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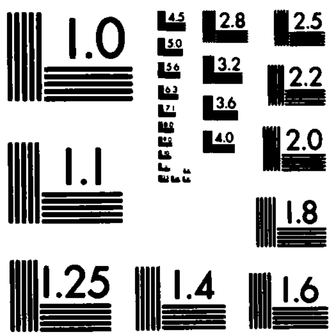

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THE TRAVELLING SALESMAN PROBLEM IN GRAPHS

WITH 3-EDGE CUTSETS

by

G. Cornuéjols*
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August, 1983

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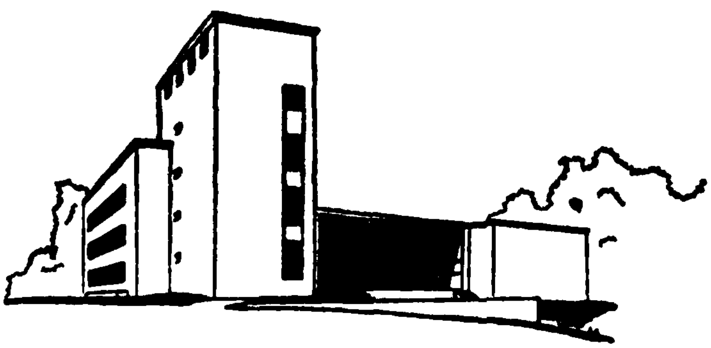

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This report was preared as part of the activities of the Management Sciences Research Group, Carnegie-Mellon University, Under Contract No. N00014-82-K-0329 NR 047-048 with the U. Office of Naval Research and in part by NSF grants ENG 79-02506 and ECS 8205425 and in part by the Natural Sciences and Engineering Research Council of Canada. Reproduction in whole or in part is permitted for any purpose of the U. S. Government.

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ABSTRACT

In this paper we analyze decomposition properties of a graph which, when they occur, permit a polynomial solution of the travelling salesman problem and a description of the travelling salesman polytope by a system of linear equalities and inequalities. The central notion is that of a 3-edge cutset, namely a set of 3 edges which, when removed, disconnects the graph. Conversely, our approach can be used to construct classes of graphs for which there exists a polynomial algorithm for the travelling salesman problem. We illustrate the approach on two examples, Halin graphs and prismatic graphs.

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1. Introduction

Many NP-hard graph problems become polynomially solvable for special classes of graphs. For example, the problem of finding a maximum weight stable set in a general graph is NP-hard, but is polynomially solvable for series-parallel graphs (Boulala, Uhry [2]), for claw-free graphs (Minty [23], see also Sbihi [27]) and for perfect graphs (Grötschel, Lovász and Schrijver [15]). Many optimization problems on graphs can be formulated as integer programs. There is a polyhedron naturally associated with any integer program, namely the convex hull of the feasible integer solutions. Following the pioneering work of Edmonds on matchings [8] and matroids [9], the development of a polynomial algorithm for such a problem has often been very closely related to the obtaining of a system of linear equalities and inequalities whose solution set is the corresponding polyhedron. Knowledge of such a linear system is useful for two reasons: First, if the size of the system is small, then general linear programming methods can be used to solve the integer program. Second, even when this linear system is large, linear programming duality provides a good optimality criterion.

Recently Grötschel, Lovász and Schrijver [15] have shown that there is indeed a close connection between the problem of finding a polynomial optimization algorithm and the problem of finding a linear description of the underlying polyhedron. They showed, by making use of the polynomially bounded Schor-Kachian ellipsoidal algorithm for linear programming, that there exists a polynomially bounded algorithm for an optimization problem if and only if there is a polynomially bounded algorithm for the associated separation problem: given a point x, either verify that it belongs in the polyhedron or else find a hyperplane that separates it from the polyhedron. (See [15] for a precise description of this result.)

In fact, their algorithm for finding a maximum weighted stable set in a perfect graph makes use of an earlier theorem of Chvatal which gives a complete linear system sufficient to define the convex hull of the incidence vectors (or characteristic vectors) of the stable sets. The Boulala-Uhry algorithm for series-parallel graphs is developed in connection with a defining linear system for the stable set polyhedra for these graphs. Curiously, the Minty algorithm for maximum weight stable sets in claw-free graphs does not provide an explicit linear description of the associated polyhedra, and it would be fair to say that the obtaining of such a linear system is one of the outstanding open problems of polyhedral combinatorics. Giles and Trotter [11] have shown that such a system will be less "simple" than for the other classes for which such a system is known.

At present, perhaps surprisingly, only very few classes of travelling salesman problems (TSP's) are known to be polynomially solvable (see Gilmore and Gomory [12], Syslo [28], Lawler [20], Garfinkel [10], Cutler [7] and Ratliff and Rosenthal [26]) . They are generally cases of the asymmetric TSP and assume a special cost structure. To our knowledge there is no nontrivial class of graphs for which the TSP polytope has been explicitly described by a system of linear inequalities.

In this paper we show that the TSP can be solved polynomially for a class of graphs which includes, as special cases, Halin graphs and prismatic graphs. This algorithm is based on some more general decomposition properties which are presented in Section 2. (It is interesting to relate these decomposition ideas to those of Boulala and Uhry [2].) Sections 3 gives the examples of Halin and prismatic graphs whereas Section 4 deals with polyhedral theorems. The last two sections contain additional results and extensions. The remainder of this section will be devoted to some basic definitions and notation.

We let $G=(V,E)$ denote a graph with vertex set V and edge set E. If an edge $e\in E$ is incident with vertices $u,v\in V$, then we write $e=(u,v)$. For any $S\subseteq V$ we let $\delta(S)$ denote the set of edges with one end-vertex in S and the other in $V-S$, i.e., $\delta(S)=\{(u,v)\in E: u\in S, v\notin S\}$. For $v\in V$ we abbreviate $\delta(\{v\})$ by $\delta(v)$. When S and $V-S$ each contain at least two vertices, we call $\delta(S)$ a *nontrivial edge cutset*, or simply an *edge cutset*, and call S and $V-S$ its *shores*. (For $v\in V$, we call $\delta(v)$ a *trivial edge cutset*.) For $S\subseteq V$, we let $G\times S$ denote the graph obtained by *shrinking* (or contracting) S. That is, the vertices of $G\times S$ are all vertices of $V-S$, plus a new pseudovertex $\bar{S}$ obtained by identifying all vertices of S. The edges of $G\times S$ are defined as follows (Figure 1):

(i) An edge with both ends in S disappears;

(ii) An edge with both ends in $V-S$ remains unchanged;

(iii) An edge of $\delta(S)$ now joins the incident vertex of $V-S$ and the pseudo-vertex $\bar{S}$.

For any $S\subseteq V$ we let $\gamma(S)$ denote the set of edges of G having both ends in S and we let $G[S]$ denote the subgraph of G induced by the vertex set S, i.e., $G[S]=(S,\gamma(S))$. If $x=(x(e):e\in E)$ is any function from E into R then for any $\bar{E}\subseteq E$ we define $x(\bar{E})=\sum(x(e):e\in\bar{E})$.

A *Hamilton cycle* of G is a cycle passing through each vertex exactly once. A *Hamilton path* between vertices u and v is a path joining u and v

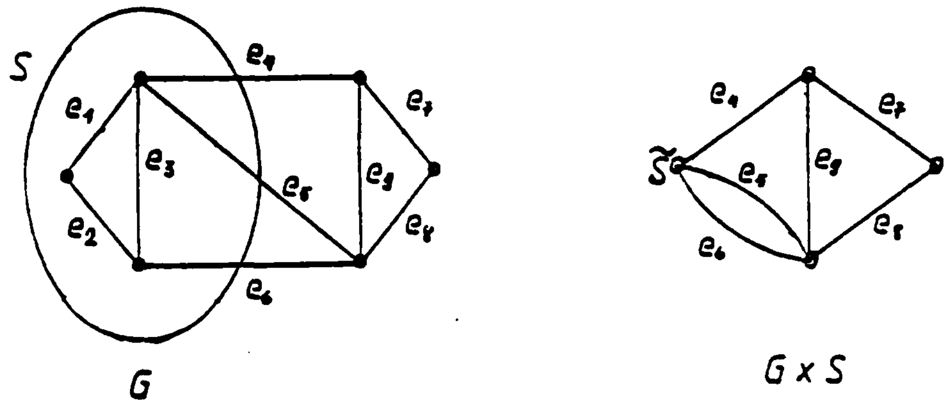

Figure 1

which passes exactly once through each vertex of G. A graph is called *Hamiltonian* if it contains at least one Hamilton cycle. A *tour* is the incidence vector of the edges belonging to a Hamilton cycle. Thus a tour is a 0-1 vector $x=(x(e):e\in E)$ having $|E|$ components, such that the set of edges having $x(e)=1$ is the edge set of a Hamilton cycle.

The *travelling salesman polytope*, $TSP(G)$, is the convex hull of the tours of G. That is, $TSP(G)$ consists of all those $y\in R^E$ which can be expressed as a convex combination of tours of G. Therefore, $TSP(G)$ is the smallest convex polytope in R^E which contains all tours of G. Every extreme point of $TSP(G)$ will be a tour, and conversely, every tour is an extreme point of $TSP(G)$.

Now suppose that we are given a vector $l=(l(e):e\in E)$ of real edge costs. The cost of a set $\bar{E}\subseteq E$ is simply $l(\bar{E})=\sum(l(e):e\in\bar{E})$. The *travelling salesman problem* (abbreviated by TSP) is the following: Given a graph $G=(V,E)$ and a vector $l=(l(e):e\in E)$ of edge costs, find a Hamilton cycle for which the cost of the edge set is minimized. This problem is well known to be NP-hard for general graphs. It has attracted a great deal of attention in the last two decades and references on the subject are too numerous to be listed here completely. One can consult for example Little et al [21], Held and Karp [19] and more recently, Grötschel [14], Crowder and Padberg [6].

Solving a *TSP* is equivalent to minimizing $\sum(l(e)x(e): e \in E)$ over all $x \in TSP(G)$, for the minimum of an linear objective function over a polytope is always attained at an extreme point, in our case a tour. Because of the fact that the number of Hamilton cycles in a graph can grow exponentially with the number of vertices, the above definition of $TSP(G)$ does not provide a reasonable means for solving the travelling salesman problem. However a classical result of polyhedral theory is that every polytope is the solution set to a finite system of linear inequalities. For several classes of graphs described in this paper, the size of such a system for $TSP(G)$ grows linearly with the number of vertices, even though the number of tours grows exponentially. Thus the polyhedral results presented here enable us to solve *TSP*'s in these classes of graphs using standard linear programming algorithms as an alternative to the direct algorithms presented in the paper.

However a complete description of $TSP(G)$ by a system of linear inequalities is not known for general graphs. It is known that the number of essential inequalities in such a system can grow at least exponentially with the number of vertices. A partial description can be found in Grötschel and Padberg [16], Grötschel [13], Cornuejols and Pulleyblank [5] and Grötschel and Pulleyblank [17].

2. Some Basic Results

The basic reduction:

Let $\{e,f,g\}$ be a 3-edge cutset and let S and $V-S$ be its two shores. Assume (see Figure 2) that there are distinct vertices u,v,w, of S incident with e,f,g respectively. Let L_{uv},L_{uw},L_{vw} be the costs of the optimal Hamilton paths between u and v, u and w and v and w respectively in $G[S]$, where we set this cost to some large value L if no such Hamilton path exists. (For example, we can let L be twice the sum of the absolute values of the edge costs plus one.)

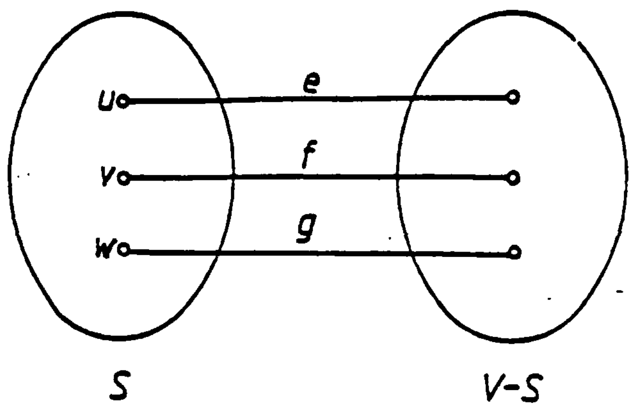

Figure 2

Let α,β,ϵ be defined by the following system of equations:

$$\alpha+\beta=L_{uv}$$

$$\beta+\epsilon=L_{vw}$$

$$\alpha+\epsilon=L_{uw}.$$

This system has the unique solution

$$\alpha=\frac{1}{2}(L_{uv}-L_{vw}+L_{uw})$$

$$\beta = \frac{1}{2}(L_{vw} - L_{uw} + L_{uv})$$

$$\epsilon = \frac{1}{2}(L_{uw} - L_{uv} + L_{vw}).$$

Now define the vector l' of edge costs for $G \times S$ by letting $l' \equiv l$ for all edges except e, f and g and by letting $l'(e) \equiv l(e) + \alpha$, $l'(f) \equiv l(f) + \beta$, $l'(g) \equiv l(g) + \epsilon$. If $G \times S$ has no Hamilton cycle or if the cost of an optimum Hamilton cycle of $G \times S$ with respect to l' is at least $L/2$ then G has no Hamilton cycle. Otherwise the cost of an optimum Hamilton cycle of $G \times S$, with respect to l', is the cost of an optimum Hamilton cycle of G with respect to l. Furthermore the optimum cycle for $G \times S$ can be extended to an optimum Hamilton cycle of G, with respect to l, by taking an appropriate optimum Hamilton path of $G[S]$.

Thus, *effectively*, the basic reduction reduces the *TSP* for G to four *TSP*'s, three for $G \times (V-S)$ to compute L_{uv}, L_{uw} and L_{vw}, and one for $G \times S$. This rather simple observation does, however, have some useful algorithmic consequences as observed below.

A *3-connected graph* is a graph such that at least 3 vertices must be removed in order to disconnect it. Given a vertex set $S \subseteq V$, let $\bar{S} = V - S$.

LEMMA 2.1. Given a 3-connected graph G and vertex sets S_1 and S_2 such that $|\delta(S_1)| = |\delta(S_2)| = 3$, then at least one of S_1 and $\bar{S}_1$ is contained in one of S_2 and $\bar{S}_2$.

PROOF: Suppose the conclusion is false. Then the sets $S_1 \cap S_2$, $S_1 \cap \bar{S}_2$, $\bar{S}_1 \cap S_2$ and $\bar{S}_1 \cap \bar{S}_2$ are all nonempty and so, since G is 3-connected, each has at least three edges in the coboundary. Let J_1 be the set of edges joining nodes of $S_1 \cap S_2$ and $\bar{S}_1 \cap \bar{S}_2$ and let J_2 be the set of edges joining nodes of $\bar{S}_1 \cap S_2$ and $S_1 \cap \bar{S}_2$. Then $|\delta(S_1)| + |\delta(S_2)| - |J_1| - |J_2| = \frac{1}{2}(|\delta(S_1 \cap S_2)| + |\delta(\bar{S}_1 \cap S_2)| + |\delta(S_1 \cap \bar{S}_2)| + |\delta(\bar{S}_1 \cap \bar{S}_2)|) \geq 6$. Since

$|\delta(S_1)|+|\delta(S_2)|=6$ we must have therefore that $J_1=J_2=\emptyset$ and $|\delta(S_1\cap S_2)|=|\delta(S_1\cap\bar{S}_2)|=|\delta(\bar{S}_1\cap S_2)|=|\delta(\bar{S}_1\cap\bar{S}_2)|=3$. But this means that $\delta(S_1\cap S_2)$ can contain at most one edge of one of $\delta(S_1)$ or $\delta(S_2)$, say $\delta(S_1)$. Therefore $\delta(\bar{S}_1\cap S_2)$ contains at most one edge of $\delta(S_1)$. Then since $\delta(S_1\cap S_2)$ must contain at least two edges of $\delta(S_2)$, $\delta(\bar{S}_1\cap S_2)$ contains at most one edge of $\delta(S_2)$. Since $J_2=\emptyset$ we have, therefore, that $|\delta(\bar{S}_1\cap S_2)|\leq 2$, contradictory to G being 3-connected. □

We say that a class **C** of 3-connected graphs is *fully reducible* if it satisfies the following:

(i) if $G\in\mathbf{C}$ has a 3-edge cutset with shores S and $\bar{S}$, then both $G\times S$ and $G\times\bar{S}$ are in **C**; and

(ii) the *TSP* can be solved in polynomial time for the graphs in **C** which do not have a 3-edge cutset. We call such graphs *irreducible*.

The basic reduction introduced in this section enables us to polynomially solve the *TSP* for the graphs in any fully reducible class.

The basic algorithm:

Input: A 3-connected graph $G=(V,E)$ belonging to a fully reducible class **C** and a vector of edge costs.

Output: A minimum cost Hamilton cycle.

Step 1. If G contains no 3-edge cutset, then the *TSP* is polynomially solvable in G, since $G\in\mathbf{C}$. Solve the TSP and stop.

Otherwise find a 3-edge cutset with a shore S which is minimal with respect to set inclusion. Go to Step 2.

Step 2. Solve three *TSP*'s in $G\times\bar{S}$ imposing in turn that each pair of edges of $\delta(S)$ be in the solution. We can force a pair of edges in $\delta(S)$ to be in the solution by giving a large cost to the third edge of $\delta(S)$. (Note that the minimality of S insures that $G\times\bar{S}$ will contain no 3-edge cutset, so solving TSP's in $G\times\bar{S}$ is polynomial.)

Step 3. Modify the edge costs of the members of $\delta(S)$ as described in the basic reduction and recursively apply the algorithm to $G\times S$. Extend the optimum tour in $G\times S$ found by the algorithm to an optimum tour of G by combining it with the appropriate tour of $G\times\bar{S}$ found in Step 2. Stop.

Note that Step 1 can be performed in polynomial time by simply trying all sets of edges of cardinality three.

This algorithm provides a framework which can be specialized for specific fully reducible classes. The actual time for such a class will depend on how efficiently we can find the 3-edge cutset required in Step 1 and how efficiently we can solve TSP's in irreducible graphs. In the next section we show that for the case of Halin graphs, the basic algorithm specializes to a linear time algorithm.

As mentioned in the introduction, the existence of a polynomial algorithm for a combinatorial optimization problem is often related to finding a system of linear inequalities which define the convex hull of its feasible solutions. When the graph G has a 3-edge cutset we have a remarkable polyhedral result relating $TSP(G)$, $TSP(G\times S)$ and $TSP(G\times\bar{S})$, where S and $\bar{S}$ are the shores of the 3-edge cutset.

The basic polyhedral theorem:

THEOREM 2.2. Suppose that G has a 3-edge cutset with shores S and $\bar{S}$. A linear system sufficient to define $TSP(G)$ is obtained by taking the union of linear systems sufficient to define $TSP(G\times S)$ and $TSP(G\times\bar{S})$.

PROOF. Every tour of G induces a tour in $G\times S$ and one in $G\times\bar{S}$, and therefore it satisfies linear systems defining $TSP(G\times S)$ and $TSP(G\times\bar{S})$.

Conversely, we need to prove that every solution $\hat{x}$ to the union of these systems can be expressed as a convex combination of tours of G.

Let x^1 be the restriction of $\hat{x}$ to $G \times S$. The vector x^1 satisfies the linear system defining $P(G \times S)$ and so can be expressed as a convex combination of tours of $G \times S$. That is, if T^1 is the set of tours of $G \times S$, then $x^1 = \sum(\lambda_t t : t \in T^1)$ where $\lambda_t \geq 0$ for all $t \in T^1$ and $\sum(\lambda_t : t \in T^1) = 1$.

Let e, f, g be the three edges of the 3-edge cutset joining S to $\bar{S}$, and let

$$\mu(e,f) = \sum(\lambda_t : t \in T^1 \text{ and } t_e = t_f = 1)$$

$$\mu(f,g) = \sum(\lambda_t : t \in T^1 \text{ and } t_f = t_g = 1)$$

$$\mu(e,g) = \sum(\lambda_t : t \in T^1 \text{ and } t_e = t_g = 1)$$

Then

$$\hat{x}_e = x_e^1 = \mu(e,f) + \mu(e,g)$$

$$\hat{x}_f = x_f^1 = \mu(e,f) + \mu(f,g) \qquad (2.1)$$

$$\hat{x}_g = x_g^1 = \mu(e,g) + \mu(f,g).$$

Note that the linear system (2.1) together with the values $\hat{x}_e$, $\hat{x}_f$ and $\hat{x}_g$ uniquely determines the values $\mu(e,f)$, $\mu(f,g)$ and $\mu(e,g)$.

Now let x^2 be the restriction of $\hat{x}$ to $G \times \bar{S}$, and let T^2 be the set of tours of $G \times \bar{S}$. Anologously we obtain $x^2 = \sum(\sigma_t t : t \in T^2)$ where $\sigma_t \geq 0$ for all $t \in T^2$ and $\sum(\sigma_t : t \in T^2) = 1$. We define

$$\nu(e,f) = \sum(\sigma_t : t \in T^2 \text{ and } t_e = t_f = 1)$$

$$\nu(f,g) = \sum(\sigma_t : t \in T^2 \text{ and } t_f = t_g = 1)$$

$$\nu(e,g) = \sum(\sigma_t : t \in T^2 \text{ and } t_e = t_f = 1).$$

Then $\nu(e,f)$, $\nu(f,g)$ and $\nu(e,g)$ satisfy the system (2.1), replacing μ with ν. So we must have $\nu(e,f)=\mu(e,f)$, $\nu(f,g)=\mu(f,g)$ and $\nu(e,f)=\mu(e,f)$. Therefore we can combine the tours t in T^1 and t' in T^2 to obtain a set T of tours of G and a set of coefficients $\eta_t \geq 0$ for all $t \in T$ satisfying $\sum(\eta_t : t \in T)=1$ and $\hat{x}=\sum(\eta_t t : t \in T)$. That is, $\hat{x}$ is a convex combination of tours of G and the proof is complete. □

This theorem enables us to obtain a complete linear description of $TSP(G)$ for any graph G in a fully reducible class provided that we know a linear description of the travelling salesman polytopes of the irreducible graphs in the class. Some examples of this are given in the next section.

3. Some Examples

One method of constructing a fully reducible class is to start with a specific set of irreducible graphs and then close the class under a composition which is the inverse of the basic reduction. First we introduce some useful irreducible graphs.

For certain graphs $G=(V,E)$, the polytope $TSP(G)$ is given by the following linear system:

$$0 \leq x_e \leq 1 \text{ for all } e \in E \tag{3.1}$$

$$x(\delta(v))=2 \text{ for all } v \in V. \tag{3.2}$$

We call such graphs *elementary*. For these graphs, not only can we solve the *TSP* as a small linear program, but we can solve this linear program polynomially by reducing it to a weighted bipartite matching problem. (In fact the linear program minimize $(lx : x$ satisfies (3.1), (3.2)) can be solved in this way for any graph.) However all elementary graphs we know have a sufficiently simple structure that the *TSP* is more easily solved by direct methods.

Conversely, we need to prove that every solution $\hat{x}$ to the union of these systems can be expressed as a convex combination of tours of G.

Let x^1 be the restriction of $\hat{x}$ to $G \times S$. The vector x^1 satisfies the linear system defining $P(G \times S)$ and so can be expressed as a convex combination of tours of $G \times S$. That is, if T^1 is the set of tours of $G \times S$, then $x^1 = \sum(\lambda_t t : t \in T^1)$ where $\lambda_t \geq 0$ for all $t \in T^1$ and $\sum(\lambda_t : t \in T^1) = 1$.

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Then

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Note that the linear system (2.1) together with the values $\hat{x}_e$, $\hat{x}_f$ and $\hat{x}_g$ uniquely determines the values $\mu(e,f)$, $\mu(f,g)$ and $\mu(e,g)$.

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The following is a useful tool in showing that a graph is elementary:

LEMMA 3.1. (cf Grötschel [14]). Let $\bar{x}$ be an extreme point of the polytope defined by (3.1), (3.2). Then $\bar{x}_e \in \{0, \frac{1}{2}, 1\}$ for all $e \in E$ and, moreover, the set of edges e with $\bar{x}_e = \frac{1}{2}$ partitions into the edge sets of an even number of vertex disjoint odd cycles.

COROLLARY 3.2. If G does not have two vertex disjoint cycles, then G is elementary.

This implies that the complete graphs K_3, K_4 and K_5 and the complete bipartite graphs $K_{2,2}$ and $K_{3,3}$ (or, in fact, $K_{3,n}$ for $n \geq 2$) are all elementary. Moreover, we have the following:

LEMMA 3.3. If $G=(V,E)$ is elementary then for any $E' \subseteq E$, the graph $G'=(V,E')$ is elementary.

PROOF. A linear system sufficient to define $TSP(G')$ is obtained by deleting the variables corresponding to $e \in E-E'$ from the system (3.1), (3.2) for G. This is equivalent to adjoining the equations $x_e=0$ for all $e \in E-E'$ to (3.1),(3.2) for G. (Polyhedrally, $TSP(G')$ is a face of $TSP(G)$.) □

In particular, any graph on five or fewer nodes is elementary.

A useful infinite class of elementary graphs is the class of wheels. For $k \geq 3$, the *wheel* W_k consints of a cycle containing k vertices, called *rim* vertices, plus a *centre* vertex adjacent to each rim vertex. (See Figure 3) The fact that wheels are elementary follows from Corollary 3.2. In Section 5 we give other examples of elementary graphs.

Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be two graphs. Let $v_1 \in V_1$ and $v_2 \in V_2$ be two vertices of degree three, say $\delta(v_1)=\{e_1,f_1,g_1\}$ and

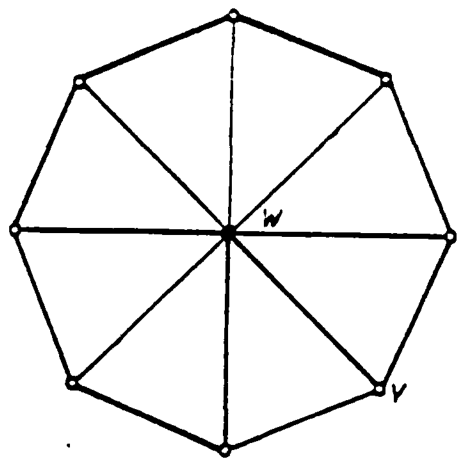

Figure 3. A wheel

$\delta(v_2)=\{e_2,f_2,g_2\}$. We say that the graph G is *obtained from* G_1 *and* G_2 *by 3-splicing* if it is constructed as follows: The vertex set of G is $(V_1-\{v_1\})\cup(V_2-\{v_2\})$. The edge set of G is $E_1\cup E_2$ where the edges e_1 and e_2 are identified, as well as the edges f_1 and f_2, and g_1 and g_2. Therefore in G each of these three edges has one end in V_1 and the other in V_2. (See Figure 4.)

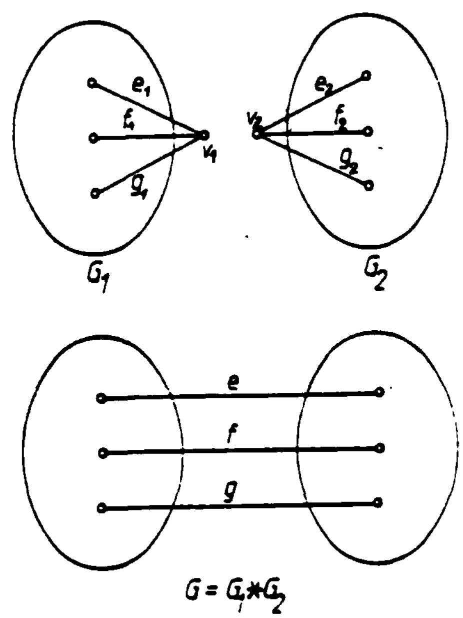

Figure 4. 3-splicing

We will write $G = G_1^{e_1 f_1 g_1} * G_2^{e_2 f_2 g_2}$ or sometimes simply $G = G_1 * G_2$ or $G = G_1^{v_1} * G_2^{v_2}$ or $G = G_1^{v_1} * G_2$. Notice that if $G = G_1^{v_1} * G_2^{v_2}$, then G_1 is isomorphic to $G \times (V_2 - \{v_2\})$ and G_2 is isomorphic to $G \times (V_1 - \{v_1\})$.

We define the *closure* $cl(\mathbf{C})$ of a class $\mathbf{C}$ of graphs with respect to 3-splicing as follows: $G \in cl(\mathbf{C})$ if either $G \in \mathbf{C}$ or there exist $G_1, G_2 \in cl(\mathbf{C})$ such that $G = G_1 * G_2$. The graphs in $cl(\mathbf{C})$ which are irreducible with respect to our basic reduction will all belong to $\mathbf{C}$. If $\mathbf{C}$ is a class of graphs for which we can polynomial solve *TSP*'s, then $cl(\mathbf{C})$ will be a fully reducible class of graphs.

If a graph G has no degree three vertex, then it cannot be used in a 3-splice. If a graph G is not Hamiltonian, and it is used in a 3-splice, then the result will not be Hamiltonian. Hence the graphs which are useful to us as "building blocks" are those which are Hamiltonian and have at least one degree three vertex, for example wheels, K_4, $K_{3,3}$ and some subgraphs of K_5.

For all these examples, the *TSP* is easily solved. Note that for the wheel W_k there are precisely k different Hamilton cycles, depending on between which consecutive pair of rim vertices we visit the centre. A minimum cost Hamilton cycle can be found in linear time by computing the minimum, over all pairs r, r' of consecutive rim vertices, of $l(c,r) + l(c,r') - l(r,r')$ where c is the centre of the wheel.

THEOREM 3.4. Let $\mathbf{C}$ be the class of elementary graphs. Then for any $G = (V, E) \in cl(\mathbf{C})$, $TSP(G)$ is defined by

$$0 \leq x_e \leq 1 \text{ for all } e \in E, \tag{3.4}$$

$$x(\delta(v)) = 2 \text{ for all } v \in V, \tag{3.5}$$

$$x(C) = 2 \text{ for all 3-edge cutsets } C \text{ of } G. \tag{3.6}$$

PROOF. All the constraints are valid. It follows from repeated application of Theorem 2.2 (the basic polyhedral theorem) that (3.4) - (3.6) is sufficient. Note that when we perform a 3-splice $G = G_1^{v_1} * G_2^{v_2}$, the constraints (3.5) for v_1 in G_1 and for v_2 in G_2 become the constraint (3.6) for the 3-edge cutset created by 3-splicing. □

Let **W** be the class of wheels. Then 3.4 applies to cl(**W**). A subclass of cl(**W**) has received some study on the literature. These are the so called Halin graphs, or roofless polyhedra which are described as follows:

A Halin graph $H = T \cup C$ is obtained by taking a tree T having no vertices of degree two, embedding it in the plane in a planar fashion then adding new edges to form a cycle C containing all the leaves (degree one vertices) of the tree in such a way that the resulting graph is planar. (See Figure 5). These graphs were introduced by Halin [18] as an example of a class of planar edge minimal 3-connected graphs. They are exactly those 3-connected planar graphs for which one face shares one edge with every other face. Bondy and Lovász (see [22]) showed that these graphs are Hamiltonian and moreover, the deletion of any vertex leaves a Hamiltonian graph. It can also be verified that for each edge e, there exists a Hamilton cycle containing e and another that does not contain e. They were studied from a point of view of matching theory by Lovász and Plumer [22], Pulleyblank [25] and Naddef and Pulleyblank [24]. Recently Syslo and Proskurowski [29] have shown that several NP-complete problems are polynomially solvable for these graphs.

Since Halin graphs are 3-connected and planar, there is a planar embedding of such a graph which is unique, up to the choice of the infinite face. Thus determining whether or not an arbitrary graph is a Halin graph requires simply finding a planar embedding, verifying 3-connectivity, and then seeing if there exists a face such that the deletion of the edges in its boundary leaves a tree. Therefore Halin graphs can be recognized in

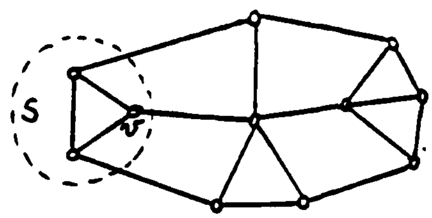

Figure 5. A Halin graph.

polynomial time.

We can see that the class **H** of Halin graphs is contained in $cl(\mathbf{W})$ as follows. Let $G=T\cup C$ be a Halin graph. If T is a star then G is a wheel. If T is not a star, then there exists a nonleaf vertex v of T adjacent only to leaves of T plus one other nonleaf vertex. Let S consist of v plus all adjacent leaves of T. Then $|\delta(S)|=3$ and $G\times S$ is a Halin graph and $G\times\bar{S}$ is a wheel. Thus G can be obtained by 3-splicing a wheel to a smaller Halin graph. Therefore, by induction, $G\in cl(\mathbf{W})$. For a more detailed description of these ideas for Halin graphs, see [4].

It is easy to see that we can construct precisely the class of Halin graphs from wheels if we restrict 3-splicing as follows: For any Halin graph $G=T\cup C$ we call a vertex v a *rim vertex* if v is in C. (This is consistent with our definition for wheels.) Similarly edge e is a *rim edge* if e is in C.

We only permit splicing between rim vertices, with the additional condition that rim edges must be identified with rim edges.

Now we describe how the basic algorithm specializes for Halin graphs. As we already noted, the *TSP* in a wheel can be solved in linear time. In order to find 3-edge cutsets with minimal shores we do the following. Represent T as a tree rooted at some nonleaf vertex r, order the children of each vertex on the basis of the planar embedding of T, and then perform a postorder scan of T. (In other words, each node is processed after all its children have been processed.) When a node is processed there are three possibilities: if it is a leaf of T it is bypassed; if it is a nonleaf vertex v different from r then v together with the adjacent leaves form a suitable set S; if v is equal to r then G is a wheel. Note that when we recursively apply the basic algorithm to $G \times S$, we can start our postorder scan with the vertex obtained by shrinking S.

Finally, note that $G \times \bar{S}$ will be a wheel W_l. Therefore finding a minimum cost Hamilton cycle which uses a prescribed pair of edges incident with a rim vertex u is easy. If either edge joins u to the centre vertex, there is a unique possibility. If neither edge joins u to the centre, there are $l-2$ possibilities.

Therefore the basic algorithm, specialized to the case of Halin graphs can be implemented in time linear in the number of vertices. See [4] for more details.

A 2- *factor* of a graph is a set of vertex-disjoint cycles which span the vertices (i.e. every vertex of the graph belongs to exactly one cycle of the 2-factor). Edmonds [8] has described the convex hull of the incidence vectors of the 2-factors of a graph (the 2- *factor polytope*) by a system of linear inequalities. In order to describe this system, we require a definition. A *blossom* $B=(S,J)$ is a subgraph of G consisting of $S \subseteq V$ having $|S| \geq 3$ and an odd cardinality set $J \subseteq \delta(S)$ of edges such that each $e \in J$ is incident with a different vertex of S. We let $P^2(G)$ denote the 2-factor polytope of G.

THEOREM 3.5. (Edmonds [8]) For an arbitrary graph $G=(V,E)$, $P^2(G)$ is defined by

$$0 \leq x_e \leq 1 \text{ for all } e \in E \tag{3.7}$$

$$x(\delta(v))=2 \text{ for all } v \in V \tag{3.8}$$

$$x(\gamma(S))+x(J) \leq |S|+(|J|-1)/2 \text{ for every blossom } (S,J) \text{ of } G. \tag{3.9}$$

In addition to this theorem Edmonds gave a polynomial algorithm to find a minimum cost 2-factor in a graph. Therefore graphs for which every 2-factor is a Hamilton cycle constitute interesting building blocks for 3-splicing. Note that the fully reducible class obtained as the closure of these graphs contains that generated as the closure of elementary graphs. In particular it contains Halin graphs.

Recently Cornuéjols, Hartvigsen and Pulleyblank [3] have described the convex hull of the *triangle-free 2-factors* of a graph (i.e. Those 2-factors in which every cycle has length at least 4) and have given a polynomial algorithm to find a minimum cost such 2-factor. Thus, the class of graphs where every triangle-free 2-factor is a Hamilton cycle can be closed under 3-splicing to form a fully reducible class. This class contains those introduced earlier in this section.

Now we turn to another fully reducible class of graphs, called prismatic graphs. A graph G_p is a *prism* if it consists of two vertex disjoint cycles of length p, say with vertices $(u_1,\ldots,u_p)$ and $(v_1,\ldots,v_p)$, where in addition each pair $u_i v_i$ is joined by an edge. (See Figure 6.) Note that prisms are Hamiltonian and regular of degree 3 and so they generate Hamiltonian graphs by 3-splicing. A graph is called *prismatic* if it belongs to the closure of prisms under 3-splicing. (See Figure 6.)

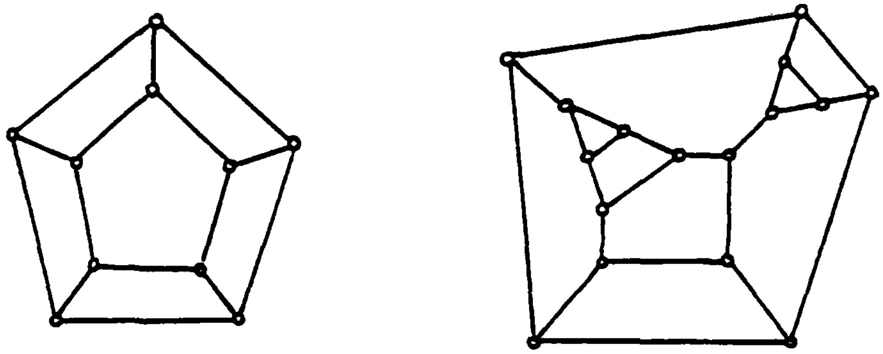

Figure 6. The prism G_5 and a prismatic graph $G_4 * G_3 * G_3$.

When p is odd the prism G_p has exactly p Hamilton cycles, all using a pair of edges $(u_i v_i)(u_{i+1} v_{i+1})$ and no other edge $(u_k v_k)$, $k \neq i, i+1$. (Here and in the remainder of this section, the indices of the vertices are assumed to be defined modulo p, e.g. $u_{p+1} \equiv u_1$.) On the other hand, when p is even, the prism G_p has $p+2$ Hamilton cycles, namely the p Hamilton cycles described above for odd p plus two new cycles H_1 and H_2 containing all the edges $(u_i v_i)$ and every other edge of the cycles $\{u_1, \ldots, u_p\}$ and $\{v_1, \ldots, v_p\}$. Whether p is odd or even, a minimum cost Hamilton cycle can be found in linear time.

Next we will show that $TSP(G)$ is particularly simple for a prism. A *simplex* is a polytope such that, for any extreme point, there is a hyperplane that does not contain it but contains all the other extreme points. For example triangles and tetrahedra are simplices whereas squares and cubes are not. As a consequence, in a defining linear system for a simplex, the number of inequalities required is equal to the number of extreme points of the simplex. This number is always relatively small. In fact, as will become clear in Section 4, the total number of relations needed in a linear system defining a simplex (number of equalities and inequalities) is equal to the number of variables plus one (except for isolated points, which are simplices and can be defined with just as many equalities as variables.) Every polytope which is not a simplex requires more relations.

Suppose we are given a polytope $S \subseteq R^n$ having a set W of extreme points and satisfying a linear system $Ax \leq b, Cx = d$. Then S is a simplex and our defining linear system is a minimal defining linear system if and only if the following conditions hold:

(3.10) C consists of $n - |W| + 1$ linearly independent rows;

(3.11) A contains $|W|$ rows and for each $\hat{w} \in W$ there is an inequality $ax \leq \beta$ from $Ax \leq b$ such that $aw = \beta$ for all $w \in W \setminus \hat{w}$ and $a\hat{w} < \beta$.

For (3.11) ensures that W is affinely independent and hence S is a simplex of dimension $|W|-1$. By (3.10), $Cx = d$ is a minimal set of equations. An inequality $ax \leq \beta$ valid for S is essential if and only if there are $dim(S)$ affinely independent members x of S satisfying $ax = \beta$. By (3.11) we have all such essential inequalities, and conversely, all our inequalities are essential. (See also Section 4.)

THEOREM 3.6. If G_p is a prism, then $TSP(G_p)$ is a simplex. When $p \geq 3$ is odd, a minimal defining linear system is:

$$x(\delta(w)) = 2 \text{ for every vertex } w \text{ of } G_p,$$

$$\sum_{i=1}^{p} x(u_i v_i) = 2$$

$$x(u_k u_{k+1}) \leq 1 \text{ for } k = 1,2,\ldots p.$$

When $p \geq 4$ is even, a minimal defining linear system is:

$$x(\delta(w)) = 2 \text{ for all vertices } w \text{ of } G_p \text{ except one;}$$

$$\sum_{i=1}^{p} x(u_i v_i) + (p-2)[x(u_k u_{k+1}) + x(v_k v_{k+1})] \leq 2p-2 \text{ for } k = 1,2,\ldots,p;$$

$$\sum_{i=1}^{p} x(u_i v_i) + (p-2)x(H_j) \geq p(p-1) \text{ for } j=1,2,$$

where H_1 and H_2 are as described above.

PROOF. First suppose that p is odd. Then G_p is nonbipartite and it is well known (and easy to verify) that the equations $x(\delta(w))=2$ for all vertices w of G_p are linearly independent. If we let $\hat{x}$ be the incidence vector of the two disjoint cycles of length p, then $\hat{x}$ satisfies all of these equations but $\sum_{i=1}^{p} \hat{x}(u_i v_i)=0$. Therefore the equation $\sum_{i=1}^{p} x(u_i v_i)=2$ is linearly independent of these other equations so (3.10) holds. (Note that G_p has $3p$ edges and p distinct Hamilton cycles.) Moreover, (3.11) follows immediately, since for each edge $(u_k u_{k+1})$ there is a unique Hamilton cycle which does not use this edge.

Now suppose that p is even. Then G_p is bipartite and so if we take all the degree constraints $x(\delta(w))=2$ except one, we have (3.10) satisfied. Consider a Hamilton cycle which uses only two edges of the form $(u_i v_i)$, say $(u_j v_j)$, $(u_{j+1} v_{j+1})$. Then it satisfies all of the first p inequalities with equality except the one corresponding to $k=j$. It also satisfies the last two inequalities as equations. The Hamilton cycles H_j both satisfy the first p inequalities as equalities and each also satisfies as an equation the one of the last two inequalities corresponding to the other. Therefore (3.11) is satisfied. □

4. Polyhedral Theorems.

Theorem 2.2 (the basic polyhedral theorem) showed that we could obtain a linear system sufficient to define $TSP(G)$ from linear systems sufficient to define $TSP(G\times S)$ and $TSP(G\times\bar{S})$ where $\delta(S)=\delta(\bar{S})$ is a 3-edge cutset of G. In this section we show that stronger results are true. First we review certain facts concerning linear systems and their solution sets. (See Bachem and Grötschel [1]) for a good introduction to polyhedral theory.)

Let $P=\{x\in R^n: Ax\leq b, Cx=d\}$. If we are interested in using linear programming techniques to solve an optimization problem

Minimize $(lx: x\in P)$

it is often desirable to have our defining linear system as small as possible. In particular, we want to eliminate any redundant equations or inequalities. First we assume

(4.1) no inequality from the set $Ax\leq b$ is satisfied with equality by all $x\in D$. In other words, if any of the inequalities can be made into equations without changing the solution set we do so.

A consequence of assumption (4.1) is that any equation which is satisfied by all $x\in P$ is a linear combination of the equalities $Cx=d$. Subject to (4.1) a fundamental result of polyhedral theory is that the defining linear system is minimal if and only if

(4.2) the rows of C are linearly independent,

(4.3) for any inequality $ax\leq\beta$ from the system $Ax\leq b$, there exists $\hat{x}\in P$ satisfying $a\hat{x}=\beta$

(4.4) for any two inequalities $a^1x\leq\beta^1$ and $a^2x\leq\beta^2$ from the system $Ax\leq b$, there exists $\hat{x}\in P$ satisfying $a^1\hat{x}=\beta^1$ and $a^2\hat{x}<\beta^2$.

A *face* of a polyhedron P is defined to be the set of all $x \in P$ satisfying $cx = \gamma$ for some inequality $cx \leq \gamma$ satisfied by all members of P. A *facet* is a maximal (with respect to set inclusion) nonempty face of P which is different from P. Conditions (4.3) and (4.4) require that, for each inequality $ax \leq \beta$ from $Ax \leq b$, $\{x \in P : ax = \beta\}$ must be a facet of P.

The *dimension* of P, denoted by $dim(P)$ is defined (subject to (4.1) and (4.2)) as the difference between the number of columns and rows of C, with the convention that the dimension of the empty set is taken to be -1. A set X of vectors is *affinely independent* if for any $\hat{x} \in X$, the set $\{x - \hat{x} : x \in X - \{\hat{x}\}\}$ is linearly independent and the *affine rank* of a set is the size of a largest affinely independent subset. We say that $\hat{x}$ is an *affine combination* of X if there exists $\alpha \in R^X$ satisfying $\sum(\alpha_x : x \in X) = 1$ and $\hat{x} = \sum(\alpha_x x : x \in X)$. (Recall that for any set S, R^S denotes the set of all real vectors indexed by S.) It can be seen that $dim(P)$ is one less than the affine rank of P and that a nonempty face F of P is a facet if and only if $dim(F) = dim(P) - 1$.

Let J be a 3-edge cutset in a graph $G = (V,E)$. Let $\hat{J}$ denote the set of those $j \in J$ such that G has a Hamilton cycle which does not use j. In other words, for each pair of edges of J which belongs to a Hamilton cycle of G, the other edge is in $\hat{J}$. If G has no Hamilton cycles, then $\hat{J} = \emptyset$. If, as is the case for Halin graphs and prismatic graphs, every edge is missed by some Hamilton cycle, then $\hat{J} = J$.

When we wish to obtain minimality results for a linear representation of $TSP(G)$, this set $\hat{J}$ plays an important role. We define a restricted travelling salesman polytope as follows: for any $E' \subseteq E$ we let $TSP(G,E')$ denote the convex hull of those tours of G which use all the edges of E', (Thus $TSP(G,\emptyset) = TSP(G)$.)

We now show that if $J = \delta(S)$ is a 3-edge cutset in $G = (V,E)$ then we can combine minimal defining linear systems for $TSP(G\times S, J-\hat{J})$ and $TSP(G\times \overline{S}, J-\hat{J})$ and obtain an essentially minimal defining system for $TSP(G)$. There are two technical problems. First, the equations $x(J)=2$ and $x_j=1$ for $j\in J-\hat{J}$ need not occur explicitly in either linear system and yet will be obtainable as linear combinations of equations defining both $TSP(G\times S, J-\hat{J})$ and $TSP(G\times \overline{S}, J-\hat{J})$. Therefore combining the equations in these systems results in some redundancies which must be eliminated. Second, there is one situation in which an inequality is facet inducing for one of two smaller systems and yet not facet inducing for $TSP(G)$. This is when it induces the same facet as $x_j\leq 1$ for some $j\in\hat{J}$ in one of the subsystems, but this inequality is not facet inducing in the other.

In the applications we make of this Theorem, these problems will be minimized. Every edge will appear in some Hamilton cycle and not appear in another. Consequently we will always have $\hat{J}=J$. Moreover, $x_j\leq 1$ will be facet inducing for all j and the equation $x(J)=2$ will occur explicitly in both subsystems.

THEOREM 4.1. Let $J=\delta(S)$ be a 3-edge cutset in a graph G for which $TSP(G)\neq\emptyset$. Let $P_1=TSP(G\times S, J-\hat{J})$ and let $P_2=TSP(G\times \overline{S}, J-\hat{J})$. For $i=1,2$, let

$$A^i x\leq b^i$$

$$x(J)=2$$

$$x_j=1 \text{ for } j\in J-\hat{J}$$

$$C^i x=d^i$$

be a minimal defining linear system for P_i, which satisfies (4.1). Then

(4.5) the union of these linear systems gives a linear system satisfying (4.1) whose solution set is $TSP(G)$;

(4.6) the equations $x(J)=2, x_j=1$ for all $j\in J-\hat{J}, C^1x=d^1, C^2x=d^2$ are a linearly independent basis of the equations satisfied by all $x\in TSP(G)$;

(4.7) an inequality from $A^1x\leq b^1$ or $A^2x\leq b^2$ is facet inducing, and hence essential, for $TSP(G)$ unless it induces the same facet as $x_j\leq 1$ for some $j\in\hat{J}$, for one of the two subsystems, and this inequality $x_j\leq 1$ does not induce a facet of the polyhedron defined by the other system.

PROOF. We first establish (4.5). Let P_1 and P_2 be the set of all vectors in R^E which satisfy $\{x(S)=2, C^1x=d^1, A^1x\leq b^1\}$ and $\{x(S)=2, C^2x=d^2, A^2x\leq b^2\}$ respectively. Then $TSP(G)\subseteq P_1\cap P_2$. Let $\bar{P}_1$ and $\bar{P}_2$ be the sets of all vectors in R^E which satisfy linear systems sufficient to define $TSP(G\times S)$ and $TSP(G\times\bar{S})$ respectively. Then $\bar{P}_1\supseteq P_1$ and $\bar{P}_2\supseteq P_2$ and, by Theorem 2.1, $TSP(G)=\bar{P}_1\cap\bar{P}_2$. Therefore $TSP(G)=P_1\cap P_2$ and so the union of those two linear systems defines $TSP(G)$.

Now we prove (4.6) and (4.7). Let $X^1\subseteq R^{E-\gamma(S)}$ be a maximal affinely independent set of tours of P_1 and let $X^2\subseteq R^{E-\gamma(\bar{S})}$ be analagously defined for P_2. For each $j\in\hat{J}$ we choose a tour $\bar{x}^j\in X^1$ such that $\bar{x}^j_j=0$ and a tour $\bar{\bar{x}}^j\in X^2$ such that $\bar{\bar{x}}^j_j=0$.

For any tours x of $G\times S$ and $\bar{x}$ of $G\times\bar{S}$ such that $x_j=\bar{x}_j$ for all $j\in J$, we define the *splice* of x and $\bar{x}$ to be the tour x' of G defined by

$$x'_j=\begin{cases}\bar{x}_j & \text{for } j\in\gamma(S)\\ x_j & \text{for } j\in\gamma(\bar{S})\\ x_j=\bar{x}_j & \text{for } j\in J=\delta(S)\end{cases}.$$

We now define the following sets of tours:

$\tilde{X}^1$ is the set of all splices of tours of $X^1-\{\bar{x}^j:j\in\hat{J}\}$ with the appropriate tour of $\{\bar{\bar{x}}^j:j\in\hat{J}\}$;

$\tilde{X}^2$ is the set of all splices of tours of $X^2-\{\bar{\bar{x}}^j:j\in\hat{J}\}$ with the appropriate tour of $\{\bar{x}^j:j\in\hat{J}\}$

$\tilde{X}^3$ is the set of splices of $\bar{\bar{x}}^j$ and $\bar{x}^j$ for all $j\in\hat{J}$.

Note that

$$|\tilde{X}^1\cup\tilde{X}^2\cup\tilde{X}^3|=|X^1|+|X^2|-|\hat{J}|. \tag{4.8}$$

CLAIM 1: $\tilde{X}_1\cup\tilde{X}_2\cup\tilde{X}_3$ *forms an affinely independent basis of the tours of* G.

First, we show that they are affinely independent. Suppose there exists $\alpha\in R^{\tilde{X}^1\cup\tilde{X}^2\cup\tilde{X}^3}$ such that

$$\sum(\alpha_x x:x\in\tilde{X}^1\cup\tilde{X}^2\cup\tilde{X}^3)=0.$$

Since X^1 is affinely independent, we must have $\alpha_x=0$ for all $x\in\tilde{X}^1$ and $\sum(\alpha_x:x\in\tilde{X}^2\cup\tilde{X}^3)=0$. Similarly, since X^2 is affinely independent, we must have $\alpha_x=0$ for all $x\in\tilde{X}^2$. Thus we must have $(\sum\alpha_x x:x\in\tilde{X}^3)=0$, which implies $\alpha_x=0$ for $x\in\tilde{X}^3$, since the members of $\tilde{X}^3$ are affinely independent. Therefore $\alpha=0$ so $\tilde{X}^1\cup\tilde{X}^2\cup\tilde{X}^3$ is linearly independent.

Now let $\hat{x}$ be a tour in G and let $\hat{j}$ be the edge j of $\hat{J}$ for which $\hat{x}_j=0$. Let $\hat{x}^1$ and $\hat{x}^2$ be the tours of $G\times S$ and $G\times\bar{S}$ respectively induced by $\hat{x}$. By the maximality of X^1 and X^2 there exist $\alpha^1\in R^{X^1}$ and $\alpha^2\in R^{X^2}$ such that

$$\sum(\alpha_x^1:x\in X^1)=1 \text{ and } \hat{x}^1=\sum(\alpha_x^1 x:x\in X^1)$$

and

$$\sum(\alpha_x^2 : x \in X^2) = 1 \text{ and } \hat{x}^2 = \sum(\alpha_x^2 x : x \in X^2).$$

Note that this implies that for any $x \in X^1$ such that $\alpha_x^1 \neq 0$, $x_j = 0$, and similarly for X^2. In other words, the only tours having α_x^1 or α_x^2 nonzero are those agreeing with $\hat{x}$ on J. Now for any $x \in X^1$, if $x \neq \bar{x}^j$ for any $j \in \hat{J}$, we define x' to be corresponding tour of $\tilde{X}^1$. If $x = \bar{x}^j$ for some $j \in \hat{J}$ we define x' to be the corresponding tour of $\tilde{X}^3$. The vector $\sum(\alpha_x^1 x' : x \in X^1)$ is identical with $\hat{x}$ on $E-\gamma(S)$ and equals $\bar{\bar{x}}^{\hat{j}}$ on $\gamma(S)$. We define x' analagously for $x \in X^2$ and have $\sum(\alpha_x^2 x' : x \in X^2)$ is identical with $\hat{x}$ on $E-\gamma(\bar{S})$ and equals $\bar{x}^{\hat{j}}$ on $\gamma(\bar{S})$. Therefore

$$\hat{x} = \sum(\alpha_x^1 x' : x \in X^1) + \sum(\alpha_x^2 x' : x \in X^2) - x^{\hat{j}}$$

where $x^{\hat{j}}$ is the splice of $\bar{\bar{x}}^{\hat{j}}$ and $\bar{x}^{\hat{j}}$. Since $\sum(\alpha_x^1 : x \in X^1) + \sum(\alpha_x^2 : x \in X^2) - 1 = 1$, we have expressed $\hat{x}$ as an affine combination of members of $\tilde{X}^1 \cup \tilde{X}^2 \cup \tilde{X}^3$ as required. Thus Claim 1 is established.

Note that (4.1) is clearly satisfied for our combined linear system, for consider an inequality $ax \leq \beta$, which belongs to $A^1 x \leq b^1$, say. Some tour $\hat{x} \in P_1$ satisfies $a\hat{x} < \beta$ so there will be a tour x' of G obtained by splicing some tour of P_2 with $\hat{x}$ which also satisfies $ax' < \beta$.

It follows from Claim 1 and (4.8) that

$$dim(TSP(G)) = dim(P_1) + dim(P_2) - |\hat{J}| + 1.$$

Since we have assumed minimal defining systems for P_i, the equations $x(J) = 2$, $x_j = 1$ for $j \in J - \hat{J}$, $C^i x = d^i$ are linearly independent. Let r_i be the number of these equations for P_i. By the definition of dimension, we have $dim(P_1) + r_1 = |E-\gamma(S)|$ and $dim(P_2) + r_2 = |E-\gamma(\bar{S})|$. The linear rank of the combined system is $|E| - dim(TSP(G)) = r_1 + r_2 - 1 - |J - \hat{J}|$ since $|E| = |E-\gamma(S)| + |E-\gamma(\bar{S})| - |J|$. But this is exactly the number of equations so they are linearly independent and hence all essential which establishes (4.6), since our combined system satisfies (4.1).

Now let $ax \leq \beta$ be an inequality from the system $A^1 x \leq b^1$. Then there exists a set Y^1 of $dim(P_1)$ affinely independent tours in P_1 all satisfying $ax = \beta$. Moreover, if the facet of P_1 induced by $ax \leq \beta$ is different from the facets induced by $x_j \leq 1$ for all $j \in \hat{J}$ then for each $j \in \hat{J}$ there exists a tour $x^j \in Y^1$ satisfying $x_j^j = 0$. We now splice the tours of Y^1 and X^2 in the same manner as we did previously with X^1 and X^2 to obtain $dim(P_1) + dim(P_2) + 1 - |\hat{J}|$ affinely independent tours of G, all satisfying $ax = \beta$. Therefore $dim(TSP(G))$ affinely independent members of $TSP(G)$ all satisfy $ax = \beta$ so the inequality is facet inducing and essential.

Suppose $ax \leq \beta$ induces the same facet as $x_j \leq 1$ for some $j \in \hat{J}$. If $x_j \leq 1$ is also facet inducing for P_2, then we can find sets Y_1 and Y_2 of $dim(P_1)$ and $dim(P_2)$ tours respectively of P_1 and P_2, all satisfying $x_j = 1$. Let $k \in \hat{J} - \{j\}$. Suppose every $x \in Y_1 \cup Y_2$ also satisfies $x_k = 1$. If there existed a tour $\bar{x}$ in either P_1 or P_2 which satisfied $\bar{x}_k = 1$ but $x_j = 0$, then we would have contradicted $\{x \in P_1 \text{ or } P_2 : x_j = 1\}$ being a maximal proper face, i.e. a facet. Therefore every tour x in P_1 or P_2 which has $x_k = 1$ must also have $x_j = 1$. But every tour x in P_1 or P_2 must have $x_j = 1$ or $x_k = 1$, hence must have $x_k = x_j = 1$. This means that the inequality $x_j \leq 1$ is not facet inducing for P_1 or P_2, a contradiction. Therefore, for each $k \in \hat{J} - \{j\}$ there exists $x^k \in Y_1$ such that $x_k^k = 0$, and similarly for Y_2. So as before we can splice Y_1 and Y_2 and obtain $dim(P_1) + dim(P_2) - (|\hat{J}| - 1) = dim(TSP(G))$ affinely independent tours of G all satisfying $x_j = 1$. Therefore the inequality is facet inducing.

Finally, suppose $ax \leq \beta$ induces the same facet as $x_j \leq 1$ for $j \in \hat{J}$, but $x_j \leq 1$ is not facet inducing for P_2. Then there is some other inequality $a' x \leq \beta'$ from $A^2 x \leq b^2$ such that $\{x \in P_2 : a' x = \beta'\} \supset \{x \in P_2 : x_j = 1\}$. If we extend the tours in these two sets to tours of G in all possible ways, we see that $ax \leq \beta$ does not induce a maximal nonempty face of $TSP(G)$, i.e., the inequality is not facet inducing and so is inessential.

This completes the proof of (4.7) and the theorem. □

COROLLARY 4.2. Let $J = \delta(S)$ be a 3-edge cutset in a graph G for which $TSP(G) \neq \emptyset$. Then

$$dim(TSP(G)) = dim(TSP(G \times S, J-\hat{J})) + dim(TSP(G \times \bar{S}, J-\hat{J})) - |\hat{J}| + 1.$$

(We actually proved this explicitly in the course of proving Theorem 4.1.)

Now we apply these results to Halin graphs. Recall that for a wheel W_k we defined a *rim edge* to be an edge not incident with the centre vertex.

PROPOSITION 4.3. The following is a minimal defining linear system for $TSP(W_k)$ for a wheel W_k:

$$x_j \leq 1 \text{ for every rim edge } j$$

$$x(\delta(v)) = 2 \text{ for every vertex } v.$$

PROOF. The wheel W_k has exactly k different Hamilton cycles, each one omitting a different rim edge. It is easy to verify that (3.10) and (3.11) are satisfied so $TSP(W_k)$ is a simplex and the given linear system is both minimal and sufficient. □

We remark that there are two notable omissions in our list of inequalities in Proposition 4.3. First we do not require $x_j \leq 1$ for a non rim edge j. However this inequality can be deduced as follows. Suppose j joins vertices u and w. Then the inequality $x_j \leq 1$ can be obtained by adding 1/2 times the equations $x(\delta(v))$ for $v \in \{u,w\}$; -1/2 times the equations $x(\delta(v))$ for $v \in V(W_k) - \{u,w\}$; and the inequalities $x_k \leq 1$ for all $k \in E(W_k) - (\delta(u) \cup \delta(w))$. Moreover, all of the above are part of our linear system. Second we do not have inequalities $x_j \geq 0$ for any of our edges. These can be derived from the fact that every edge is incident with a degree three vertex and each edge k incident with the vertex has the inequality $x_k \leq 1$ either explicitly in or derivable from our linear system.

THEOREM 4.4. Let $H = T \cup C$ be a Halin graph. The following is a minimal linear system sufficient to define $TSP(H)$:

$$x_j \leq 1 \text{ for all } j \in E(C);$$

$$x(\delta(v)) = 2 \text{ for all } v \in V(H);$$

$$x(J) = 2 \text{ for every (nontrivial) 3-edge cutset } J \text{ of } H.$$

PROOF. We prove by induction on the number p of nonleaf vertices of T. If $p=1$ then H is a wheel and the result follows from Prop. 4.3. If p > 1 then the result follows by induction and Theorem 4.1 applied to any (nontrivial) 3-edge cutset J of H. □

COROLLARY 4.5. If $H = T \cup C$ is a Halin graph such that T has p leaf vertices and q nonleaf vertices then $dim(TSP(H)) = p - q$.

PROOF. There is a bijection between the (nontrivial) 3-edge cutsets of H and the edges of T which are not incident with a leaf. Therefore the number of equations in a minimal defining system for H is $(p+q)+(q-1) = p+2q-1$. The number of edges in H is $(p+q-1)+p = 2p+q-1$. Therefore $dim(TSP(H)) = p-q$. □

In the previous section we discussed those graphs obtainable by 3-splicing from elementary graphs, and observed how Theorem 2.2 provided an easy means of obtaining a sufficient linear system to define their TSP's. We now show how Theorem 4.2 can be used to give converse results.

Let **E** be the class of those graphs $G = (V, E)$ satisfying

(4.9) for each edge j of G there is a Hamilton cycle of G which does not use j;

(4.10) $TSP(G)$ is defined by the following linear system:

$$0 \leq x_j \leq 1 \text{ for all } j \in E, \tag{4.11}$$

$$x(\delta(v)) = 2 \text{ for all } v \in V \tag{4.12}$$

$$x(J) = 2 \text{ for every 3-edge cutset } J. \tag{4.13}$$

First we show that $\mathbb{E} \subseteq cl$ (Elementary graphs).

THEOREM 4.6. Let G be a graph having a 3-edge cutset $J = \delta(S)$. Then $G \in \mathbb{E}$ if and only if both $G \times S$ and $G \times \bar{S}$ are in $\mathbb{E}$.

PROOF. If both $G \times S$ and $G \times \bar{S}$ are in $\mathbb{E}$, then (4.10) follows from Theorem 2.2, and it is easy to see that (4.9) is satisfied. Conversely, suppose $G \times S \notin \mathbb{E}$. Then there must be a facet not induced by an inequality of the form (4.11) or else a valid equation linearly independent of (4.12) and (4.13). Thus an inequality or equation not of the form (4.11) - (4.13) is essential for $TSP(G \times S)$. Therefore, by Theorem 4.2, it is also essential for $TSP(G)$. Hence $G \notin \mathbb{E}$, a contradiction. □

Therefore characterizing the graphs in $\mathbb{E}$ reduces to the problem of characterizing graphs $G = (V,E)$ such that $|\delta(S)| \geq 4$ for every $S \subset V$ satisfying $2 \leq |S| \leq |V| - 2$ and for which $TSP(G)$ is defined by (4.11) and (4.12). Some results in this direction are presented in Section 5.

Finally we show that not only do we have algorithmic and polyhedral reductions over 3-edge cutsets, but we can also construct optimum solutions to the dual linear program of minimizing lx for $x \in TSP(G)$ from optimum dual solutions for $G \times S$ and $G \times \bar{S}$ where $\delta(S)$ is a 3-edge cutset.

For convenience, we assume that $J = \hat{J}$, i.e., each pair of edges of J belongs to a Hamilton cycle. (This assumption could be removed using the same ideas as in Theorem 4.1) Suppose that $TSP(G\times S)$ and $TSP(G\times \bar{S})$ are defined by the linear systems $(A^1x\leq b^1, C^1x=d^1)$ and $(A^2x\leq b^2, C^2x=d^2)$ respectively. The dual linear program to minimizing lx subject to this linear system (i.e. to solving the TSP for edge costs l) is the following:

$$\text{Maximize } -\eta^1b^1-\eta^2b^2+\rho^1d^1+\rho^2d^2$$

subject to

$$-\eta^1A^1-\eta^2A^2+\rho^1C^1+\rho^2C^2=l$$

$$\eta^1,\eta^2\geq 0.$$

We proceed as in the basic reduction. Let u,v and w be the vertices of S incident with edges of J and let u',v' and w' be the adjacent vertices in $\bar{S}$. (Note that our $J=\hat{J}$ assumption ensures that these vertices are distinct.) We compute L_{uv}, L_{uw} and L_{vw} to be the minimum costs of Hamilton paths between u and v, u and w, and v and w respectively in $G[S]$. (Again, by virtue of our $J=\hat{J}$ assumption, these values are well defined). Again, we compute

$$\alpha=(L_{uv}+L_{uw}-L_{vw})/2$$

$$\beta=(L_{uv}+L_{vw}-L_{uw})/2$$

$$\epsilon=(L_{uw}+L_{vw}-L_{uv})/2.$$

We now define vectors l^1 and l^2 of edge costs as follows, where $e=(u,u')$, $f=(v,v')$, $g=(w,w')$. We define l_j^1 for $j\in E-\gamma(\bar{S})$ by

$$l^1(j)=\begin{cases} l(j) & \text{if } j\in E-\gamma(\bar{S})-\{e,f,g\} \\ l(e)-\alpha & \text{if } j=e \\ l(f)-\beta & \text{if } j=f \\ l(g)-\epsilon & \text{if } j=g \end{cases}$$

We define l_j^2 for $j \in E-\gamma(\bar{S})$ by

$$l^2(j) = \begin{cases} l(j) & \text{if } j \in E-\gamma(S)-\{e,f,g\} \\ l(e)+\alpha & \text{if } j=e \\ l(f)+\beta & \text{if } j=f \\ l(g)+\epsilon & \text{if } j=g \end{cases}$$

Note that l^2 is just the normal cost vector we use for $G \times S$ in the basic reduction. However by subtracting α, β, ϵ to construct l^1, we have ensured that the minimum cost (with respect to l^1) tours in $G \times \bar{S}$ using each pair of edges of G will have the same objective value, namely zero.

Let $(\bar{\pi}^1, \bar{\rho}^1)$ and $(\bar{\pi}^2, \bar{\rho}^2)$ be optimum dual solutions to minimizing $l^1 x$ subject to $(A^1 x \le b^1, C^1 x = d^1)$ and to minimizing $l^2 x$ subject to $(A^2 x \le b^2, C^2 x = d^2)$ respectively. Then, since the optimum value of the first linear program is zero, we have

$$-\bar{\pi}^1 b^1 + \bar{\rho}^1 d^1 = 0.$$

The optimum value of the second linear program is $\min\{lx : x \in TSP(G)\}$ so

$$-\bar{\pi}^2 b^2 + \bar{\rho}^2 d^2 = \min\{lx : x \in TSP(G)\}.$$

If we extend l^1 and l^2 to be vectors defined on E by letting any undefined components take on the value zero, then $l = l^1 + l^2$. Therefore

$$-\bar{\pi}^1 A^1 - \bar{\pi}^2 A^2 + \bar{\rho}^1 C^1 + \bar{\rho}^2 C^2 = l,$$

$$\bar{\pi}^1, \bar{\pi}^2 \ge 0,$$

$$\bar{\pi}^1 b^1 + \bar{\pi}^2 b^2 - \bar{\rho}^1 d^1 - \bar{\rho}^2 d^2 = \min\{lx : x \in TSP(G)\}.$$

Therefore $\bar{\pi}^1, \bar{\pi}^2, \bar{\rho}^1, \bar{\rho}^2$ comprise an optimum feasible solution to the dual linear program as required.

We illustrate this process for Halin graphs. The only problem is to compute an optimum solution to the dual problem for a wheel $W_k=(V,E)$ having centre $\hat{v}$ and arc costs l. Let $\hat{u}$ and $\hat{w}$ be adjacent rim vertices such that $l(\hat{v},\hat{u})+l(\hat{v},\hat{w})-l(\hat{u},\hat{w})$ is minimized, over all adjacent pairs of rim vertices. Our primal linear program is

$$\text{minimize } lx$$

subject to

$$x(\delta(v))=2 \text{ for all } v\in V,$$

$$x_j\leq 1 \text{ for every edge } j \text{ joining two rim vertices.}$$

The dual problem is

$$\text{maximize } \sum_{v\in V} 2\rho(v)-\sum(\eta(j): j\in E \text{ joins two rim vertices}).$$

We define an optimum dual solution $\overline{\eta},\overline{\rho}$ as follows:

$$\overline{\rho}(\hat{v}):=(l(\hat{v},\hat{u})+l(\hat{v},\hat{w})-l(\hat{u},\hat{w}))/2$$

$$\overline{\rho}(v):=l(v,u)-\overline{\rho}(\hat{v}) \text{ for } v\in V-\{\hat{v}\},$$

$$\overline{\eta}(u,w):=\overline{\rho}(u)+\overline{\rho}(w)-l(u,w) \text{ for each pair}(u,w)$$

$$\text{of adjacent rim vertices.}$$

It is routinely checked that $(\overline{\eta},\overline{\rho})$ is feasible and that it gives the same objective value as the Hamilton cycle of W_k which uses the edges $(\hat{u},\hat{v})$ and $(\hat{v},\hat{w})$.

For any arbitrary Halin graph, we can use the general method described above, together with this specific optimum for a wheel, to construct an optimum dual solution. Note that this provides a second proof of the validity of our basic reduction - a solution to a primal linear program is optimal if and only if these exists a feasible solution to the dual linear program giving the same objective value.

5. Additional results on elementary graphs

As stated in Section 3, the travelling salesman problem can be solved in polynomial time for an elementary graph $G=(V,E)$. Remember that, for these graphs, the polytope $TSP(G)$ is defined by the linear system

$$0 \leq x_j \leq 1 \text{ for all } j \in E, \tag{5.1}$$

$$x(\delta(v))=2 \text{ for all } v \in V. \tag{5.2}$$

If this system satisfies assumption (4.1), the graph G is called *basic elementary*. An equivalent way of stating assumption (4.1) is

(5.3) for every edge $j \in E$, there is at least one Hamilton cycle which contains j and at least one which misses j.

Examples introduced earlier such as wheels, $K_{3,3}, K_4, K_5$, K_5 minus one edge and K_5 minus two nonadjacent edges are basic elementary graphs.

PROPOSITION 5.1. A basic elementary graph with n vertices contains at least $n/2+1$ affinely independent Hamilton cycles.

PROOF. By assumption (4.1), the dimension of $TSP(G)$ is $|E|$ minus the number of linearly independent equations (5.2). Note that $|E| \geq \frac{3n}{2}$ since every vertex of G has degree at least 3. [Edges incident with a vertex of degree 1 or 2 would violate (5.3).] The number of equations (5.2) is at most n. So the dimension of the polytope $TSP(G)$ is at least $\frac{n}{2}$ and so the affine rank is at least $n/2+1$. □

The next theorem shows the importance of bipartite basic elementary graphs.

THEOREM 5.2. If G_1 and G_2 are two basic elementary graphs and G_1 is bipartite, then $G_1 * G_2$ is also a basic elementary graph. Conversely, if a basic elementary graph G has a 3-edge cutset with shores S_1 and S_2, then either $G \times S_1$ or $G \times S_2$ is bipartite. Furthermore both $G \times S_1$ and $G \times S_2$ are basic elementary graphs.

PROOF. Suppose $G_1=(V_1,E_1)$ is bipartite. Then the constraint $x(\delta(v_1))=2$ for a vertex $v_1 \in V_1$ is a linear combination of $x(\delta(v))=2$ for the vertices $v \in S_1 \equiv V_1-\{v_1\}$. As a consequence, in $G_1 * G_2$, the 3-edge cutset constraint $x(\delta(S_1))=2$ is implied by the degree constraints for $v \in S_1$ and thus can be omitted. Now it follows from Theorem 2.2 that, if G_1 and G_2 are elementary, then $G_1 * G_2$ also elementary. If G_1 and G_2 both satisfy (5.3), then consider any edge j of $G_1 * G_2$. Without loss of generality assume that edge j belongs to G_1. By (5.3), there is a Hamilton cycle of G_1 that contains j and another one that misses j. Any such Hamilton cycle contains two edges of $\delta(S_1)$ and misses the third edge, say j_1. By (5.3), there is a Hamilton cycle of G_2 that misses j_1 and therefore any Hamilton cycle of G_1 can be completed into a Hamilton cycle of $G_1 * G_2$. That proves that $G_1 * G_2$ satisfies property (5.3).

Conversely, if G is a basic elementary graph, then the assumption (4.1) implies that the valid equality $x(\delta(S_1))=2$ must be a linear combination of the degree constraints $x(\delta(v))=2$ for $v \in V$, i.e. $x(\delta(S_1))-2 \equiv \sum_{v \in V} \alpha_v [x(\delta(v))-2]$. Let $\bar{V}=\{v \in V: \alpha_v \neq 0\}$. If $\bar{V}=S_1$ or S_2, then $G \times \bar{V}$ is bipartite as required. Otherwise $\bar{V}=V$, since $\delta(\bar{V}) \subseteq \delta(S_1)=\delta(S_2)$ and $G[S_1]$ and $G[S_2]$ are connected. Then note that all the coefficients α_v for $v \in S_1$ (respectively $v \in S_2$), must be equal in absolute value. (This follows from the fact that the linear combination of degree constraints must add to 0 for every edge of $G[S_1]$.) Furthermore $G[S_1]$ is bipartite, the bipartition being given by the sign of α_v. Let $\delta(S_1)=\{e,f,g\}$ and denote by u_e,u_f,u_g (respectively v_e,v_f,v_g) the vertices of S_1 (resp. S_2) incident with e,f and g. If $\alpha_{u_e}=\alpha_{u_f}=\alpha_{u_g}$, then $G\times S_2$ is bipartite since u_e,u_f and u_g belong to the same side of the bipartition of $G[S_1]$. Otherwise, assume without loss of generality that $\alpha_{u_e}=-\alpha_{u_f}$. Then $\alpha_{u_e}+\alpha_{v_e}=1$ and $\alpha_{u_f}+\alpha_{v_f}=1$ imply $\alpha_{v_e}+\alpha_{v_f}=2$. Since these two coefficients are equal in absolute value, we must have $\alpha_{v_e}=\alpha_{v_f}=1$. This implies $\alpha_{u_e}=0$ and $\alpha_{u_f}=0$ which contradicts $\bar{V}=V$. □

Theorem 5.2 gives a way of generating infinite families of basic elementary graphs, by recursively 3-splicing a basic elementary bipartite graph onto a basic elementary graph. Unfortunately we only know two irreducible basic elementary bipartite graphs, namely $K_{3,3}$ and the 3-6 cage (see Figure 7.)

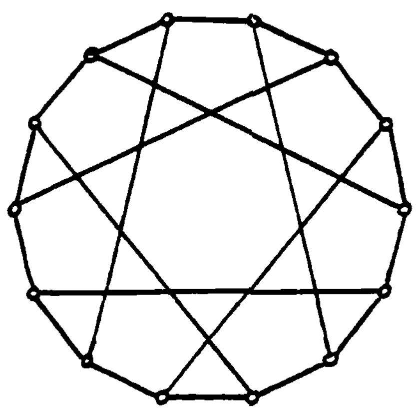

Figure 7. The 3-6 cage

By specializing Lemma 3.1 to bipartite graphs, we have that all the extreme points of the polytope defined by the linear system (5.1) - (5.2) are 2-factors. Thus, to show that the 3-6 cage is elementary, we have to check that all its 2-factors are Hamilton cycles. Since every cycle contains at least 6 vertices, it suffices to check that the vertices of the 3-6 cage cannot be partitioned into a cycle of length 6 and a cycle of length 8. This verification is left to the reader. It is also easy to verify that condition (5.3) is satisfied.

In the remainder of this section we describe an infinite class of basic elementary graphs which contains the wheels.

LEMMA 5.3. Let B be a bipartite basic elementary graph and u a degree-3 vertex of B. Let B' be constructed from B by inserting new vertices v and w on two edges of B incident with u, and joining v and w by an edge. Then B' is a basic elementary graph.

PROOF. It is clear from the construction of B' and the fact that B is bipartite that B' cannot have two disjoint odd cycles. Thus by Lemma 3.1 every extreme point of (5.1) - (5.2) is a 2-factor. Now consider a 2-factor x of B'. If the edge (v,w) is not in the 2-factor, then x induces a 2-factor in B, and therefore it must be Hamiltonian since be B is elementary. So assume that the edge (v,w) is in the 2-factor x. Note that the triangle (u,v,w) cannot be a cycle of x because the removal of these 3 vertices would leave a bipartite graph with one more vertex on one side of the bipartition. Therefore exactly one of the edges (u,v) or (u,w) must be in the 2-factor (a consequence of the fact that u has degree 3). So x is a Hamiltonian cycle. □

By Theorem 5.2, every graph obtained as $[(W_k * B_1) * \ldots] * B_p$ is basic elementary if $B_i, 1 \leq i \leq p$, stands for $K_{3,3}$, the 3-6 cage or some other bipartite basic elementary graph. A larger class **W** can be obtained as follows.

(i) $W_3 = K_4 \in \mathbf{W}$. Let w be the center of W_3.

(ii) If $G \in \mathbf{W}$ and B is a bipartite basic elementary graph, then $G^v * B \in \mathbf{W}$ for any $v \neq w$.

(iii) If $G \in \mathbf{W}$, then the graph obtained from G by joining w to a new vertex z' placed on some edge (z,u) where z is adjacent to w and $u \neq w$, also belongs to $\mathbf{W}$.

Note that, through operations (ii) and (iii), the *center* w of any graph in $\mathbf{W}$ remains well defined. Of course the wheels belong to $\mathbf{W}$, from (i) and repeated application of (iii). Then, by repeated application of (ii) the graphs $[(W_k * B_1) * ...] * B_p$ can be obtained. It is interesting that some or all of their 3-edge cutsets can then be removed by application of (iii). (See Figure 8).

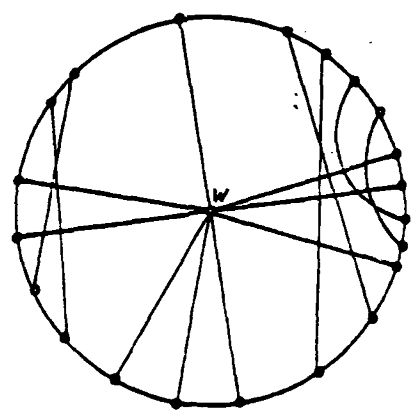

Figure 8. An irreducible graph in W

PROPOSITION 5.4. Every graph in the class $\mathbf{W}$ is a basic elementary graph.

PROOF. Assume not and let G be the smallest graph in $\mathbf{W}$ which is not basic elementary. Consider an extreme point x of (5,1) - (5,2) which is not the incidence vector of a Hamilton cycle.

Assume $G = G_1 * B$ with $G_1 \in W$ and B a bipartite basic elementary graph. Let e, f, g be the edges in the 3-edge cutset joining G_1 and B. The constraint $x_e + x_f + x_g = 2$ is implied by the fact that B is bipartite. If $x_e = x_f = 1$ and $x_g = 0$, then the facts that every 2-factor of B is Hamiltonian and that x is a non-Hamiltonian 2-factor of G imply that the 2-factor induced by x on G_1 is non-Hamiltonian. This contradicts the minimality of G. If $x_e = x_f = \frac{1}{2}$ and $x_g = 1$, then the facts that B contains no odd cycle and that the 2-factor x of G has at least two odd cycles assigned the value $\frac{1}{2}$ imply that the 2-factor induced by x on G_1 has at least two odd cycles with $\frac{1}{2}$'s . Again this contradicts the minimality of G.

Note that $x(\delta(w)) = 2$ and the fact that $x_e = 0, \frac{1}{2}$ or 1 on every edge, with the $\frac{1}{2}$'s occuring on vertex disjoint odd cycles implies that at most 3 edges incident with w have $x_e > 0$. Assume that some duplicated edge e (step (iii)) has the value $x_e = 0$. Then the graph without the duplicated edge also has an extreme point which is not Hamilton cycle. This contradicts the minimality of G. Thus in G, step (iii) has been applied at most twice.

Therefore G must be obtained as $[(W_3^{v_1} * B_1) * \cdots]^{v_p} * B_p$ followed by one or two duplications of the vertex v_p, where $v_i, 2 \leq i \leq p$, is the vertex of B_{i-1} which is adjacent to w. This is equivalent to $W_3^v * B$ followed by one or two duplications of v.

Consider the case of one duplication, say edge (w,v) is duplicated into (w,u). Since x must be positive on these two edges, it has to be zero on at least one of the original edges in W_3. Remove it as well as its end vertex other than w. Then we have a graph B' obtained from B as described in Lemma 5.3. So G is basic elementary.

In the case of two duplications, two of the original edges of W_3 must have $x_e=0$. Removing them leaves a graph with a 2-edge cutset. In this graph every 2-factor is a Hamilton cycle as a consequence of the fact that B is elementary. □

6. Extensions

This paper deals only with one graph reduction. Specifically 3-edge cutsets are used to break up travelling salesman problem into four smaller problems. But other reductions could also be studied. One obvious direction is to reduce a graph using its k-edge cutsets, for any given k. When $k=2$, this works nicely. Let S and $\bar{S}$ be the shores of a 2-edge cutset. Then minimum cost Hamilton cycles for $G \times S$ and $G \times \bar{S}$ can simply be patched to produce an optimal solution for G. The basic polyhedral theorem (Theorem 2.2) also holds. When $k \geq 4$, a reduction into $G \times S$ and $G \times \bar{S}$ does not seem to work since a Hamilton cycle of G may use 4 or more edges of the k-edge cutset. Even if every Hamilton cycle were to use only two edges of the k-edge cutset, one could not in general use cost reductions $\alpha, \beta, \gamma, \ldots$, (recall the basic reduction in Section 2) because they would have to satisfy $\left\{ {k \atop 2} \right\}$ equations with only k unknowns.

A more promising direction for investigation is to consider vertex cutsets instead of edge cutsets. Suppose $\{u,v\}$ is a 2-vertex cutset of G. Let $G[S_1]$ and $G[S_2]$ be the two connected components of $G[V-\{u,v\}]$. (Note that three connected components cannot occur if G is a Hamiltonian graph.) Then solutions of the travelling salesman problem in $G \times S_1$ and $G \times S_2$ can be patched into a solution for G. A simple polyhedral theorem also exists.

The most appealing generalization of our basic reduction occurs by allowing both edges and vertices in a cutset. (See Figure 9.)

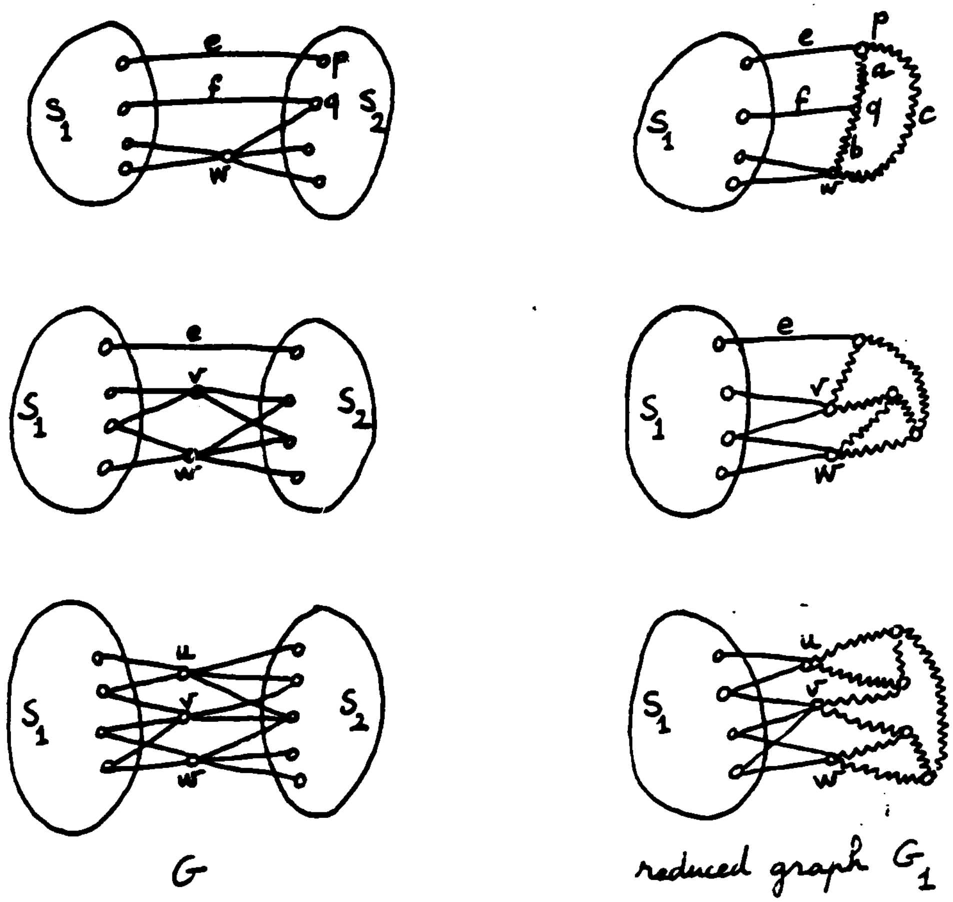

Figure 9. Reductions

First consider the case where G has a cutset consisting of two edges and one vertex, say edges e and f and vertex w. We will show that the travelling salesman problem on G can be reduced to five smaller problems (assuming both S_1 and S_2 contain at least 3 vertices,) Let $p \in S_2$ and $q \in S_2$ be the endpoints of edges e and f respectively. Compute the values

H_{pq} = minimum cost of a Hamilton path from p to q in $G[S_2]$

L_{pq} = minimum cost of a Hamilton path from p to q in $G[S_2 \cup \{w\}]$

L_{pw} = minimum cost of a Hamilton path from p to w in $G[S_2 \cup \{w\}]$

L_{qw} = minimum cost of a Hamilton path from q to w in $G[S_2 \cup \{w\}]$.

Now consider the reduced graph G_1 given in Figure 9. In order to reflect in G_1 the cost of the four Hamilton paths computed above, we introduce costs α, β, γ for the edges a, b, c respectively and we modify the costs $l(e)$ and $l(f)$ to $l'(e) = l(e) + \epsilon$ and $l'(f) = l(f) + \eta$. A proper set of values $\{\alpha, \beta, \gamma, \epsilon, \eta\}$ is given by the system

$$\begin{cases} H_{pq} = \alpha + \epsilon + \eta \\ L_{pq} = \beta + \gamma + \epsilon + \eta \\ L_{pw} = \alpha + \beta + \epsilon \\ L_{qw} = \alpha + \gamma + \eta. \end{cases}$$

A solution to this system is

$$\alpha = \frac{1}{2}(-L_{pq} + l_{pw} + L_{qw})$$

$$\beta = \frac{1}{4}(-2H_{pq} + L_{pq} + 3L_{pw} - L_{qw})$$

$$\gamma = \frac{1}{4}(-2H_{pq} + L_{pq} - L_{pw} + 3L_{qw})$$

$$\epsilon = \eta = \frac{1}{4}(2H_{pq} + L_{pq} - L_{pw} - L_{qw}).$$

With these costs, a minimum cost Hamilton cycle in G_1 is also a minimum cost solution of the travelling salesman problem in G.

The graph G_1 was obtained from G by replacing the graph $G[S_2 \cup \{w\}]$ by three edges a, b, c. Similarly we can define a reduced graph G_2 from G by replacing the graph $G[S_1 \cup \{w\}]$ by three edges, say a', b'

and c'. Assume that we know linear systems defining $TSP(G_1)$ and $TSP(G_2)$. Following the proof of Theorem 2.2 it can be shown that a linear system defining $TSP(G)$ is simply the union of linear systems for $TSP(G_1)$ and $TSP(G_2)$. Note however that the linear system just given for $TSP(G)$ contains variables $x_a, x_b, x_c, x_{a'}, x_{b'}$ and $x_{c'}$ which do not correspond to edges of G. To get a linear system only in terms of the edge variables of G we need to eliminate those 6 variables. It turns out that this can be done since the following equations are valid (they are degree constraints)

$$\begin{cases} x_a + x_b = 2 - x_f \\ x_a + x_c = 2 - x_e \\ x_b + x_c = 2 - x(W_1) \end{cases} \quad \text{and} \quad \begin{cases} x_{a'} + x_{b'} = 2 - x_f \\ x_{a'} + x_{c'} = 2 - x_e \\ x_{b'} + x_{c'} = 2 - x(W_2) \end{cases}$$

where W_i is the set of edges of G joining S_i to the vertex w, for $i=1,2$.

Now we will mention briefly the cases where the cutset consists of one edge and two vertices or where it consists of 3 vertices. The travelling salesman problem in G can then be reduced to six or seven problems on smaller graphs, respectively. A valid reduced graph G_1 which accomplishes this is given in Figure 9. The reader can easily figure out the costs that must be associated with its wiggly edges. Again, if G_2 is defined as G_1 except that S_2 is replaced by S_1, it is still true that a linear system defining $TSP(G)$ is obtained as the union of linear systems for $TSP(G_1)$ and $TSP(G_2)$. However now there are not enough valid equations to eliminate the variables which are associated with edges of G. If we insist on eliminating these variables we must perform a Fourier - Motzkin elimination (see e.g. [1]). Then the size of the system defining $TSP(G)$ can increase exponentially in the number of eliminated variables.

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