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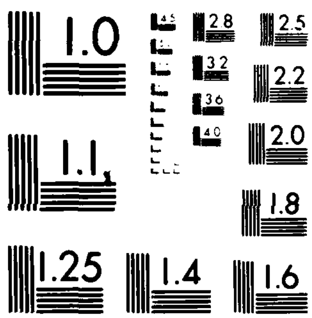

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TESTS FOR AN INCREASING TREND IN THE INTENSITY OF A POISSON PROCESS: A POWER STUDY[1]

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Key Words: Poisson processes; Increasing intensities; Weibull Poisson processes; Laplace test; Likelihood ratio tests

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1. INTRODUCTION

Non-homogeneous Poisson processes (NHPP) provide models for a variety of physical phenomena. For instance, if at each failure, a system is repaired to its condition at the time of failure and placed in service again, then the failures are often modelled by a NHPP provided the repair times can be neglected. In some of these situations, it may be reasonable to assume that the intensity, $\lambda(\cdot)$, is nondecreasing and so tests of H_0: $\lambda(\cdot)$ is constant versus H_1: $\lambda(\cdot)$ is increasing are of interest. This would indicate whether the simple homogeneous Poisson process (HPP) may be adequate, or whether a more general NHPP model is required.

Suppose that a NHPP is observed for T* units of time with n failures and failure times $0 < T_1 < T_2 < \ldots < T_n \leq T^*$. (Of course, the number of failures is a random variable, which we denote by N.) We compare several of the tests which are available in the literature for H_0 versus H_1 in this time truncated framework, and similar conclusions should hold for their counterparts based on data truncated on the number of failures. One of the earliest tests is attributed to Laplace and is based on the statistic $L = \sum_{i=1}^{n} T_i/T^*$. Under H_0, the T_i/T^* are distributed as the order statistics from a uniform distribution on (0,1), so L behaves as the sum of uniform random variables and in particular, has an approximate normal distribution with mean n/2 and variance n/12. Hence, H_0 is rejected if $L > n/2 + z_{1-\alpha}(n/12)^{1/2}$, where $z_{1-\alpha}$ is the

$1-\alpha$ th quantile of a standard normal distribution. Cox (1955) showed that this test is appropriate for testing $\beta = 0$ versus $\beta > 0$ in $\lambda(t) = \alpha e^{\beta t}$. Bartholomew (1956) gave an expression for its power (see also Bates (1955)) and showed that it compares favorably with the one-sided Kolmogorov-Smirnov test. Ascher and Feingold (1978) further discuss its use in the study of repairable systems.

Another family of intensity functions, which is quite flexible, is $\lambda(t) = (\beta/\theta)(t/\theta)^{\beta-1}$ for $\beta,\theta > 0$. Because this is the failure rate for the Weibull distribution, the corresponding process has been called the Weibull Poisson Process (WPP). Inferences for a WPP are discussed in Crow (1974, 1982), Saw (1975), Finkelstein (1976), Lee and Lee (1978), Engelhardt and Bain (1978) and Bain and Engelhardt (1980). The earlier paper by Crow gives tests for β with θ a nuisance parameter. In testing $\beta=1$ versus $\beta>1$, which is equivalent to testing H_0 versus H_1, $\beta=1$ is rejected for small values of $Z = 2\sum_{i=1}^{n}\log(T^*/T_i)$, which has a chi-squared distribution with 2n degrees of freedom under H_0 and is UMPU in this WPP setting.

Boswell (1966) developed the likelihood ratio test (LRT), conditional on n failures, for H_0 versus H_1 and an arbitrary NHPP. The maximum likelihood estimator of $\lambda(\cdot)$ under $H_0 \cup H_1$ is shown to be zero on $[0,T_1)$ and constant on $[T_k,T_{k+1})$ for $k = 1,2,\ldots,n$ with $T_{n+1} = T^*$ and

$$\hat{\lambda}(T_k) = \max_{1\leq\alpha\leq k}\min_{k\leq\beta\leq n}(\beta-\alpha+1)/(T_{\beta+1}-T_{\alpha}).$$

A computation algorithm is also given there (cf. p. 1567). The LRT rejects for large values of

$$W = 2\{\sum_{k=1}^{n}\log(\hat{\lambda}(T_k)) + \log(T^*/n)\},$$

and letting $\chi^2(k)$ denote a chi-squared variable with k degrees of freedom

$$P[W\geq w] \approx \sum_{k=1}^{n}P(k,n)P[\chi^2(k+1) \geq w]$$

for large n, where the P(k,n) are given in Table A.5 of Barlow et al. (1972).

If the intensity is increasing the inter-failure times, T_k-T_{k-1}, k = 1,2,...,n ($T_0=0$), should tend to decrease. Hence, nonparametric tests for trends using either Kendall's tau or Spearman's rho could be considered for testing H_0 versus H_1. (For a discussion of these tests as tests of trend, see Hollander and Wolfe (1973, p. 190).)

Barlow et al. (1972, p. 197) observed that the failure times could be divided into two parts and the ratio $(n-d)T_d/(d(T_n-T_d))$ used as a test statistic for trend. Of course, an increasing intensity should correspond to a larger value of this ratio. If the intensity is constant, then conditional on n failures, T_k can be expressed as $\sum_{j=1}^{k}Y_j/\sum_{j=1}^{n+1}Y_j$ where the Y_j are independent exponential variables with a common mean. Hence, H_0 is rejected if this ratio, which we denote by F, exceeds the $1-\alpha$ th quantile of the F distribution with 2d and 2(n-d) degrees of freedom. We considered the test with d = [n/2] and the one with $T_d \leq T^*/2 < T_{d+1}$. However, the latter test is only applicable if both of the intervals (0,T*/2] and (T*/2,T*] contain failures and the former test is applicable whenever $n \geq 2$. For small truncation times this difference can be appreciable and so in that which follows, we only consider $d = [n/2]$.

Section 2 contains the results of a study which compares the powers of the tests described above and in Section 3, our recommendations based on this study are given.

2. POWER COMPARISONS

Some of the tests discussed in the last section are clearly appropriate for certain families of intensities, for instance the test based on Z is UMPU in the WPP setting, and it would be of interest to study their power functions for a broad range of intensities. On the other hand, some of the tests, such as W, are designed to be more omnibus and information concerning the loss of power incurred when using them instead of a test which is optimal in a particular situation would be useful.

Because of the complex nature of some of the power functions involved a Monte Carlo study was conducted. The tests based on L,Z,W,F Kendall's tau (which we denote by K) and Spearman's rho (which we denote by S) are compared. The following result is the key to generating a NHPP with intensity $\lambda(\cdot)$. If $\Lambda(t) = \int_0^t \lambda(s)ds$ is strictly increasing and $\{S_k\}$ denotes the failure times for a HPP with $\lambda=1$, then $\{\Lambda^{-1}(S_k)\}$ are the failure times for NHPP with intensity $\lambda(\cdot)$. ($\Lambda(t)$ is the expected number of failures in $(0,t]$.) Intensity functions which are exponential, of the Weibull type, logarithmic and those that increase but have a horizontal asymptote are considered. A particular value of T* may be appropriate for one intensity function but not for another, that is it may give rise to a reasonable amount of data for one intensity function and no data (or a very large sample size) for another. So the truncation time will be allowed to vary with $\lambda(\cdot)$ and in particular T* is chosen so that the expected sample size, E(N), is 10, 20 or 40.

If $\lambda(\cdot)$ is an intensity function with corresponding mean function, $\Lambda(\cdot)$ and $\lambda_\theta(t) = \lambda(t/\theta)/\theta$, then $\Lambda_\theta(t) = \Lambda(\theta t)$. Hence, the failure times for the process with intensity $\lambda_\theta(\cdot)$ can be expressed as $\Lambda^{-1}(S_j)/\theta$ where the S_j are the failure times for a Poisson process with $\lambda=1$. All of the tests considered are scale invariant and so we need only consider $\theta=1$. In particular, we consider processes with intensities of the form $\lambda(t) = \beta t^{\beta-1}$ with $\beta=1,2,4$, $\lambda(t) = \alpha e^t$ with $\alpha= .5,1,2$ $\lambda(t) = (x^{\alpha-1}/(\alpha-1)!)/\sum_{j=0}^{\alpha-1}(x^j/j!)$, which is the failure rate of a gamma distribution, with $\alpha=2,4$ and $\lambda(t) = \log(t+1)$. The gamma intensity was chosen because it increases slowly and in fact approaches a constant as $t \to \infty$. However, for such alternatives the data beyond some point in time resembles that for a constant intensity. The tests considered do not have enough power in such cases to make interesting comparisons. If one wishes to discriminate against this kind of an intensity several replications of the process should be observed for an initial time interval rather than observing one process. For this reason the study of the GPP was not completed and no further results concerning it are given here.

Table 1 contains Monte Carlo estimates of the powers for the tests and alternatives described above. These estimates are based on 5000 iterations. Tests with a nominal significance level of $\alpha = .05$ are considered, however the LRT is known to have larger true significance level. For instance, with

with $E(N) = 10$ and a target level of .05 its estimated significance level is .064. Experimenting we found that a nominal level of .01 yields an estimated level of approximately .05 for the LRT and the range of sample sizes being considered. Clearly, this is a disadvantage of the LRT in this situation. As can be seen from Table 1 the significance levels for the other tests are very close to .05. Of course, the Z and F tests are exact.

To give some idea of the accuracy in these estimated powers, we consider the power of Z for $\beta > 1$. Since $\Lambda(t) = t^{\beta}$, the failure times have the same joint distributions as $S_j^{1/\beta}$, with the S_j as before, and $T^* = (E(N))^{1/\beta}$. Conditional on n failures, the distribution of Z is the same as Z/β under $\lambda(t) = 1$. So the power of Z at β is

$$\sum_{n=1}^{\infty} e^{-E(N)}(E(N))^n P[\chi^2(2n) \leq \beta\chi^2_{\alpha}(2n)]/(n!(1-e^{-E(N)})),$$

where $\chi^2_{\alpha}(2n)$ is the α-th quantile of a chi-squared distribution with 2n degrees of freedom. The first row in Table 1 gives the exact powers computed from this formula. The largest discrepancy between these exact and estimated powers is .004.

Similar comparisons can be made for the L test since it is based on approximate normal distributions. For a NHPP, given that there are n failures in the interval $(0,T^*]$, the failure times are distributed as the order statistics of a random sample of size n from the density $f(t) = \lambda(t)/\Lambda(T^*)$, $0 < t < T^*$ and $f(t) = 0$ otherwise. Since L can be expressed as the sum of the unordered observations from this density divided by T^*, it has conditional mean

$$\mu_n = E(L|N=n) = n\int_0^{T^*} t\lambda(t)dt/(T^*\Lambda(T^*))$$

and conditional variance

$$\sigma_n^2 = V(L|N=n) = n\{\int_0^{T^*} t^2\lambda(t)dt/(T^{*2}\Lambda(T^*)) - \mu_n^2\}.$$

Hence, the power of L is approximately

$$\sum_{n=1}^{\infty} e^{-\Lambda(T^*)}(\Lambda(T^*))^n\{1-\Phi((n/2-\mu_n+z_{1-\alpha}(n/12)^{1/2})/\sigma_n)\}/(n!(1-e^{-\Lambda(T^*)})).$$

However, in the WPP case the formulas for μ_n and σ_n^2 simplify to

$$\mu_n = n\beta/(\beta+1) \text{ and } \sigma_n^2 = n[\beta/(\beta+2)-(\beta/(\beta+1))^2].$$

The fourth row in Table 1, which is labeled L (approx.), gives the approximate powers computed using the formulas and the fifth row (labeled L(est.)) gives the Monte Carlo estimates of these powers. The largest discrepancy between the approximate and Monte Carlo values occurs for E(N) = 10 and β=2 and the difference in this case seems to be due primarily to the normal approximation. One could obtain a further check on accuracy. Bartholomew (1955) gives the power of L for exponential intensities, conditional on n failures, and this could be used with the law of total probability to obtain a formula for the power of L for such alternatives. Because of the complex nature of the conditional power this has not been carried out.

To aid in the comparison of these tests, relative efficiencies are estimated. For each case considered, for instance the WPP with β=2 and E(N)=10 is one case, the relative efficiency of a test is its power divided by the largest of the six powers for that case. Table 2 contains the estimated relative efficiencies, that is the ratio of the estimated powers. The Z test was the most powerful for the logarithmic and Weibull alternatives. (In the case WPP, $[illegible]=2$ and $E(N) = 20$ the fact that the estimated relative efficiency of Z is less than one is due to Monte Carlo error. Recall Z is UMPU in this setting and the true power is .9999.) The L test was most powerful for the exponential intensities.

It should be noted that the minimum of the relative efficiencies of Z in our study is .914. (We denote minimum relative efficiency by MRE.) This is the largest MRE of the six tests studied and so we recommend the Z test if one wishes to discriminate against intensities in the range considered. The MRE for the LRT(W) is .819, for L it is .830, for Spearman's rho (S) it is .530 and for Kendall's tau (K) it is .521 and for the F test it is .522. If one is not concerned with the possibility of slowly increasing intensities, such as the logarithmic intensity, then the L test could be used which has an estimated MRE over that range of .95. However, the increase in MRE is not large and it seems to be best to use Z and protect against the possibility of slowly increasing intensities. For the situations considered here the nonparametric tests can not be recommended, but if the distribution assumption on inter-failure times is in question, one might wish to consider one of them. (There is very little difference in their MREs.) The F test performs better for large E(N) than for smaller E(N), but its performance for slowly increasing intensities is not strong. The LRT was designed to discriminate against all nondecreasing intensities which are not constant. If one were concerned about very nonregular intensities W could be considered, but such intensities were not considered here.

3. CONCLUSIONS

In testing for an increasing intensity in a Poisson process, the Z test performs quite well for the range of alternatives studied here, that is for logarithmic, Weibull and exponential intensities. In fact, its efficiency relative to the other five tests considered is at least 90% and for the logarithmic and Weibull intensities it is the most powerful of the six. For alternatives of the type studied here the Z test is recommended.

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BIBLIOGRAPHY

1. Ascher, H. E. and Feingold, H. (1978). "Application of Laplace's test to repairable system reliability". Proc. Int. Conf. on Reliability and Maintainability (Société Pour La Diffusion Des Sciences Et Des Arts, France), 219-225.

2. Bain, L. J. and Engelhardt, M. E. (1980). "Inferences on the parameters and current system reliability for a time truncated Weibull Process". Technometrics 22, 421-426.

3. Barlow, R. E., Bartholomew, D. J., Bremner, J. M. and Brunk, H. D. (1972). Statistical Inferences Under Order Restrictions, Wiley, New York.

4. Bartholomew, D. J. (1956). "Tests for randomness in a series of events when the alternative is a trend". JRSS,B. 18 234-239.

5. Bates, G. E. (1955). "Joint distributions of time intervals for the occurrence of successive accidents in a generalized Polya Scheme". Ann. Math. Statist. 26, 705-720.

6. Boswell, M. T. (1966). "Estimating and testing trend in a stochastic process of Poisson type". Ann. Math. Statist. 37, 1564-1573.

7. Cox, D. R. (1955). "Some statistical methods connected with series of events". JRSS,B. 17, 129-164.

8. Crow, L. H. (1974). "Reliability and analysis for complex repairable systems". Reliability and Biometry, F. Proschan and R. J. Serfling, editors. SIAM, Philadelphia, 379-410.

9. Crow, L. H. (1982). "Confidence intervals procedures for the Weibull process with applications to reliability growth". Technometrics 24, 67-72.

10. Engelhardt, M. E. and Bain, L. J. (1978). "Prediction intervals for the Weibull process". Technometric 20, 167-169.

11. Finkelstein, J. (1976). "Confidence bounds on the parameters of the Weibull process". Technometrics 18, 115-117.

12. Hollander, M. and Wolfe, D. A. (1973). Nonparametric Statistical Methods, Wiley, New York.

13. Lee, L. and Lee, S. K. (1978). "Some results on inference for the Weibull process", Technometrics 20, 41-45.

14. Saw, J. G. (1975). "Tests on the intensity of a Poisson process". Comm. Stat. 4, 777-782.

TABLE 1. Estimated powers for testing H_0 versus H_1

Weibull Intensity: $\lambda(t) = \beta t^{\beta-1}$

	β								
	1.0			2.0			4.0		
Test \ E(N)	10	20	40	10	20	40	10	20	40
Z(exact)	.0500	.0500	.0500	.6209	.8989	.9957	.9785	.9999	1.0000
Z(est.)	.052	.053	.052	.624	.902	.995	.976	.996	1.000
W[1]	.052	.048	.050	.511	.797	.978	.953	.999	1.000
L(approx.)	.0500	.0500	.0500	.5721	.8558	.9893	.9787	.9999	1.0000
L(est.)	.054	.050	.047	.593	.855	.989	.973	1.000	1.000
S	.049	.050	.049	.331	.534	.783	.542	.774	.958
K	.046	.052	.052	.325	.539	.786	.541	.782	.957
F	.052	.046	.046	.381	.614	.905	.852	.990	1.000

Exponential Intensity: $\lambda(t) = \alpha e^t$

	α								
	0.5			1.0			2.0		
Test \ E(N)	10	20	40	10	20	40	10	20	40
Z	.762	.981	.997	.608	.940	.996	.436	.844	.994
W[1]	.729	.984	1.000	.582	.939	1.000	.400	.818	.998
L	.805	.993	1.000	.656	.972	1.000	.477	.892	1.000
S	.484	.832	.986	.395	.778	.981	.290	.670	.967
K	.476	.833	.987	.383	.777	.981	.274	.668	.966
F	.620	.956	1.000	.481	.892	1.000	.343	.746	.995

Logarithmic Intensity: $\lambda(t) = \log(t+1)$

E(N) \ Test	Z	W[1]	L	S	K	F
10	.362	.311	.321	.205	.202	.206
20	.529	.444	.439	.294	.298	.276
40	.770	.650	.643	.426	.429	.422

1. The nominal level for the W test was .04.

TABLE 2. Estimated relative efficiencies.

Test \ E(N)	Logarithmic Intensity $\lambda(t)=\log(t+1)$			Weibull Intensity: $\lambda(t)=\beta t^{\beta-1}$, β = 2.0			β = 4.0		
	10	20	40	10	20	40	10	20	40
Z	1.000	1.000	1.000	1.000	1.000	1.000	1.000	.996	1.000
W[1]	.859	.839	.844	.819	.884	.983	.976	.999	1.000
L	.888	.830	.835	.950	.948	.994	.997	1.000	1.000
S	.566	.556	.553	.530	.592	.787	.555	.774	.958
K	.558	.563	.557	.521	.598	.790	.554	.782	.957
F	.569	.522	.548	.611	.681	.910	.873	.990	1.000

Exponential Intensity: $\lambda(t) = \alpha e^{t}$

Test \ E(N)	α = 0.5			α = 1.0			α = 2.0		
	10	20	40	10	20	40	10	20	40
Z	.947	.988	.997	.927	.967	.996	.914	.946	.994
W[1]	.906	.991	1.000	.887	.964	1.000	.839	.917	.998
L	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
S	.601	.838	.986	.602	.800	.981	.608	.751	.967
K	.591	.839	.987	.584	.799	.981	.574	.749	.966
F	.770	.963	1.000	.733	.918	1.000	.719	.836	.995

[1]The nominal level for the W test was .04.

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