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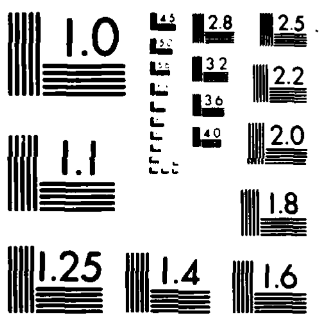

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Chapel Hill, North Carolina

AN ITERATED LOGARITHM LAW RESULT
FOR EXTREME VALUES FROM GAUSSIAN SEQUENCES

William P. McCormick

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Abstract

Let $\{X_n, n\geq 1\}$ be a stationary Gaussian sequence with $EX_1=0$, $EX_1^2=1$ and $r_n=EX_1X_{n+1}$. Let $Z_n^{(i)}$ denote the ith maximum of $X_1,\ldots, X_n$ and $a_n=(\ell n\ell nn)(2\ell nn)^{-1/2}$, $b_n=(2\ell nn)^{1/2} - (\ell n(4\pi\ell nn))/(2(2\ell nn)^{1/2})$. Then assuming $r_n(\ell nn)^2=0(1)$ the set of almost sure limit points of the vectors $((Z_n^{(1)}-b_n)a_n^{-1}, (Z_n^{(2)}-b_n)a_n^{-1},\ldots(Z_n^{(\ell)}-b_n)a_n^{-1})$ is determined. The number of components $\ell=\ell(n)\to\infty$ as $n\to\infty$. This extends a result of Hebbar.

Keywords: Iterated logarithm, Gaussian sequence, almost sure limit set.

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1. Introduction

Let $\{X_n, n\geq 1\}$ be a stationary Gaussian sequence with $EX_1=0$, $EX_1^2=1$ and $r_n=EX_1X_{n+1}$. Let $Z_n^{(i)}$ denote the ith maximum of $X_1,\ldots, X_n$ that is $Z_n^{(i)}$ equals the n-i+1 order statistic. Set $a_n = \ell n\ell nn/\sqrt{2\ell nn}$ and $b_n = \sqrt{2\ell nn} - \ell n(4\pi\ell nn)/2\sqrt{2\ell nn}$. In [1] Hebbar considers the set of almost sure limit points of the sequence of vectors $\{(\frac{Z_n^{(1)}-b_n}{a_n}, \frac{Z_n^{(2)}-b_n}{a_n},\ldots, \frac{Z_n^{(\ell)}-b_n}{a_n}), n\geq 1\}$. He shows that under the assumption $r_n(\ell nn)^{2+\varepsilon} = 0(1)$ for some $\varepsilon>0$ the above sequence has almost sure limit set equal to $\{(x_1,x_2,\ldots, x_\ell): 0\leq x_\ell\leq\ldots\leq x_1$ and $\sum_1^\ell x_i\leq 1\}$. In the present paper we strengthen this result in two directions. We relax the condition on r_n to $r_n(\ell nn)^2=0(1)$ and further we allow the number ℓ of extreme values considered to grow to infinity with n. Let $v_n^{(i)} = \frac{Z_n^{(i)}-b_n}{a_n}$. Then we consider the points in $\mathbb{R}^\infty$ given by $(v_n^{(1)},\ldots, v_n^{(\ell)},0,0\ldots)$ where $\ell=\ell(n)\to\infty$ as $n\to\infty$. In $\mathbb{R}^\infty$ we consider two modes of convergence--pointwise convergence and ℓ_1-convergence. With $\ell(n)$ suitably bounded we show that the almost sure limit set in $\mathbb{R}^\infty$ is given by

$$A = \{(x_1,x_2\ldots): \quad 0\leq x_{i+1}\leq x_i,\ i=1,2,\ldots, \text{ and } \sum_1^\infty x_i\leq 1\}$$

2. Almost sure limit set

We consider two modes of convergence in $\mathbb{R}^\infty$, pointwise which is metrized by $d(\underline{x},\underline{y}) = \sum_{n=1}^{\infty} (\frac{|x_n-y_n|}{1+|x_n-y_n|})2^{-n}$ and ℓ_1. Let us observe that a point $\underline{x}$ is a limit point of a sequence $\underline{x}_n$ with respect to pointwise convergence if and only if for each fixed ℓ, $(x_1,\ldots, x_\ell)$ is a limit point of $(x_n^{(1)},\ldots, x_n^{(\ell)})$. Therefore with regard to pointwise convergence our extension of Hebbar's result is precisely to weaken the mixing condition on r_n since finite dimensional results suffice to prove this case. Furthermore in this case we consider the almost sure limit points of the sequence $\{(v_n^{(1)}, v_n^{(2)}, \ldots, v_n^{(n)}, 0, 0, \ldots), n \geq 1\}$ that is we take $\ell(n) = n$.

However when we consider the random element $(v_n^{(1)}, \ldots, v_n^{(\ell)}, 0, 0, \ldots)$ as a point in ℓ_1 then we must take into account the rate at which $\ell(n)$ grows with n. In this case we prove an iterated logarithm law result with $\ell(n) = [\ell n_3 n]$. In the following we consider the ℓ_1 case only since the pointwise convergence case immediately follows.

The proof closely follows the method in [1] although additional detail is required to accommodate the infinite dimensional setting. However Lemma 6 in [1] receives an entirely different proof here that depends on an extension of a result of Mittal [2].

Remark: Let $\underline{x} = (x_1, x_2, \ldots) \in A$ and assume $x_1 > 0$. Define the following sequences $\lambda_k = [\ell n(\frac{1}{x_1} \ell nk)]$, $s_k = \sum_1^{\lambda_k} x_i$, $s = \sum_1^{\infty} x_i$ (assume $s<1$) and $\alpha_k = [\exp(k^{\frac{1}{s_k}})]$. Our program will be to show that the sequence $(v_{\alpha_k}^{(1)}, v_{\alpha_k}^{(2)}, \ldots, v_{\alpha_k}^{(\lambda_k)}, 0, 0 \ldots)$, $k \geq 1$ has $\underline{x}$ as a limit point almost surely. Then since $\ell_{\alpha_k} \leq \lambda_k$ and $\ell_{\alpha_k} \to \infty$ as $k \to \infty$ it follows easily that $\underline{x}$ is a limit point of $(v_{\alpha_k}^{(1)}, v_{\alpha_k}^{(2)}, \ldots, v_{\alpha_k}^{(\ell_{\alpha_k})}, 0, 0 \ldots)$. In the lemmas which follow it will be assumed that $r_n (\ell nn)^2 = 0(1)$ and that $s = \sum_1^{\infty} x_i < 1$.

<u>Lemma 1</u>. For any $\varepsilon > 0$ we have

$$(2.1) \quad P\{\sum_{i=1}^{\lambda_k} (v_{\alpha_k}^{(i)} - x_i) > \varepsilon \quad \text{and} \quad v_{\alpha_k}^{(i)} > x_i,\ i=1,\ldots,\lambda_k,\ \text{i.o.}\} = 0 .$$

<u>Proof:</u> To establish (2.1) it suffices to prove

$$(2.2) \quad P\{\max_{1 \leq i \leq \lambda_k} (v_{\alpha_k}^{(i)} - x_i) > \varepsilon/\lambda_k,\ \min_{1 \leq i \leq \lambda_k} (v_{\alpha_k}^{(i)} - x_i) > 0,\ \text{i.o.}\} = 0 .$$

Further by Borel Cantelli to establish (2.2) it suffices to show

$$(2.3)\quad \sum_k [\lambda_k \max_{1\le j\le\lambda_k} P\{v_{\alpha_k}^{(j)} > x_j + \varepsilon/\lambda_k,\ v_{\alpha_k}^{(i)} > x_i,\ 1\le i\le\lambda_k\}] < \infty .$$

Let $\{(y_{\alpha_k}^{(1)}, y_{\alpha_k}^{(2)}, \ldots, y_{\alpha_k}^{(\lambda_k)}), k\ge 1\}$ be any triangular array with $y_{\alpha_k}^{(i)} \ge x_i$, $1\le i\le\lambda_k$ and $\max_{1\le i\le\lambda_k} (y_{\alpha_k}^{(i)} - x_i) > \varepsilon/\lambda_k$. Let $\eta_{\alpha_k}^{(i)} = b_{\alpha_k} + y_{\alpha_k}^{(i)} a_{\alpha_k}$. Then we establish (2.3) by showing that

$$(2.4)\quad \sum_{k=1}^{\infty} \lambda_k P\{Z_{\alpha_k}^{(i)} \ge \eta_{\alpha_k}^{(i)},\ i=1,\ldots,\lambda_k\} < \infty .$$

Let $Z_n^{*(i)}$ be the ith maximum of a sample of size n of i.i.d. standard normal random variables. Then in order to show (2.4) it suffices to show

$$(2.5)\quad \sum_1^{\infty} \lambda_k P\{Z_{\alpha_k}^{*(i)} \ge \eta_{\alpha_k}^{(i)},\ i=1,\ldots,\lambda_k\} < \infty \quad \text{and}$$

$$(2.6)\quad \sum_1^{\infty} \lambda_k |P\{Z_{\alpha_k}^{*(i)} \ge \eta_{\alpha_k}^{(i)},\ i=1,\ldots,\lambda_k\} - P\{Z_{\alpha_k}^{(i)} \ge \eta_{\alpha_k}^{(i)},\ i=1,\ldots,\lambda_k\}| < \infty .$$

In considering (2.5) observe that

$$P\{Z_{\alpha_k}^{*(i)} \ge \eta_{\alpha_k}^{(i)},\ i=1,\ldots,\lambda_k\}$$

$$(2.7)\qquad = P\{Z_{\alpha_k}^{*(1)} \ge \eta_{\alpha_k}^{(1)}\}$$

$$- \sum_{i=2}^{\lambda_k} P\{Z_{\alpha_k}^{*(1)} > \eta_{\alpha_k}^{(1)}, \ldots, Z_{\alpha_k}^{*(i-1)} > \eta_{\alpha_k}^{(i-1)},\ Z_{\alpha_k}^{*(i)} \le \eta_{\alpha_k}^{(i)}\}$$

Further it can be easily checked that

$$(2.8)\quad P\{Z_{\alpha_k}^{*(1)} > \eta_{\alpha_k}^{(1)}\} = k^{-\frac{y_{\alpha_k}^{(1)}}{s_k}} + O(\tfrac{1}{\alpha_k}) \quad \text{and}$$

$$(2.9)\quad P\{Z_{\alpha_k}^{*(1)} > \eta_{\alpha_k}^{(1)},\ldots, Z_{\alpha_k}^{*(i-1)} > \eta_{\alpha_k}^{(i-1)}, Z_{\alpha_k}^{*(i)} \le \eta_{\alpha_k}^{(i)}\}$$

$$= k^{-(\frac{1}{s_k}\sum_1^{i-1} y_{\alpha_k}^{(t)})} - k^{-(\frac{1}{s_k}\sum_1^{i} y_{\alpha_k}^{(t)})} + 0(\frac{1}{\alpha_k})$$

Thus by (2.8) and (2.9) we obtain that (2.7) equals $k^{-\frac{1}{s_k}\sum_1^{\lambda_k} y_{\alpha_k}^{(t)}} + 0(\frac{\lambda_k}{\alpha_k})$

Thus since $\sum_1^{\lambda_k} y_{\alpha_k}^{(t)} > s_k + \epsilon\lambda_k^{-1}$ and $s_k \nearrow s = \sum_1^{\infty} x_i \le 1$, (2.5) is established.

Next we consider (2.6). First observe that

$$|P\{Z_{\alpha_k}^{*(i)} \ge \eta_{\alpha_k}^{(i)}, i=1,\ldots,\lambda_k\} - P\{Z_{\alpha_k}^{(i)} \ge \eta_{\alpha_k}^{(i)}, i=1,\ldots,\lambda_k\}|$$

$$\le |P\{\eta_{\alpha_k}^{(1)} \le Z_{\alpha_k}^{*(1)} \le z_{\alpha_k}\} - P\{\eta_{\alpha_k}^{(1)} \le Z_{\alpha_k}^{(1)} \le z_{\alpha_k}\}|$$

$$(2.10)\qquad + \sum_{i=2}^{\lambda_k} |P\{\eta_{\alpha_k}^{(1)} \le Z_{\alpha_k}^{*(1)} \le z_{\alpha_k},\ldots, \eta_{\alpha_k}^{(i-1)} \le Z_{\alpha_k}^{*(i-1)} \le z_{\alpha_k}, Z_{\alpha_k}^{*(i)} \le \eta_{\alpha_k}^{(i)}\}$$

$$- P\{\eta_{\alpha_k}^{(1)} \le Z_{\alpha_k}^{(1)} \le z_{\alpha_k},\ldots, \eta_{\alpha_k}^{(i-1)} \le Z_{\alpha_k}^{(i-1)} \le z_{\alpha_k}, Z_{\alpha_k}^{(i)} \le \eta_{\alpha_k}^{(i)}\}|$$

$$+ P\{Z_{\alpha_k}^{*(1)} > z_{\alpha_k}\} + P\{Z_{\alpha_k}^{(1)} > z_{\alpha_k}\}$$

where $z_{\alpha_k} = 2\sqrt{\ell n\alpha_k}$.

It can be checked that

$$(2.11)\qquad \begin{aligned} &P\{Z_{\alpha_k}^{*(1)} > z_{\alpha_k}\} = 0(\frac{1}{\alpha_k}) \quad \text{and} \\ &|P\{Z_{\alpha_k}^{*(1)} > z_{\alpha_k}\} - P\{Z_{\alpha_k}^{(1)} > z_{\alpha_k}\}| \le \alpha_k^{-2(\frac{1-\bar{r}_1}{1+\bar{r}_1})} \end{aligned}$$

where $\bar{r}_x = \sup_{i>x} |r_i|$. Thus by (2.11) $\lambda_k(P\{Z_{\alpha_k}^{*(1)} > z_{\alpha_k}\} + P\{Z_{\alpha_k}^{(1)} > z_{\alpha_k}\})$ is summable on k.

Similarly it is easily checked that

$$(2.12)\quad |P\{Z^{*(1)}_{\alpha_k} \le \eta^{(1)}_{\alpha_k}\} - P\{Z^{(1)}_{\alpha_k} \le \eta^{(1)}_{\alpha_k}\}|\lambda_k \le (\text{CONST.})k^{-\left(\frac{1+2y^{(1)}_{\alpha_k}}{s_k}\right)}\lambda_k .$$

Since $s_k<1$ and $y^{(1)}_{\alpha_k} \ge x_1 > 0$ for all k, the series in (2.12) is summable.

Now consider a term of the form

$$|P\{\eta^{(1)}_{\alpha_k} \le Z^{*(1)}_{\alpha_k} \le z_{\alpha_k},\dots, \eta^{(i-1)}_{\alpha_k} \le Z^{*(i-1)}_{\alpha_k} \le z_{\alpha_k},\ Z^{*(i)}_{\alpha_k} \le \eta^{(i)}_{\alpha_k}\}$$

$$(2.13)\quad - P\{\eta^{(1)}_{\alpha_k} \le Z^{(1)}_{\alpha_k} \le z_{\alpha_k},\dots, \eta^{(i-1)}_{\alpha_k} \le Z^{(i-1)}_{\alpha_k} \le z_{\alpha_k},\ Z^{(i)}_{\alpha_k} \le \eta^{(i)}_{\alpha_k}\}|$$

$$\le S = \sum_{t_1,\dots,t_{i-1}} |P\{\eta^{(1)}_{\alpha_k} \le X^*_{t_1} \le z_{\alpha_k},\dots, \eta^{(i-1)}_{\alpha_k} \le X^*_{t_{i-1}} \le z_{\alpha_k},\ X^*_t \le \eta^{(i)}_{\alpha_k}$$

for all $t \ne t_1,\dots, t_{i-1},\ 1\le t\le\alpha_k\}|$

where $\{X^*_1,X^*_2,\dots\}$ denotes an i.i.d. sequence of standard normal random variables and where the summation is over all $1 \le t_1,\dots, t_{i-1} \le \alpha_k$ and $t_u \ne t_v$ of $u \ne v$.

Let $0<\theta<1$ be fixed and to be specified later. We write

$$S = S_0 + S_1 + \dots + S_{i-2}$$

where S_u denotes the sum over all $t_1,\dots, t_{i-1}$ such that when the t's are ordered $t_{(1)} <\dots< t_{(i-1)}$ there are exactly u indices h where $t_{(h+1)} - t_{(h)} < \alpha_k^{\theta}$.

Consider S_0. We have

$$|P\{\eta^{(1)}_{\alpha_k} < X^*_{t_1} < z_{\alpha_k},\dots, \eta^{(i-1)}_{\alpha_k} < X^*_{t_{i-1}} < z_{\alpha_k},\ X^*_t \le \eta^{(i)}_{\alpha_k},\ t\ne t_1,\dots, t_{i-1} \text{ and } 1\le t\le\alpha_k\}$$

$$- P\{\eta^{(1)}_{\alpha_k} < X_{t_1} < z_{\alpha_k},\dots, \eta^{(i-1)}_{\alpha_k} < X_{t_{i-1}} < z_{\alpha_k},\ X_t \le \eta^{(i)}_{\alpha_k}, t\ne t_1,\dots, t_{i-1} \text{ and } 1\le t\le\alpha_k\}|$$

$$\le (\text{Const.})(T_0 + \sum_{0\le u\ne v\le i-1} T_{u,v})$$

where $T_0 = \sum^{(0)}_{s,t} |r|\phi(\eta^{(i)}_{\alpha_k}, \eta^{(i)}_{\alpha_k}, |r|)$

$$\cdot\ P\{\eta^{(1)}_{\alpha_k} < X_{t_1} < z_{\alpha_k},\dots, \eta^{(i-1)}_{\alpha_k} < X_{t_{i-1}} < z_{\alpha_k} | X_s=\eta^{(i)}_{\alpha_k},\ X_t=\eta^{(i)}_{\alpha_k}\}$$

where $r = r_{s,t}$ and $\sum^{(0)}$ is summation over all $s \neq t$ and $s,t \neq t_1,\ldots, t_{i-1}$, $1 \le s,t \le \alpha_k$ and where $\phi(\cdot,\cdot,r)$ denotes the bivariate normal density with zero means, unit variances and correlation r. Further for $v>0$,

$$T_{0,v} = \sum_s^{(0,v)} |r|\phi(\eta_{\alpha_k}^{(i)}, \eta_{\alpha_k}^{(v)}, |r|)$$

$$\cdot\, P\{\eta_{\alpha_k}^{(1)} < X_{t_1} < z_{\alpha_k}, \ldots, \eta_{\alpha_k}^{(v-1)} < X_{t_{v-1}} < z_{\alpha_k}, \eta_{\alpha_k}^{(v+1)} < X_{t_{v+1}} < z_{\alpha_k},$$

$$\ldots \eta_{\alpha_k}^{(i-1)} < X_{t_{i-1}} < z_{\alpha_k} | X_{t_v} = \eta_{\alpha_k}^{(v)}, X_s = \eta_{\alpha_k}^{(i)}\}$$

where the sum is over $s \neq t_1,\ldots, t_{i-1}$ and $1 \le s \le \alpha_k$ and $r = r_{st_v}$.

$T_{u,0}$ is defined in exactly the same way and finally for $u,v>0$

$$T_{u,v} = |r|\phi(\eta_{\alpha_k}^{(u)}, \eta_{\alpha_k}^{(v)}, |r|)$$

$$P\{\eta_{\alpha_k}^{(1)} < X_{t_1} < z_{\alpha_k}, \ldots, \eta_{\alpha_k}^{(u-1)} < X_{t_{u-1}} < z_{\alpha_k}, \eta_{\alpha_k}^{(u+1)} < X_{t_{u+1}} < z_{\alpha_k},$$

$$\ldots \eta_{\alpha_k}^{(v-1)} < X_{t_{v-1}} < z_{\alpha_k}, \eta_{\alpha_k}^{(v+1)} < X_{t_{v+1}} < z_{\alpha_k}, \ldots,$$

$$\eta_{\alpha_k}^{(i-1)} \le X_{t_{i-1}} \le z_{\alpha_k} | X_{t_u} = \eta_{\alpha_k}^{(u)}, X_{t_v} = \eta_{\alpha_k}^{(v)}\}.$$

We will give details only for the sum T_0 since the other sums are handled in the same way. For T_0 *first consider the case when*

$$(2.14) \quad \min\{|s-t_u|, |t-t_u|, u=1,\ldots, i-1\} > \alpha_k^\theta$$

In evaluating T_0 we need to evaluate

$$(2.15) \quad P\{\eta_{\alpha_k}^{(1)} \le X_{t_1} \le z_{\alpha_k}, \ldots, \eta_{\alpha_k}^{(i-1)} \le X_{t_{i-1}} \le z_{\alpha_k} | X_s = \eta_{\alpha_k}^{(i)}, X_t = \eta_{\alpha_k}^{(i)}\}$$

when

(2.16) $t_{(h+1)} - t_{(h)} > \alpha_k^\theta$, $h=1,\ldots, i-2$

and (2.14) hold. Now subject to (2.14) we have that

(2.17) $E(X_{t_i} | X_s = \eta_{\alpha_k}^{(i)},\ X_t = \eta_{\alpha_k}^{(i)}) = 0((\ell n\alpha_k)^{-3/2})$ and

$$\mathrm{CORR}(X_{t_u}, X_{t_v} | X_s, X_t) = r_{t_u,t_v} + 0((\ell n\alpha_k)^{-4})$$

Therefore by (2.17) the probability in (2.15) is at most

(2.18) $P\{\eta_{\alpha_k}^{(u)} - c(\ell_n\alpha_k)^{-3/2} \le X_{t_u} \le z_{\alpha_k} + c(\ell n\alpha_k)^{-3/2},\ u=1,\ldots, i-1\}$

for some constant c not depending on k. Conditioning on X_{t_1} yields that (2.18) is at most

$$(1-\Phi(b_{\alpha_k}(1-c\bar{r})))P\{b_{\alpha_k}(1-c\bar{r})^2 \le X_{t_u} \le z_{\alpha_k}(1-c\bar{r})^2,\ u=2,\ldots, i-1\}$$

where $\bar{r} = \bar{r}_{\alpha_k^\theta}$.

Iterating the procedure yields that (2.18) is at most

(2.19) $\prod_{u=1}^{i-1} [1-\Phi(b_{\alpha_k}(1-c\bar{r})^u]$

Finally since $i \le \lambda_k = [\ell n(\frac{\ell nk}{x_1})]$ and $(1-c\bar{r})^u \ge 1 - 2\lambda_k c\bar{r}$, (2.19) is at most

(2.20) $[1-\Phi(b_{\alpha_k} - c\frac{\ell nk}{(\ell n\alpha_k)^{3/2}})]^{(i-1)}$. c is some constant.

In the same way it can be checked that if for some u_0, $|s-t_{u_0}| < \alpha_k^\theta$ but $|t-t_u| > \alpha_k^\theta$, $u=1,\ldots, i-1$ or the same case with s and t interchanged then (2.15) is at most

(2.21) $(1-\Phi(\gamma b_{\alpha_k}))[1-\Phi(b_{\alpha_k} - c\,\frac{\ell nk}{(\ell n\alpha_k)^{3/2}})]^{(i-2)}$

where $\gamma > 0$.

And finally if both $|s-t_{u_0}| < \alpha_k^\theta$ and $|t-t_{v_0}| < \alpha_k^\theta$ for some u_0 and v_0 then (2.15) is at most

$$(2.22)\quad (1-\Phi(\gamma b_{\alpha_k}))^2[1-\Phi(b_{\alpha_k}-c\frac{\ell nk}{(\ell n\alpha_k)^{3/2}})]^{(i-3)}$$

Thus we have that provided (2.16) holds

$$(2.23)\quad T_0 \le \frac{(\text{CONST.})}{(\ell n\alpha_k)^{(2y_{\alpha_k}^{(i)}+1)}}[1-\Phi(b_{\alpha_k}-c\frac{\ell nk}{(\ell n\alpha_k)^{3/2}})]^{(i-1)} = \frac{(\text{CONST.})}{(\ell n\alpha_k)^{(2y_{\alpha_k}^{(i)}+1)}}\frac{1}{\alpha_k^{(i-1)}}$$

Similarly if (2.16) holds we find

$$\text{(i)}\quad T_{0,v} \le \frac{(\text{CONST.})}{(\ell n\alpha_k)^{(y_{\alpha_k}^{(i)}+y_{\alpha_k}^{(v)}+1)}}\frac{1}{\alpha_k^{(i-1)}}$$

$$(2.24)\quad \text{(ii)}\quad T_{u,0} \le \frac{(\text{CONST.})}{(\ell n\alpha_k)^{(y_{\alpha_k}^{(i)}+y_{\alpha_k}^{(u)}+1)}}\frac{1}{\alpha_k^{(i-1)}}$$

$$\text{(iii)}\quad T_{u,v} \le \frac{(\text{CONST.})}{(\ell n\alpha_k)^{(y_{\alpha_k}^{(u)}+y_{\alpha_k}^{(v)}+1)}}\frac{1}{\alpha_k^{(i-1)}}$$

Thus by (2.23) and (2.24)

$$(2.25)\quad S_0 \le [\frac{(\text{CONST.})}{\alpha_k^{(i-1)}}\frac{(\ell n_2 k)^2}{k^{(1+2x_i)/s_k}}](\alpha_k^{(i-1)}) \le \frac{1}{k^{1+e}}$$

for some $e > 0$.

Next consider S_h. For simplicity let us consider a summand in (2.13) when $t_1<t_2<\ldots< t_{i-1}$ and

(2.26) $0 < t_2 - t_1,\ t_3 - t_2, \ldots,\ t_{n+1} - t_n \le \alpha_k^\theta$ and $t_{u+1} - t_u > \alpha_k^\theta$ $u=h+1, \ldots, i-1$.

Then

$$|P\{\eta_{\alpha_k}^{(1)} \le X_{t_1}^* \le z_{\alpha_k}, \ldots, \eta_{\alpha_k}^{(i-1)} \le X_{t_{i-1}}^* \le z_\alpha,\ X_t \le \eta_{\alpha_k}^{(i)}$$

for all $t \ne t_1, \ldots, t_{i-1},\ 1 \le t \le \alpha_k\}$

$$- P\{\eta_{\alpha_k}^{(1)} \le X_{t_1} \le z_{\alpha_k}, \ldots, \eta_{\alpha_k}^{(i-1)} \le X_{t_{i-1}} \le z_\alpha,\ X_t \le \eta_{\alpha_k}^{(i)}$$

for all $t \ne t_1, \ldots, t_{i-1},\ 1 \le t \le \alpha_k\}|$

$$\le T_0 + \sum_{0 \le u \ne v \le i-1} T_{u,v}$$

where the $T_{u,v}$ have the same meaning as before except now condition (2.26) holds. Consider T_0. We need to evaluate

$$P\{\eta_{\alpha_k}^{(1)} < X_{t_1} < z_{\alpha_k}, \ldots, \eta_{\alpha_k}^{(i-1)} < X_{t_{i-1}} < z_{\alpha_k} | X_s = \eta_{\alpha_k}^{(i)} X_t = \eta_{\alpha_k}^{(i)}\}$$

Suppose that (2.14) holds. Then the above conditional probability is at most the expression given at (2.18). Therefore let us consider

(2.27) $P\{\eta_{\alpha_k}^{(u)} \le X_{t_u} \le z_{\alpha_k},\ u=1, \ldots, i-1\}$

subject to condition (2.26).

Let $K = K(\alpha_k) = [\exp(\sqrt{\ln \alpha_k})]$. Suppose that in addition to (2.26) we have that

(2.28) $t_2 - t_1, \ldots, t_{m+1} - t_m \le K$, and $K < t_{m+2} - t_{m+1}, \ldots, t_{i-1} - t_{i-2}$.

Then given (2.26) and (2.28), (2.27) equals at most

$$\prod_{u=0}^{m} (1-\Phi(b_{\alpha_k}(1- c\bar{r}_1)^u)) \cdot \prod_{u=m+1}^{h} (1-\Phi(b_{\alpha_k}(1-c\bar{r}_K)^u)) \cdot \prod_{u=h+1}^{i=2} (1-\Phi(b_{\alpha_k}(1-c\bar{r}_{\alpha_k^\theta})^u))$$

$$\le (\text{CONST.})\alpha_k^{-Q}$$

where $Q = 1 + \delta_1 - \delta_1^{m+1} + (h-m)\delta_K^{m+1} + (i-2-h)$ and $\delta_1 = 1 - c\bar{r}_1$, $\delta_K = 1 - c\bar{r}_K$ where c is some constant and without loss of generality we can assume $c\bar{r}_1 < 1$ because if necessary we can work with the sequence $\{X_{mn}, n\geq 1\}$ where m is some fixed integer.

Then as before we find that

$$\text{(i)}\quad T_0 \leq \frac{(\text{CONST.})}{(\ell n\alpha_k)^{(2y^{(i)}_{\alpha_k}+1)}} \frac{1}{\alpha_k^Q}$$

$$(2.29)\qquad \text{(ii)}\quad T_{u,0} \leq \frac{(\text{CONST.})}{(\ell n\alpha_k)^{(y^{(u)}_{\alpha_k}+y^{(i)}_{\alpha_k}+1)}} \frac{1}{\alpha_k^Q}$$

and

$$\text{(iii)}\quad T_{u,v} \leq \frac{(\text{CONST.})}{(\ell n\alpha_k)^{(y^{(u)}_{\alpha_k}+y^{(v)}_{\alpha_k}+1)}} \frac{1}{\alpha_k^Q}$$

Thus by the inequalities in (2.29) we have that

$$S_h \leq \sum_{m=0}^{h} \left(\frac{(\text{CONST.})}{\alpha_k^Q}\right)\left(\frac{(\ell n_2 k)^2}{k^{\frac{1+2x_i}{s_k}}}\right)\left(\alpha_k^{[i-h-1+\theta(h-m)]}K^m\right)$$

from which it can be easily checked that

$$(2.30)\qquad S_n \leq \alpha_k^{-f} \quad \text{for some } f > 0 \text{ not depending on } h.$$

Finally from (2.25) and (2.30) we have that

$$S \leq \frac{(\text{CONST.})}{k^{1+e}} \qquad k \geq 1 \text{ for some } e > 0\ .$$

Hence (2.6) holds completing the proof of lemma 1.

Lemma 2. Let $c_n^{(i)} = b_n + x_i a_n$. Then

(2.31) $P\{Z_{\alpha_k}^{(i)} > c_{\alpha_k}^{(i)},\ i=1,\ldots,\ \lambda_k)$ i.o.$\} = 1$.

Proof: Let $\beta_k = [1/2\ \alpha_k]$ and let $\tilde{Z}\alpha_k^{(i)}$ be the ith maximum of the random variables $X_{\alpha_k-\beta_k+1},\ldots,X_{\alpha_k}$. Let z_{α_k} be as in Lemma 1 and define $I_k = \prod_{i=1}^{\lambda_k} I_{[c_{\alpha_k}^{(i)} < \tilde{Z}_{\alpha_k}^{(i)} < z_{\alpha_k}]}$.

Let $J_n = \sum_{k=[n^a]}^{n} I_k$ where $0<a<1$ is a fixed real number. Then to show (2.31) it suffices to show

$$\text{(2.32)}\quad \begin{aligned} &\text{(i)}\quad EJ_n \to \infty \text{ as } n \to \infty \text{ and} \\ &\text{(ii)}\quad J_n/EJ_n \xrightarrow{P} 1 \text{ as } n \to \infty . \end{aligned}$$

The proof of (2.32) follows the method of proof of Lemma 3 in [1] with changes similar to those in our Lemma 1. Therefore the details of this proof will be omitted.

Remark: The sequence $(v_{\alpha_k}^{(1)},\ldots,v_{\alpha_k}^{(\lambda_k)},0,0\ldots)$ has $\underline{x}$ as an almost sure limit point. To see this let $N_n = \{\omega\colon \sum_1^{\lambda_k} (v_{\alpha_k}^{(i)} - x_i) > 1/n,\ v_{\alpha_k}^{(i)} > x_i,\ i=1,\ldots,\ \lambda_k$ i.o.$\}$ and $N = \bigcup_{n=1}^{\infty} N_n$. Let $A = \{\omega\colon\ v_{\alpha_k}^{(i)} > x_i,\ i=1,\ldots,\ \lambda_k$ i.o.$\}$ and $\Lambda = A \cap N^c$. It is easy to check that if $\omega \in \Lambda$, then $(v_{\alpha_k}^{(1)}(\omega),\ldots,v_{\alpha_k}^{(\lambda_k)}(\omega),0,0\ldots)$ has $\underline{x}$ as a limit point and by Lemmas 1 and 2 $P(\Lambda) = 1$. Therefore by the Remark preceding Lemma 1, we have that $\underline{x}$ is an almost sure limit point of $(v_{\alpha_k}^{(1)},\ldots,v_{\alpha_k}^{(\ell\alpha_k)},\ 0,0,\ldots)$.

Lemma 3. Let $\underline{x} = (x_1,x_2,\ldots)$ be any point in $\mathbb{R}^\infty$ with $0 \le x_{i+1} \le x_i$, $i=1,2,\ldots$ and $\sum_1^\infty x_i > 1$. Then $\underline{x}$ cannot be an a.s. limit point of the sequence $(v_n^{(1)},\ldots,v_n^{(\ell_n)},0,0\ldots)$

Proof: Let m be such that $s_m = \sum_1^m x_i > 1$, and $x_m > 0$. Let $z_i = (\frac{s_m+1}{2s_m})x_i$, $i=1,\ldots, m$. Then since $\sum_1^m z_i > 1$, it follows as in [1] that $P\{v_n^{(i)} > z_i,\ i=1,\ldots, m \ \text{ i.o.}\} = 0$. Let $N = \{\omega\colon\ v_n^{(1)} > z_1,\ldots, v_n^{(m)} > z_m,\ \text{ i.o.}\}$. Then if $\omega \in N^c$, $\underline{x}$ cannot be a limit point of $(v_n^{(1)},\ldots, v_n^{(\ell n)},0,0,\ldots)$ because for all n sufficiently large

$$\sum_1^{\ell(n)} |v_n^{(i)}-x_i| \geq \min_{1\leq i\leq m} (x_i-z_i) = (\frac{s_m+1}{2s_m})x_m \ .$$

A useful uniform bound on the tail probabilities of the normalized maxima for a Gaussian sequence is provided by Lemma 1 in [2]. We state a version of this result which is suited to our problem.

Lemma 4. Let $c_n = \sqrt{2\ell nn}$. Let $\{X_{k,n}\}$, $k=1,\ldots,n$, $n=1,2,\ldots$ be a triangular array of standard normal random variables. Then setting $r_n(i,j) = EX_{i,n}X_{j,n}$, $M_n = \max_{1\leq k\leq n} X_{k,n}$ and $\delta_n(x) = \sup_{|i-j|\geq x} |r_n(i,j)|$ we have

$$e^{tA^2} P\{c_n(M_n-b_n) \leq -A\} = o(1) \quad \text{as } A \to \infty$$

uniformly in n for all t in a neighborhood of zero provided

(i) $\overline{\lim_{n\to\infty}}\ \delta_n(1) < 1$

(ii) $\delta_n(n^\alpha)\ell nn = 0(1)$ for some fixed $0<\alpha<1$.

Lemma 5. For any fixed positive integer ℓ and $\varepsilon > 0$, $P\{Z_n^{(\ell)} < b_n - \varepsilon a_n,\ \text{ i.o.}\} = 0$. It is easily checked that it is sufficient to show $P\{Z_{n_k}^{(\ell)} < b_{n_{k+1}} - \varepsilon a_{n_{k+1}},\ \text{i.o.}\}=0$.

Also since for k sufficiently large $b_{n_{k+1}} - \varepsilon a_{n_{k+1}} < b_{n_k} - \varepsilon/2\ a_{n_k}$ it is enough to show

(2.33) $P\{Z_{n_k}^{(\ell)} < b_{n_k} - \varepsilon\ a_{n_k},\ \text{i.0.}\} = 0$.

Observe that

$$P\{Z_n^{(\ell)} < b_n - \varepsilon\, a_n\} \le P\{Z_n^{(1)} > b_n + 2a_n\}$$

(2.34)

$$+ \sum_{i=2}^{\ell} P\{Z_n^{(i)} < b_n - \varepsilon\, a_n < Z_n^{(i-1)} < Z_n^{(1)} < b_n + 2a_n\}$$

Now

$$(2.35)\quad P\{Z_n^{(1)} > b_n + 2a_n\} \le n(1-\Phi(b_n + 2a_n)) = \frac{1}{(\ell nn)^2}$$

Further we have that

$$P\{Z_n^{(i)} < b_n - \varepsilon\, a_n < Z_n^{(i-1)} < Z_n^{(1)} < b_n + 2a_n\}$$

(2.36)

$$= \sum_{t_1,\dots,t_{i-1}} P\{X_j \le b_n - \varepsilon\, a_n,\ j \ne t_1,\dots,\ t_{i-1},\ 1 \le j \le n$$

and $\quad b_n - \varepsilon\, a_n < X_{t_u} < b_n + 2a_n,\ u=1,\dots,\ i-1\}$

For a fixed $0<\theta<1$ and n sufficiently large

$$P\{X_j \le b_n - \varepsilon\, a_n,\ j \ne t_1,\dots,\ t_{i-1},\ 1 \le j \le n$$

and

$$b_n - \varepsilon\, a_n < X_{t_u} < b_n + 2a_n,\ u=1,\dots,\ i-1\}$$

$$\le \int_{b_n-\varepsilon a_n}^{b_n+2a_n} \cdots \int_{b_n-\varepsilon a_n}^{b_n+2a_n} P\{X_j \le b_n - \varepsilon\, a_n,\ |j-t_u| > n^{\theta}\ ,$$

(2.37) $\quad u = 1,\dots,\ i-1,\ 1 \le j \le n \,|\, X_{t_u} = x_u,\ u = 1,\dots,\ i-1\}$

$$dP\{X_{t_1} \varepsilon dx_1,\dots,\ X_{t_{i-1}} \varepsilon dx_{i-1}\}$$

$$\le P\{\tilde{X}_j \le b_n - \varepsilon/2\ a_n,\ |j-t_u| > n^{\theta},\ u=1,\dots,\ i-1,\ 1 \le j \le n\}$$

$$P\{b_n - \varepsilon\, a_n \le X_{t_u} \le b_n + 2a_n,\ u=1,\dots,\ i-1\}$$

where $CORR(\tilde{X}_j,\tilde{X}_k) = r_{jk} + 0(\bar{r}^{2}_{n^\theta})$.

Let $1 \le t_{1,n},\ldots, t_{i-1,n} \le n$ be chosen to maximize

$$P\{\tilde{X}_j \le b_n - \varepsilon/2\ a_n,\ |j-t_u| > n^\theta,\ u=1,\ldots,\ i-1,\ 1\le j\le n\}$$

Let $Y_{1,m}Y_{2,m}\ldots Y_{m,m}$ represent the $\tilde{X}_j$, $|j-t_u| > n^\theta$, $u=1,\ldots,$ $i-1$, $1\le j\le n$ in their natural order and let $M_m = \max_{1\le k\le m} Y_k$. Note $n - 2(i-1)n^\theta \le m \le n$. Then the sum in (2.36) is at most

$$(2.38)\quad P\{M_m \le b_m - \varepsilon/4\ a_m\} \cdot \sum_{t_1,\ldots,t_{i-1}} P\{b_n - \varepsilon\ a_n \le X_{t_u} \le b_n + 2a_n,\ u=1,\ldots,i-1\}$$

It is easily checked that the $Y_{k,m}$ satisfy the hypothesis of Lemma 4. Hence

$$(2.39)\quad P\{M_m \le b_m - \varepsilon/4\ a_m\} \le e^{-c(\ell n\ell nn)^2}$$

for some constant $c > 0$ not depending on n.

Further if $\min\{|t_u-t_v|:\ 1\le u<v\le i-1\} \ge n^\theta$, then following the approach in Lemma 1 one can check that

$$P\{b_n - \varepsilon\ a_n \le X_{t_u} \le b_n + 2a_n,\ u=1,\ldots,\ i-1\} \le (1-\Phi(b_n-i\varepsilon a_n))^{i-1}$$

$$(2.40)\qquad = (1/n\ e^{i\varepsilon\ell n\ell nn})^{i-1}$$

While if there are exactly h indices say $u_1,\ldots,\ u_h$ such that when the t's are ordered $t_{(u_1+1)} - t_{(u_1)} < n^\theta,\ldots,\ t_{(u_h+1)} - t_{(u_h)} < n^\theta$ then

$$P\{b_n - \varepsilon\ a_n \le X_{t_u} \le b_n + 2a_n,\ u=1,\ldots,\ i-1\}$$

$$\le (1-\Phi(b_n\delta))^h(1-\Phi(b_n-i\varepsilon a_n))^{i-h-1}$$

where $0<\delta<1$ is some constant not depending on n

$$(2.41)\quad \le (1/n)^{h\delta^2+i-h-1}\ e^{\varepsilon i^2(\ell n\ell nn)}$$

Therefore by choosing $\theta < \delta^2$ we have by (2.39), (2.40) and (2.41) that (2.38) is at most $e^{-c(\ell n\ell nn)^2}$ for some $c > 0$. Therefore by (2.34), (2.35) and the above we see that

$$P\{Z_n^{(\ell)} < b_n - \varepsilon\, a_n\} \le \frac{1}{(\ell nn)^2} \quad .$$

Since this series evaluated at the n_k is summable on k, (2.33) holds completing the proof of Lemma 5.

Theorem 1. Under the assumptions of Lemma 1 the almost sure limit points of the sequence $(v_n^{(1)},\ldots, v_n^{(\ell n)},0,0,\ldots)$ in ℓ_1 coincide with the set

$$A = \{(x_1,x_2,\ldots): \;\; 0 \le x_{i+1} \le x_i,\; i=1,2,\ldots,\; \sum_1^\infty x_i \le 1\}$$

Proof: Lemmas 1 and 2 establish that each point of A is an almost sure limit point while Lemmas 3 and 5 establish that no point in A^c can be an almost sure limit point.

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