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PURDUE UNIVERSITY

DEPARTMENT OF STATISTICS

ON BAYES AND EMPIRICAL BAYES RULES
FOR SELECTING GOOD POPULATIONS*

by

Shanti S. Gupta and Lii-Yuh Leu
Purdue University

Technical Report #83-37

Department of Statistics
Purdue University

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Abstract

This paper deals with the problem of selecting all populations which are close to a control or standard. A general Bayes rule for the above problem is derived. Empirical Bayes rules are derived when the populations are assumed to be uniformly distributed. Under some conditions on the marginal and prior distributions, the rate of convergence of the empirical Bayes risk to the minimum Bayes risk is investigated. The rate of convergence is shown to be $n^{-\delta/3}$ for some δ, $0 < \delta < 2$.

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1. Introduction

Empirical Bayes rules have been considered for multiple decision problems by Deely (1965), Van Ryzin (1970), Van Ryzin and Susarla (1977), Singh (1977), and Gupta and Hsiao (1981). Most of the papers are concerned with the selection of the best population where best is usually defined in terms of the largest or smallest unknown parameter. Gupta and Hsiao (1981) considered the problem which is concerned with the selection of populations better than a control. In some practical applications, one may be interested in selecting populations which are close to a control. We will consider this kind of problem in this paper.

In Section 2, we propose a general Bayes rule for selecting good populations. In Section 3, assuming that the populations are uniformly distributed, empirical Bayes rules are derived for both the known control parameter and the unknown control parameter cases. Under some conditions on the marginal and prior distributions, the rate of convergence of the empirical Bayes risk to the minimum Bayes risk is investigated. The rate of convergence is shown to be $n^{-\delta/3}$ for some δ, $0 < \delta < 2$.

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2. A General Bayes Rule for Selecting Good Populations

Let $\pi_0, \pi_1, \ldots, \pi_k$ be (k+1) independent populations which are characterized by parameters $\theta_0, \theta_1, \ldots, \theta_k$, respectively. Assume that π_0 is the control population with parameter θ_0 which may be known or unknown. When θ_0 is unknown, let $\underset{\sim}{\theta} = (\theta_0, \theta_1, \ldots, \theta_k)$ and $\underset{\sim}{X} = (X_0, X_1, \ldots, X_k)$ where X_i is an observation from π_i, $i = 0,1,\ldots,k$. When θ_0 is known, no observation from population π_0 is taken, and θ_0, X_0 are deleted from $\underset{\sim}{\theta}$ and $\underset{\sim}{X}$, respectively. When there is no confusion, $\underset{\sim}{\theta}$ and $\underset{\sim}{X}$ are used to represent either case. We define population π_i to be a good population if $|\theta_i - \theta_0| < \Delta$ and a bad population if $|\theta_i - \theta_0| \geq \Delta$, where $\Delta > 0$ is a pre-assigned constant. Our goal is to find a Bayes rule which selects all good populations and rejects bad ones. We assume that given θ_i, X_i has probability density function $f(x_i|\theta_i)$ with respect to a σ-finite measure μ, for $i = 0,1,\ldots,k$, and $\underset{\sim}{\theta}$ has a prior distribution $G(\underset{\sim}{\theta}) = \prod_{i=0}^{k} G_i(\theta_i)$ on the parameter space Ω. Let $\mathcal{A} = \{s | s \subseteq \{1,2,\ldots,k\}\}$ be the action space and let

$$(2.1) \quad L(\underset{\sim}{\theta}, s) = \sum_{i \in s} \{c_1(\theta_0 - \Delta - \theta_i) I_{\{\theta_i \leq \theta_0 - \Delta\}}(\theta_i) + c_2(\theta_i - \theta_0 - \Delta) I_{\{\theta_0 + \Delta \leq \theta_i\}}(\theta_i)\} + \sum_{i \notin s} \{c_3(\theta_i - \theta_0 + \Delta) I_{\{\theta_0 - \Delta < \theta_i \leq \theta_0\}}(\theta_i) + c_4(\theta_0 + \Delta - \theta_i) I_{\{\theta_0 < \theta_i < \theta_0 + \Delta\}}(\theta_i)\}$$

be the loss function defined on $\Omega \times \mathcal{A}$, where c_i, $i = 1,2,3,4$ are positive constants and I is the indicator function. The Bayes risk with respect to G can be expressed as

$$(2.2) \qquad r(G,s) = \int_{\mathcal{X}} \int_{\Omega} L(\underset{\sim}{\theta}, s) f(\underset{\sim}{x}|\underset{\sim}{\theta}) dG(\underset{\sim}{\theta}) d\mu(\underset{\sim}{x}),$$

where $\mathcal{X}$ is the sample space and $f(\underset{\sim}{x}|\underset{\sim}{\theta}) = \prod_i f(x_i|\theta_i)$.

Since the action space is finite, attention can be restricted to the non-randomized rules for deriving the Bayes rules. For a non-randomized decision function δ: $\mathcal{X} \to \mathbb{G}$, the corresponding Bayes risk with respect to G is given by

$$(2.3) \qquad r(G,\delta) = \int_{\mathcal{X}} \int_{\Omega} L(\underline{\theta},\delta(\underline{x}))f(\underline{x}|\underline{\theta})dG(\underline{\theta})d\mu(\underline{x}).$$

In the sequel we consider the special case where $c_1 = c_2 = c_3 = c_4 =$ constant which can be taken to be unity without loss of generality. If ϕ is the empty set, we have

$$L(\underline{\theta},\phi) = \sum_{i=1}^{k}\{(\theta_i-\theta_0+\Delta)I_{\{\theta_0-\Delta<\theta_i\leq\theta_0\}}(\theta_i) + (\theta_0+\Delta-\theta_i)I_{\{\theta_0<\theta_i<\theta_0+\Delta\}}(\theta_i)\},$$

and (2.1) can be expressed as

$$(2.4) \qquad L(\underline{\theta},s) = L(\underline{\theta},\phi) + \sum_{i\in s}\{(\theta_0-\Delta-\theta_i)I_{\{\theta_i\leq\theta_0\}}(\theta_i) + (\theta_i-\theta_0-\Delta)I_{\{\theta_0<\theta_i\}}(\theta_i)\}.$$

Hence, for any δ, we have

$$(2.5) \qquad r(G,\delta) - r(G,\phi)$$
$$= \int_{\mathcal{X}} \sum_{i\in\delta(\underline{x})}\{\int_{\Omega}(\theta_0-\Delta-\theta_i)f(\underline{x}|\underline{\theta})dG(\underline{\theta}) + 2\int_{\{\theta_0<\theta_i\}}(\theta_i-\theta_0)f(\underline{x}|\underline{\theta})dG(\underline{\theta})d\mu(\underline{x})\}.$$

From (2.5), $\delta_B(\underline{x})$ is given by $i \in \delta_B(\underline{x})$ if

$$(2.6) \qquad \int_{\Omega}(\theta_0-\Delta-\theta_i)f(\underline{x}|\underline{\theta})dG(\underline{\theta})+2\int_{\{\theta_0<\theta_i\}}(\theta_i-\theta_0)f(\underline{x}|\underline{\theta})dG(\underline{\theta}) < 0,$$

then $\delta_B(\underline{x})$ is a Bayes rule with respect to G.

Let $m_i(x_i) = \int_\Omega f(x_i|\theta_i)dG_i(\theta_i)$ be the marginal distribution of X_i, $\pi(\theta_i|x_i)$ be the posterior distribution of θ_i given $X_i = x_i$, and $E(\theta_i|x_i)$ be the expected value of θ_i given $X_i = x_i$. If $m_i(x_i) > 0$ for all x_i, then (2.6) is equivalent to

(2.7) $\quad (\theta_0-\Delta)-E(\theta_i|x_i) + 2\int_{\{\theta_0<\theta_i\}}(\theta_i-\theta_0)\pi(\theta_i|x_i)d\theta_i < 0$

if θ_0 is known, or

(2.8) $\quad E(\theta_0|x_0)-E(\theta_i|x_i)-\Delta+2\int_{\{\theta_0<\theta_i\}}(\theta_i-\theta_0)\pi(\theta_i|x_i)\pi(\theta_0|x_0)d\theta_i d\theta_0 < 0$

if θ_0 is unknown.

From the above discussion, we have the following main result:

Theorem 2.1. Under the loss function (2.4), the Bayes rule $\delta_B(\underline{x})$ with respect to G is given by

(a) If θ_0 is known, then $i \in \delta_B(\underline{x})$ if the inequality (2.7) holds.

(b) If θ_0 is unknown, then $i \in \delta_B(\underline{x})$ if the inequality (2.8) holds

An Application:

Suppose that

(2.9) $\quad f(x_i|\theta_i) = e^{-\theta_i}\theta_i^{x_i}/(x_i!),\ x_i = 0,1,\ldots,\theta_i > 0$

and θ_i has a prior distribution $g_i(\theta_i) = G_i'(\theta_i)$ which is given by

(2.10) $\quad g_i(\theta_i) = \beta_i^{\alpha_i}\theta_i^{\alpha_i-1}e^{-\beta_i\theta_i}/\Gamma(\alpha_i)I_{(0,\infty)}(\theta_i)$,

where $\alpha_i > 0$ and $\beta_i > 0$ are known. Then

(2.11) $\quad \pi(\theta_i|x_i) = (1+\beta_i)^{x_i+\alpha_i}\theta_i^{x_i+\alpha_i-1}e^{-\theta_i(1+\beta_i)}/\Gamma(x_i+\alpha_i)$

and

(2.12) $\quad E(\theta_i|x_i) = (x_i+\alpha_i)/(1+\beta_i)$.

<u>Lemma 2.2</u>. If $\pi(\theta_i|x_i)$ is defined by (2.11), then

$$\int_{\{\theta_0<\theta_i\}}(\theta_i-\theta_0)\pi(\theta_i|x_i)d\theta_i$$

$$= \frac{x_i+\alpha_i}{1+\beta_i}\{1-\Gamma(\theta_0(1+\beta_i);x_i+\alpha_i+1)\}-\theta_0\{1-\Gamma(\theta_0(1+\beta_i);x_i+\alpha_i)\},$$

where $\theta_0 > 0$ is known and

$$\Gamma(a;\ \alpha) = \int_0^a \frac{x^{\alpha-1}}{\Gamma(\alpha)} e^{-x}dx, \quad a > 0, \quad \alpha > 0.$$

Proof. Proof is simple and hence omitted.

<u>Lemma 2.3</u>. If $\pi(\theta_i|x_i)$ is defined by (2.11) and $\beta_i = \beta$, $i = 0,1,\ldots,k$ and θ_0 is unknown, then

$$\int_{\{\theta_0<\theta_i\}}(\theta_i-\theta_0)\pi(\theta_i|x_i)\pi(\theta_0|x_0)d\theta_i d\theta_0$$

$$= \frac{(x_i+\alpha_i+x_0+\alpha_0)}{1+\beta} I(\tfrac{1}{2};\ x_0+\alpha_0,x_i+\alpha_i) - \frac{2(x_0+\alpha_0)}{1+\beta} I(\tfrac{1}{2};\ x_0+\alpha_0+1,x_i+\alpha_i),$$

where

$$I(z;\ \alpha,\beta) = \int_0^z \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}dx, \quad \alpha > 0, \quad \beta > 0,$$

and

$$B(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta).$$

Proof. $\int_{\{\theta_0<\theta_i\}}(\theta_i-\theta_0)\pi(\theta_i|x_i)\pi(\theta_0|x_0)d\theta_i d\theta_0$

$$= \int_0^\infty \int_{\theta_0}^\infty (\theta_i-\theta_0) \frac{(1+\beta)^{x_i+\alpha_i+x_0+\alpha_0}}{\Gamma(x_i+\alpha_i)\Gamma(x_0+\alpha_0)} \theta_i^{x_i+\alpha_i-1} \theta_0^{x_0+\alpha_0-1} e^{-(\theta_i+\theta_0)(1+\beta)} d\theta_i d\theta_0$$

$$= \int_0^\infty \int_1^\infty \frac{(1+\beta)^{x_i+\alpha_i+x_0+\alpha_0}}{\Gamma(x_i+\alpha_i)\Gamma(x_0+\alpha_0)} \theta_0^{x_i+\alpha_i+x_0+\alpha_0}(u-1)u^{x_i+\alpha_i-1} e^{-(1+u)\theta_0(1+\beta)} du d\theta_0$$

$$= \int_1^\infty \frac{(x_i+\alpha_i+x_0+\alpha_0)}{(1+\beta)B(x_0+\alpha_0,x_i+\alpha_i)} (u-1)u^{x_i+\alpha_i-1}(1+u)^{-(x_i+\alpha_i+x_0+\alpha_0+1)}du$$

$$= \frac{(x_i+\alpha_i+x_0+\alpha_0)}{(1+\beta)B(x_0+\alpha_0,x_i+\alpha_i)} \int_0^1 (1-v)(v+1)^{-(x_i+\alpha_i+x_0+\alpha_0+1)} v^{x_0+\alpha_0-1}dv$$

$$= \frac{(x_i+\alpha_i+x_0+\alpha_0)}{(1+\beta)B(x_0+\alpha_0,x_i+\alpha_i)} \int_0^{\frac{1}{2}}(1-2s)(1-s)^{x_i+\alpha_i-1}s^{x_0+\alpha_0-1}ds$$

$$= \frac{(x_i+\alpha_i+x_0+\alpha_0)}{1+\beta} I(\tfrac{1}{2}; x_0+\alpha_0,x_i+\alpha_i) - \frac{2(x_0+\alpha_0)}{1+\beta} I(\tfrac{1}{2}; x_0+\alpha_0+1; x_i+\alpha_i).$$

From Theorem 2.1, Lemma 2.2 and Lemma 2.3, we have the following theorem:

Theorem 2.4. If $f(x_i|\theta_i)$ is defined by (2.9) and $g_i(\theta_i)$ is defined by (2.10). Under the loss function (2.4), the Bayes rule $\delta_B(\underline{x})$ is given by

(a) If θ_0 is known, then $i \in \delta_B(\underline{x})$ if

$$\frac{x_i+\alpha_i}{1+\beta_i}\{1-2\Gamma(\theta_0(1+\beta_i); x_i+\alpha_i+1)\} - \theta_0\{1-2\Gamma(\theta_0(1+\beta_i); x_i+\alpha_i)\} < \Delta.$$

(b) If θ_0 is unknown and $\beta_i = \beta$, $i = 0,1,\dots,k$, then $i \in \delta_B(\underline{x})$ if

$$\frac{x_i+\alpha_i}{1+\beta}\{2I(\tfrac{1}{2}; x_0+\alpha_0,x_i+\alpha_i)-1\} + \frac{x_0+\alpha_0}{1+\beta}\{1 +$$

$$2I(\tfrac{1}{2}; x_0+\alpha_0,x_i+\alpha_i)-4I(\tfrac{1}{2}; x_0+\alpha_0+1,x_i+\alpha_i)\} < \Delta.$$

3. Empirical Bayes Rules for Uniform Populations

In this section we will assume that X_i has probability density function $f(x_i|\theta_i) = \frac{1}{\theta_i} I_{(0,\theta_i)}(x_i)$, where $\theta_i > 0$ is unknown. Suppose that $\underline{\theta}$ has a prior distribution $G(\underline{\theta}) = \pi_i G_i(\theta_i)$ on Ω and G_i has a continuous probability density function g_i and g_i is positive. Let $m_i(x_i)$, $M_i(x_i)$ be the marginal pdf and cdf of X_i, respectively. Then

$$(3.1) \qquad m_i(x_i) = \int_{x_i}^{\infty} \frac{1}{\theta_i}\, dG_i(\theta_i)$$

and

$$(3.2) \qquad M_i(x_i) = x_i m_i(x_i) + G_i(x_i).$$

From (3.2), we have

$$(3.3) \qquad G_i(x_i) = M_i(x_i) - x_i m_i(x_i).$$

It follows that

$$(3.4) \qquad \int_a^b \frac{1}{\theta_i}\, dG_i(\theta_i) = m_i(a) - m_i(b)$$

and

$$\int_a^{\infty} \frac{1}{\theta_i}\, dG_i(\theta_i) = m_i(a)$$

for any $x_i \leq a < b < \infty$.

3.1. θ_0 known

In the case where θ_0 is known, let

$$(3.5) \qquad \Delta_{G_i}(x_i) = (\theta_0-\Delta)m_i(x_i) - \int_{x_i}^{\infty} dG_i(\theta_i) + 2\int_{(x_i,\infty)\cap(\theta_0,\infty)} (\theta_i-\theta_0) f(x_i|\theta_i)\, dG_i(\theta_i).$$

From (2.6), we have $i \in \delta_B(\underset{\sim}{x})$ if $\Delta_{G_i}(x_i) < 0$. If $x_i \leq \theta_0$,

$$\begin{aligned}(3.6) \qquad \Delta_{G_i}(x_i) &= (\theta_0-\Delta)m_i(x_i) - \int_{x_i}^{\infty} dG_i(\theta_i) + 2\int_{\theta_0}^{\infty}(\theta_i-\theta_0)\frac{1}{\theta_i}\, dG_i(\theta_i) \\ &= (\theta_0-\Delta-x_i)m_i(x_i) + 1 - 2M_i(\theta_0) + M_i(x_i) \\ &= \Delta_{1,G_i}(x_i) \quad \text{(say).}\end{aligned}$$

If $x_i > \theta_0$,

$$(3.7)\quad \Delta_{G_i}(x_i) = (\theta_0-\Delta)m_i(x_i)-\int_{x_i}^{\infty} dG_i(\theta_i)+2\int_{x_i}^{\infty}(\theta_i-\theta_0)\frac{1}{\theta_i}\,dG_i(\theta_i)$$

$$= (x_i-\theta_0-\Delta)m_i(x_i)+1-M_i(x_i)$$

$$= \Delta_{2,G_i}(x_i)\quad \text{(say)}.$$

Therefore

$$(3.8)\quad \delta_B(\underline{x}) = \{i\,|\,x_i \leq \theta_0, \Delta_{1,G_i}(x_i) < 0\}\cup\{i\,|\,x_i > \theta_0, \Delta_{2,G_i}(x_i) < 0\}.$$

Remarks:

(1) $\Delta_{G_i}(x_i)$ is strictly decreasing for $0 < x_i < \theta_0-\Delta$, strictly increasing for $\theta_0-\Delta < x_i < \theta_0+\Delta$, and strictly decreasing for $\theta_0+\Delta < x_i$ (we assume that $\theta_0-\Delta > 0$).

(2) If $x_i \geq \theta_0+\Delta$, then $\Delta_{G_i}(x_i) \geq 0$. Hence $i \notin \delta_B(\underline{x})$ if $x_i \geq \theta_0+\Delta$.

(3) If G_i is such that $1-2M_i(\theta_0) + M_i(\theta_0-\Delta) \geq 0$, then $\delta_B(\underline{x}) = \phi$. Otherwise, $i \in \delta_B(\underline{x})$ if $(\theta_0-\Delta)-d_1 < x_i < (\theta_0+\Delta)-d_2$, for some positive real numbers d_1 and d_2. Hence this type of selection rules are Bayes rules relative to some prior distribution.

If G is unknown, the Bayes rules are not obtainable. In this case, we consider a sequence $(\underline{X}_1,\underline{\Lambda}_1)$, $(\underline{X}_2,\underline{\Lambda}_2),\ldots,$ which are independent pairs of random vectors, each $\underline{\Lambda}_i$ is distributed as G on Ω and $\underline{X}_i = (X_{i1},\ldots,X_{ik})$ has conditional density function $f(\underline{x}|\underline{\theta})$ given $\underline{\Lambda}_i = \underline{\theta}$. The empirical Bayes approach, which was introduced by Robbins (1955), attempts to construct a decision rule concerning $\underline{\Lambda}_{n+1}$ at stage n+1 based on $\underline{X}_1,\ldots,\underline{X}_{n+1}$. The risk at stage n+1 taking action $\delta_n(\underline{x};\ \underline{x}_1,\ldots,\underline{x}_n) = \delta_n(\underline{x})$ is given by

$$(3.9)\qquad r_n(G,\delta_n) = \int_{\mathcal{X}} E_n\{\sum_{i\in\delta_n(\underline{x})}[\int_{\Omega}(\theta_0-\Delta-\theta_i)f(\underline{x}|\underline{\theta})dG(\underline{\theta}) +$$

$$2\int_{(\theta_0,\infty)}(\theta_i-\theta_0)f(\underline{x}|\underline{\theta})dG(\underline{\theta})]\}d\underline{x} + r(G,\phi),$$

where E_n denotes the expectation with respect to the n independent random variables $\underline{X}_1,\ldots,\underline{X}_n$ each with common density function

$$m(\underline{x}) = \int_\Omega f(\underline{x}|\underline{\theta})dG(\underline{\theta}) = \prod_{i=1}^{k} m_i(x_i).$$

<u>Definition 3.1</u>. The sequence of procedures $\{\delta_n\}$ is said to be asymptotically optimal (a.o.) relative to G if $r_n(G,\delta_n) - r(G) = o(1)$ as $n \to \infty$, where $r(G) = \inf_\delta r(G,\delta)$.

In order to find an a.o. sequences of rules, let $\delta_{1,B}(\underline{x}) = \{i|x_i \leq \theta_0, \Delta_{1,G_i}(x_i) < 0\}$ and $\delta_{2,B}(\underline{x}) = \{i|\theta_0 < x_i < \theta_0+\Delta, \Delta_{2,G_i}(x_i) < 0\}$. From (3.8) and Remark (2), we have $\delta_B(\underline{x}) = \delta_{1,B}(\underline{x}) \cup \delta_{2,B}(\underline{x})$. For any $i = 1,2,\ldots,k$ and $\ell = 1,2$, let $\Delta_{\ell,i,n}(x_i) = \Delta_{\ell,i}(x_i;\ x_{1i},\ldots,x_{ni})$, $n = 1,2,\ldots$ be two sequences of real-valued measurable functions. We define

$$\delta_n(\underline{x}) = \delta_{1,n}(\underline{x}) \cup \delta_{2,n}(\underline{x}), \tag{3.10}$$

where

$$\delta_{1,n}(\underline{x}) = \{i|x_i \leq \theta_0,\ \Delta_{1,i,n}(x_i) < 0\}$$

and

$$\delta_{2,n}(\underline{x}) = \{i|\theta_0 < x_i < \theta_0+\Delta, \Delta_{2,i,n}(x_i) < 0\}.$$

We have the following theorem:

<u>Theorem 3.1</u>. **If $\int_0^\infty \theta dG_i(\theta) < \infty$, $i = 1,2,\ldots,k$ and $\Delta_{1,i,n}(x_i) \xrightarrow{P} \Delta_{1,G_i}(x_i)$, for almost all $x_i \leq \theta_0$ and $\Delta_{2,i,n}(x_i) \xrightarrow{P} \Delta_{2,G_i}(x_i)$, for almost all $\theta_0 < x_i < \theta_0+\Delta$, where "$\xrightarrow{P}$" means convergence in probability. Then $\{\delta_n(\underline{x})\}$ defined by (3.10) is a.o. relative to G.**

Proof. $0 \leq \int_{\Omega} L(\underset{\sim}{\theta},\delta_n(\underset{\sim}{x}))f(\underset{\sim}{x}|\underset{\sim}{\theta})dG(\underset{\sim}{\theta}) - \int_{\Omega} L(\underset{\sim}{\theta},\delta_B(\underset{\sim}{x}))f(\underset{\sim}{x}|\underset{\sim}{\theta})dG(\underset{\sim}{\theta})$

$$(3.11)\quad = \{\sum_{i\in\delta_{1,n}(\underset{\sim}{x})} \Delta_{1,G_i}(x_i)\prod_{\substack{j=1\\j\neq i}}^{k} m_j(x_j) - \sum_{i\in\delta_{1,B}(\underset{\sim}{x})} \Delta_{1,G_i}(x_i)\prod_{\substack{j=1\\j\neq i}}^{k} m_j(x_j)\}$$

$$+ \{\sum_{i\in\delta_{2,n}(\underset{\sim}{x})} \Delta_{2,G_i}(x_i)\prod_{\substack{j=1\\j\neq i}}^{k} m_j(x_j) - \sum_{i\in\delta_{2,B}(\underset{\sim}{x})} \Delta_{2,G_i}(x_i)\prod_{\substack{j=1\\j\neq i}}^{k} m_j(x_j)\}.$$

The first term of (3.11) can be expressed as

$$(3.12)\quad \{\sum_{i\in\delta_{1,n}(\underset{\sim}{x})} \Delta_{1,G_i}(x_i)\prod_{\substack{j=1\\j\neq i}}^{k} m_j(x_j) - \sum_{i\in\delta_{1,n}(\underset{\sim}{x})} \Delta_{1,i,n}(x_i)\prod_{\substack{j=1\\j\neq i}}^{k} m_j(x_j)\}$$

$$+ \{\sum_{i\in\delta_{1,n}(\underset{\sim}{x})} \Delta_{1,i,n}(x_i)\prod_{\substack{j=1\\j\neq i}}^{k} m_j(x_j) - \sum_{i\in\delta_{1,B}(\underset{\sim}{x})} \Delta_{1,i,n}(x_i)\prod_{\substack{j=1\\j\neq i}}^{k} m_j(x_j)\}$$

$$+ \{\sum_{i\in\delta_{1,B}(\underset{\sim}{x})} \Delta_{1,i,n}(x_i)\prod_{\substack{j=1\\j\neq i}}^{k} m_j(x_j) - \sum_{i\in\delta_{1,B}(\underset{\sim}{x})} \Delta_{1,G_i}(x_i)\prod_{\substack{j=1\\j\neq i}}^{k} m_j(x_j)\}$$

$$\leq \sum_{i\in\delta_{1,n}(\underset{\sim}{x})} (\Delta_{1,G_i}(x_i) - \Delta_{1,i,n}(x_i))\prod_{\substack{j=1\\j\neq i}}^{k} m_j(x_j)$$

$$+ \sum_{i\in\delta_{1,B}(\underset{\sim}{x})} (\Delta_{1,i,n}(x_i) - \Delta_{1,G_i}(x_i))\prod_{\substack{j=1\\j\neq i}}^{k} m_j(x_j).$$

Since by the definition of $\delta_{1,n}(\underset{\sim}{x})$, the second sum of (3.12) is less than or equal to zero.

The second term of (3.11) has a similar result.

Hence, if $\Delta_{\ell,i,n}(x_i) \xrightarrow{P} \Delta_{\ell,G_i}(x_i)$, $\ell = 1,2$, then

$$0 \leq \int_{\Omega} L(\underset{\sim}{\theta},\delta_n(\underset{\sim}{x}))f(\underset{\sim}{x}|\underset{\sim}{\theta})dG(\underset{\sim}{\theta}) - \int_{\Omega} L(\underset{\sim}{\theta},\delta_B(\underset{\sim}{x}))f(\underset{\sim}{x}|\underset{\sim}{\theta})dG(\underset{\sim}{\theta})$$

$$\leq 2\sum_{i=1}^{k} |\Delta_{1,G_i}(x_i) - \Delta_{1,i,n}(x_i)| \prod_{\substack{j=1\\ j\neq i}}^{k} m_j(x_j) + 2\sum_{i=1}^{k} \Delta_{2,G_i}(x_i) - \Delta_{2,i,n}(x_i)| \prod_{\substack{j=1\\ j\neq i}}^{k} m_j(x_j)$$

$$\leq 4\varepsilon \sum_{i=1}^{k} (\prod_{\substack{j=1\\ j\neq i}}^{k} m_j(x_j))$$

with probability near 1, for large n. Hence

$$\int_{\Omega} L(\underset{\sim}{\theta},\delta_n(\underset{\sim}{x}))f(\underset{\sim}{x}|\underset{\sim}{\theta})dG(\underset{\sim}{\theta}) \xrightarrow{P} \int_{\Omega} L(\underset{\sim}{\theta},\delta_B(\underset{\sim}{x}))f(\underset{\sim}{x}|\underset{\sim}{\theta})dG(\underset{\sim}{\theta})$$

for almost all $\underset{\sim}{x}$.

By Corollary 1 of Robbins (1964), $\{\delta_n(\underset{\sim}{x})\}$ is a.o. relative to G.

From Theorem 3.1, our problem is reduced to finding consistent estimators of $\Delta_{1,G_i}(x_i)$ and $\Delta_{2,G_i}(x_i)$. Let

(3.13) $$M_{in}(x_i) = \frac{1}{n}\sum_{j=1}^{n} I_{(-\infty,x_i]}(X_{ji}),$$

then $M_{in}(x_i) \xrightarrow{P} M_i(x_i)$ for all $x_i > 0$. Next, let $\varphi(x) \geq 0$ be a Borel function satisfying the following conditions:

(3.14) (i) $\sup_{-\infty<x<\infty} \varphi(x) < \infty$, (ii) $\int_{-\infty}^{\infty} \varphi(x)dx = 1$, and (iii) $\lim_{x\to\infty} x\varphi(x) = 0$

and $\{h(n)\}$ be a sequence of positive constants satisfying the following conditions:

(3.15) (i) $h(n) \to 0$ as $n \to \infty$ and (ii) $nh(n) \to \infty$ as $n \to \infty$.

We define

(3.16) $$m_{in}(x) = \frac{1}{nh(n)}\sum_{j=1}^{n} \varphi(\frac{x-X_{ji}}{h(n)}),$$

then $m_{in}(x) \xrightarrow{P} m_i(x)$ for all x (see Parzen (1962)). For $i = 1,2,\ldots,k$, let

(3.17) $\Delta_{1,i,n}(x_i) = (\theta_0-\Delta-x_i)m_{in}(x_i)+1-2M_{in}(\theta_0)+M_{in}(x_i)$

and

(3.18) $\Delta_{2,i,n}(x_i) = (x_i-\theta_0-\Delta)m_{in}(x_i)+1-M_{in}(x_i)$.

Then

$$\Delta_{1,i,n}(x_i) \xrightarrow{P} \Delta_{1,G_i}(x_i) \quad \text{for all } x_i \le \theta_0$$

and

$$\Delta_{2,i,n}(x_i) \xrightarrow{P} \Delta_{2,G_i}(x_i) \text{ for all } \theta_0 < x_i < \theta_0+\Delta.$$

Finally, we define

$$\delta_n(\underline{x}) = \{i \mid x_i \le \theta_0, \Delta_{1,i,n}(x_i) < 0\} \cup \{i \mid \theta_0 < x_i < \theta_0+\Delta,\ \Delta_{2,i,n}(x_i) < 0\}.$$

Then $\{\delta_n(\underline{x})\}$ is a.o. relative to G.

3.2. θ_0 unknown

If θ_0 is unknown, let π_0 be the control population and X_0 be the random variable from π_0. We assume that X_0 has conditional pdf $f(x_0|\theta_0) = \frac{1}{\theta_0} I_{(0,\theta_0)}(x_0)$, $\theta_0 > 0$. In this case $\Omega = \{\underline{\theta} = (\theta_0,\theta_1,\ldots,\theta_k) \mid \theta_i > 0,\ i = 0,1,\ldots,k\}$, $\mathcal{X} = \{\underline{x} = (x_0,x_1,\ldots,x_k) \mid x_i > 0,\ i = 0,1,\ldots,k\}$, $G(\underline{\theta}) = \prod_{i=0}^{k} G_i(\theta_i)$, $f(\underline{x}|\underline{\theta}) = \prod_{i=0}^{k} f(x_i|\theta_i)$, and at stage n we observed $\underline{x}_n = (x_{n0},x_{n1},\ldots,x_{nk})$. Under the loss function (2.4), the Bayes rule $\delta_B(\underline{x})$ is given by

$$i \in \delta_B(\underline{x}) \text{ if } \Delta_{G_0,G_i}(x_0,x_i) < 0$$

where

$$\Delta_{G_0,G_i}(x_0,x_i) = \int\theta_0 f(x_0|\theta_0)dG_0(\theta_0)m_i(x_i)-\Delta m_0(x_0)m_i(x_i)-$$
$$\int\theta_i f(x_i|\theta_i)dG_i(\theta_i)m_0(x_0)+2\int_{\{\theta_0<\theta_i\}}(\theta_i-\theta_0)f(x_i|\theta_i)f(x_0|\theta_0)dG_i(\theta_i)dG_0(\theta_0).$$

Using formula (3.4) if $0 < x_i \leq x_0$, we have

$$\int_{\{\theta_0<\theta_i\}} (\theta_i-\theta_0)f(x_i|\theta_i)f(x_0|\theta_0)dG_i(\theta_i)dG_0(\theta_0)$$

$$= m_0(x_0) - \int_{x_0}^{\infty} \frac{M_i(\theta_0)}{\theta_0}\, dG_0(\theta_0),$$

and if $0 < x_0 < x_i$, we have

$$\int_{\{\theta_0<\theta_i\}} (\theta_i-\theta_0)f(x_i|\theta_i)f(x_0|\theta_0)dG_i(\theta_i)dG_0(\theta_0)$$

$$= (1-G_i(x_i))(m_0(x_0)-m_0(x_i)) - m_i(x_i)(G_0(x_i)-G_0(x_0))$$

$$- \int_{x_i}^{\infty} \frac{M_i(\theta_0)}{\theta_0}\, dG_0(\theta_0) + m_0(x_i).$$

Hence

$$\Delta_{G_0,G_i}(x_0,x_i) = m_i(x_i)(1-M_0(x_0)) + (1+M_i(x_i))m_0(x_0) + (x_0-x_i-\Delta)m_i(x_i)m_0(x_0)-2\int_{x_0}^{\infty} \frac{M_i(\theta_0)}{\theta_0}\, dG_0(\theta_0) \tag{3.19}$$

$$= \Delta_{1,G_0,G_i}(x_0,x_i) \text{ (say), if } 0 < x_i \leq x_0$$

and

$$\Delta_{G_0,G_i}(x_0,x_i) = (1-M_i(x_i))m_0(x_0)+(1+M_0(x_0)-2M_0(x_i))m_i(x_i) + (x_i-x_0-\Delta)m_i(x_i)m_0(x_0)+2M_i(x_i)m_0(x_i)-2\int_{x_i}^{\infty} \frac{M_i(\theta_0)}{\theta_0}\, dG_0(\theta_0) \tag{3.20}$$

$$= \Delta_{2,G_0,G_i}(x_0,x_i) \text{ (say), if } 0 < x_0 < x_i.$$

Thus

$$\delta_B(\underline{x}) = \delta_{1,B}(\underline{x}) \cup \delta_{2,B}(\underline{x}) \tag{3.21}$$

where

$$\delta_{1,B}(\underline{x}) = \{i|0 < x_i \le x_0, \Delta_{1,G_0,G_i}(x_0,x_i) < 0\}$$

and

$$\delta_{2,B}(\underline{x}) = \{i|0 < x_0 < x_i, \Delta_{2,G_0,G_i}(x_0,x_i) < 0\}.$$

Similar to Theorem 3.1, we have the following result.

Theorem 3.2. If $\int_0^\infty \theta dG_i(\theta) < \infty$, $i = 0,1,\ldots,k$ and for all $1 \le i \le k$, $\Delta_{1,i,n}(x_0,x_i) \xrightarrow{P} \Delta_{1,G_0,G_i}(x_0,x_i)$ for $x_i \le x_0$ and $\Delta_{2,i,n}(x_0,x_i) \xrightarrow{P} \Delta_{2,G_0,G_i}(x_0,x_i)$ for $x_0 < x_i$. Let $\delta_n(\underline{x}) = \{i|x_i \le x_0, \Delta_{1,i,n}(x_0,x_i) < 0\} \cup \{i|x_0 < x_i, \Delta_{2,i,n}(x_0,x_i) < 0\}$, then $\{\delta_n(\underline{x})\}$ is a.o. relative to G.

Hence our problem is to find a consistent estimator of $\int_a^\infty \frac{M_i(\theta_0)}{\theta_0} dG_0(\theta_0)$ for $x_0 \le a$.

Theorem 3.3. Let $M_{in}(x)$ and $m_{in}(x)$ be defined by (3.13) and (3.16), respectively. Then

$$\int_a^\infty \frac{M_{in}(\theta_0)}{\theta_0} dG_{0n}(\theta_0) \xrightarrow{P} \int_a^\infty \frac{M_i(\theta_0)}{\theta_0} dG_0(\theta_0) \quad \text{for } x_0 \le a,$$

where $G_{0n}(\theta_0) = M_{0n}(\theta_0) - \theta_0 m_{0n}(\theta_0)$.

Proof.
$$\left|\int_a^\infty \frac{M_{in}(\theta_0)}{\theta_0} dG_{0n}(\theta_0) - \int_a^\infty \frac{M_i(\theta_0)}{\theta_0} dG_{0n}(\theta_0)\right|$$
$$\le \int_a^\infty \frac{|M_{in}(\theta_0)-M_i(\theta_0)|}{\theta_0} dG_{0n}(\theta_0)$$
$$\le \frac{1}{a} \sup_{-\infty<x<\infty} |M_{in}(x)-M_i(x)| \le \varepsilon$$

with probability near 1, for large n, by Glivenko-Cantelli Theorem. Since

$\frac{M_i(\theta_0)}{\theta_0}$ is bounded continuous and $G_{0n}(\theta_0) \overset{P}{\to} G_0(\theta_0)$, we have

$\int_a^\infty \frac{M_i(\theta_0)}{\theta_0}\, dG_{0n}(\theta_0) \overset{P}{\to} \int_a^\infty \frac{M_i(\theta_0)}{\theta_0}\, dG_0(\theta_0)$.

Thus

$$\left|\int_a^\infty \frac{M_{in}(\theta_0)}{\theta_0}\, dG_{0n}(\theta_0) - \int_a^\infty \frac{M_i(\theta_0)}{\theta_0}\, dG_0(\theta_0)\right|$$

$$\leq \left|\int_a^\infty \frac{M_{in}(\theta_0)}{\theta_0}\, dG_{0n}(\theta_0) - \int_a^\infty \frac{M_i(\theta_0)}{\theta_0}\, dG_{0n}(\theta_0)\right| +$$

$$\left|\int_a^\infty \frac{M_i(\theta_0)}{\theta_0}\, dG_{0n}(\theta_0) - \int_a^\infty \frac{M_i(\theta_0)}{\theta_0}\, dG_0(\theta_0)\right|$$

$\leq \varepsilon$ with probability near 1, for large n.

From Theorem 3.3, if we define

$$\Delta_{1,i,n}(x_0,x_i) = m_{in}(x_i)(1-M_{0n}(x_0)) + m_{0n}(x_0)(1+M_{in}(x_i))$$

$$+ (x_0-x_i-\Delta)m_{in}(x_i) \quad m_{0n}(x_0) - 2\int_{x_0}^\infty \frac{M_{in}(\theta_0)}{\theta_0}\, dG_{0n}(\theta_0),$$

where $G_{0n}(\theta_0) = M_{0n}(\theta_0) - \theta_0 m_{0n}(\theta_0)$ and $M_{in}(x)$, $m_{in}(x)$ are defined by (3.13) and (3.16), respectively, and

$$\Delta_{2,i,n}(x_0,x_i) = m_{0n}(x_0)(1-M_{in}(x_i)) + m_{in}(x_i)(1+M_{0n}(x_0)-2M_{0n}(x_i))$$

$$+ (x_i-x_0-\Delta)m_{0n}(x_0)m_{in}(x_i) + 2M_{in}(x_i)m_{0n}(x_i) - 2\int_{x_i}^\infty \frac{M_{in}(\theta_0)}{\theta_0}\, dG_{0n}(\theta_0).$$

Then

$$\Delta_{\ell,i,n}(x_0,x_i) \overset{P}{\to} \Delta_{\ell,G_0,G_i}(x_0,x_i),\ \ell = 1,2.$$

Now, let

$$\delta_n(\underline{x}) = \{i \mid x_i \leq x_0, \Delta_{1,i,n}(x_0,x_i) < 0\} \cup \{i \mid x_0 < x_i, \Delta_{2,i,n}(x_0,x_i) < 0\}.$$

From Theorem 3.2, we have $\{\delta_n(\underline{x})\}$ is a.o. relative to G.

3.3. Rate of Convergence of the Empirical Bayes Rules

In this section we will consider the rate of convergence of the empirical Bayes rules derived in Section 3.1.

Definition 3.2. The sequence of procedures $\{\delta_n\}$ is said to be asymptotically optimal of order α_n relative to G if $r_n(G,\delta_n)-r(G) = O(\alpha_n)$ as $n \to \infty$, where $\lim\limits_{n\to\infty} \alpha_n = 0$.

The main result (Theorem 3.8) of this section is based on a series of lemmas.

Lemma 3.4. Let $\Delta_{1,G_i}(x_i)$, $\Delta_{2,G_i}(x_i)$, $\Delta_{1,i,n}(x_i)$ and $\Delta_{2,i,n}(x_i)$ be defined by (3.6), (3.7), (3.17) and (3.18) respectively. Then $0 \le r_n(G,\delta_n)-r(G)$

$$\le \sum_{i=1}^{k} \int_0^{\theta_0} |\Delta_{1,G_i}(x_i)|^{1-\delta} E|\Delta_{1,i,n}(x_i)-\Delta_{1,G_i}(x_i)|^{\delta} dx_i +$$

$$\sum_{i=1}^{k} \int_{\theta_0}^{\theta_0+\Delta} |\Delta_{2,G_i}(x_i)|^{1-\delta} E|\Delta_{2,i,n}(x_i)-\Delta_{2,G_i}(x_i)|^{\delta} dx_i, \quad \delta > 0.$$

Proof. The proof is similar to that of Lemma 3 of Van Ryzin and Susarla (1977) and hence omitted.

Lemma 3.5. Let $\varphi(x)$ satisfy the conditions (i) $\varphi(x) = 0$ if $x \notin (0,a)$ for some finite $a > 0$, (ii) $\int_0^a \varphi(x)dx = 1$, and (iii) $\sup\limits_x|\varphi(x)| < \infty$ and define

$m_{in}(x_i) = \frac{1}{nh(n)} \sum\limits_{j=1}^{n} \varphi(\frac{X_{ji}-x_i}{h(n)})$, where $\{h(n)\}$ satisfy the conditions (3.15) (see Johns and Van Ryzin (1972)). Then

$|Em_{in}(x_i)-m_i(x_i)| \le h(n)f_\varepsilon(x_i)\int_0^a|u\varphi(u)|du$, for large n, where

$f_\varepsilon(x_i) = \sup\limits_{0\le y\le\varepsilon} |m_i'(x_i+y)|, \ \varepsilon > 0.$

Proof. $Em_{in}(x_i) - m_i(x_i)$

$$= \frac{1}{h(n)} \int \varphi\left(\frac{y-x_i}{h(n)}\right) m_i(y)\,dy - m_i(x_i)$$

$$= \int_0^a \varphi(u)[m_i(x_i+uh(n))-m_i(x_i)]du$$

$$= \int_0^a \varphi(u)[uh(n)m_i'(x_i+\eta_n(x_i,u)]du$$

where $0 < \eta_n(x_i,u) < uh(n)$.

For $\varepsilon > 0$, let n be large enough so that $uh(n) \leq \varepsilon$, then

$$|Em_{in}(x_i)-m_i(x_i)| \leq h(n)f_\varepsilon(x_i)\int_0^a |u\ \varphi(u)|du.$$

Lemma 3.6. Under the conditions of Lemma 3.5, we have

$$\text{var } m_{in}(x_i) \leq \frac{1}{nh(n)} m_i(x_i)\int_0^a \varphi^2(u)du.$$

Proof. $\text{Var } m_{in}(x_i) = \text{var}\{\frac{1}{nh(n)} \sum_{j=1}^{n} \varphi(\frac{X_{ji}-x_i}{h(n)})\}$

$$\leq \frac{1}{nh(n)} \int_0^a \varphi^2(u)m_i(x_i+uh(n))du$$

$$\leq \frac{1}{nh(n)} m_i(x_i)\int_0^a \varphi^2(u)du, \text{ since } m_i(x_i)\downarrow.$$

Remark: From Lemma 3.5 and Lemma 3.6, we have

$$m_{in}(x_i) \overset{P}{\to} m_i(x_i) \quad \text{if} \quad f_\varepsilon(x_i) < \infty.$$

Lemma 3.7. Under the conditions of Lemma 3.5, we have

(a) $\text{Var } \Delta_{1,i,n}(x_i) = O((\theta_0-\Delta-x_i)^2 m_i(x_i) \frac{1}{nh(n)})$,

(b) $\text{Var } \Delta_{2,i,n}(x_i) = O((x_i-\theta_0-\Delta)^2 m_i(x_i) \frac{1}{nh(n)})$.

Proof. (a) $\text{Var}\ \Delta_{1,i,n}(x_i)$

$$\leq 2\{(\theta_0-\Delta-x_i)^2\text{Var}\ m_{in}(x_i) + \text{var}(M_{in}(x_i)-2M_{in}(\theta_0))\}$$

$$\leq 2\{(\theta_0-\Delta-x_i)^2 m_i(x_i)\ \frac{1}{nh(n)}\int_0^a \varphi^2(u)du + \frac{5}{2n}\}\ \text{(By Lemma 3.6)}$$

$$\leq M\frac{1}{nh(n)}(\theta_0-\Delta-x_i)^2 m_i(x_i),\ \text{for some}\ M > 0.$$

Similarly, we have the result (b).

Theorem 3.8. Under the conditions of Lemma 3.5. If

(i) $\int_0^{\theta_0} |\Delta_{1,G_i}(x_i)|^{1-\delta}|\theta_0-\Delta-x_i|^\delta m_i^{\delta/2}(x_i)dx_i < \infty,$

(ii) $\int_{\theta_0}^{\theta_0+\Delta} |\Delta_{2,G_i}(x_i)|^{1-\delta}|x_i-\theta_0-\Delta|^\delta m_i^{\delta/2}(x_i)dx_i < \infty,$

(iii) $\int_0^{\theta_0} |\Delta_{1,G_i}(x_i)|^{1-\delta}|\theta_0-\Delta-x_i|^\delta f_\varepsilon^\delta(x_i)dx_i < \infty,$

and

(iv) $\int_{\theta_0}^{\theta_0+\Delta} |\Delta_{2,G_i}(x_i)|^{1-\delta}|x_i-\theta_0-\Delta|^\delta f_\varepsilon^\delta(x_i)dx_i < \infty,$

where $0 < \delta < 2$, then

$$r_n(G,\delta_n)-r(G) = O(\max\{(\frac{1}{nh(n)})^{\delta/2},\ (h(n))^\delta\})\ \text{as}\ n \to \infty.$$

Proof. For $0 < \delta < 2$, by Hölder inequality and Lemma 3.4, we have

$$0 \leq r_n(G,\delta_n)-r(G)$$

$$\leq \sum_{i=1}^k\{\max(1,2^{\delta-1})[\int_0^{\theta_0}|\Delta_{1,G_i}(x_i)|^{1-\delta}(\text{Var}\ \Delta_{1,i,n}(x_i))^{\delta/2}dx_i +$$

$$\int_0^{\theta_0} |\Delta_{1,G_i}(x_i)|^{1-\delta} |(\theta_0-\Delta-x_i)(Em_{in}(x_i)-m_i(x_i))|^{\delta} dx_i]\} +$$

$$\sum_{i=1}^{k} \{\max(1,2^{\delta-1})[\int_{\theta_0}^{\theta_0+\Delta} |\Delta_{2,G_i}(x_i)|^{1-\delta}(\text{var } \Delta_{2,i,n}(x_i))^{\delta/2} dx_i +$$

$$\int_{\theta_0}^{\theta_0+\Delta} |\Delta_{2,G_i}(x_i)|^{1-\delta} |(x_i-\theta_0-\Delta)(Em_{in}(x_i)-m_i(x_i))|^{\delta} dx_i]\}.$$

By Lemma 3.7, we have

$$\int_0^{\theta_0} |\Delta_{1,G_i}(x_i)|^{1-\delta}(\text{var } \Delta_{1,i,n}(x_i))^{\delta/2} dx_i = O((nh(n))^{-\delta/2})$$

and

$$\int_{\theta_0}^{\theta_0+\Delta} |\Delta_{2,G_i}(x_i)|^{1-\delta}(\text{var } \Delta_{2,i,n}(x_i))^{\delta/2} dx_i = O((nh(n))^{-\delta/2}).$$

By Lemma 3.5, we have

$$\int_0^{\theta_0} |\Delta_{1,G_i}(x_i)|^{1-\delta} |\theta_0-\Delta-x_i|^{\delta} |E\ m_{in}(x_i)-m_i(x_i)|^{\delta} dx_i = O((h(n))^{\delta})$$

and

$$\int_{\theta_0}^{\theta_0+\Delta} |\Delta_{2,G_i}(x_i)|^{1-\delta} |(x_i-\theta_0-\Delta)(E\ m_{in}(x_i)-m_i(x_i))|^{\delta} dx_i = O((h(n))^{\delta}).$$

Hence

$$r_n(G,\delta_n)-r(G) = O(\max\{(nh(n))^{-\delta/2},\ (h(n))^{\delta}\}) \text{ as } n \to \infty.$$

<u>Corollary 3.9</u>. Under the conditions of Theorem 3.8. If we take $h(n) = n^{-\alpha}$, $0 < \alpha < 1$, then the optimal choice of α is 1/3 and $r_n(G,\delta_n)-r(G) = O(n^{-\delta/3})$ as $n \to \infty$.

<u>Remark</u>: If the prior distribution G_i is such that $g_i(x)/x$ and $m_i(x)$ are both bounded on $(0, \theta_0+\Delta+\varepsilon)$, it is easy to check that the conditions of Theorem 3.8 are satisfied for $0 < \delta \leq 1$.

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This paper deals with the problem of selecting all populations which are close to a control or standard. A general Bayes rule for the above problem is derived. Empirical Bayes rules are derived when the populations are assumed to be uniformly distributed. Under some conditions on the marginal and prior distributions, the rate of convergence of the empirical Bayes risk to the minimum Bayes risk is investigated. The rate of convergence is shown to be $n^{-\delta/3}$ for some δ, $0 < \delta < 2$.

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