,.122715382857E-1,.613588464915E-2,0
300   READ C(*)
310   K=1024/N
320   FOR J=0 TO N/4
330   C(J)=C(K*J)
340   NEXT J
350   N1=N/4
360   N2=N1+1
370   N3=N2+1
```

```
380   N4=N1+N3
390   Log2n=INT(1.4427*LOG(N)+.5)
400   FOR I1=1 TO Log2n
410   I2=2^(Log2n-I1)
420   I3=2*I2
430   I4=N/I3
440   FOR I5=1 TO I2
450   I6=(I5-1)*I4+1
460   IF I6<=N2  HEN 500
470   N6=-C(N4-I6-1)
480   N7=-C(I6-N1-1)
490   GOTO 520
500   N6=C(I6-1)
510   N7=-C(N3-I6-1)
520   FOR I7=0 TO N-I3 STEP I3
530   I8=I7+I5
540   I9=I8+I2
550   N8=X(I8-1)-X(I9-1)
560   N9=Y(I8-1)-Y(I9-1)
570   X(I8-1)=X(I8-1)+X(I9-1)
580   Y(I8-1)=Y(I8-1)+Y(I9-1)
590   X(I9-1)=N6*N8-N7*N9
600   Y(I9-1)=N6*N9+N7*N8
610   NEXT I7
620   NEXT I5
630   NEXT I1
640   I1=Log2n+1
650   FOR I2=1 TO 10                    ' 2^10=1024
660   C(I2-1)=1
670   IF I2>Log2n THEN 690
680   C(I2-1)=2^(I1-I2)
690   NEXT I2
700   K=1
710   FOR I1=1 TO C(9)
720   FOR I2=I1 TO C(8) STEP C(9)
730   FOR I3=I2 TO C(7) STEP C(8)
740   FOR I4=I3 TO C(6) STEP C(7)
750   FOR I5=I4 TO C(5) STEP C(6)
760   FOR I6=I5 TO C(4) STEP C(5)
770   FOR I7=I6 TO C(3) STEP C(4)
780   FOR I8=I7 TO C(2) STEP C(3)
790   FOR I9=I8 TO C(1) STEP C(2)
800   FOR I10=I9 TO C(0) STEP C(1)
810   J=I10
820   IF K>J THEN 890
830   A=X(K-1)
840   X(K-1)=X(J-1)
850   X(J-1)=A
860   A=Y(K-1)
870   Y(K-1)=Y(J-1)
880   Y(J-1)=A
890   K=K+1
900   NEXT I10
910   NEXT I9
920   NEXT I8
930   NEXT I7
940   NEXT I6
950   NEXT I5
960   NEXT I4
970   NEXT I3
980   NEXT I2
990   NEXT I1
1000  SUBEND
```

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NAVELESYSYCOM (ELEX 03, PME-124, ELEX 612)	3
NAVSEASYSCOM (SEA-63R (2), -63D, -63Y, PMS-411, -409)	6
NAVAIRDEVCEN, Warminster	4
NAVAIRDEVCEN, Key West	1
NOSC	4
NOSC, Library (Code 6565)	1
NAVWPNSCEN	2
NCSC	2
NAVCIVENGRLAB	1
NAVSWC	4
NAVSWC, White Oak Lab.	4
DWTNSRDC, Annapolis	2
DWTNSRDC, Bethesda	2
NAVTRAEQUIPCENT (Tech. Library)	1
NAVPGSCOL	4
APL/UW, Seattle	1
ARL/Penn State	2
ARL, University of Texas	1
DTIC	2
DARPA (CDR K. Evans)	1
Woods Hole Oceanographic Institution	1
MPL/Scripps	1