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VISCOUS EFFECTS IN THE WEDEMEYER MODEL OF SPIN-UP FROM REST

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June 1983

US ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND

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The problem of impulsive spin-up from rest of a liquid filling a right-circular cylinder is treated. A numerical solution is obtained to the non-linear partial differential equation that governs the spin-up flow by the Wedemeyer model. An implicit, two-time level, second order, iterative finite difference technique is used. Approximations to an impulsive start are discussed. An analytical procedure is applied at the first time step of the calculation to avoid errors caused by a discontinuity in the boundary conditions.

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Conditions for the validity of the model are discussed and a detailed discussion is presented of the Ekman compatibility condition relating radial to azimuthal velocity. Representative results are shown demonstrating the effects of the problem parameters and different Ekman compatibility conditions on spin-up profiles, i.e., curves of azimuthal velocity vs radial coordinate. For several cases the spin-up results are validated by comparison with those of the Navier-Stokes equations.

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I. INTRODUCTION

For many years it has been observed that liquid-filled projectiles have a proclivity for unusual flight behavior, often being unstable even though the same projectile with a solid payload is stable. The spin of the projectile imparts rotation to the liquid which affects the dynamic behavior of the shell. Typically, the fluid is contained in a right-circular cylinder. In the problem considered here, the fluid fills the cylinder. The coordinate system and notation are shown in Figure 1.

The fluid motions can be separated into two classes: solid-body rotation and spin-up. The meaning of the former is obvious. The latter refers to the transient state produced by changing the fluid motion from solid rotation at one spin (which may be zero) to that at a greater spin. The physics of spin-up from rest, the problem to be considered here, is different in some important respects from that of spin-up from a state of solid-body rotation.[1,2] In particular, spin-up from rest is inherently a nonlinear problem, whereas spin-up between two finite states of solid-body rotation can be treated as a linear problem if the change in rotation is small. Spin-down is, in general, an unstable flow and often becomes turbulent. Relatively little work has been done on spin-down; see Reference 3 for a discussion of this problem and some pertinent references on experimental work.

The basic reference on spin-up from rest is Wedemeyer's paper;[4] our work is an extension of his. Wedemeyer's model was based on an order-of-magnitude analysis rather than on formal matched asymptotic expansions. A more formal treatment was given by Greenspan[1] for a two-term expansion. In both cases a phenomenological approach was required to complete the theory.

Wedemeyer's model yielded a nonlinear partial differential equation of the diffusion type for the azimuthal velocity, V, as a function of time and radial coordinate but not of axial coordinate. Without the diffusion terms the solution for V is elementary; Wedemeyer omitted them in his approximate analysis. The main objective of this report is to present the numerical

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1. *H.P. Greenspan, The Theory of Rotating Fluids, Cambridge University Press, London and New York, 1968.*

2. *E.R. Benton and A. Clark, Jr., "Spin-Up," article in Annual Review of Fluid Mechanics, Vol. 6, Annual Reviews, Inc., Palo Alto, California, 1974.*

3. *G.P. Neitzel and Stephen H. Davis, "Energy Stability Theory of Decelerating Swirl Flows," The Physics of Fluids, Vol. 23, No. 3, March 1980, pp. 432-437.*

4. *E.H. Wedemeyer, "The Unsteady Flow Within a Spinning Cylinder," BRL Report No. 1225, October 1963 (AD 431846). Also Journal of Fluid Mechanics, Vol. 20, Part 3, 1964, pp. 383-399.*

solution of Wedemeyer's equation. We shall state the necessary equations and boundary conditions and shall describe the numerical procedure. Also, we shall present results together with some discussion of the accuracy of the solution. Applications to projectile problems will be emphasized.

One of the main applications of spin-up theory is in the study of the eigenvalue problem, i.e., the determination of frequencies and decay rates of the waves in a rotating fluid.[5] Spin-up theory is also required to solve the forced oscillation, or moment, problem. Previously this theory was used to study the spin decay of a liquid-filled projectile[6] after ejection from the gun.

Some appreciation of spin-up effects on a projectile flight can be gained by considering orders of magnitude. Let a and c be the radius and half-height of the cylinder, Ω the spin of the projectile in rad/sec (say at the muzzle), ν the kinematic viscosity of the liquid, and

$$Re = \Omega a^2/\nu, \quad E = \nu/\Omega c^2, \tag{1.1}$$

the Reynolds number and Ekman number. Note that E is sometimes defined using the height, 2c, rather than c, and that for historical reasons we shall use Re in this report rather than the more conventional E. If the Ekman layers, i.e., endwall boundary layers, are laminar, the characteristic time for spin-up is

$$\begin{aligned} \bar{t}_s &= (2c/a)\, Re^{1/2}/\Omega \\ &= 2/E^{1/2}\Omega \text{ (sec)}. \end{aligned} \tag{1.2}$$

This expression is derivable from linear spin-up theory[1,2] and from the Wedemeyer solution without diffusion. We also use a nondimensional spin-up time $t_s = \Omega \bar{t}_s$. For $Re > 10^5$, approximately, the Ekman layer may be turbulent, in which case the characteristic spin-up time can be estimated by

$$\bar{t}_{st} = (28.6\ c/a)\, Re^{1/5}/\Omega. \tag{1.3}$$

This result can be obtained from the Wedemeyer solution without diffusion for turbulent Ekman layers. Basically, these are time scales for the spin-up process and do not necessarily give a measure of the closeness of the process to solid-body rotation which can be measured in several ways, e.g.,

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5. *R. Sedney and N. Gerber, "Oscillations of a Liquid in a Rotating Cylinder: Part II. Spin-Up," BRL Technical Report ARBRL-TR-02489, May 1983.*

6. *C.W. Kitchens, Jr., N. Gerber, and R. Sedney, "Spin-Decay of Liquid-Filled Projectiles," Journal of Spacecraft and Rockets, Vol. 15, No. 6, November-December 1978, pp. 348-354.*

the time for the angular momentum, flow rate across a meridian plane, or other property to attain a value within 10%, say, of the solid-body value. These measures require detailed calculations and so are not very convenient. A rule of thumb is that solid-body rotation is reached at about 4 $\bar{t}_s$ after an impulsive start of the cylinder.

For projectile flight these characteristic spin-up times should be compared with $\bar{t}_{fl}$, the time of flight of the projectile. If $\bar{t}_s$ or $\bar{t}_{st}$ << $\bar{t}_{fl}$, spin-up effects can be disregarded and solid-body rotation can be assumed. If $\bar{t}_s$ or $\bar{t}_{st} \approx \bar{t}_{fl}/10$, or larger, spin-up effects probably have to be considered. To put these estimates in perspective, consider the parameters for two 155mm projectiles. For one of these, Case 1, c/a = 3.120, Re = 4×10^4, Ω = 754 rad/sec giving $\bar{t}_s$ = 1.65 sec. For Case 2, c/a = 5.200, Re = 2×10^6, Ω = 754 rad/sec giving $\bar{t}_{st}$ = 3.52 sec. For both, $\bar{t}_{fl}$ = 40 sec, and the above criteria indicate that spin-up effects may be important, especially for Case 2. The extent to which the projectile motion is, in fact, affected by the internal liquid motion requires a solution to the forced oscillation, or moment, problem. Reference 7 treats the moment problem for solid-body rotation flow. The present report is restricted to the unperturbed time-dependent axisymmetric fluid motion.

II. THE SPIN-UP MODEL

A. Time Scales.

Consider the motion of a fluid which fills a cylinder, initially at rest, which is "rapidly" brought to a constant angular velocity. The fluid motion is axisymmetric and time dependent. The conditions for which the Wedemeyer spin-up model[4] is valid can be presented in terms of the time scales involved. It was derived on the basis of an impulsively started cylindrical container, a theoretical concept which will be discussed below; for now this time scale is taken to be zero. Only the laminar Ekman layer case will be discussed. There are three time scales that govern the spin-up process. In nondimensional form these are

$$\bar{t}_\Omega = 2\pi\Omega^{-1}$$

$$\bar{t}_s = 2\, E^{-1/2}\, \Omega^{-1}$$

$$\bar{t}_\nu = (a/c)^2\, E^{-1}\, \Omega^{-1}.$$

7. *N. Gerber and R. Sedney, "Moment on a Liquid-Filled Spinning and Nutating Projectile: Solid Body Rotation," BRL Technical Report ARBRL-TR-02470, February 1983 (AD A125332).*

The first of these is the time for the container to rotate through one revolution. It can be shown that $\bar{t}_\Omega$ is the time scale for the formation of quasi-steady Ekman layers on the endwalls, which layers are of thickness $O(\nu/\Omega)^{1/2}$. Actually, the Ekman layers are quasi-steady after about 2 radians of rotation, i.e., $\Omega\bar{t} = 2$. Wedemeyer used this result; a more complete study of Ekman layer formation is given by Benton.[8] The thickness of the Ekman layers is assumed to be small compared to either c or a so that $E^{1/2}$ or $Re^{-1/2}$ is required to be small. The second time scale is (1.2), anticipated to be the characteristic spin-up time; spin-up occurs for $\bar{t}/\bar{t}_s = O(1)$. The time scale $\bar{t}_\nu$ is the time for velocity gradients or vorticity to diffuse viscously across the radius of the cylinder. For $c/a = O(1)$, $\bar{t}_\nu$ is an order of magnitude greater than $\bar{t}_s$ so that viscous diffusion plays a small role in spin-up. For an infinite cylinder it is the only mechanism for spin-up; for large c/a, $\bar{t}_\nu$ can become comparable to or less than $\bar{t}_s$.

For an impulsive start, the conditions for the validity of the spin-up model are

$$\bar{t}_\Omega \ll \bar{t}_s \ll \bar{t}_\nu,$$

$$E^{-1/2} \gg 1 \quad \text{and} \quad c/a = O(1).$$

B. Approximation to Impulsive Start.

The degree to which an impulsive start can be approximated in an experiment depends on the system parameters. For our application the angular acceleration is large during the time when the projectile is in the gun; the small deceleration in flight, the spin decay,[6] is neglected. For artillery projectiles this acceleration time is typically 0.020 sec. A time history of projectile spin, $\bar{\Omega}(\bar{t})$, is shown in Figure 2. To quantify the departure from an ideal impulse several approaches are possible; here time scales are used.

For a finite angular acceleration of the cylinder let the spin be $\bar{\Omega}(\bar{t})$, as in Figure 2, e.g. The time $\bar{t}_\Omega$ is used as a reference. A characteristic time $\bar{t}_c$ is introduced, the time when the acceleration becomes zero. The acceleration or impulse time scale is taken to be

8. E.R. Benton, "On the Flow Due to a Rotating Disk," Journal of Fluid Mechanics, Vol. 24, Part 4, 1966, pp. 781-800.

$$\bar{t}_I = (d\bar{\Omega}/d\bar{t})^{-1/2}$$

which was used by Weidman[9,10]; $\bar{t}_I = 0$ for a true impulse. For comparative purposes $\bar{\Omega}(\bar{t})$ is taken to be linear for $0 \leq \bar{t} \leq \bar{t}_c$ and $\bar{\Omega}(\bar{t}) = \Omega$ for $\bar{t} > \bar{t}_c$. In most spin-up experiments enough information to define $\bar{\Omega}(\bar{t})$ more precisely is not available; a linear approximation to $\bar{\Omega}(\bar{t})$ in Figure 2 is reasonable.

In Table 1 the above time scales are given for three experimental arrangements that have been used in spin-up studies. (In some other experiments insufficient information was available to define the time scales.) The first is the gun tube, using the parameters of Figure 2; the second is the spin generator from which Wedemeyer[4] obtained observations of secondary flow during spin-up; the third is the most nearly impulsive case in Weidman's experiments, Figure 2c of Reference 10.

TABLE 1. TIME SCALES FOR IMPULSIVE START, MS

	t_I	$\bar{t}_\Omega$	$\bar{t}_c$
Gun tube	5.6	10.8	18
Spin generator	26.6	22.2	200
Weidman	218	57.6	5,200

To approximate an impulsive start for the spin-up problem, two criteria must be satisfied: $\bar{t}_I \ll \bar{t}_c$ as a condition on the mechanical system and $\bar{t}_I \ll \bar{t}_\Omega$ to insure that the impulse time scale is small compared to the time

9. P.D. Weidman, "On the Spin-Up and Spin-Down of a Rotating Fluid: Part 1. Extending the Wedemeyer Model," Journal of Fluid Mechanics, Vol. 77, Part 4, 1976, pp. 685-708.

10. P.D. Weidman, "On the Spin-Up and Spin-Down of a Rotating Fluid: Part 2. Measurements and Stability," Journal of Fluid Mechanics, Vol. 77, Part 4, 1976, pp. 709-735.

of formation of the quasi-steady Ekman layers. Table 1 shows that the gun tube gives the best approximation to an impulsive start, but only marginally according to the criteria. The spin generator does not give a impulsive start and Weidman's case is far from it.

Since the Wedemeyer model assumes an impulsive start, deviations from it as shown in Table 1 would require modification of the model. The necessary analysis has not been done for either small or large departures from an impulsive start. In References 9 and 11 the same type of modification of the Wedemeyer model was made to account for a nonimpulsive start. The value of Ω was replaced by the variable $\bar{\Omega}$ $(\bar{t})$. This should be a reasonable approximation for an almost impulsive start. In Reference 11 its validity was checked by comparison with solutions to the Navier-Stokes equations.

C. The Physics of Spin-Up.

In time of $O(\bar{t}_\Omega)$ quasi-steady Ekman layers are formed on the endwalls. For small time and near the center of the endwalls these layers are essentially the same as the boundary layer on a steady rotating disk, the von Karman disk problem. The suction exerted by these layers draws external fluid into the Ekman layers and imparts rotation to this fluid. With no pressure gradient acting, the fluid spirals out to larger radii, where the Ekman layers eject fluid. This fluid is now rotating. This is the basic mechanism for spin-up.

D. Spin-Up Theory.

The notation here will be the same as in Reference 5. Lengths, velocities, pressure, and time are made nondimensional by a, $a\Omega$, $\rho\Omega^2a^2$, and Ω^{-1}, respectively, where ρ is the liquid density and Ω is the constant spin rate of the cylinder. In the inertial frame cylindrical coordinates r, θ, z are used, with the origin of z at the center of the cylinder, and velocities U, V, W, respectively. Dimensionless time is t. Derivatives are indicated by subscripts. Wedemeyer[4] showed that the flow can be divided into two regions: the quasi-steady Ekman layers at the endwalls and the rest of the flow, called the core flow. Wedemeyer did not point out that a boundary layer, i.e., Stewartson layer, must be inserted at the cylinder wall. Starting with the Navier-Stokes equations for axisymmetric flow,

$$U^*_t + U^* U^*_r + W^* U^*_z - V^{*2}/r = -P^*_r + Re^{-1} (\nabla^2 U^* - U^*/r^2) \qquad (2.1a)$$

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11. *C.W. Kitchens, Jr., and N. Gerber, "Prediction of Spin-Decay of Liquid-Filled Projectiles," BRL Report 1996, July 1977 (AD A043275).*

$$V^*_t + U^* V^*_r + W^* V^*_z + U^* V^*/r = Re^{-1} (\nabla^2 V^* - V^*/r^2) \tag{2.1b}$$

$$W^*_t + U^* W^*_r + W^* W^*_z = -P^*_z + Re^{-1} \nabla^2 W^* \tag{2.1c}$$

$$(rU^*)_r + rW^*_z = 0 \tag{2.1d}$$

$$\nabla^2() = ()_{rr} + (1/r)\,()_r + ()_{zz}, \tag{2.1e}$$

where the asterisk indicates the exact solution, he used order-of-magnitude arguments as indicated in Sections IIA and IIC to simplify them in the core flow. The approximations reduce the three momentum equations, (2.1, a,b,c), to

$$V_t + U\,(V_r + V/r) = Re^{-1} [V_{rr} + (V/r)_r] \tag{2.2}$$

and

$$U_z = V_z = P_z = 0, \tag{2.3}$$

where the asterisk is dropped for this approximate solution. For $Re \to \infty$ Wedemeyer proposed neglecting the diffusion terms in (2.2) and arrived at

$$V_{w_t} + U_w\,(V_{w_r} + V_w/r) = 0, \tag{2.4}$$

where the sub w indicates this approximation. In applying his model Wedemeyer used (2.4) rather than (2.2). Although he showed the crucial importance of the Ekman layers to the spin-up process, his model did not require a solution for the flow in these layers. He did discuss certain properties of the solution which were required in his model for the core flow.

A more formal approach to this problem was given by Greenspan.[1] In his treatment, lengths are made dimensionless by 2c rather than a, which is more natural for the spin-up process; time is made dimensionless by t_s so that the new time is $t' = t/t_s = t/[(2c/a)\,Re^{1/2}]$. An expansion in the small parameter

$$(1/t_s) = (a/2c)\,Re^{-1/2} \tag{2.5}$$

is assumed. The form of this expansion

$$\begin{aligned} U^* &= \quad\;\; (1/t_s)\,U_1 + \ldots \\ V^* &= V_0 + (1/t_s)\,V_1 + \ldots \\ W^* &= \quad\;\; (1/t_s)\,W_1 + \ldots \\ P^* &= P_0 + (1/t_s)\,P_1 + \ldots \end{aligned} \tag{2.6}$$

follows from the knowledge that in the core flow U^* and W^* are $O(1/t_s)$ and V^* is $O(1)$ if $1/t_s$ is small. See Reference 1 for details. The independent variables are r, z, t'. Substituting the expansion into (2.1) after transforming to t' yields

$$V_{o_{t'}} + U_1 (V_{o_r} + V_o/r) = 0$$

$$V_o{}^2/r = P_{o_r} \tag{2.7}$$

$$P_{o_z} = V_{o_z} = U_{1_z} = 0.$$

These are the same as (2.3) and (2.4) if we set $V_o = V_w$, $U_w = (1/t_s)\, U_1$, and $t' = t/t_s$. Although the formalism of matched asymptotic expansions was not used, the first two terms of the outer expansion are apparently given by (2.6). The Ekman layer solution would be obtained from the inner expansion, which was not considered in Reference 1.

To solve (2.2), (2.4), or (2.7) a relationship between U and V is necessary. This is called the Ekman compatibility condition, abbreviated EC, because the Ekman layer suction must be made compatible with the core flow. Wedemeyer used the facts that the Ekman layers are quasi-steady after one revolution and that the radial mass flux in the core flow must be balanced by that in the Ekman layers to obtain some conditions on the U, V relationship. At this point he was forced to take a phenomenological approach. The relationship is known at $t \to 0$ and $t \to \infty$ and he proposed a linear interpolation between them to obtain an approximate relationship for any t. He tested this idea in some other problems where the solution was known and decided it was satisfactory. Some confusion has appeared in the literature because this step was misinterpreted; further discussion of this is given below. His result is

$$\begin{aligned} U &= k_\ell\, (V - r) \\ k_\ell &= \kappa\, (a/c)\, Re^{-1/2} \\ &= \kappa\, E^{1/2} \qquad = 2\kappa/t_s \end{aligned} \tag{2.8}$$

for laminar Ekman layers. Wedemeyer proposed $\kappa = 0.443$ but Greenspan suggested $\kappa = 0.5$; the latter often gives results in better agreement with numerical solutions to the Navier-Stokes equations. For turbulent Ekman layers

$$\begin{aligned} U &= -k_t\, (r-V)^{8/5} \\ k_t &= .035\, (a/c)\, Re^{-1/5} \qquad \cong 1/t_{st}, \end{aligned} \tag{2.9}$$

where t_{st} is the nondimensional, turbulent spin-up time. The core flow is assumed to be laminar so that turbulent stresses are not introduced in the right-hand side of (2.2).

Using (2.8), (2.4) can be solved explicitly for $V(r,t)$ with $V(r,0) = 0$ and $V(1,t) = 1$:

$$V_w = (re^{2k_\ell t} - 1/r)/(e^{2k_\ell t} - 1) \quad \text{for } r \geq e^{-k_\ell t}$$
$$= 0 \quad \text{for } r < e^{-k_\ell t}, \tag{2.10}$$

so that $r = e^{-k_\ell t}$ separates rotating and nonrotating fluid; it is called the front. Although V_w is continuous there, a discontinuity in shear, i.e., V_{w_r}, exists because diffusion is neglected. Using (2.9) a numerical integration is necessary, but the character of the solution is the same as when (2.8) is used. The radial velocity is obtained from (2.8) or (2.9) and W is obtained from the continuity equation

$$W = -(z/r)\,(rU)_r$$

At $r = 1$, $W \neq 0$ so that a Stewartson layer must be inserted there, as mentioned before; this has yet to be done. Also $W \neq 0$ at the endwalls $z = \pm c/a$ and W is discontinuous at the front.

Wedemeyer pointed out that the shear discontinuity in the solution to (2.4), see (2.10), would be smoothed out if the diffusion terms were retained, as in (2.2). But this is a nonlinear second order equation and must be integrated by finite difference methods. This is discussed in Section III. For laminar Ekman layers, the spin-up velocity profile is

$$V = f\,(r,\ k_\ell t,\ k_\ell\,Re),$$

where $k_\ell t = t'$ if $\kappa = 0.5$; for the turbulent case k_ℓ is replaced by k_t. These functional forms follow directly from (2.2) and (2.8) or (2.9) and represent the most efficient way to display the results.

E. Discussion of Ekman Compatibility Condition.

The original forms of the EC given by Wedemeyer, for laminar and turbulent Ekman layers, were discussed in Section IID. It is important to realize that an exact EC does not exist; it is necessarily approximate. This can be deduced theoretically and is shown from finite difference solutions of the spin-up problem using the Navier-Stokes equations.[12] Three attempts to obtain EC which give improved approximations to V will be described.

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12. *C.W. Kitchens, Jr., "Ekman Compatibility Conditions in Wedemeyer Spin-Up Model," The Physics of Fluids, Vol. 23, No.5, May 1980, pp. 1062-1064.*

From his finite difference solutions Kitchens[12] obtained the radial mass flow in the Ekman layer as a function of V/r for several r and t. The mass flow can be used to form the EC. A unique relation was not found, rather a band of points. Based on these numerical data, Kitchens deduced an approximate, monotonic, nonlinear EC. For $0 \leq V/r \leq 0.75$, U is a cubic in V/r and for $0.75 \leq V/r \leq 1$, it is a linear function. Limited tests of this EC show only small improvement in V compared to using (2.8).

The following EC is proposed as an extension of (2.8):

$$U = -[r/(Re^{1/2}\ c/a)]\ [.500\ (1-V/r) - .057\ (1-V/r)^3] \tag{2.11}$$

This relation satisfies (9) in Reference 4 which holds for $V/r = 0$ at $t = 0$ and the condition $U\ (V/r = 1) = 0$, for $t \to \infty$, the two conditions imposed by Wedemeyer in arriving at his linear expression. However, it also satisfies (14) in Reference 4, in slope and curvature, when the core flow is almost solid-body rotation. Only one test of this EC has been made; it showed that V was significantly closer to the results from Navier-Stokes calculations than those using other EC's. This EC is the only one which is a rational extension of Wedemeyer's linear expression.

The last EC to be discussed is of a somewhat different nature, and its origin is not clear. It utilizes the solutions of Rogers and Lance[13] to the boundary layer flow over an infinite disk rotating with angular velocity Ω and outer flow in solid-body rotation with angular velocity ω. The main result used from this paper is the computed radial mass flux in the boundary layer/ $r\ Re^{-1/2} \equiv \dot{m}$ vs ω/Ω. It is plotted in Figure 3, and we shall call this the RL curve. Using arguments somewhat different from Wedemeyer's, Greenspan[1] derived the EC (2.8) using a linear approximation to the RL curve. Possibly because of a suggestion of Greenspan,[1] page 169, several investigators have attempted to improve (2.8) by using an analytical fit to the RL curve; these include Goller and Ranov,[14] Venezian[15] Watkins and Hussey,[16] and Weidman.[9] This EC is determined from Figure 3 using the alternate labels on the scales, $U\ (c/a)\ Re^{1/2}/r$ vs V/r. Weidman[9] uses a 7th order polynominal to approximate the RL curve.

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13. *M. H. Rogers and G.N. Lance, "The Rotationally Symmetric Flow of a Viscous Fluid in the Presence of an Infinite Rotating Disk," Journal of Fluid Mechanics, Vol. 7, Part 2, 1960, pp. 617-631.*

14. *H. Goller and J. Ranov, "Unsteady Rotating Flow in a Cylinder with a Free Surface," Journal of Basic Engineering, TRANS. ASME, Vol. 90, Series D, December 1968, pp. 445-454.*

15. *G. Venezian, Topics in Ocean Engineering, Vol. 2, pp. 87-96, Gulf Publishing Co., Houston, Texas, 1970.*

16. *W.B. Watkins and R. G. Hussey, "Spin-Up from Rest: Limitations of the Wedemeyer Model," The Physics of Fluids, Vol. 16, No. 9, September 1973, pp. 1530-1531.*

Note that Figure 3 appears in Wedemeyer's paper[4] but not for the purpose of determining an EC. After presenting his arguments for (2.8), he tested it against two other similar problems with known solutions. One of these was the solutions of Rogers and Lance.[13] There has been some confusion on this matter because it was assumed that Wedemeyer obtained (2.8) as a linear approximation to the RL curve.

As pointed out by Weidman, one of the assumptions needed to apply the Rogers and Lance results to derive an EC is that the fluid in the core is "locally in solid-body rotation." It appears that this assumption is rather unrealistic. Some arguments can be constructed to show that this EC is implausible; but it cannot be proved that it is incorrect, because an exact EC does not exist. Some qualitative arguments were given by Benton;[17] later, he provided some arguments of a more quantitative nature[18] which were based on the computations described in this report. One can discuss the consequences of using the EC obtained from the RL curve for either (2.2), the spin-up equation with diffusion, or (2.4), without diffusion. Some examples of results from the numerical solution of (2.2), using this EC, will be given later. The solution of (2.4) with the EC using the RL curve has some unusual properties which actually provide the most compelling reason for rejecting this EC. Since (2.4) is a first order, quasi-linear partial differential equation, it can be solved by quadrature; this is true if (2.9) is used, e.g.. For this purpose the method of characteristics is used, there being a one parameter family of characteristics. Using the EC from the RL curve, Weidman[9] showed that the characteristics intersect near the front, implying that the solution to (2.4) is double-valued: for a fixed t, V is double-valued over a range of r, in the neighborhood of the r for which V = 0. He showed that this result was directly attributable to the fact that the RL curve is not monotonic. Clearly this is an unacceptable result. Weidman says that there must be a discontinuity in V to resolve this dilemma. This remains a conjecture since he did not determine the position and strength of the discontinuity.

Using the EC from the RL curve in the spin-up equation with diffusion, (2.2), the numerical solution is not double-valued or discontinuous. One such result will be shown later. It will be seen that when the limit with no diffusion is approached, $k_\ell \, Re \to \infty$, there is a tendency for the V vs r curve to develop a vertical tangent. This might be an indication of an approach to a discontinuity. Both a vertical tangent in the theory with diffusion or a discontinuity without diffusion are physically unacceptable and violate the assumptions of the model. Since the double-valued/discontinuous nature of the solution is caused by the nonmonotonic RL curve, it seems evident that the EC based on it should be discarded. Any suggestion that these anomalies indicate a breakdown of the Wedemeyer model is unjustified, of course. The numerical solutions of (2.2), rather than (2.4), using this EC does give reasonable-

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17. *E.R. Benton, Memo to COTR, Task Order 74-461, BRL, January 1975.*

18. *E.R. Benton, "Vorticity Dynamics in Spin-Up from Rest," The Physics of Fluids, Vol. 22, No. 7, July 1979, pp. 1250-1251.*

looking $V(r,t)$ as long as the limit with no diffusion is not approached, i.e., if k_ℓ Re is not large. For reasons given above this EC is not used except for showing some comparative results.

III. METHOD OF SOLUTION

A. Finite Difference Method.

The initial and boundary conditions for (2.2) are:

$$V = 0 \quad \text{for } t \leq 0, \quad 0 \leq r < 1 \tag{3.1}$$

$$V = 1 \quad \text{for } r = 1, \quad t > 0 \tag{3.2}$$

$$V = 0 \quad \text{for } r = 0, \quad t > 0 \tag{3.3}$$

Since (2.2) is a parabolic, quasi-linear equation, it can be solved by marching in time; i.e., $V(r,t)$ is obtained at $t = k\Delta t$, where $k = 1, 2, \ldots$ and Δt is the time increment used in approximating V_t. The grid used in the finite difference scheme is shown in Figure 4. The scheme is an implicit one and the discretization error is second order in both r and t, since central differences are used in approximating all derivatives.

The interval $0 \leq r \leq 1$ is divided into N segments of equal length $\Delta r = 1/N$. V is calculated at the N-1 nodal points $r_j = j\,\Delta r$, $j = 1, 2, \ldots, N-1$. Given V at $t = k\Delta t$, for all j, the solution at $(k+1)\Delta t$ is obtained. At that time level, consider a typical point Q in Figure 4. The finite difference approximation to (2.2) is centered at P. Thus,

$$V_Q - V_R = (1/2)\,(F_R + F_Q)\,\Delta t + 0\,(\Delta t)^3 \tag{3.4}$$

where

$$F \equiv Re^{-1}\,[V_{rr} + (V/r)_r] - U\,(V_r + V/r) \tag{3.5}$$

The following difference approximations are made at the point R ($= j\Delta r$, $k\Delta t$):

$$V_r = (V_B - V_A)/(2\,\Delta r) \tag{3.6a}$$

$$V_{rr} = (V_B - 2V_R + V_A)/(\Delta r)^2 \tag{3.6b}$$

$$(V/r)_r = [(V/r)_B - (V/r)_A]/(2\,\Delta r) \qquad (j > 1) \tag{3.6c}$$

$$(V/r)_r = (1/2)\,(V_B - 2V_R)/(\Delta r)^2 \qquad (j = 1). \tag{3.6d}$$

The special form of (3.6d) is required because $(V/r)_A$ is indeterminate at $r = \Delta r$. Identical approximations are made at point Q, with points A and B replaced by points S and T, respectively, in (3.6).

In (3.4) and (3.5) U_Q, related to V_Q by the EC, requires an iterative process. An initial guess for V_Q is used to calculate U_Q. The calculated V_Q from (3.4) is then used to obtain a new U_Q, etc., until the desired degree of convergence is obtained at all points for $t = (k+1)\,\Delta t$. The initial guess for V_Q is obtained by extrapolating from V_M and V_R. Because of the implicit nature of the finite difference scheme the values of V at all nodal points at $t = (k+1)\Delta t$ must be obtained simultaneously. However, the matrix of the set of N linear algebraic equations is tridiagonal so that a straightforward algorithm can be used to solve the equations, which have the form

$$a_j\, v_{j-1,\,k+1} + b_j\, v_{j,\,k+1} + c_j\, v_{j+1,\,k+1} = d_j. \qquad (3.7)$$

The expressions for a_j, b_j, c_j, and d_j are given in the Appendix.

B. Accuracy of Results.

Convergence criteria were set to insure that the numerical solution was accurate to at least three decimal places. For a number of cases, covering a large range of k_ℓ Re, or k_t Re, calculations were made for different combinations of Δr and Δt to determine maximum permissible interval sizes. The Δt was kept small enough so that no more than four iterations were required for convergence at any time step. The $\Delta r = 1/N$ required to compute V to an accuracy of, say, 1% or less depends on r and t and on the parameter k_ℓ Re, or k_t Re. Some representative results showing the accuracy of V are presented in Tables 2 and 3 for the parameters of Cases 1 and 2, respectively. The percentage error is largest for small r; V at $r = 0.2$ is chosen for Tables 2 and 3. For k_ℓ Re = 28 in Table 2, N = 100 is large enough for an accuracy of 1% or less when $t \geq 600$. For k_ℓ Re = 736 in Table 2, $N \geq 200$ is required for t = 6,000. When this solution is used as the basic flow in our eigenvalue calculations, the vorticity $\zeta = (rV)_r/2r$ is also required. The error in it is usually largest near r = 1. A convenient test on the accuracy of ζ is available and discussed in Section III C. The CPU time for solving (2.2) varies with the parameter but is usually less than one minute on the VAX.

Ultimately, the adequacy of the numerical solution, and of the model itself, must be judged by comparisons with solutions to the Navier-Stokes equations. Some examples of these comparisons will be given in the next section.

TABLE 2. EFFECT OF N ON SPIN-UP VELOCITY PROFILE, CASE 1, k_{ℓ} Re = 28.3

	t = 1,200,	r = 0.20
N	V	ΔV/V = [V(N) - V(100)]/V(100)
20	.07558	3.49%
40	.07355	0.71%
80	.07309	0.08%
100	.07303	0.00%

	t = 600,	r = 0.20
N	V	ΔV/V = [V(N) - V(100)]/V(100)
20	.00240	72.6%
40	.00160	15.2%
80	.00141	1.4%
100	.00139	0.0%

TABLE 3. EFFECT OF N ON SPIN-UP VELOCITY PROFILE, CASE 2, k_t Re = 736

	t = 9,000,	r = 0.20
N	V	ΔV/V = [V(N) - V(400)]/V(400)
50	.08626	1.90%
100	.08517	0.63%
200	.08476	0.14%
400	.08464	0.00%

	t = 6,000,	r = 0.20
N	V	ΔV/V = [V(N) - V(1000)]/V(1000)
50	.01748	21.2%
100	.01534	6.4%
200	.01461	1.3%
400	.01440	-0.14%
800	.01443	0.07%
1000	.01442	0.00%

C. Discontinuity Due to Impulsive Start.

For an impulsive start, conditions (3.1) and (3.2) require a discontinuity in V at r = 1, t = 0. If the finite difference scheme just described is used without modification, the value of V(1,0) would be required but is not available. If values such as 0, 1, or 1/2 are used, a significant error is made in a small neighborhood of r = 1. Because of the diffusive nature of (2.2), this error will decrease as t increases. The error may be significant at the earliest time for which a solution is required. To reduce this error a local solution is needed to resolve the discontinuity.

Introducing new coordinates

$$R = (1 - r)\, t'^{-1/2}, \qquad T = t',$$

a solution for small t' was obtained in the form

$$V = \tilde{V}_0\ (R) + T^{1/2}\ \tilde{V}_1\ (R) + \ldots$$

The first term is given by

$$\tilde{V}_0 = \text{erfc}\ (R/2\varepsilon^{1/2}) = \text{erfc}\ ([1-r]\ [Re/t]^{1/2}/2)$$

$$\varepsilon = 1/(k_\ell\ Re).$$

For r near unity, this is the same as the solution for the impulsively started, infinite plate (the Rayleigh problem) as expected. This solution satisfies (3.1) and (3.2) but not (3.3). It is used to provide the initial condition which is applied at the first time step $t = \Delta t$. Since (3.3) is not satisfied, an error is made at r = 0. A limit is placed on this error: $V(0, \Delta t) < 10^{-6}$, which is satisfied if $(Re/\Delta t)^{1/2} > 7$. The latter inequality must be satisfied for the calculation to proceed. In practice it has never been violated. The second term, $\tilde{V}_1$, has not been needed in our calculations.

Using this initial condition we obtain V/r vs r shown in Figure 5 for Re = 4974, c/a = 3.30 and t = 20 and 50. Also shown are the results obtained if V(1,0) = 1 is assumed; there is a relatively large error for t = 20. The intersection of the dashed lines is a consequence of the error. In Figure 6 the nondimensional vorticity $\zeta = (rV)_r/2r$ vs r is shown for t = 100 and 600. The two sets of curves are again obtained using this initial condition and V(1,0) = 1. For $r \geq 0.95$ and t = 100 the errors resulting from use of V(1,0) = 1 are quite large. The vorticity plot has the advantage that a check is readily available. It can be shown from (2.2) and (2.8) that $\zeta_r\ (r = 1) = 0$. The slope of the solid curve at r = 1 for t = 100 is slightly negative; for t = 600 it is zero. The slopes of the dashed curves at r = 1 are quite different from zero.

IV. RESULTS

The main features of the theory will be illustrated by some examples. Usually the parameters are chosen to be of interest in liquid-filled projectile applications, but sometimes they are chosen to make a particular point.

The Wedemeyer model for spin-up simplifies the Navier-Stokes equations so that a more tractable mathematical problem is obtained while retaining the essential physics of the problem. Since it is now feasible to solve the Navier-Stokes equations by finite difference techniques, it may be that the model is not needed; it is premature to draw that conclusion. However, the results of the model can be compared with accurate finite difference solutions.[19] One such comparison is shown in Figure 7. Using the program of Reference 19, V vs r was computed for $Re = 10^3$, $t = 62.5$ and for $Re = 10^4$, $t = 62.5$ and 125, both for $c/a = 1$. The results at $z = 0$ are plotted. The circled points are solutions of (2.2) and (2.8). For these parameters the differences between the spin-up model and the Navier-Stokes solutions are of the order of a few percent, except possibly for very small V. The differences are not always so small; we have no general rules for what values of the parameters give such small differences.

Consider the solution to the spin-up equation for the parameters of the two cases of 155mm projectiles, given in the Introduction. For Case 1, with $Re = 4 \times 10^4$, we should expect laminar Ekman layers. The solution is shown in Figure 8 for three values of t. One of them is $t_s = 1245$. For $t = 4800$, approximately $4\ t_s$, solid-body rotation is achieved, essentially. For Case 2, with $Re = 2 \times 10^6$, we should expect turbulent Ekman layers. The solution to (2.2) with (2.9) gives the velocity profile shown in Figure 9 for three values of t including $t_{st} = 2700$. For $t = 10{,}000$, approximately $4\ t_{st}$, solid-body rotation has not been achieved to the same degree as in Figure 8.

The Ekman compatibility conditions (2.8) and (2.9) were proposed by Wedemeyer for either laminar or turbulent Ekman layers, there being no provision for transition from one to the other. Quite different results can be obtained using (2.8) or (2.9). This is illustrated in Figure 10 with $Re = 3 \times 10^5$, a value at which either condition might apply. The results using (2.8) and (2.9) are shown for two values of c/a and t. The difference is large for $c/a = 1.0$ but small for $c/a = 0.05$, an extremely small aspect ratio.

The need for including diffusion in the spin-up model was discussed in Section II. The differences in V, with and without diffusion, are illustrated in Figure 11 for $Re = 6.08 \times 10^5$, $c/a = 2.679$ and $t = 800$ and 3200. The differences can be smaller or larger, depending on the parameters; generally they are smaller at larger values of k_ℓ Re. For given t, the largest

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19. *C.W. Kitchens, Jr., "Navier-Stokes Solutions for Spin-Up from Rest in a Cylindrical Container," Technical Report ARBRL-TR-02193, September 1979 (AD A077115).*

difference occurs at the front and this increases as t decreases. Even though Re is large, laminar Ekman layers are assumed in this example.

A similar comparison is shown in Figure 12 for turbulent Ekman layers for $Re = 2 \times 10^6$ and $c/a = 5.2$. The results including diffusion are the same as those in Figure 9 for $t = 1300$ and 2700. Without diffusion, V depends only on r and $k_t t$. The differences are smaller than for the laminar case.

The effect on the velocity profile of using the EC from the RL curve (Figure 3) is shown in Figure 13. A rather abrupt change in curvature is obtained for small V; recall that the maximum in the RL curve occurs for small V. To obtain this result it was necessary to use $\Delta r = 1/200$; a value of $\Delta r = 1/50$ did not show the abrupt change in curvature. The purpose of this calculation was to illustrate the approach to the no-diffusion limit in which double-valued, or possibly discontinuous, V is predicted by Weidman's theory.[9] Thus the rather large value k_ℓ Re = 5000. It is not unreasonable to conclude that the curve in Figure 13 is tending toward one with a vertical tangent, which might be related to the predicted double-valued or discontinuous V. As the slope of the tangent increases, the calculation would become increasingly more difficult. For comparison the V calculated from (2.2) with (2.8) is also shown. The difference between the two is about 0.1 at $r = 0.4$, which is quite large; for smaller k_ℓ Re the differences are smaller. The results obtained by using (2.8) show no unusual behavior. This comparison gives further evidence for not using the EC from the RL curve.

V. APPROXIMATIONS FOR DIFFUSION EFFECTS

Inclusion of diffusion effects in the solution to the spin-up equation was amply demonstrated in the previous section, but the numerical results do not provide much understanding of the physical process. Without diffusion a shear discontinuity exists at the front which implies that viscous diffusion, neglected in arriving at (2.4), cannot be neglected in the neighborhood of the front. A local solution, including diffusion, can be sought to examine the structure of the shear layer. This was done by Venezian,[15] who did a boundary layer type analysis of (2.2) on the moving front. His result will be shown below as part of a more general solution. He found that the moving shear discontinuity is a layer of thickness $O(E^{1/4})$. His interpretation is that the $E^{1/4}$ Stewartson layer that exists at the sidewall in linear problems breaks away from the sidewall for this strongly nonlinear case and propagates into the interior. The $E^{1/3}$ Stewartson layer apparently remains attached; it is the one mentioned just before (2.1a).

Inclusion of diffusion over the entire radius can be studied using matched asymptotic expansions. It is convenient to work with the circulation $\Gamma = rV$. Using $\kappa = 0.5$ and introducing the time t', (2.2) and (2.8) yield

$$\Gamma_{t'} - (r - \Gamma/r)\, \Gamma_r = \varepsilon\, (\Gamma_{rr} - \Gamma_r/r) \qquad (5.1)$$

where

$$\varepsilon = (k_\ell \, Re)^{-1}.$$

Coordinates centered on the front are introduced

$$x = (r^2 e^{2t'} - 1)/(e^{2t'} - 1)$$

$$y = e^{2t'} - 1.$$

After transforming (5.1) to the (x,y) coordinates, it is solved by expansions in ε. The first term in the outer solution is (2.10); the first term in the inner solution is Venezian's solution. Using a 2-term inner expansion and a 4-term outer expansion, the composite solution is

$$\Gamma = \varepsilon^{1/2} \, 2(2/\pi)^{1/2} \, y^{-1/2} \, e^{-S^2/8} \, [\mathrm{erfc}\,(S/8^{1/2})]^{-1} + \varepsilon \, H\,(S) \tag{5.2}$$

where

$$S = x(y/\varepsilon)^{1/2}.$$

The first term is essentially the same as Venezian's solution. H (S) satisfies a linear, 2nd order, ordinary differential equation. It was most convenient to solve that equation numerically, although asymptotic forms were also derived. The solution H (S) is shown in Figure 14.

In Figure 15 V vs r is shown for $\varepsilon = 1/50$ at $t' = 0.80$. The solution (2.10) with the shear discontinuity is shown and the solution to (2.2) with (2.8). Venezian's result, the first term of (5.2), is also presented; his solution is very successful in correcting (2.10) in the neighborhood of the front and for larger radii. As r decreases from the radius of the front, his solution deviates increasingly from the solution to (2.2) and has a minimum. Of course it was intended to apply only near the front. Also shown are a few points calculated from (5.2). To plotting accuracy they are the same as the solution to (2.2) even though the value of ε is not very small. Thus diffusion can be included analytically, giving results essentially the same as (2.2). The front, or shear discontinuity, is at $S = 0$ and from (5.2) its thickness is $O(\varepsilon^{1/2})$ or $O(E^{1/4})$.

VI. CONCLUSIONS

The Wedemeyer model for spin-up of a fluid contained in a cylinder which is impulsively rotated about its axis has been implemented, and numerical solutions to the spin-up equation have been obtained. The approximation to an impulsive start was discussed. A critique of the Ekman compatibility condition based on the Rogers and Lance boundary layer mass flow relation was given; there are compelling reasons for not using it.

The finite difference method of solving the spin-up equation is presented in detail and the results of error studies given. Error due to a discontinuity in boundary conditions, caused by the impulsive start, were avoided by developing a local solution.

A number of results were presented to illustrate the main features of the theory and calculations. Comparison of the results with those from finite difference solutions to the Navier-Stokes equations validate the spin-up theory and numerical method. The effects of including diffusion are adequately shown by various examples but an analytical approach is also derived.

ACKNOWLEDGMENT

The authors wish to thank Ms Joan Bartos for her work on a large number of tasks which made the program used here operational.

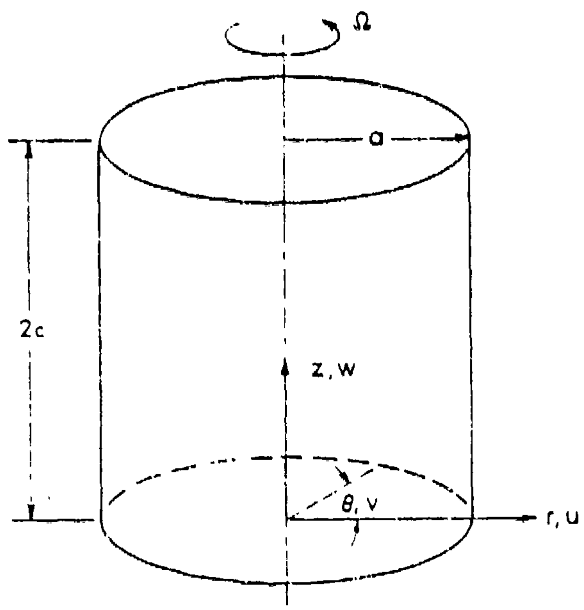

Figure 1. Notation and Coordinate System for the Spinning Cylinder.

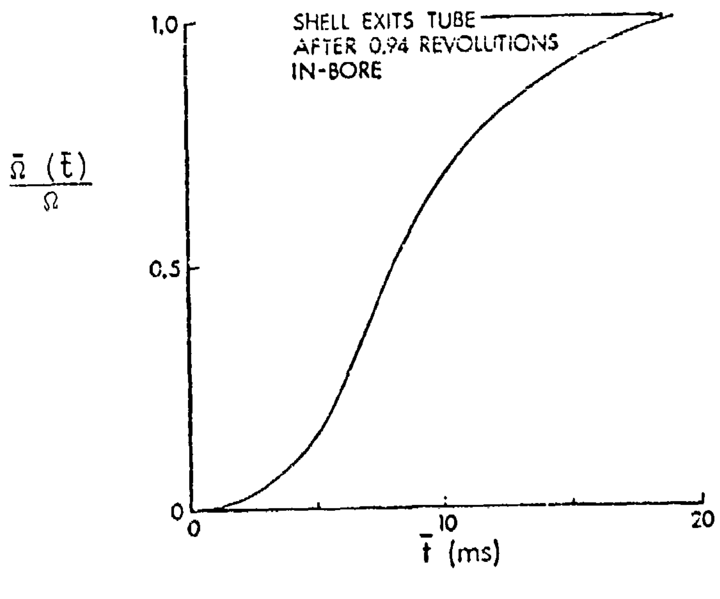

Figure 2. Normalized In-Bore Spin History for an M687 Shell Launched From the M109 Howitzer at Charge 3, Ω = 92.3 Rev/s.

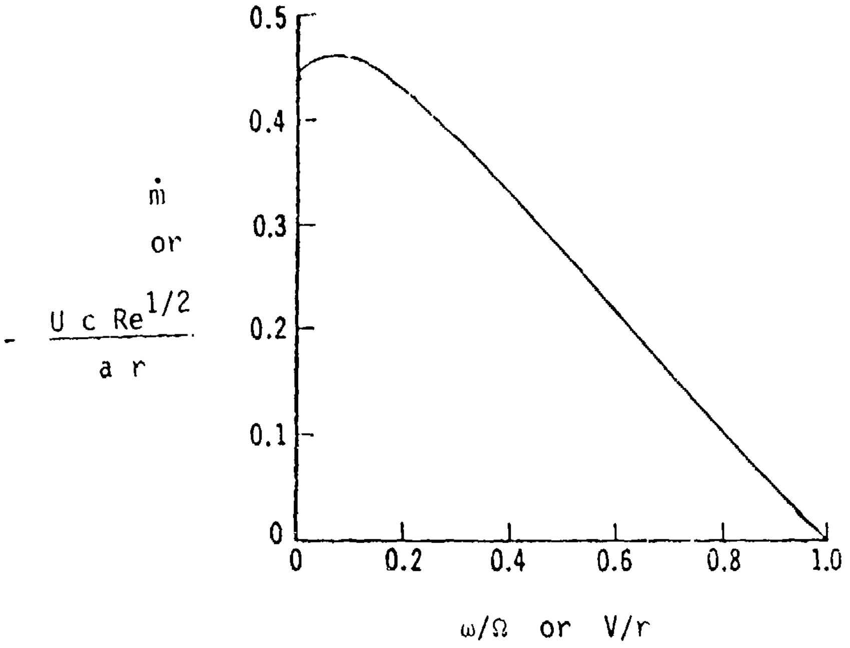

Figure 3. Normalized Radial Mass Flux in the Boundary Layer vs Ratio of Outer Flow Spin to Disk Spin, Reference 13. Also U vs V/r when the RL Curve is Used to Define an EC.

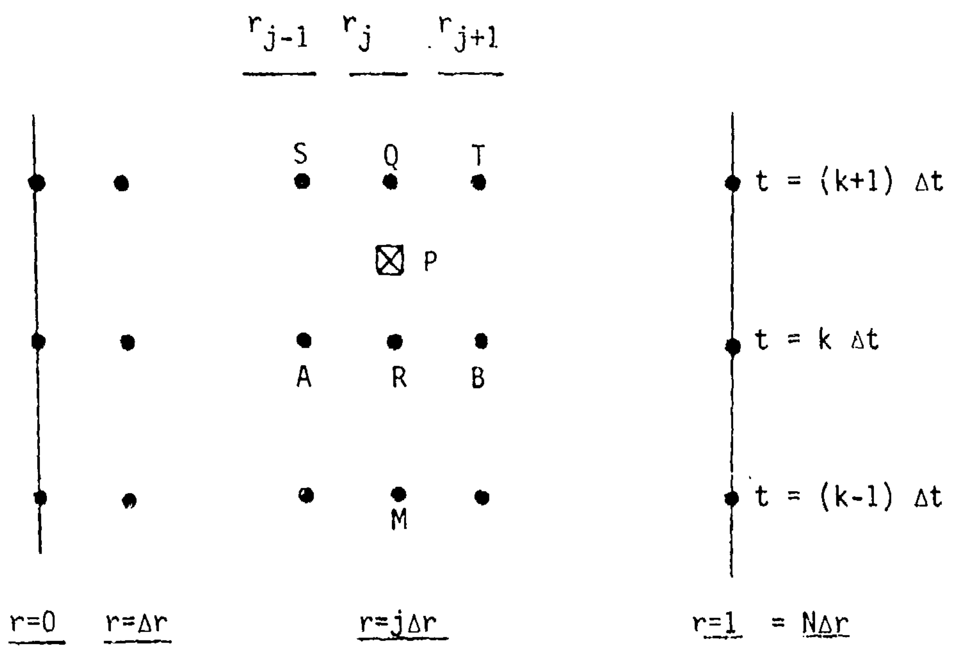

Figure 4. Grid Scheme in (r,t) Plane.

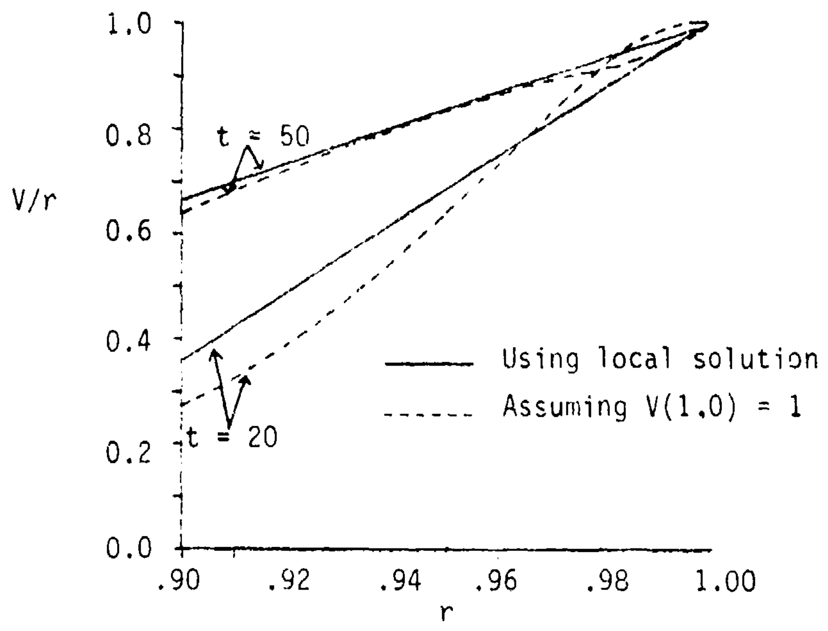

Figure 5. V/r vs r for Re = 4974, c/a = 3.30 at t = 20 and 50 Using Local Solution or Assuming V(1,0)≈1.

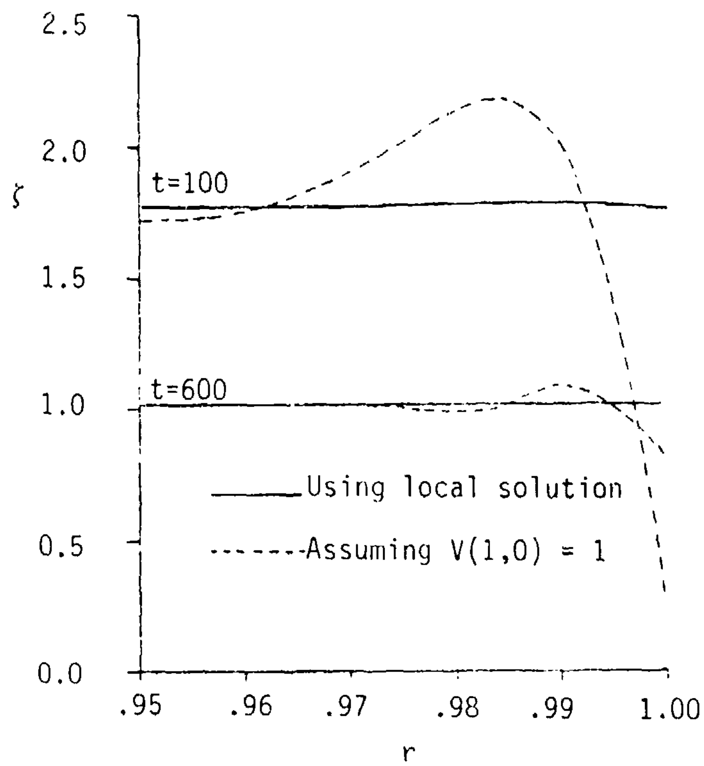

Figure 6. Vorticity ζ vs r for Re = 4974, c/a = 3.30 at t = 100 and 600 Using Local Solution or Assuming V(1,0) = 1.

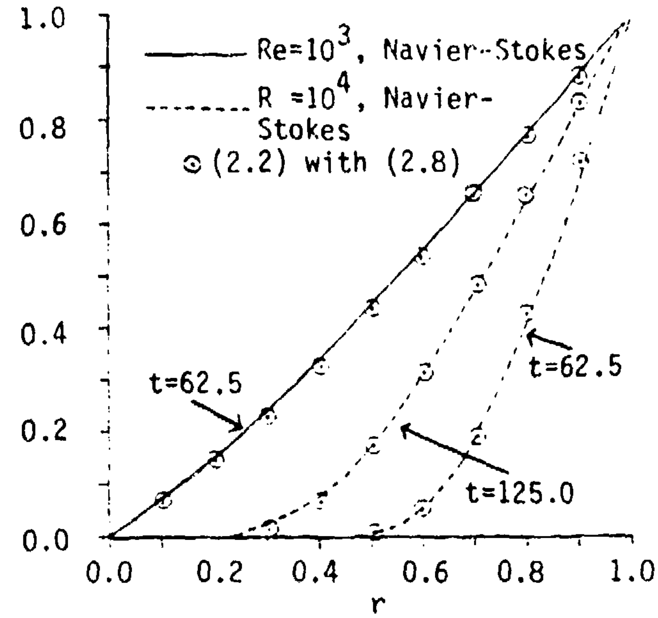

Figure 7. V vs r for Two Re and t With c/a = 1. The Curves are From Navier-Stokes Solutions; Circled Points are Solutions of (2.2) and (2.8).

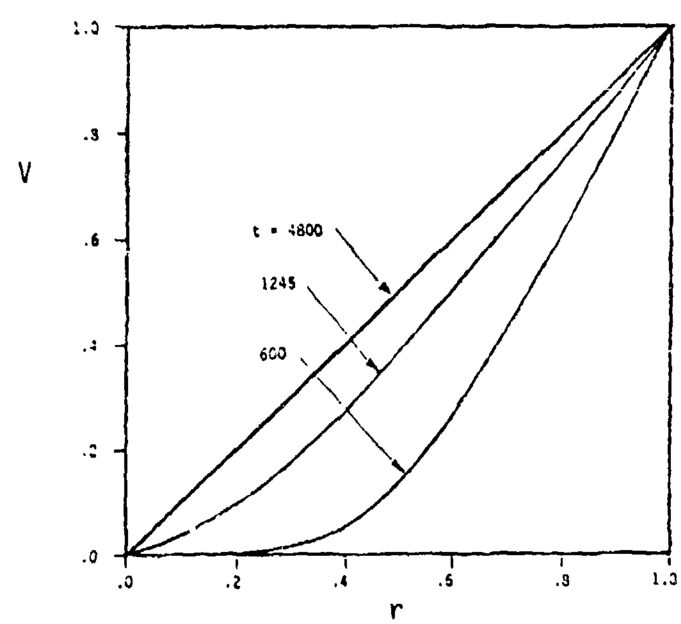

Figure 8. V vs r for Case 1 With c/a = 3.120 Re = 4×10^4 at Three Values of t; t_s = 1245, Laminar Ekman Layers.

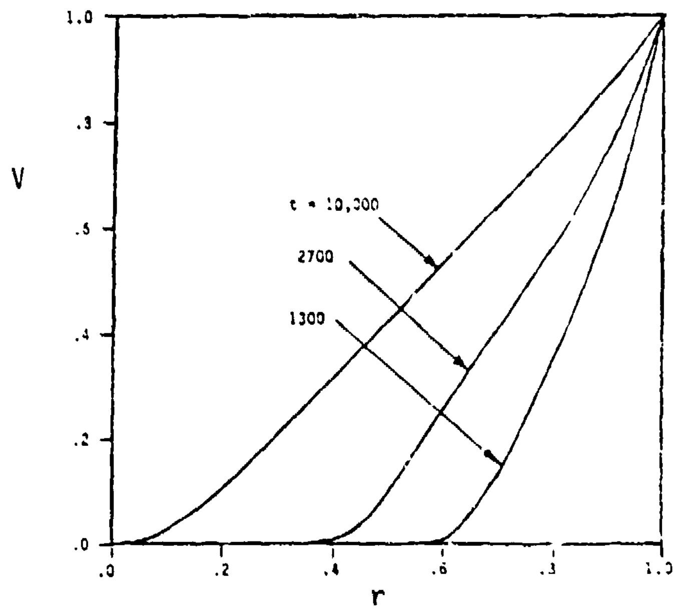

Figure 9. V vs r for Case 2 With c/a = 5.200, Re = 2×10^6 at Three Values of t; t_{st} = 2700, Turbulent Ekman Layers.

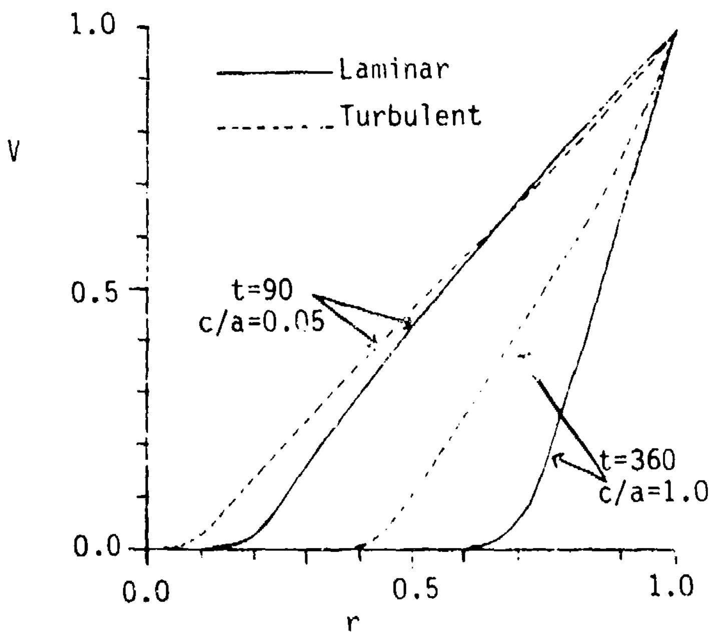

Figure 10. V vs r for Re = 3×10^5 at Two Values of c/a, t With Laminar or Turbulent Ekman Layers.

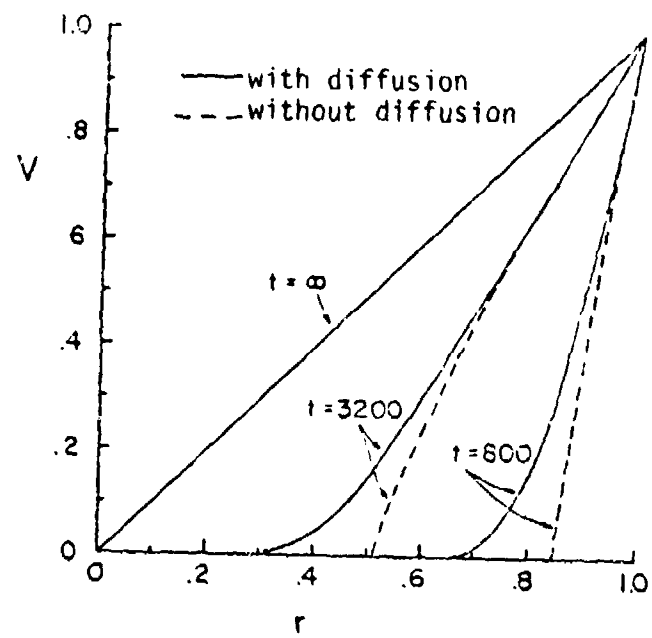

Figure 11. V vs r With Diffusion, (2.2), and Without Diffusion, (2.4), for Re = 6.08×10^5, c/a = 2.679 at Two Values of t, Laminar Ekman Layers.

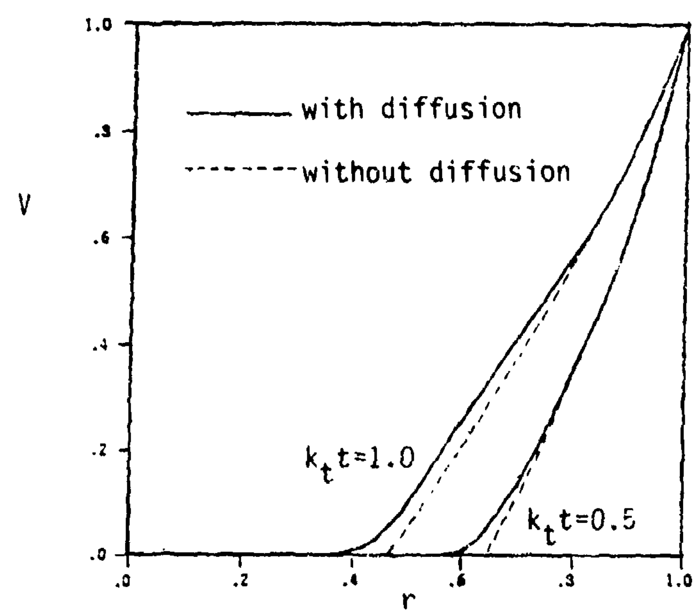

Figure 12. V vs r With Diffusion, (2.2), and Without Diffusion, (2.4), for Re = 2×10^6, c/a = 5.200 at Two Values of $k_t t$; $k_t = 1/t_{st}$ = 1/2700, Turbulent Ekman Layers.

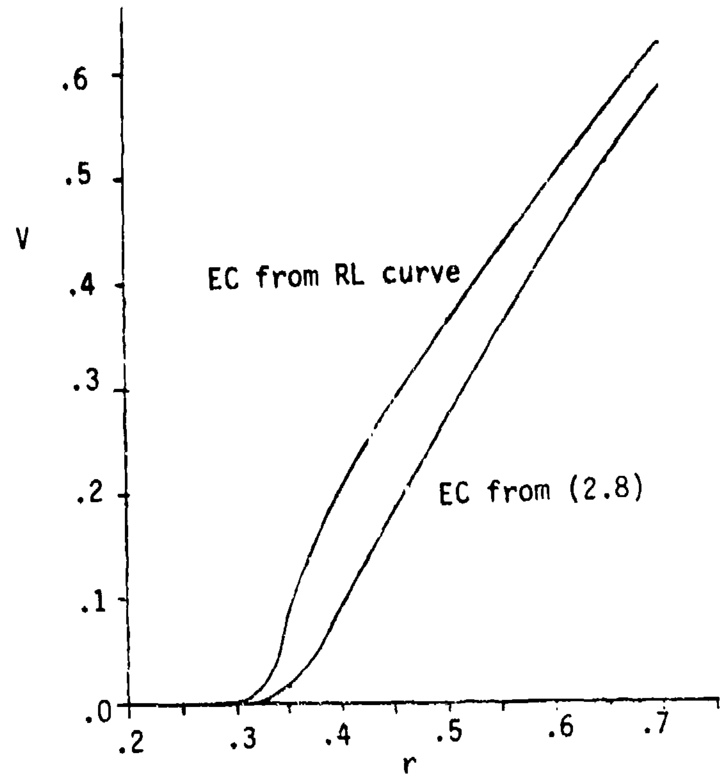

Figure 13. V vs r Using the EC From the RL Curve, Figure 3, and the EC From (2.8) for k_ℓ Re = 5,000, c/a = 0.0485, t = 60, and Δr = 1/200.

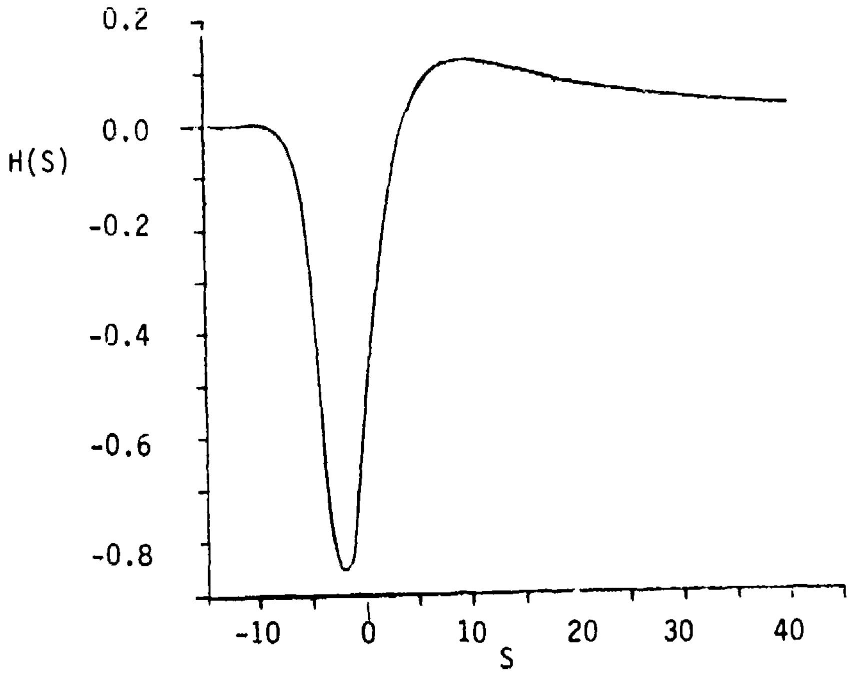

Figure 14. H (S) vs S, the Function in (5.2).

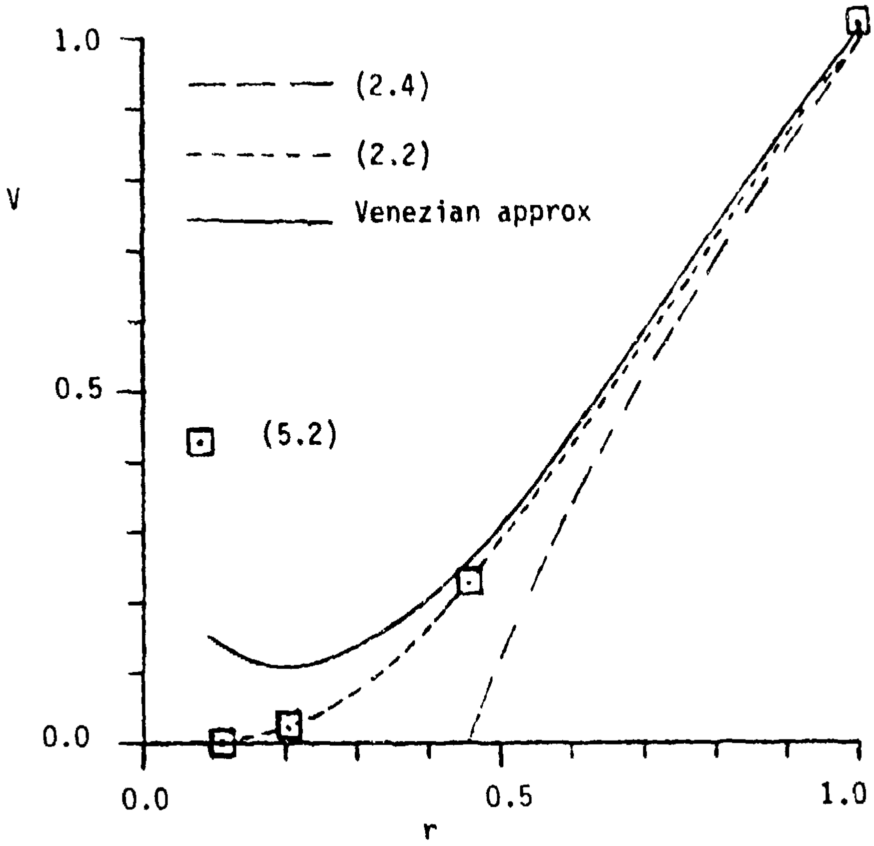

Figure 15. V vs r for $\varepsilon = 1/50$ at $t' = 0.80$ From (2.2) and (2.4). The Venezian Approximation is the First Term of (5.2). The Points are From Both Terms of (5.2).

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APPENDIX A

Coefficients in Eq. (3.7)

APPENDIX A: COEFFICIENTS IN EQ. (3.7)

$$u_{p_j} \equiv -k_\ell \, [j \, \Delta r - (1/2)\,(v_{j,k} + v_{j,k+1})] \qquad \text{-- laminar Ekman layer}$$

$$u_{t_j} \equiv -k_t \, [j \, \Delta r - (1/2)\,(v_{j,k} + v_{j,k+1})]^{8/5} \qquad \text{-- turbulent Ekman layer}$$

$$\Delta_j \equiv (\Delta t/\Delta r)u_{p_j} \,, \qquad D \equiv [\Delta t/(\Delta r)^2]/Re$$

$$a_1 = a_N = 0$$

$$a_j = -(\Delta_j/4) + [\{4(j-1)\}^{-1} - (1/2)]D \qquad \text{for } j \neq 1,N$$

$$b_1 = 1 + (\Delta_1/2) + (3D/2)$$

$$b_N = 1$$

$$b_j = 1 + [\Delta_j/(2j)] + D \qquad \text{for } j \neq 1,N$$

$$c_1 = (\Delta_1/4) - (3D/4)$$

$$c_N = 0$$

$$c_j = (\Delta_j/4) - [\{4(j+1)\}^{-1} + (1/2)]D \qquad \text{for } j \neq 1,N$$

$$d_1 = v_{1,k} - (\Delta_1/4)\,(v_{2,k} + 2v_{1,k}) + (D/4)\,(3v_{2,k} - 6v_{1,k})$$

$$d_N = 1$$

$$d_j = v_{j-1,k}\,\{(\Delta_j/4) + (D/4)\,[2-(j-1)^{-1}]\} + v_{j,k}\,\{1-D-(\Delta_j/[2j])\}$$

$$+ v_{j+1,k}\,\{-(\Delta_j/4) + (D/4)\,[2+(j+1)^{-1}]\} \qquad \text{for } j \neq 1,N$$

LIST OF SYMBOLS

a	cross-sectional radius of cylinder
c	half-height of cylinder
$\mathcal{E}$	$\equiv \nu/\Omega\, c^2$, Ekman number
EC	abbreviation for "Ekman compatibility condition"
j	index indicating r-coordinate of nodal point in finite difference solution ($r = j\ \Delta r$)
k	index indicating time in finite-difference calculation ($t = k\ \Delta t$)
k_ℓ	$\equiv \kappa\ (a/c)\ Re^{-1/2}$ (see (2.8))
k_t	$\equiv 0.035\ (a/c)\ Re^{-1/5}$ (see (2.9))
$\dot{m}$	radial mass flux in boundary layer/$r\ Re^{-1/2}$
N	number of subintervals in r in finite-difference solution
P	pressure/$(\rho a^2 \Omega^2)$ in Wedemeyer model (see (2.3))
P_o	zeroth order approximation, in $(1/t_s)$, to P^* (see (2.6))
P_1	coefficient of first order approximation to P^* in (2.6)
P^*	pressure/$(\rho a^2 \Omega^2)$ in solution to Navier-Stokes equations (see (2.1))
r	radial coordinate/a
R	$\equiv (1 - r)/t'^{1/2}$
Re	$\equiv a^2\Omega/\nu$, Reynolds number
RL	abbreviation for "Rogers-Lance"
S	$\equiv x\ (y/\varepsilon)^{1/2}$
t	time $\times\ \Omega$
t_s	$\bar{t}_s\ \Omega$
t'	$\equiv t/t_s$
$\bar{t}$	time
$\bar{t}_c$	time required for spin of cylinder to reach Ω; i.e., $\bar{\Omega}\ (\bar{t}_c) = \Omega$

$\bar{t}_{fl}$	characteristic time of flight of projectile
$\bar{t}_I$	$\equiv (d\,\bar{\Omega}/dt)^{-1/2}$, acceleration or impulse time
$\bar{t}_s$	$\equiv (2\ c/a)\ Re^{1/2}/\Omega$, characteristic spin-up time for laminar Ekman layer
$\bar{t}_{st}$	$\equiv (28.6\ c/a)\ Re^{1/5}/\Omega$, characteristic spin-up time for turbulent Ekman layer
$\bar{t}_\nu$	$\equiv (a/c)^2/(E\Omega)$, time scale for viscous diffusion of vorticity
$\bar{t}_\Omega$	$\equiv 2\pi/\Omega$, time for one revolution
U, V, W	radial, azimuthal, axial velocity components $\times\ 1/(a\Omega)$ of Wedemeyer model spin-up flow with diffusion (see (2.2) and (2.3))
U_w, V_w	radial and azimuthal velocity components $\times\ 1/(a\Omega)$ of Wedemeyer model spin-up flow without diffusion (see (2.4))
U_1, V_1, W_1	coefficients in first order approximations, in $1/t_s$, to U^*, V^*, W^* (see (2.6))
U^*, V^*, W^*	radial, azimuthal, axial velocity components $\times\ 1/(a\Omega)$ of Navier-Stokes flow (see (2.1))
V_o	zeroth approximation to V^* (see (2.6))
$\tilde{V}_o, \tilde{V}_1$	functions of R in early time solution in Chapter IIIC
x	$\equiv (r^2 e^{2t'} - 1)/(e^{2t'} - 1)$
y	$\equiv e^{2t'} - 1$
z	axial coordinate ($z = 0$ at cylinder midplane)
Γ	$\equiv rV$, circulation
Δr	$= 1/N$, r-interval size in finite-difference solution
Δt	t-interval size in finite-difference solution
ε	$1/(k_\ell\ Re)$
ζ	$\equiv (rV)_r/(2r)$, nondimensional vorticity
θ	azimuthal angle
κ	constant in expression for radial velocity with laminar Ekman layer (see (2.8))

ν	kinematic viscosity of fluid
ρ	density of fluid
Ω	angular velocity of spinning cylinder
$\bar{\Omega}(\bar{t})$	time history of projectile spin

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