Accurate Efficient Evaluation of Cumulative or Exceedance Probability Distributions Directly From Characteristic Functions

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Preface

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An accurate and efficient method of evaluating the entire cumulative or exceedance probability distribution, via one fast Fourier transform of the sampled characteristic function, is presented. The sampling rate applied to the characteristic function results in aliasing of the probability density function, while the limited extent of the sampling gives rise to a systematic disturbance in the calculated probability distribution. Both types of errors are easily recognizable and can be controlled by a trial and error procedure whereby the calculated distributions are plotted for observation and modification.
The size of the fast Fourier transform determines the number of distribution values available, but has no effect upon the accuracy of the result. Regardless of the number of characteristic function evaluations required for accurate results, the storage required is just that corresponding to the size of the fast Fourier transform.

A program for the procedure is presented, and the inputs required of the user are indicated. Several representative examples and plots illustrate the utility of the approach.
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<tr>
<td>$x$</td>
<td>random variable of primary interest</td>
</tr>
<tr>
<td>$f_x(\xi)$</td>
<td>characteristic function of random variable $x$</td>
</tr>
<tr>
<td>$p_x(v)$</td>
<td>probability density function of random variable $x$</td>
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<tr>
<td>$b$</td>
<td>bias or shift added to $x$</td>
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<td>$y$</td>
<td>random variable $y = x + b$</td>
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<td>$f_y(\xi)$</td>
<td>characteristic function of random variable $y$</td>
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<td>$\mu_x$</td>
<td>mean of random variable $x$</td>
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<td>$g(\xi, v)$</td>
<td>auxiliary function (6)</td>
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<td>$\Im$</td>
<td>imaginary part</td>
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<td>$\mu_y$</td>
<td>mean of random variable $y$</td>
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<td>$C(v)$</td>
<td>right-hand side of (8)</td>
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<td>$\Delta$</td>
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<td>$\tilde{p}_y(v)$</td>
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<td>$M$</td>
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<td>sequence of characteristic function samples; (20)</td>
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<td>collapsed sequence; (21)</td>
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<td>$L$</td>
<td>limit on integral of characteristic function; (28)</td>
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<td>$N$</td>
<td>number of nonzero $z_n$; $N = L/\Delta$</td>
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<td>$\bar{\text{ensemble average}}$</td>
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ACCURATE EFFICIENT EVALUATION OF CUMULATIVE OR EXCEEDANCE PROBABILITY DISTRIBUTIONS DIRECTLY FROM CHARACTERISTIC FUNCTIONS

INTRODUCTION

The performance of a signal processor can often be evaluated in terms of the characteristic function of the decision variable, either numerically or in closed form; see for example, refs. 1 and 2. However, a closed form for the corresponding probability density function or cumulative distribution function is seldom available, and numerical procedures must be employed. Several such procedures have been published in the literature, refs. 3-8. However they have limited accuracy or they require extensive storage or analytical manipulations and calculations.

We present a technique which is limited in accuracy only by the round-off noise of the computer or by the errors of the special functions required in the characteristic function calculation. The amount of storage depends only on the number of cumulative or exceedance distribution function values requested and does not influence the accuracy of the final probability values. The entire cumulative and exceedance distribution function values result as the output of one fast Fourier transform (FFT). The size of the FFT dictates the storage required and the spacing of the calculated probability values, but not their accuracy.

The addition and subtraction of integrand functions given in ref. 7 can be entirely circumvented and yet enable use of an FFT, through proper manipulation of the origin contribution of the characteristic function. Specific connections with past results will be noted at appropriate points in the derivations.
DERIVATION OF PROCEDURE

Shifted Random Variable

The primary random variable of interest is the real quantity \( x \) with given characteristic function \( f_x(\xi) \) which is related to the probability density function \( p_x(x) \) of random variable \( x \) via Fourier transform:

\[
f_x(\xi) = \int dv \exp(i\xi v) p_x(v). \tag{1}
\]

We define secondary random variable \( y \) as

\[
y = x + b, \tag{2}
\]

where bias (shift) \( b \) is a constant, chosen such that random variable \( y \) has insignificant probability of being less than zero. However, we also pick \( b \) as small as possible, so that the characteristic function of \( y \),

\[
f_y(\xi) = f_x(\xi) \exp(ib\xi), \tag{3}
\]

will vary slowly with \( \xi \). In fact, \( b \) can be negative, as for example if \( x \) were limited to values larger than some positive threshold. The approach here is not limited to positive random variables \( x \), as were some of the results in ref. 7, but is applicable to any random variable distribution.

By way of example, for an exponential probability density function for random variable \( x \), we choose \( b=0 \); while for a Gaussian random variable, \( b=\mu_x+3\sigma_x \) yields a probability less than 1E-15 of \( y \) being negative. The probability density function of random variable \( y \) therefore appears as depicted in figure 1.

---

* Integrals and sums without limits are over \((-\infty, \infty)\).
Figure 1. Probability Density Function of Secondary Random Variable $y$

The cumulative distribution functions of random variables $y$ and $x$ are related according to

$$\int_{-\infty}^{v} dt p_{y}(t) = p_{y}(v) = p_{x}(v-b); \quad p_{x}(v) = p_{y}(v+b).$$

(4)

Thus we can inspect $p_{x}(v)$ in the neighborhood of $v=-b$ (the lower edge of interest of $x$) by looking at cumulative distribution function $p_{y}(v)$ in the neighborhood of $v=0$. More precisely, we will investigate $p_{y}(v)$ for values of $v$ greater than zero, since this is the region of significant variation of $p_{y}(v)$; this is called the positive neighborhood of $v=0$.

Approximation to Cumulative Distribution Function

From ref. 4, eq. 7, we have the cumulative distribution function of random variable $y$ in terms of the characteristic function according to

$$p_{y}(v) = \frac{1}{2} \left[ \int_{0}^{\infty} g(\xi, v) \right].$$

(5)

where we have defined auxiliary function

$$g(\xi, v) = \text{Im} \left\{ \exp(-i\xi v) \frac{f(y)}{v} \right\}.$$  

(6)

Observe for later use that

$$g(0, v) = \lim_{\xi \to 0} \text{Im} \left\{ \left(1-i\xi v\right) \frac{1-i\xi v}{v} \right\} = \frac{u_y - v}{v},$$

(7)

where $u_y$ is the mean of random variable $y$. 

3
For \( v \) in the neighborhood of zero, \( \exp(-i\xi v) \) in (6) varies slowly with \( \xi \), and we have the approximation, via the Trapezoidal rule, to (5) as

\[
P_y(v) = \frac{1}{2} - \frac{\Delta}{2} g(0^+, v) - \sum_{n=1}^{\infty} \Delta g(n\Delta, v) \equiv C(v),
\]

where the right-hand side of (8) has been defined as \( C(v) \). Here, \( \Delta \) is the sampling interval in \( \xi \), and is small enough to track changes in \( \exp(-i\xi v) \cdot f_y(\xi)/\xi \). We choose the Trapezoidal rule in (8) over other integration rules, such as Simpson's rule, because it results in minimum aliasing for Fourier transforms relative to all other rules; see appendix A for elaboration and proof.

Observe from (8) that

\[
P_y(0) = C(0) \quad \text{means} \quad C(0) = 0,
\]

since \( P_y(0) \) is insignificant by the choice of \( b \) in (2); this relation will be used later.

**Relationship of Approximation**

Although we want to evaluate the exact cumulative distribution function \( P_y(v) \), we have instead arrived at an approximation \( C(v) \) via (8). How are these two related? To determine the relationship, we manipulate (6)-(8) as follows:

\[
C(v) = \frac{1}{2} \cdot \frac{\xi}{2} \frac{v-u}{v} - \sum_{n=1}^{\infty} \Im \left\{ \exp(-i\Delta v) \frac{f_y(n\Delta)}{v_n} \right\} = \frac{1}{2} \cdot \frac{\xi}{2} \frac{v-u}{v} - \Im \left\{ \sum_{n=1}^{\infty} \exp(-i\Delta v) \frac{f_y(n\Delta)}{v_n} \right\} =
\]

\[
= \frac{1}{2} \cdot \frac{\xi}{2} \frac{v-u}{v} - \frac{1}{12} \sum_{n \neq 0} \exp(-i\Delta v) \frac{f_y(n\Delta)}{v_n}.
\]
The removal of the imaginary operation from within the summation in (10) is a crucial step; it does not create a problem in divergence since \( n > 0 \). This is in contrast with the integral of (5) and (6), where removal of the imaginary operation would create a divergent integral. This postponement of the removal of the imaginary operation, until after the approximation to the integral was developed in (8), is the major difference with the results in ref. 7.

Taking a derivative of (11), we obtain

\[
C'(v) = \frac{\alpha}{2\pi} + \frac{\alpha}{2\pi} \sum_{n-0} \exp(-i\alpha v) f_y(n+\alpha) =
\]

\[
= \frac{\alpha}{2\pi} \sum_{n} \exp(-i\alpha v) f_y(n+\alpha) =
\]

\[
= \frac{1}{2\pi} \int dx \exp(-ixv) f_y(x) \delta(x) =
\]

\[
= p_y(v) \delta_{2\pi}(v) = \sum_{n} p_y(v-n \frac{2\pi}{\alpha}) \equiv \bar{p}_y(v),
\] (12)

where infinite impulse train

\[
\delta_{2\pi}(x) = \sum_{n} \delta(x-n\pi),
\] (13)

where \( \otimes \) denotes convolution, and where we have used the relation

\[
\frac{1}{2\pi} \int dt \exp(-i\omega t) \delta_{\alpha}(t) = \delta_{2\pi}(\omega).
\] (14)

This last result follows from ref. 9, p. 28, rule 11, with \( u(t) = \delta(t), T=\alpha, F=1/T, \) and \( \omega=2\pi f \). Relation (12) indicates that \( C'(v) \) is an infinitely aliased version of the probability density function \( p_y(v) \), with resultant period \( 2\pi/\alpha \) in \( v \). For small enough sampling increment \( \alpha \) in (8), there will be very little overlap of the displaced versions of \( p_y \) in (12), thereby yielding the good approximation
\[ \bar{p}_y(v) = p_y(v) \text{ for } 0 \leq v \leq \frac{2\pi}{\Delta}. \]  

(15)

The situation for relation (12) is depicted in figure 2.

![Figure 2. Infinitely Aliased Probability Density Function \( \bar{p}_y(v) \)](image)

There now follows from (12),

\[ C(v) = C(0) + \int_0^v du \, \bar{p}_y(u) = C(0) + \bar{p}_y(v), \]  

(16)

where \( C(0) \) is given by (10) as

\[ C(0) = \frac{1}{2} - \frac{\Delta u}{2\pi} - \text{Im} \left\{ \sum_{n=1}^{+\infty} \frac{f_y(n\Delta)}{\pi n} \right\}. \]  

(17)

Relation (16) is an exact relation, showing that \( C(v) \) is the integral of the infinitely aliased version of \( p_y(v) \), starting at \( v=0 \), plus an additive constant which is substantially zero; see (9).

So for \( v \) in the positive neighborhood of zero, (4), (8), (16), and (9) yield

\[ P_x(v-b) = p_y(v) = C(v) = C(0) + \bar{p}_y(v) = \bar{p}_y(v). \]  

(18)

Thus the quantity we want, the left-most term in (18), is well-approximated by calculated quantity \( C(v) \), which itself is approximately the integral of the infinitely aliased version of \( p_y(v) \).
Calculation of $C(v)$

Let $v = \frac{2\pi k}{M\Delta}$ in (10), where $M$ and $k$ are arbitrary integers. Then

$$C\left(\frac{2\pi k}{M\Delta}\right) = \frac{1}{2} + \frac{k}{M} - \Delta \mu - \frac{1}{\pi} \text{Im} \left\{ \sum_{n=1}^{+\infty} \exp(-i2\pi nk/M) \frac{f_y(n\Delta)}{\pi n} \right\}$$

$$= \frac{1}{2} + \frac{k}{M} - \frac{1}{\pi} \text{Im} \left\{ \sum_{n=0}^{+\infty} \exp(-i2\pi nk/M) z_n \right\}, \quad (19)$$

where we define complex sequence

$$z_n = \begin{cases} i \frac{1}{2} \Delta \mu_y & \text{for } n=0 \\ f_y(n\Delta)/n & \text{for } n \geq 1 \end{cases}. \quad (20)$$

Now define collapsed sequence (ref. 7, pp. 13-16) as

$$z_n^{(\text{c})} = \sum_{j=0}^{M-1} z_{n+Mj} \quad \text{for } 0 \leq n \leq M-1. \quad (21)$$

Then since $z_n$ receives the same weight as $z_{n+Mj}$ in (19), regardless of the value of $k$, (19) can be expressed as

$$C\left(\frac{2\pi k}{M\Delta}\right) = \frac{1}{2} + \frac{k}{M} - \frac{1}{\pi} \text{Im} \left\{ \sum_{n=0}^{M-1} \exp(-i2\pi nk/M) z_n^{(\text{c})} \right\}. \quad (22)$$

Relation (22) is exact and valid for all $k$. Since we are only interested in the positive neighborhood of $v=0$ in (18), we confine attention in (22) to $0 \leq k \leq M-1$. Relation (22) can then be accomplished by an $M$-point FFT if $M$ is chosen to be a power of 2. Notice that storage only for the $M$ complex numbers $\{\hat{z}_n\}$ in (21) is required, even though the $\{z_n\}$ sequence in (20) is of infinite length.

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* Values for other $k$ are available from (22) when we observe that

$$C\left(\frac{2\pi k(M+K)}{M\Delta}\right) = 1 + C\left(\frac{2\pi k}{M\Delta}\right) \quad \text{for all } k.$$
Observe that the size of $M$ in no way affects the error of the calculation of $C\left(\frac{2\pi k}{M\Delta}\right)$ or estimation of $P_y(v)$. Rather, $M$ specifies the spacing at which $C\left(\frac{2\pi k}{M\Delta}\right)$ is calculated, and can be coarse if desired. The accuracy of the estimate of $P_y(v)$ is governed thus far by $\Delta$, through the aliasing depicted in figure 2.

Reference to (18) now yields

$$P_x(\frac{2\pi k}{M\alpha} - b) = C\left(\frac{2\pi k}{M\alpha}\right) \quad \text{for} \quad 0 \leq k \leq M-1,$$

(23)

where the latter quantity is given by (22). Thus the $M$-point FFT sweeps out the argument range $(-b, -b + 2\pi / \Delta)$ for the cumulative distribution function $P_x$.

If we want the exceedance distribution function of $y$ instead of the cumulative distribution function, we use (18) and (22) to get

$$1 - C\left(\frac{2\pi k}{M\alpha}\right) = \frac{1}{2} - \frac{k}{M} + \frac{1}{\pi} \text{Im} \left\{ \sum_{n=0}^{M-1} \exp(-i2\pi nk/M) \right\} \quad \text{for} \quad 0 \leq k \leq M-1.$$

(24)

(By the footnote to (22), we have $1 - C(2\pi / \Delta) = -C(0)$.)

Since $u_y$ must be known in (20) in order to use this approach, we need the mean $u_x$ of random variable $x$, since from (2)

$$u_y = u_x + b.$$  

(25)

The quantity $u_x$ can be found analytically from characteristic function $f_x(y)$ according to

$$f_x'(0) = iu_x;$$  

(26)

see (1).
In addition to the error caused by aliasing associated with nonzero sampling increment $\Delta$, an additional error occurs because we cannot calculate all the coefficients $\{z_n\}$ in (20) and (21) out to $n=\infty$. Rather, we terminate the calculation at integer $n=N$, such that $|z_n|$ is sufficiently small as to be negligible for $n \geq N$. Letting

$$L = N\Delta,$$  \hspace{1cm} (27)

this is equivalent to ignoring the contribution to (5) of the tail error

$$-\int_{L}^{+\infty} g(\xi, v) d\xi = -\text{Im} \int_{L}^{+\infty} d\xi \exp(-i\xi v) \frac{f_y(\xi)}{\pi \xi}.$$  \hspace{1cm} (28)

If the asymptotic behavior of $f_y(\xi)$ for large $\xi$ is known, this error can sometimes be evaluated in closed form and used to ascertain an adequate value of $L$. Instead, we have observed that tail error (28) causes a characteristic low-level sinusoidal variation in the calculated cumulative distribution function for small $v$ near 0, and in the calculated exceedance distribution function for large $v$ near $2\pi/\Delta$. When this sinusoidal variation is deemed excessive, $L$ can be increased until the effect disappears or decreases to acceptable levels. This trial and error approach avoids the necessity of analytically upper-bounding the magnitude of error (28), which is often very tedious and generally pessimistic.

So there are two errors to be concerned with: aliasing due to nonzero sampling interval $\Delta$ and tail error due to non infinite limit $L$. Later examples will demonstrate how the errors manifest themselves in the cumulative and exceedance distribution functions and how they can be controlled by a trial and error approach.
Relation to Requicha's Method, ref. 5

From ref. 5, eqs. 7, 9, 10, the cumulative distribution function is given by an expression that can be manipulated into the form (using current notation)

\[ F_k = \frac{k}{M} - \frac{1}{\pi} \text{Im} \left\{ \sum_{n=1}^{M/2} \exp\left(-i2\pi kn/M\right) \frac{f_y(n\Delta)}{n} \right\} + \frac{1}{\pi} \text{Im} \left\{ \sum_{n=1}^{M/2} \frac{f_y(n\Delta)}{n} \right\}. \]  

(29)

Although this is similar to the upper line of (19) here, it differs in several important respects:

1. \( F_k \) does not use mean \( \mu_y \) at all; it is therefore not using a direct approximation to the specified integral in (5) and (6).

2. From (29), there follows \( F_0 = 0, F_M = 1 \); however, these results are not strictly true for the actual cumulative distribution function at these end points, thereby leading to poor estimates in the neighborhoods of these points. This is due to the arbitrary origin established in ref. 5, eq. 6.

3. The sums in (29) utilize characteristic function samples \( f_y(n\Delta) \) only for \( n < M/2 \), where \( M \) is the size of the FFT. This is a very severe and unnecessary restriction; in fact, the sum on \( n \) in (29) ought to be conducted to the point where the tail contribution, (28), is negligible, regardless of the value of \( M \).

4. In ref. 5, if eq. 4 is substituted into eq. 1, and the summation limits are extended to \( \pm \infty \), we get exactly the second line of (12) here. When the probability density function is integrated to get the cumulative distribution function in ref. 5, eq. 6, the resultant cumulative distribution function is arbitrarily set to zero at \( v=0 \). We instead have from (9) and (17),

\[ p_y(0) = C(0) = \frac{1}{\Delta} - \frac{\Delta u}{\Delta v} - \frac{1}{\pi} \text{Im} \left\{ \sum_{n=1}^{\infty} \frac{f_y(n\Delta)}{n} \right\}, \]  

(30)

which is small, but not necessarily zero. This consideration is very important on the tails of the cumulative and exceedance distribution functions.
Summary of Procedure

The cumulative distribution function of $y$ is given by

$$P_y \left( \frac{2\pi k}{M\Delta} \right) = C \left( \frac{2\pi k}{M\Delta} \right) = \frac{1}{2} + \frac{k}{M} - \frac{1}{\pi} \Im \left\{ \sum_{n=0}^{M-1} \exp(-i2\pi nk/M) \hat{z}_n \right\}$$

for $0 \leq k \leq M-1,$ (31)

where $M$ is the size of the FFT and storage employed. Also

$$\hat{z}_n = \sum_{j=0}^{\infty} z_{n+Mj} \quad \text{for} \quad 0 \leq n \leq M-1,$$ (32)

where

$$z_n = \begin{cases} \frac{1}{2} \Delta u_y & \text{for} \quad n=0 \\ f_y(n\Delta)/n & \text{for} \quad 1 \leq n \leq N \\ 0 & \text{for} \quad n > N \end{cases}$$ (33)

(The value for $n=N$ should be scaled by 1/2 for the Trapezoidal rule). The zero values for $z_n$, when $n > N$, serve to terminate the collapsed sum in (32) at a finite upper limit. The value of $N$ is given by the integer part of $L/\Delta$, where $\Delta$ and $L$ must be chosen so as to minimize aliasing and tail error, respectively. The characteristic function of random variable $y$ needed in (33) is given by

$$f_y(f) = f_x(f) \exp(ibf),$$ (34)

in terms of the characteristic function of the primary random variable $x$, where shift $b$ must be chosen such that $y = b+x$ is positive with probability virtually 1. The mean $\mu_y = b+\mu_x$ can be determined analytically from knowledge of characteristic function $f_x(f)$. Finally, the exceedance distribution function for random variable $y$ is obtained by subtracting (31) from 1.
EXAMPLES

Programs for the following five examples are listed in appendix B.

1. Chi-Square

A chi-square variate of 2K degrees of freedom has probability density function (ref. 10)

\[ p_x(v) = \frac{v^{K-1} \exp(-v/2)}{2^K(K-1)!} \quad \text{for } v > 0 \]  

(35)

and characteristic function

\[ f_x(\xi) = (1-\xi^2)^{-K}. \]  

(36)

Since random variable \( x \) is obviously nonnegative by (35), we can choose shift \( b=0 \); i.e. \( y=x \). A plot of the cumulative and exceedance distribution functions of random variable \( y \) obtained from characteristic function (36) with \( K=4 \) is given in figure 3 for \( 0 < v < 2\pi/\Delta \). The values of \( \Delta \) and \( L \) have been chosen such that aliasing and tail error are insignificant.

The ordinate scale for figure 3 is a logarithmic one. The lower right end of the exceedance distribution function curve decreases smoothly to the region \( 10^{-1} \), where round-off noise is encountered. The exceedance distribution function values continue to decrease with \( v \) until, finally, negative values (due to round-off noise) are generated. For negative probability values, the logarithm of the absolute value is plotted, but mirrored below the \( 10^{-12} \) level. These values have no physical significance, of course; they are plotted to illustrate the level of accuracy attainable by this procedure with appropriate choices of \( \Delta \) and \( L \).

For this example, \( N=L/\Delta=2666 \), while \( M=256 \). Thus collapsing, according to (21) or (32), by over a factor of 10 has been employed and a small size FFT has been utilized. Nevertheless the error realized for the cumulative and exceedance distribution functions is in the \( 10^{-12} \) range, the limit of accuracy of the Hewlett Packard 9845B Desk Calculator used here. Finer spacing in the distribution outputs is achievable by merely increasing \( M \).
Figure 3. Chi-Square; L=200, $\Delta=0.075$, b=0, M=256
2. Gaussian

The characteristic function for a zero-mean unit-variance random variable is

\[ f_x(\xi) = \exp(-\xi^2/2), \quad (37) \]

and the probability density function and cumulative distribution function are (ref. 11, eq. 10.5.3)

\[ p_x(v) = (2\pi)^{-1/2} \exp(-v^2/2), \quad P_x(v) = \Phi(v). \quad (38) \]

For \( b = 5\pi/2 \), using (4),

\[ P_y(0) = P_x(-b) = \Phi(-b) = 2E-15. \quad (39) \]

which is negligible, as desired.

Plots of the cumulative and exceedance distribution functions for random variable \( y \) are given in figure 4 for \( L=7, \Delta=.3 \). The logarithmic ordinate gives rise to the characteristic parabolic shape on the tails of the distributions. Once again, the probabilities decrease to the level of the round-off noise and fluctuate around \( 1E-12 \) near the edges of the fundamental aliased interval \((0,2\pi/\Delta)\). The fact that the cumulative distribution function of \( y \) starts in the round-off noise at \( v=0 \) indicates that \( b=5\pi/2 \) was large enough to guarantee \( y > 0 \) with probability virtually 1. Also indicated on the figure is the origin for random variable \( x \). We have, from (4),

\[ P_x(u) = P_y(u+b); \quad (40) \]

thus for example

\[ \text{Prob}(x < 0) = P_x(0) = P_y(b) = .5. \quad (41) \]
Figure 4. Gaussian; L=7, Δ=.3, b=2.5π, M=256
In figure 5, the only change is to decrease limit L from 7 to 6. The tail error mentioned in (28) et seq. then dominates the round-off noise and has a sinusoidal variation. Aliasing is not a problem, as witnessed by the fact that the cumulative and exceedance distribution functions of random variable y have decayed below 1E-12 well before the edges of the interval are reached.

When limit L is restored to 7, and sampling increment ∆ is increased to .5, aliasing becomes significant, as shown in figure 6. The exceedance distribution function has not yet decayed to the round-off noise level at v = 2π/∆, and the cumulative distribution function shows a large negative probability region near v = 0. Shift b has been maintained at the value 5π/2, corresponding to (39).

When L and ∆ are restored to their values 7 and .3 as for figure 4, but b is decreased to 5π/3, the probability of y becoming negative is, from (4) and (38), \[ f(5\pi/3) = 0.82E-7 \]. This is reflected in the cumulative distribution function for y in figure 7 at v = 0, where the probability value is well above the round-off noise level. Also, the exceedance distribution function develops significantly negative values near v = 2π/∆.

Accurate evaluation of the cumulative and exceedance distribution functions can only be achieved when L, ∆, and b are properly chosen. Probably the optimum combination for the Gaussian variate is displayed in figure 8, where ∆ has been increased to .4, the distributions are centered on the fundamental aliased interval (0, 2π/∆) by choice of b, and L is taken at 7 to avoid tail error.

3. Smirnov

The limiting characteristic function of a measure of goodness of fit based on the sample distribution function was derived by Smirnov and is given by (ref. 12, eq. 30.104)

\[ f(\gamma) = \left( \frac{s}{\sin(s)} \right)^{1/2} \text{ where } s = (1+i)\gamma^* \text{ for } \gamma > 0. \]
Figure 5. Gaussian; L=6, Δ=.3, b=2.5π, M=256
Figure 6. Gaussian; L=7, Δ=.5, b=2.5π, M=256
Figure 7. Gaussian; \( L=7, \Delta = .3, b=5\pi/3, N=256 \)
Figure 8. Gaussian; $L=7$, $\Delta = .4$, $b=2.5\pi$, $M=256$
An expansion about \( f=0 \) yields

\[
f_x(f) = 1 + i \frac{1}{6} f - \frac{1}{40} f^2; \quad \text{i.e.,} \quad u_x = 1/6, \quad \sigma_x^2 = 1/45. \quad (43)
\]

And since the goodness of fit is always positive, random variable \( x \) is positive and we can choose

\[
b=0. \quad (44)
\]

Since

\[
\sin((1+i)\sqrt{f}) \sim i^{1/2} \exp(i\sqrt{f}(1-i)) \quad \text{as} \quad f \to \infty, \quad (45)
\]

it follows that

\[
f_x(f) \sim 2^{3/4} f^{1/4} \exp\left(-\frac{1}{2} \sqrt{f} + i\left(\frac{1}{2} \sqrt{f} - \frac{\pi}{4}\right)\right) \quad \text{as} \quad f \to \infty. \quad (46)
\]

The phase of this term rotates according to \( \sqrt{f}/2 \); if we were to choose \( b=0 \), \( f_y(f) \) would rotate faster than \( f_x(f) \) (linear with \( f \) rather than \( \sqrt{f} \)). This could necessitate a faster sampling rate, which is undesirable.

The cumulative and exceedance distribution functions are plotted in figure 9. \( L \) and \( a \) have been chosen so as to avoid tail error and aliasing. The exceedance distribution function is seen to decay exponentially until it reaches approximately \( 2E-11 \); the bump in the curve at this point is a manifestation of the limited accuracy of the trigonometric functions built into the calculator employed. Larger values of \( v \) lead to round-off noise around the \( 1E-12 \) level.

A comparison of results for this characteristic function, with Requicha's method described in (29) et seq., is given in figure 10 for FFT size \( M=1024 \). The plot labeled with \( N=L=512 \) is precisely Requicha's method. Aliasing is known to be insignificant for \( a=1 \), as seen by reference to figure 9 and observing that extrapolation of the straight line section of the exceedance distribution function would result in probability values near \( 1E-13 \) at \( v=2\pi/4 \). The dashed portion of the \( N=L=512 \) curve in figure 10 in fact
Figure 9. Smirnov; L = 3000, Δ = 1, b = 0, M = 256
Figure 10. Comparison with Requicha's Method;
\[ \Delta=1, \ b=0, \ M=1024 \]
corresponds to negative probability estimates; these grossly inaccurate results are due to an inadequate value of limit L, leading to large tail error.

When N is simply increased to 1023, the middle curve in figure 10 results from Requicha's method. Again, negative estimates are indicated by the dashed portion of this curve, although two orders of magnitude smaller than above. The reasons for these errors have been delineated in (29) et seq.

The bottom-most curve in figure 10 (solid curve) is that obtained by the method proposed in this report for L = 1023. Exceedance distribution function estimates in the 1E-10 range are obtained, but the error returns to the 1E-8 range at v=2π/A. No negative probability values occur. Also, by simply increasing limit L, while keeping FFT size M fixed, the error can be reduced significantly further, as already witnessed by figure 9.

4. Noncentral Chi-Square

Here the random variable x is given by

\[ x = \sum_{k=1}^{K} (g_k + d_k)^2 \]  \hspace{1cm} (47)

where \( \{d_k\} \) are constants, and \( \{g_k\} \) are independent Gaussian random variables with zero-mean and unit variance. The characteristic function of x is

\[ f_x(\xi) = \frac{1-i\xi}{2} \exp \left( \frac{1}{1-2\xi} \right), \]  \hspace{1cm} (48)

where deflection d is defined according to

\[ d^2 = \sum_{k=1}^{K} d_k^2. \]  \hspace{1cm} (49)

We actually consider a more general characteristic function than (48), namely
\[ f_x(\xi) = (1 - i\xi)^{-\nu} \exp \left( \frac{1d^2}{1 - 12\xi} \right) = \exp \left( \frac{1d^2}{1 - 12\xi} - \nu \ln(1 - i\xi) \right), \tag{50} \]

where \( \nu \) is an arbitrary positive real constant. Suppose that we use the principal value logarithm for \( \ln(z) \), where the branch cut lies along the negative real axis of the complex \( z \) plane (ref. 13, sect. 4.1.1). Then since the argument of the logarithm in (50) never crosses the branch cut, form (50) gives the correct characteristic function values automatically for all real \( \xi \), and any \( \nu \).

The probability density function and exceedance distribution function corresponding to (50) are (ref. 14, 6.631 4)

\[ p_x(v) = \frac{1}{2} \exp \left( - \frac{4^2}{2} \right) \left( \frac{\sqrt{\nu}}{\alpha} \right)^{v-1} I_{v-1}(d\sqrt{\nu}) \quad \text{for} \quad v > 0, \]

\[ 1 - p_x(v) = \int_{\xi>0} dt \exp \left( - \frac{d^2 + t^2}{2} \right) \left( \frac{t}{\alpha} \right)^{v-1} I_{v-1}(dt) = Q_0(d, \sqrt{\nu}) \quad \text{for} \quad v > 0. \tag{51} \]

Since the probability density function in (51) is never negative (ref. 13, sect. 9.6.1), (50) is a legal characteristic function. Also because random variable \( x \) is always positive according to (51), we choose shift \( b = 0 \). Plots of the exceedance distribution function, as determined from characteristic function (50) are displayed for various values of \( d \) in figure 11. The values of \( L \) were chosen for each \( d \) value so as to control the tail error below the \( 1 \times 10^{-10} \) level plotted. Direct calculation of the exceedance distribution function directly from (51) would be a formidable task for arbitrary \( \nu \) values.

5. **Product of Correlated Gaussian Variates**

Let

\[ x = st \tag{52} \]

where \( s \) and \( t \) are zero-mean unit-variance Gaussian random variables with correlation coefficient \( \rho \). The joint probability density function of \( s \) and \( t \) is
Figure 11. Non-Central Chi-Square; $\Delta = 0.05$, $b = 0$, $M = 256$
The characteristic function of $x$ is then

$$f_x(\xi) = \exp(i\xi x) = \int du \, dv \, \exp(i\xi uv) \, p_{st}(u,v) =$$

$$= \left[1 - i2\rho \xi + (1-\rho^2) \xi^2 \right]^{-1/2} \left[1 - i(1+\rho) \xi \right]^{-1/2} \left[1 + i(1-\rho) \xi \right]^{-1/2},$$

via repeated use of ref. 14, eq. 3.323 2. The corresponding probability density function of $x$ is

$$p_x(v) = \int \frac{dv}{|v|} \, p_{st}(y,v) = -\frac{1}{\pi \exp \left(\frac{\rho v}{1-\rho^2}\right)} \frac{1}{\sqrt{1-\rho^2}} \frac{1}{\sqrt{1-\rho^2}} \, k_0 \left(\frac{|v|}{\sqrt{1-\rho^2}}\right)$$

for all $v$, via ref. 14, eq. 3.478 4.

(If we transform this probability density function according to (1) and use ref. 14, eq. 6.611 9 and ref. 13, eq. 4.4.15, we get precisely (54). Alternatively, if we transform (54) and modify the contour to wrap around the branch line along the imaginary axis and then use ref. 14, eq. 3.388 2, we get (55). Or we can use ref. 14, eq. 3.754 2.)

We actually consider a more general characteristic function than (54), namely

$$f_x(\xi) = \left[1 - i2\rho \xi + (1-\rho^2) \xi^2 \right]^{-\nu} \exp \left(-\nu \ln \left[1 - i2\rho \xi + (1-\rho^2) \xi^2 \right] \right) =$$

$$= \left[\frac{1}{1-\rho^2} + \left\{\frac{1}{1-\rho^2} \xi - i \frac{\rho}{\sqrt{1-\rho^2}}\right\}^2 \right]^{-\nu}. \quad (56)$$

The mean of this random variable $x$ is given by

$$u_x = 2v\rho. \quad (57)$$
The probability density function corresponding to (56) is

\[ p_x(v) = \frac{1}{2\pi} \int d\xi \exp(-i\xi v) f_x(\xi) = \]

\[ = \frac{1}{2\pi \sqrt{1-\rho^2}} \int_{-\infty}^{\infty} dy \exp \left( \frac{\rho v}{1-\rho^2} - i \frac{yv}{\sqrt{1-\rho^2}} \right) \left( \frac{1}{1-\rho^2} + y^2 \right)^{-\frac{1}{2}} \]  

(58)

where we let

\[ y = \sqrt{1-\rho^2} \xi - \frac{i\rho}{\sqrt{1-\rho^2}}. \]  

(59)

We can move the contour in (58) to the real y-axis, because the branch points of the integrand are at \( y = \pm i\sqrt{1-\rho^2} \) which are outside the path of integration, since \( |\rho| < 1 \). Then using ref. 14, eq. 3.771 2 and ref. 13, eq. 6.1.17, we obtain

\[ p_x(v) = \left( \frac{\rho v}{1-\rho^2} \right)^{-\frac{1}{2}} \exp \left( \frac{\rho v}{1-\rho^2} \right) K_{-\frac{1}{2}} \left( \frac{\rho v}{1-\rho^2} \right) \]  

for all v.  

(60)

Since this probability density function is never negative (ref. 13, sect. 9.6.1), (56) is a legal characteristic function. If we Fourier transform (60) via ref. 14, 6.699 12, we get (56) directly.

There is no simple relation for the cumulative distribution function of this random variable. Nevertheless, it is a simple matter to evaluate directly from characteristic function (56). The \( \ln \) in (56) causes no problems since its argument never crosses the branch cut. A plot for \( \nu=7.7 \) and \( \omega=-.3 \) is displayed in figure 12. The rate of decay of the distribution is different for each tail. The round-off noise is clearly visible at both ends of the range of v values.
Figure 12. Gaussian Product; $L=5, \Delta=.06, b=50\pi/3, M=256$
We now have the capability to handle the following type of statistical problem in a fairly easy fashion. Consider random variable

$$x = \sum_{k=1}^{K} r_k^v$$

(61)

where \(\{r_k\}\) are arbitrary random variables, statistically independent of each other, and with different distributions. Power \(v_k\) is arbitrary (except that \(v_k\) must be a positive integer for those \(r_k\) that can become negative). Let the probability density function of random variable \(r_k\) be \(p_k(v)\). Then the characteristic function of \(r_k\) is

$$g_k(\xi) = \exp \left( i \xi r_k^v \right) = \int dv \exp \left( i \xi v_k \right) p_k(v) =$$

$$= \frac{1}{v_k} \int \frac{dt}{t} \exp(i \xi t) t^{1/v_k} p_k(t^{1/v_k}).$$

(62)

If (62) is not integrable in closed form, it can be evaluated by means of an FFT (one for each \(k\) if the probability density functions or \(v_k\) are all different). Then the characteristic function of random variable \(x\) in (61) is given by

$$f_x(\xi) = \prod_{k=1}^{K} g_k(\xi).$$

(63)

Now the techniques of this report are directly applicable to (63).

An additional example is afforded by
\[ x = \sum_{k=1}^{K} a_k v_k^2 + \left( \sum_{k=1}^{K} b_k v_k \right)^2 + \sum_{k=1}^{K} \gamma_k v_k, \quad (64) \]

where \( \{a_k\}, \{b_k\}, \) and \( \{\gamma_k\} \) are constants, and \( \{v_k\} \) are independent random variables with arbitrary probability density functions. The characteristic function of \( x \) is

\[ f_x(\xi) = \mathbb{E}(\exp(\imath \xi x)) = \int dV p_V(V) \exp (\imath \xi \sum a_k v_k^2 + (\sum b_k v_k)^2 + \sum \gamma_k v_k)), \quad (65) \]

where \( V = (v_1, v_2, ..., v_K). \) Now since

\[ \left( \frac{ia}{\pi} \right)^{1/2} \int dy \exp(-\imath ax^2 + \imath by) = \exp \left( \frac{ib^2}{4a} \right) \text{ for } a \neq 0, \quad (66) \]

we identify \( a = \xi/4, \) \( b = \xi \sum b_k v_k, \) eliminate the square in the exponent, and express (65) as

\[ f_x(\xi) = \int dV p_V(V) \exp (\imath \xi \sum a_k v_k^2 + \imath \xi \sum \gamma_k v_k)) \]

\[ \left( \frac{\xi}{\pi} \right)^{1/2} \int dy \exp \left( \frac{\imath \xi y^2}{2} + \imath \xi y \sum b_k v_k \right) \]

\[ = \left( \frac{\xi}{\pi} \right)^{1/2} \int dy \exp \left( \frac{\imath \xi y^2}{4} \right) \prod_{k=1}^{K} \left\{ \int dv_k p_{k,V_k}(v_k) \exp (\imath \xi (a_k v_k^2 + \gamma_k v_k + b_k v_k)) \right\}, \quad (67) \]

where

\[ p_{k,V_k}(v_k) = \prod_{k=1}^{K} \left\{ p_k(v_k) \right\}. \quad (68) \]

The inner integrals in (67) can either be done analytically or numerically. Then the remaining single integral on \( y \) must be numerically evaluated to find characteristic function \( f_x(\xi). \) As an example, if \( v_k \) is exponentially distributed

31
\[ p_k(v) = a_k \exp(-a_k v) \text{ for } v > 0, \tag{69} \]

then the inner integrals in (67) are \( w \)-functions; see ref. 13, ch. 7. A simpler method of handling general quadratic expressions like (64) with Gaussian \( V \) is presented in ref. 15.
SUMMARY

An accurate method for efficient evaluation of the cumulative and exceedance distribution functions has been derived and applied to several examples to illustrate its utility. Choice of the sampling increment $\Delta$ applied to the characteristic function controls the aliasing problem, and selection of the limit $L$ minimizes the tail error; the effects of both of these parameters can be observed from sample plots of the distributions and can be modified if needed. Additionally, shift $b$ must be chosen so as to yield a positive random variable with probability virtually 1. The number of distribution values yielded depends on the size of the FFT employed and can be independently selected to yield the desired spacing in distribution values.
APPENDIX A. SAMPLING FOR A FOURIER TRANSFORM

Suppose we are interested in evaluating Fourier transform

$$G(f) = \int dt \exp(-i2\pi ft) g(t). \quad (A-1)$$

If we sample at interval $\Delta$ in $t$ in (A-1), and use integration weighting $w(t)$, we have the approximation to $G(f)$,

$$\tilde{G}(f) \equiv \int dt \exp(-i2\pi ft) g(t) S_\Delta(t) w(t)$$

$$= G(f) \otimes \frac{1}{\Delta} \sum_{n} S_\Delta(\frac{t-n\Delta}{\Delta}) W(f)$$

$$= \frac{1}{\Delta} \sum_{n} G(f - \frac{n}{\Delta}) \otimes W(f), \quad (A-2)$$

where infinite impulse train (sampling function)

$$S_\Delta(t) = \sum_{n} S(t-n\Delta), \quad (A-3)$$

and $\otimes$ denotes convolution.

The term

$$\frac{1}{\Delta} \sum_{n} G(f - \frac{n}{\Delta}) \quad (A-4)$$

in (A-2) is an infinitely aliased version of desired function $G(f)$; this aliasing is an unavoidable effect due to sampling at increment $\Delta$. However, to minimize any further aliasing in (A-2), we would like $W(f) = \delta(f)$, which requires $w(t) = 1$ for all $t$; strictly, all we need is

$$w(n\Delta) = 1 \quad \text{for all } n. \quad (A-5)$$
That is, the best weighting in (A-2) is uniform.

As an example, for Simpson's rule, we have weighting

\[ w(n\Delta) = \ldots, \frac{2}{3}, \frac{4}{3}, \frac{2}{3}, \frac{4}{3}, \frac{2}{3}, \ldots = 1 + \frac{1}{3}(-1)^n \text{ or } 1 - \frac{1}{3}(-1)^n, \]  

(A-6)

which can be represented as samples of time function

\[ w(t) = 1 + \frac{1}{3} \exp(i\pi t/\Delta) \text{ or } 1 - \frac{1}{3} \exp(i\pi t/\Delta). \]  

(A-7)

The corresponding transform is

\[ W(f) = \int dt \exp(-i2\pi ft) \, w(t) = \]

\[ = \mathcal{S}(f) + \frac{1}{3} \mathcal{S}(f - \frac{1}{2\Delta}) \text{ or } \mathcal{S}(f) - \frac{1}{3} \mathcal{S}(f - \frac{1}{2\Delta}). \]  

(A-8)

But this window function substituted in (A-2) results in an extra aliasing lobe in \( \hat{G}(f) \), halfway between the unavoidable major lobes of (A-4) at multiples of \( 1/\Delta \), of magnitude \( 1/3 \) as large. This effect very adversely affects the quality of \( \hat{G}(f) \) insofar as its approximation to the desired \( G(f) \) is concerned. Thus the best sampling plan in (A-2) is the equal weight structure of (A-5) when one wants to approximate the Fourier transform of (A-1). For a bounded region, this is modified to the Trapezoidal rule, i.e., half-size weights at the boundaries.
APPENDIX B. LISTINGS OF PROGRAMS FOR FIVE EXAMPLES

The following listings are programs in BASIC for the Hewlett Packard 9845B Desktop Calculator. The FFT utilized is one with the capability of a zero subscript and is listed at the end of the appendix. Mathematically, the FFT programmed is

\[
Z_m = \sum_{k=0}^{M-1} \exp(-i2\pi mk/M) z_k \quad \text{for } 0 \leq m \leq M-1,
\]

where the arrays \(\{z_k\}_{0}^{M-1}\) and \(\{Z_m\}_{0}^{M-1}\) are handled directly, including the zero-subscript terms \(z_0\) and \(Z_0\).

A detailed explanation of the first program below for Chi-Squared random variables is as follows: line 20 specifies the parameter \(K\), where \(2K\) is the number of squared-Gaussian random variables summed to yield random variable \(x\). Lines 30-60 require inputs \(L, \Delta, a, b, M\) respectively, on the part of the user. Line 110 is the input of mean \(u_x\) of random variable \(x\), as evaluated analytically from characteristic function \(f_x(\xi)\). Lines 180-210 specifically evaluate the characteristic function \(f_y(\xi)\) at general point \(\xi\). All of these lines mentioned thus far require inputs on the part of the user and are so noted in the listing by the presence of a single : on each line; the comments after a double !! are for information purposes only and need not be modified. This convention is also adopted in the remaining listings.

Lines 220-240 accomplish the collapsing operation of (32)-(33). The cumulative and exceedance distribution functions are finally evaluated and stored in arrays \(X(*)\) and \(Y(*)\) in lines 400-410.

Some further elaboration is necessary for the listing of the Smirnov characteristic function as given by (42). Since a characteristic function is a continuous function of real \(\xi\), the square root in (42) is not a principal value square root, but in fact must yield a continuous function in \(\xi\). In
order to achieve this, the argument of the square root is traced continuously from $\gamma = 0$ (line 110). If an abrupt change in phase is detected, a polarity indicator takes note of this fact (line 250) and corrects the final values of characteristic function $f_y(\gamma)$ (lines 260-270). No problems are encountered with complex $\sin(z)$ since it is analytic for all $z$.

10 ! CHI-SQUARE CHARACTERISTIC FUNCTION $1/(1-i 2 \times i)^4$
20 K=4 ! 2K=8 degrees of freedom
30 L=204 ! Limit on integral of char. function
40 Delta=.075 ! Sampling increment on char. function
50 Bs=0 ! Shift b
60 M=2^8 ! Size of FFT
70 PRINT IS 0
80 PRINT "$L = \text{";}L,"\Delta = \text{";}Delta,"b = \text{";}Bs,"M = \text{";}M$
90 REDIM X(0:M-1),Y(0:M-1)
100 DIM X(0:1023),Y(0:1023) ! Mean of random variable x
110 Mu=2*K
120 Muy=Mu+Bs
130 X(0)=0
140 Y(0)=.5*Delta*Muy
150 N=INT(L/Delta)
160 FOR Ns=1 TO N
170 Xi=Delta*Ns ! Argument $\xi$ of char. fn.
180 C=Xi+Xi ! Calculation of
190 CALL Mul(1,-C,1,-C,A,B) ! characteristic
200 CALL Mul(A,B,A,B,C,D) ! function $f_y(\xi)$
210 CALL Div(1,0,0,0,C,D,Fy,Fy)
220 Ms=Ns MOD M ! Collapsing
230 X(Ms)=X(Ms)+Fy-Ns
240 Y(Ms)=Y(Ms)+Fy1-Ns
250 NEXT Ns
260 CALL fft10z(M,X(*),Y(*)) ! 0 subscript FFT
PLOTTER IS "GRAPHICS"

SCALE 0,M,-14,0
LINE TYPE 3
GRID M/8,1
PENUP
LINE TYPE 1
B=Bs*M*Delta/(2*PI)!! Origin for random variable x
MOVE B,0
DRAW B,-14
PENUP
FOR Ks=0 TO M-1
T=Y(Ks)/PI-Ks/M
X(Ks)=.5-T!! Cumulative probability in X(*)
Y(Ks)=Pr=.5+T!! Exceedance probability in Y(*)
IF Pr>=1E-12 THEN Y=LGT(Pr)
IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
IF ABS(Pr)<1E-12 THEN Y=-12
PLOT Ks,Y
NEXT Ks
PENUP
PRINT Y(0);Y(1);Y(M-2);Y(M-1)
FOR Ks=0 TO M-1
Pr=X(Ks)
IF Pr>=1E-12 THEN Y=LGT(Pr)
IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
IF ABS(Pr)<1E-12 THEN Y=-12
PLOT Ks,Y
NEXT Ks
PENUP
PAUSE
DUMP GRAPHICS
PRINT LIN(5)
PRINTER IS 16
END

SUB Mul'X1,Y1,X2,Y2,A,B)
Z1*Z2
A=X1*X2-Y1*Y2
B=X1*Y2+X2*Y1
SUBEND

SUB Div(X1,Y1,X2,Y2,A,B)
Z1/22
T=X2*X2+Y2*Y2
A=(X1*X2*Y1*Y2)*T
B=(Y1*X2-X1*Y2)*T
SUBEND

SUB F10z(N,X(*),Y(*))!! N <= 2^10 = 1024, N=2^INTEGER
8 subscript
GAUSSIAN CHARACTERISTIC FUNCTION \( \exp(-0.5 \times \xi^2) \)

10 \( L = 7 \)  
20 \( \Delta = 0.3 \)  
30 \( B = 0.375 \times (2 \pi / \Delta) \)  
40 \( M = 2^8 \)  
50 \( \text{PRINTER IS} \)  
60 \( \text{REDIM} \)  
70 \( \text{DIM} \)  
80 \( \text{FOR} \)  
90 \( \text{NEXT} \)  
100 \( \text{mu} = 0 \)  
110 \( \text{mu} = \mu + B \)  
120 \( X(0) = 0 \)  
130 \( Y(0) = \Delta \times M \times \mu \)  
140 \( N = \text{INT}(L / \Delta) \)  
150 \( \text{FOR} \)  
160 \( \text{NEXT} \)  
170 \( \text{CALL FFT10Z}(M, X(*), Y(*)) \)  
180 \( \text{PLOTTER IS} \)  
190 \( \text{GRAPHICS} \)  
200 \( \text{SCALE} 0, M, -14, 0 \)  
210 \( \text{LINE TYPE} \)  
220 \( \text{GRID M/8, 1} \)  
230 \( \text{PENUP} \)  
240 \( \text{FOR} \)  
250 \( \text{NEXT} \)  
260 \( \text{SUB} \)  
270 \( \text{END} \)  
280 \( \text{PRINT} \)  
290 \( \text{DUMP GRAPHICS} \)  
300 \( \text{PRINT} \)
SMIRNOV CHARACTERISTIC FUNCTION \( [s/sin(s)]^{1/2} \) where \( s=(1+i)sqr(x) \)

Limit on integral of char. function
Sampling increment on char. function
Shift \( b \)
Size of FFT

PRINT "L = ";L," Delta = ";Delta," b = ";b," M = ";M
REDIM X(0:M-1),Y(0:M-1)
DIM X(0:1023),Y(0:1023)

Mean of random variable \( x \)
Argument of square root
Polarity indicator

Myx=1/6
R=0
P=1

Muy=Mux+Bs

Y(0)=.5*Delta*Muy

N=INT(L/Delta)

FOR Ns=1 TO N
Xi=Delta*Ns
A=SQR(Xi)
CALL Sin(R,A,B,C)
CALL Div(A,A,B,C,D,E)
CALL Sqr(D,E,R,B)

Ro=R
R=ATN(B/A)

IF ABS(R-Ro)>1.6 THEN P=-P

Fyr=A*P
Fyi=B*P

Ms=Ns MOD M
X(Ms)=X(Ms)+Fyr/Ns
Y(Ms)=Y(Ms)+Fyi/Ns

NEXT Ns

CALL FFT10z(M,X(*),Y(*)) ' 0 subscript FFT

GRAPHICS
SCALE 0,11,-14,0
LINE TYPE 3
GRID M/8,1
PENUP
LINE TYPE 1

B=Bs*M*Delta/(2*PI)
MOVE B,0
DRAW B,-14
PENUP
FOR Ks=0 TO M-1
T=Y(Ks)/PI-Ks/M
X(Ks)=.5-T
Y(Ks)=Pr,+.5+T

IF Pr>1E-12 THEN Y=LGT(Pr)
IF Pr<-.1E-12 THEN Y=-24-LGT(-Pr)

NEXT Ks
```plaintext
530  PENUP
540  PRINT Y(0);Y(1);Y(M-2);Y(M-1)
550  FOR Ks=0 TO M-1
560  Pr=X(Ks)
570  IF Pr>1E-12 THEN Y=LGT(Pr)
580  IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
590  IF ABS(Pr)<1E-12 THEN Y=-12
600  PLOT Ks,Y
610  NEXT Ks
620  PENUP
630  PAUSE
640  DUMP GRAPHICS
650  PRINT LIN(5)
660  PRINTER IS 16
670  END
680
690  SUB Div(X1,Y1,X2,Y2,A,B)
700  T=X2*X2+Y2*Y2
710  A=(X1*X2+Y1*Y2)/T
720  B=(Y1*X2-X1*Y2)/T
730  SUBEND
740
750  SUB Sqr(X,Y,A,B)
760  IF X<0 THEN 800
770  A=B=SQR(.5*ABS(Y))
780  IF Y<0 THEN B=-B
790  GOTO 910
800  F=SQR(SQR(X*X+Y*Y))
810  T=.5*ATN(Y/X)
820  A=F*COS(T)
830  B=F*SIN(T)
840  IF X>0 THEN 910
850  T=A
860  A=-B
870  B=T
880  IF Y>0 THEN 910
890  A=-A
900  B=-B
910  SUBEND
920
930  SUB Sin(X,Y,A,B)
940  E=EXP(Y)
950  A=.5*SIN(X)*E*(E+1/E)
960  IF ABS(Y)<.1 THEN 990
970  S=.5*(E-1/E)
980  GOTO 1010
990  S*Y*Y
1000  S=Y*(120+S*(20+S)):120
1010  B=COS(X)*S
1020  SUBEND
1030
1040  SUB Fft1024(N,X(*),Y(*))
```

B-6
10 ! NON-CENTRAL CHI-SQUARE CHARACTERISTIC FUNCTION
20 ! $\exp\left(i \cdot \frac{2 \cdot x_i}{s}\right) / s^{-\nu}$ where $s = 1-1^{2} x_i$
30 Nu=2.7 ! Power law $\nu$
40 Ds=3 ! Deflection d
50 L=500 ! Limit on integral of char. function
60 Delta=.05 ! Sampling increment on char. function
70 Bs=0 ! Shift b
80 M=2^8 ! Size of FFT
90 PRINTER IS 0
100 PRINT "L =";L,"Delta =";Delta,"b =";Bs,"M =";M
110 REDIM X(0:M-I),Y(0:M-I)
120 DIM X(0:1023),Y(0:1023)
130 D2=Ds^Ds ! Calculate parameter
140 Mux=2*Nu+D2 ! Mean of random variable x
150 Muy=Mux*Bs
160 X(0)=0
170 Y(8)=.5*Delta*Muy
180 N=INT(L/Delta)
190 FOR Ns=1 TO N
200 Xi=Delta*Ns ! Argument $x_i$ of char. fn.
210 T=Xi+Xi ! Calculation of
220 CALL Div(0,D2*Xi,1,-T,A,B) ! characteristic
230 CALL Log(1,-T,C,D) ! function
240 CALL Exp(R-Nu*C,B-Nu*D+Bs*Xi,Fyr,Fyi) ! $f_y(x_i)$
250 Ms=Ns MOD M ! Collapsing
260 X(Ms)=X(IMs)+Fyr/Ns
270 Y(Ms)=Y(Ms)+Fyi/Ns
280 NEXT Ns
290 CALL FFT10Z(M,X(*),Y(*)) ! 0 subscript FFT
300 PLOTTER IS "GRAPHICS"
310 GRAPHICS
320 SCALE 0,M,-14,0
330 LINE TYPE 3
340 GRID M/8,1
350 PENUP
360 LINE TYPE 1
370 B=Bs*M*Delta/(2*PI) ! Origin for random variable x
380 MOVE B,0
390 DRAW B,-14
400 PENUP
410 FOR Ks=0 TO M-1
420 Ty(Ks)=PI-Ks/M
430 X(Ks)*.5-T ! Cumulative probability in $X(*)$
440 Y(Ks)*Pr=.5+T ! Exceedance probability in $Y(*)$
450 IF Pr>1E-12 THEN Y=LGT(Pr)
460 IF Pr<1E-12 THEN Y=-24-LGT(-Pr)
470 IF ABS(Pr)<1E-12 THEN Y=-12
480 PLOT Ks,Y
490 NEXT Ks
500 PENUP
510 PRINT Y(0);Y(1);Y(M-2);Y(M-1)
520 FOR Ks=0 TO M-1
530 Pr=X(Ks)
540 IF Pr>1E-12 THEN Y=LGT(Pr)
550 IF Pr<-1E-12 THEN Y=-24-LGT(-Pr)
560 IF ABS(Pr)<1E-12 THEN Y=12
570 PLOT Ks,Y
580 NEXT Ks
590 PENUP
600 PAUSE
610 DUMP GRAPHICS
620 PRINT LIN(S)
630 PRINTER IS 16
640 END
650 !
660 SUB Div<X1,Y1,X2,Y2,A,B> ! Z1/22
670 T=X2*X2+Y2*Y2
680 A=(X1*X2+Y1*Y2)/T
690 B=(Y1*X2-X1*Y2)/T
700 SUBEND
710 !
720 SUB Exp<X,Y,A,B> ! EXP(Z)
730 T=EXP(X)
740 A=T*COS(Y)
750 B=T*SIN(Y)
760 SUBEND
770 !
780 SUB Log<X,Y,A,B> ! PRINCIPAL LOG(Z)
790 A=.5*LOG(X*Y+Y)
800 IF X>0 THEN 830
810 B=.5*PI*SGN(Y)
820 GOTO 850
830 B=ATN(Y/X)
840 IF X<0 THEN B=B+PI*(1-2*(Y<0))
850 SUBEND
860 !
870 SUB Fft10z<N,X(*)>Y(*)> ! N <= 2^10 = 1024, N=2^INTEGER 0 subscript

B-8
10 ! GAUSSIAN PRODUCT CHARACTERISTIC FUNCTION (56)
20 Nu=7.7       ! Power Nu
30 Rho=-.3      ! Correlation coefficient
40 L=5          ! Limit on integral of char. function
50 Delta=.06    ! Sampling increment on char. function
60 Bs=.5*(2*PI/Delta)  ! Shift b, as fraction of alias interval
70 M=2^8        ! Size of FFT
80 PRINTER IS 0
100 REDIM X(0:M-1),Y(0:M-1)
110 DIM X(0:1023),Y(0:1023)
120 T1=1-Rho*Rho Calculate
130 T2=2*Rho    ! parameters
140 Mux=2*Nu*Rho ! Mean of random variable x
150 Muy=Mux+Bs
160 X(0)=0
170 Y(0)=.5*Delta*Muy
180 N=INT(L/Delta)
190 FOR Ns=1 TO N
200 Xi=Delta*Ns  ! Argument xi of char. fn.
210 CALL Log(1+T1*Xi*Xi,-T2*Xi,A,B) ! Calculation of
220 CALL Exp(-Nu*A,Bs*Xi-Nu*B,Fy/Fy) ! characteristic function fy(xi)
230 Ms=Ns MOD M
240 X(Ms)=X(Ms)+Fy/Ns
250 Y(Ms)=Y(Ms)+Fyi/Ns
260 NEXT Ns
270 CALL Fft10z(M,X(*),Y(*))  ! 0 subscript FFT
280 PLOTTER IS "GRAPHICS"
290 GRAPHICS
300 SCALE 0,M,-14,0
310 LINE TYPE 3
320 GRID M-8,1
330 PENUP
340 LINE TYPE 1
350 B=Bs*M*Delta/(2*PI)  ! Origin for random variable x
360 MOVE B,0
370 DRAW B,-14
380 PENUP
FOR Ks=0 TO M-1
T=Y(Ks)/PI-Ks/M
X(Ks)=.5-T
!! Cumulative probability in X(*)
Y(Ks)=Pr=.5+T
!! Exceedance probability in Y(*)
IF Pr>=1E-12 THEN Y=LGT(Pr)
IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
IF ABS(Pr)<1E-12 THEN Y=-12
PLOT Ks,Y
NEXT Ks
PENUP
PRINT Y(0);Y(1);Y(M-2);Y(M-1)
FOR Ks=0 TO M-1
Pr=X(Ks)
IF Pr>=1E-12 THEN Y=LGT(Pr)
IF Pr<=-1E-12 THEN Y=-24-LGT(-Pr)
IF ABS(Pr)<1E-12 THEN Y=-12
PLOT Ks,Y
NEXT Ks
PENUP
PAUSE
DUMP GRAPHICS
PRINT LINC(5)
PRINT LINER IS 16
END

SUB Exp(X,Y,A,B)
T=EXP(X)
A=T*COS(Y)
B=T*SIN(Y)
SUBEND

SUB Log(X,Y,A,B)
A=.5*LOG(X*X+Y*Y)
IF X>0 THEN 750
B=.5*PI*SGNC(Y)
GOTO 770
B=ATN(Y/X)
IF X<0 THEN B=PI+(1-2*(Y<0))
SUBEND

SUB FFT10z(N,X(*),Y(*))
N = 2^10 = 1024, N=2^INTEGER
380 \( N4=N1+N3 \)
390 \( \log_{2}n=\text{INT}(1.4427 \times \log(n)+.5) \)
400 FOR I1=1 TO Log2n
410 I2=2^(Log2n-I1)
420 I3=2*I2
430 I4=N/I3
440 FOR I5=1 TO I2
450 I6=(I5-1)*I4+1
460 IF I6<=N2 THEN 500
470 N6=-C(N4-I6-1)
480 N7=-C(I6-N1-1)
490 GOTO 520
500 N6=C(I6-1)
510 N7=-C(N3-I6-1)
520 FOR I7=0 TO N-13 STEP 13
530 I8=I7+15
540 I9=I8+12
550 N8=X(I8-1)-X(I9-1)
560 N9=Y(I8-1)-Y(I9-1)
570 X(I8-1)=X(I8-1)+X(I9-1)
580 Y(I8-1)=Y(I8-1)+Y(I9-1)
590 X(I9-1)=N6*N8-N7*N9
600 Y(I9-1)=N6*N9+N7*N8
610 NEXT 17
620 NEXT 15
630 NEXT 11
640 I1=Log2n+1
650 FOR I2=1 TO 10
660 C(I2-1)=1
670 IF I2<=Log2n THEN 690
680 C(I2-1)=2^(I1-12)
690 NEXT 12
700 K=1
710 FOR I1=1 TO C(9)
720 FOR I2=11 TO C-9 STEP C-9
730 FOR I3=12 TO C-7 STEP C-7
740 FOR I4=13 TO C-6 STEP C-6
750 FOR I5=14 TO C-5 STEP C-5
760 FOR I6=15 TO C-4 STEP C-4
770 FOR I7=16 TO C-3 STEP C-3
780 FOR I8=17 TO C-2 STEP C-2
790 FOR I9=18 TO C-1 STEP C-1
800 FOR I10=19 TO C-0 STEP C-1
810 J=10
820 IF J=K THEN 890
830 A=X(K-1)
840 X(K-1)=X(J-1)
850 X(J-1)=A
860 A=Y(K-1)
870 Y(K-1)=Y(J-1)
880 Y(J-1)=A
890 K=K+1
900 NEXT 110
910 NEXT 19
920 NEXT 18
930 NEXT 17
940 NEXT 16
950 NEXT 15
960 NEXT 14
970 NEXT 13
980 NEXT 12
990 NEXT 11
1000 SUBEND
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