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# DAVID W. TAYLOR NAVAL SHIP
# RESEARCH AND DEVELOPMENT CENTER

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FEASIBILITY OF APPLYING THE FEARS PROGRAM
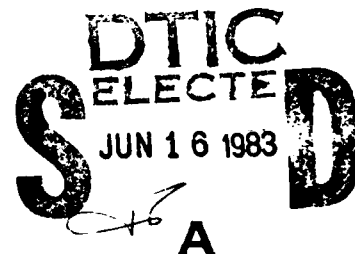TO THE PLATE BENDING PROBLEM

by

Donald A. Gignac
David W. Taylor Naval Ship R&D Center

and

Ivo Babuska
University of Maryland
Contract N00167-82-M-0743

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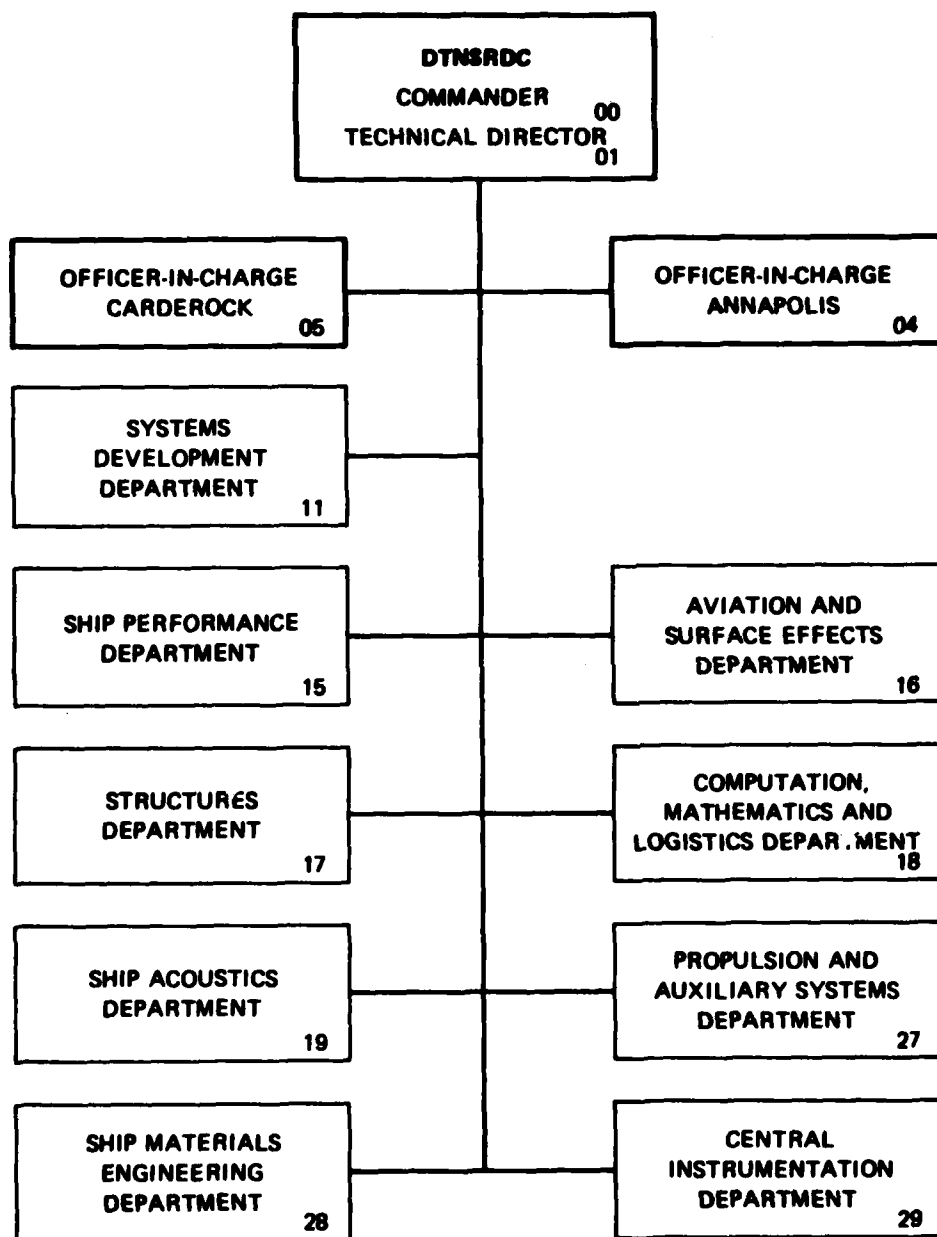DEPARTMENTAL REPORT

May 1983                                    DTNSRDC/CMLD-83/11

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| REPORT DOCUMENTATION PAGE | READ INSTRUCTIONS BEFORE COMPLETING FORM |
|---|---|

| 1. REPORT NUMBER | 2. GOVT ACCESSION NO. | 3. RECIPIENT'S CATALOG NUMBER |
|---|---|---|
| DTNSRDC/CMLD-83/11 | A129395 | |

| 4. TITLE (and Subtitle) | 5. TYPE OF REPORT & PERIOD COVERED |
|---|---|
| FEASIBILITY OF APPLYING THE FEARS PROGRAM TO THE PLATE BENDING PROBLEM | |
| | 6. PERFORMING ORG. REPORT NUMBER |

| 7. AUTHOR(s) | 8. CONTRACT OR GRANT NUMBER(s) |
|---|---|
| Donald A. Gignac Ivo Babuska | |

| 9. PERFORMING ORGANIZATION NAME AND ADDRESS | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS |
|---|---|
| David Taylor Naval Ship Research and Development Center Bethesda, Maryland 20084 | Program Element 61152 Task Area ZR0140201 Work Unit 1844-140 |

| 11. CONTROLLING OFFICE NAME AND ADDRESS | 12. REPORT DATE |
|---|---|
| | May 1983 |
| | 13. NUMBER OF PAGES |
| | 27 |

| 14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) | 15. SECURITY CLASS. (of this report) |
|---|---|
| | UNCLASSIFIED |
| | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

| | | |
|---|---|---|
| Finite Element Method | Adaptive Meshing | Shear Forces |
| Plate Bending Problem | Lame' Equations | |
| FEARS Program | Splitting Method | |
| Mesh Refinement | Biharmonic Equation | |
| Error Estimation | Moments | |

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report documents an investigation of the feasibility of the application of the Finite Element Adaptive Research Solver (FEARS) computer program to the Plate Bending Problem. Two methods of reducing this biharmonic problem to an elliptic system of two second order partial differential equations are considered. The first is the splitting method and the second is the transformation to the Lame' system of elasticity equations. The FEARS program is used to solve these reduced systems for three examples. The

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results are analyzed and discussed with regard to the computation of displacements, moment, and shear forces. While the Lame' system approach is well-founded theoretically, it poses some computational problems with regard to accuracy. The authors feel that the splitting method when monitored by error indicators is the preferred method for FEARS users even though the theory of this method has not been completely developed yet.

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## ADMINISTRATIVE INFORMATION

## INTRODUCTION

The Finite Element Adaptive Research Solver (FEARS) computer program is
an adaptive finite element solver with a posteriori error estimates.[1,2*] It
solves boundary value problems for an elliptic system of two partial differ-
ential equations with two independent variables, using first degree elements
which have continuous displacements across element boundaries. Elements with
such continuous displacements are said to be $C^0$.

The classical plate bending problem is described by a partial differential
equation of the fourth order, the biharmonic equation. It should be possible
to transform this biharmonic problem into an elliptic system of two second

* A complete list of references is given on page 21.

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order partial differential equations, but the problem of obtaining a family of such transformations has not been solved yet for the general case.

The biharmonic problem can be solved through a reduction to a system of second order equations by various methods: the Reissner-Mindlin refined plate formulation [3,4] which provides three fields of variables (displacements and two rotations), the Hermann-Mioshi method [5,6,7,8,9] and the Hermann-Johnson method [5,6,10] which provides a system of four equations. Other methods of solving biharmonic equations are described by Scholz.[11] In addition, the splitting method first considered by Glowinski[12] and Mercier[13] and further developed by others [8,9,11,14,15] provides a system of two second order equations. Finally the biharmonic problem can be transformed into the Lame' system of elasticity equations by exploiting the fact that the Airy function is biharmonic. The last two approaches provide a boundary value problem solvable by the FEARS program.

This report studies the performance of these two last approaches with respect to the FEARS program and assesses their advantages and disadvantages. The second section contains basic information about the FEARS program. The sample problem, the method of transformation to the Lame' equations, and the splitting method are described in the third section. The numerical experiments are described and some conclusions and recommendations are presented in the fourth and fifth sections respectively.

## ADAPTIVE FINITE ELEMENT SOLVER FEARS

The FEARS program has been described in two informal reports.[1,2] It solves boundary value problems for linear self-adjoint elliptic systems of partial differential equations of second order with two field variables in two dimensions. It also provides error estimates for the finite element solution with respect to the unknown exact solution of the problem.

FEARS deals with domains which are the union of the interiors of a small number of open curvilinear rectangles bounded by circular arcs or straight line segments. These curvilinear rectangles are called 2-D domains and are denoted by $D_j^2$, $j=1,\ldots,N_2$. The open arcs are called 1-D domains and are denoted by $D_j^1$, $j=1,\ldots,N_1$. The vertices of the 2-D domains are called 0-D domains and are denoted by $D_j^0$, $j=1,\ldots,N_0$.

The FEARS program solves problems described in the weak form

$$\sum_i \int_{D_i^2} \left\{ \left[\frac{\partial V}{\partial Z}\right]^T A_i \left[\frac{\partial U}{\partial Z}\right] + V^T B_i \left[\frac{\partial U}{\partial Z}\right] + \left[\frac{\partial V}{\partial Z}\right]^T B_i^T U + V^T C_i U \right\} dx_1 dx_2$$

$$+ \sum_j \int_{D_j^1} V^T \gamma_j U \, ds =$$

$$\sum_i \int_{D_i^2} \left\{ \left[\frac{\partial V}{\partial Z}\right]^T D(x_1 x_2) + V^T E(x_1 x_2) \right\} dx_1 dx_2 \qquad (1)$$

$$+ \sum_j \int_{D_j^1} \epsilon_j(s) V ds$$

where

$$U = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

and

$$\frac{\partial U}{\partial Z} = \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} \\[6pt] \dfrac{\partial u_2}{\partial x_1} \\[6pt] \dfrac{\partial u_1}{\partial x_2} \\[6pt] \dfrac{\partial u_2}{\partial x_2} \end{bmatrix}$$

$V$ and $\frac{\partial V}{\partial Z}$ are defined similarly.

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Further

$A_i$ is a 4 x 4 symmetric (constant) matrix,

$B_i$ is a 2 x 4 (constant) matrix,

$C_i$ is a 2 x 2 symmetric (constant) matrix,

$\gamma_j$ is a 2 x 2 symmetric (constant) matrix,

$D_i(x_1,x_2)$ is a 4 x 1 vector valued function with components $S_k(x_1,x_2)$
$(k = 1,2,3,4)$,

$E_i(x_1,x_2)$ is a 2 x 1 vector valued function,

$\epsilon_i(s)$ (where s is arc length) is a 1 x 2 vector valued function.

The trial space used for U and the test space used for V allow the imposition of both essential and natural boundary conditions. For a discussion of the physical significance of the matrices with examples, the reader is referred to the User's Guide.[2]

The admissible error norm for the a posteriori estimate of the error $e = u_{FE}-u_0$ (where $u_{FE}$ and $u_0$ denote the finite element and exact solutions, respectively) is $|||e|||_{2p}$ where

$$|||e|||_{2p} = \left[ \sum_i \int\int_{D_i^2} \left[ \frac{\partial e}{\partial z}^T \left( A_E \right)_i \frac{\partial e}{\partial Z} \right]^p \right]^{\frac{1}{2p}} \tag{2}$$

FEARS uses $C^0$ first degree elements of bilinear type on curvilinear rectangles (which are mapped into bilinear elements on the master square) and adaptively constructs the meshes by refinement where appropriate.

SAMPLE PROBLEM

Let $\Omega \subset R^2$ be a bounded simply connected region with a boundary $\partial\Omega$. The problem is to find a function U satisfying the biharmonic equation

$$\Delta\Delta U = f \quad \text{on } \Omega \tag{3}$$

and the boundary conditions

$$U=g, \quad \frac{\partial U}{\partial n} = h \quad \text{on } \partial\Omega \tag{4}$$

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Assume that there is a smooth function Z such that g=Z and h = $\frac{\partial Z}{\partial n}$ on $\partial\Omega$. U is the deflection of the plate. The second derivatives of U are the bending moments which are of special interest. At present only the problem of a built-in plate with boundary conditions of the type given by Equation (4) will be considered.

## REDUCTION TO THE LAME' SYSTEM OF ELASTICITY EQUATIONS

Assume that f=0 in Equation (3). Then U is a biharmonic function which can be regarded as the Airy function for the plane elasticity problem. Therefore, there exist functions $u_1, u_2$ defined on $\Omega$ such that

$$\frac{\partial^2 U}{\partial x_1^2} = \frac{\partial u_2}{\partial x_2}$$

$$\frac{\partial^2 U}{\partial x_2^2} = \frac{\partial u_1}{\partial x_1} \tag{5}$$

$$-\frac{\partial^2 U}{\partial x_1 \partial x_2} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

and the functions $u_1, u_2$ also satisfy the well-known Lame' equations (for $\nu = 0$ where $\nu$ is Poisson's ratio).

The functions g and h must be transformed into the boundary conditions for $u_1, u_2$. It is not hard to show that $u_1, u_2$ satisfy the traction condition on the boundary. However, it must be emphasized that $\Omega$ is assumed to be simply connected. (If the plate were free, then the transformed problem would provide the prescribed displacements for $u_1, u_2$ on $\partial\Omega$.)

The relationship between the traction and the functions g and h is not complicated, but it involves differentiation along the boundary. The assumption that the functions g and h are defined in terms of the function Z simplifies the transformation. The framework of the FEARS program is especially well suited to take advantage of this assumption.

The weak formulation of the problem for the functions $u_1, u_2$ is a standard one. The functions $u_1, u_2$ satisfy the variational form of the problem (virtual work condition) and are not subject to any essential boundary conditions.

More precisely, $u_1, u_2 \in H^1$ (where $H^1$ is the usual Sobolev space) are functions such that, for $v_1, v_2 \in H^1$,

$$B_1(u_1, u_2; v_1, v_2) = F(v_1, v_2) \qquad (6)$$

Here the bilinear form B is the usual form of virtual work obtained by taking $E=1$, $\nu=0$. (E is Young's modulus of elasticity and $\nu$ is Poisson's ratio.) Then

$$B_1(u_1, u_2; v_1, v_2) = \iint_\Omega \left[ \frac{\partial u_1}{\partial x_1} \frac{\partial v_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \frac{\partial v_2}{\partial x_2} \right.$$
$$\left. + \frac{1}{2}\left( \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right)\left( \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2} \right) \right] dx_1 dx_2 \qquad (7)$$

The functional $F(v_1, v_2)$ expresses the virtual work of the traction forces on the boundary. Integration by parts provides $F(v_1, v_2)$ in terms of the function Z. Thus

$$F(v_1, v_2) = \iint_\Omega \left[ \frac{\partial v_1}{\partial x_1} \frac{\partial^2 z}{\partial x_2^2} - \frac{\partial v_2}{\partial x_1} \frac{\partial^2 z}{\partial x_1 x_2} \right.$$
$$\left. - \frac{\partial v_1}{\partial x_2} \frac{\partial^2 z}{\partial x_1 \partial x_2} + \frac{\partial v_2}{\partial x_2} \frac{\partial^2 z}{\partial x_2^2} \right] dx_1 dx_2 \qquad (8)$$

This functional is of a form which the FEARS program can handle if

$$S_1 = \frac{\partial^2 z}{\partial x_2^2}$$

$$S_2 = S_3 = -\frac{\partial^2 z}{\partial x_1 \partial x_2}$$

$$S_4 = \frac{\partial^2 z}{\partial x_1^2}$$

The conditions given by Equations (6), (7), and (8) determine $u_1, u_2$ uniquely only up to rigid body motion. Accordingly, a displacement is specified at one point and a rotation through another point.

When the functions $u_1, u_2$ have been obtained, the second derivatives of U (the bending moments) are provided by Equation (5). An additional integration must be performed to obtain the values of U. If however, the bending moments are the main concern, then the functions $u_1, u_2$ and Equation (5) provide them directly.

So far it has been assumed $f=0$. Now assume that $f \neq 0$ and $g=h=0$. In this case a particular solution is needed, which often (e.g., for $f=1$) can be easily found. If this particular solution is denoted by Z, then

$$U = Z - W$$

where

$$\Delta\Delta W = 0 \text{ on } \Omega$$

and

$$Z = W, \ \frac{\partial Z}{\partial n} = \frac{\partial W}{\partial n} \text{ on } \partial\Omega.$$

This is exactly the case discussed earlier. Now

$$\frac{\partial^2 U}{\partial x_1^i \partial x_2^j} = \frac{\partial^2 Z}{\partial x_1^i \partial x_2^j} - \frac{\partial^2 W}{\partial x_1^i \partial x_2^j} \ , \ i,j = 0,1,2 \text{ such that } i+j = 2 \qquad (9)$$

This approach has the following advantages and disadvantages:

1) Advantages. The finite element method for solving elasticity problems is well-founded theoretically. The error with respect to the energy

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norm (which is essentially the $L_2$ norm for the bending moments) can be reliably estimated. The postprocessing approach allows the computation of the moments at selected points with generally better accuracy along with an error estimate. Likewise the stress intensity factors can also be computed.

2) Disadvantages. If $f \neq 0$, then a particular solution Z must be known. Obtaining the particular solution Z for a general f could be very laborious. However obtaining Z for a particular f in most applied problems is not that difficult. Often Z can be found by inspection or some artifice. Similarly the computation of the displacements (the values of U) could also be quite laborious. In addition, the accuracy of the numerical solution depends on the selection of the particular solution. Since values of the moments are obtained from Equation (9) by subtracting numbers of approximately the same magnitude, the relative error in the computation of W can result in a much

larger relative error in $\dfrac{\partial^2 U}{\partial x_1^i \partial x_2^j}$. Accordingly, it is necessary to compute

W with greater relative accuracy. If the domain $\Omega$ is not simply connected, then additional difficulties arise.


SPLITTING METHOD

Although the theory of this method has been studied by various authors,[8,9,13,14,15,16] the method is still not very well understood. Today's theoretical results presuppose assumptions which are rather restrictive and it is not clear whether these assumptions are really necessary. These basic assumptions are

    (1) The elements used are at least of degree 2.
    (2) The mesh is quasi-uniform.
    (3) The domain $\Omega$ is convex.


However, these assumptions do not seem necessary. Computational results obtained in the course of this investigation strongly suggest that the assumptions are sufficient and not necessary conditions.

For simplicity, consider the problem given by Equations (3), (4) with g=h=0. The method consists in writing Equation (3) in the form

$$\Delta u_1 = u_2$$

$$-\Delta u_2 = -f$$
(10)

where the boundary conditions are

$$u_1 = 0$$
(11)

$$\frac{\partial u_1}{\partial n} = 0$$
(12)

The bilinear form is

$$B_2(u_1,u_2; v_1,v_2) =$$
$$\iint_\Omega \left( \frac{\partial u_1}{\partial x_1} \frac{\partial v_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \frac{\partial v_2}{\partial x_2} + u_2 v_2 \right.$$
$$\left. + \frac{\partial u_2}{\partial x_1} \frac{\partial v_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \frac{\partial v_1}{\partial x_2} \right) dx_1 dx_2$$
(13)

and the functional is

$$F_2(v_1,v_2) = -\iint_\Omega f v_1 \, dx_1 dx_2$$
(14)

Then the problem can be written in the standard weak form

$$B_2(u_1,u_2; v_1,v_2) = F_2(v_1,v_2)$$
(15)

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with

$$u_1, v_1 \in \overset{0}{H}{}^1, \quad u_2, v_2 \in H^1 \quad \text{where} \quad \overset{0}{H}{}^1 = \left\{ u \in H^1 \mid u = 0 \text{ on } \partial\Omega \right\}$$

The boundary condition given by Equation (11) is essential and the boundary condition given by Equation (12) is natural. The bilinear form $B_2$ is not positive definite and Equation (15) is a "saddle point" problem.

The bilinear form $B_2$ and the functional $F_2$ can be used by the FEARS program. However, since FEARS employs elements of degree 1, the first condition is violated. The numerical examples of this investigation also violate the second and third conditions.

There are various options for error norms in the FEARS program. The norms

$$||u||_1^2 = \iint\limits_{\Omega} \left[ \left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_1}{\partial x_2}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_2}\right)^2 \right] dx_1 dx_2 \qquad (16)$$

or

$$||u||_2^2 = \iint\limits_{\Omega} \left[ \left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_1}{\partial x_2}\right)^2 \right] dx_1 dx_2 \qquad (17)$$

can be used. For a nonconvex domain the norm $||u||_1$ of the exact solution u is infinite. If the solution is smooth then either of the norms $||.||_1$ or $||.||_2$ may be used. If the solution has singular behavior, then only the norm $||.||_2$ can be used.

This approach has the following several advantages and disadvantages:

1) Advantages. The problem can be solved by FEARS without obtaining a particular solution. This approach avoids the difficulty stemming from the arbitrary selection of a particular solution and it also avoids the problem of

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significance due to subtraction encountered with the previous method. The method directly computes the displacement and its Laplacian. The assumption that the domain is simply connected is not required for this method.

2) Disadvantages. The method is not completely understood theoretically. The saddle point approach does not guarantee monotonic convergence (which occurs when the bilinear form is positive definite). At present the convergence of the method has not been established under the conditions of the FEARS computation. The second derivatives (which are the moments) must be computed by postprocessing. The error norms used by FEARS do not monitor the accuracy of the moments.

## NUMERICAL EXPERIMENTATION

Consider the three examples given by Equations (3) and (4) where $f=64$ on domains $\Omega_1$, $\Omega_2$, $\Omega_3$ and $g=h=0$ on their boundaries $\partial\Omega_1$, $\partial\Omega_2$, $\partial\Omega_3$.

Example 1 - Uniformly Loaded Circular Plate

$$\Omega_1 = \left\{ (x_1, x_2) \mid x_1 + x_2 < 1 \right\}$$

$\Omega_1$ is partitioned into the five 2-D domains shown in Figure 1.
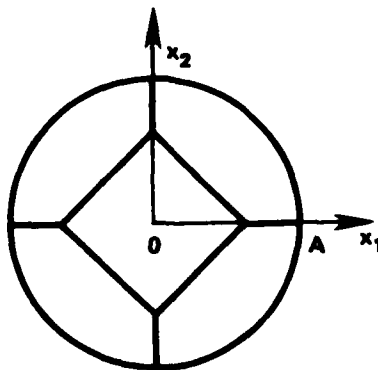


Figure 1 - Partition of Domain $\Omega_1$

The exact solution is known to be

$$U = (1-(x_1 + x_2))^2$$

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**Example 2 - Uniformly Loaded Square Plate**

$$\Omega_2 = \left\{ (x_1, x_2) \mid |x_i| < 1, \; i = 1, 2 \right\}$$

Figure 2 shows the unpartitioned domain $\Omega_2$.
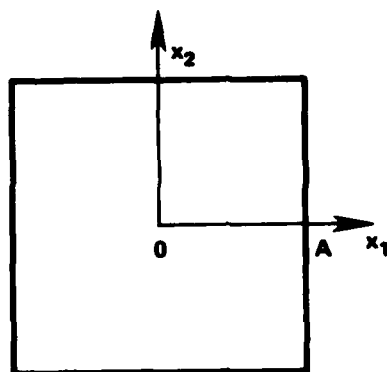
Figure 2 - Domain $\Omega_2$

The solution is not available in closed form though there are analytical expressions for U in series form.

**Example 3 - Uniformly Loaded Square Plate With a Slit**

$$\Omega_3 = \Omega_2 - \left\{ (x_1, x_2) \mid -1 < x_1 < 0, \; x_2 = 0 \right\}$$

3 is partitioned into four equal squares as shown in Figure 3.
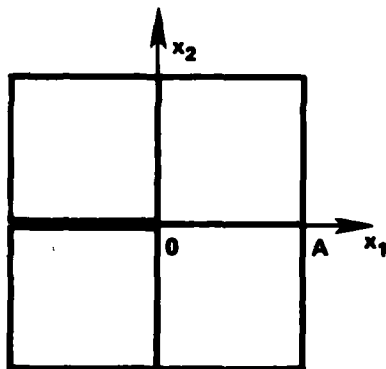
Figure 3 - Partition of Domain $\Omega_3$

The exact solution is not available for Example 3.

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The exact solution is very smooth for Example 1, fairly smooth at the corners for Example 2, and has a strong singularity at the origin for Example 3. The trial and test functions are bilinear for Examples 2 and 3, and for Example 1 they are mapped onto bilinear functions. These examples have different axes of symmetry which were utilized to a limited extent, but the material in the tables is organized as though no symmetries were present. The two approaches of the third section will now be discussed.

## REDUCTION TO THE LAME' EQUATIONS OF ELASTICITY

We use the following functions $Z$ and $S_i$.

$$Z^{(1)} = \left( x_1^2 + x_2^2 \right)^2$$

$$S_1^{(1)} = 4x_1^2 + 12x_2^2$$

$$S_2^{(1)} = S_3^{(1)} = -8x_1 x_2$$

$$S_4^{(1)} = 12x_1^2 + 4x_2^2$$

$$Z^{(2)} = 8/3x_1^4$$

$$S_1^{(2)} = S_2^{(2)} = S_3^{(2)} = 0$$

$$S_4^{(2)} = 32x_1^2$$

$$Z^{(3)} = 8x^2 x^2$$

$$S_1^{(3)} = 16x_1^2$$

$$S_2^{(3)} = S_3^{(3)} = -32x_1 x_2$$

$$S_4^{(3)} = 16x_2^2$$

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The method provides the expected results. The selection of Z barely influenced the relative accuracy of the computed function W with respect to the energy norm when the same number of elements was used. The error estimation was very effective. The main problem of the method is related to the subtraction of the values of Z and W. Table 1 shows typical results for Example 3 ($Z = Z^{(1)}$) with a mesh of 1024 elements. This table displays the values of the second derivatives of the functions Z and W at the point (.53125, .53125).

TABLE 1 – THE VALUES OF THE SECOND PARTIAL DERIVATIVES OF Z AND W
AT THE POINT (.53125, .53125) (EXAMPLE 3, $Z = Z^{(1)}$)

|   | $\dfrac{\partial^2}{\partial x_1^2}$ | $\dfrac{\partial^2}{\partial x_2^2}$ | $\dfrac{\partial^2}{\partial x_1 \partial x_2}$ |
|---|---|---|---|
| W | 5.89848 | 5.72560 | 1.81340 |
| Z | 4.51562 | 4.51562 | 2.25781 |

Now an accuracy of the order of 5% in $\dfrac{\partial^2 U}{\partial x_i x_j}$ , where U=Z–W, means that W must have an accuracy in the neighborhood of 1–2%. This restriction on W could be prohibitive. This problem arises in the most important case when $f \neq 0$. If f=0, then this problem is usually not so severe.


SPLITTING METHOD

As noted in the previous section, convergence is not theoretically guaranteed for this method. The solution was computed using both a uniform mesh and an adaptive mesh with respect to the error estimator of the $||\cdot||_2$ norm given by Equation (17). The results are displayed in Tables 2-5. In these tables the computed values of $u_1$ and $u_2$ on the line OA are given. (As the meshes are refined, more values on OA are computed.) The headings of these tables are as follows:


(1) The $x_1$- coordinate of the point $(x_1,0)$

(2) The computed value of $u_1(x_1,0)$

(3) The computed value of $u_2(x_1,0)$


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TABLE 2 - EXAMPLE 1: UNIFORM MESH

| N $x_1$ | 80 $u_1(x_1,0)$ | 80 $u_2(x_1,0)$ | 320 $u_1(x_1,0)$ | 320 $u_2(x_1,0)$ | 1280 $u_1(x_1,0)$ | 1280 $u_2(x_1,0)$ | Exact $u_1(x_1,0)$ | Exact $u_2(x_1,0)$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 1.00497 | -8.15913 | 1.00127 | -8.04037 | 1.00032 | -8.01013 | 1.000000 | -8.000000 |
| .06250 | | | | | .992521 | -7.94763 | .992203 | -7.93750 |
| .12500 | | | .970230 | -7.79035 | .969305 | -7.76012 | .968994 | -7.75000 |
| .18750 | | | | | .931220 | -7.44757 | .930923 | -7.43750 |
| .25000 | .883213 | -7.15635 | .879993 | -7.03965 | .879173 | -7.00995 | .878906 | -7.00000 |
| .31250 | | | | | .814460 | -6.44717 | .814224 | -6.43750 |
| .37500 | | | .739237 | -5.78591 | .738704 | -5.75909 | .738525 | -5.75000 |
| .43750 | | | | | .653912 | -4.94525 | .653824 | -4.93750 |
| .50000 | .562680 | -4.09640 | .562265 | -4.01785 | .562365 | -4.00290 | .562500 | -4.00000 |
| .53125 | | | | | .515320 | -3.49129 | .51599 | -3.48438 |
| .56250 | | | .467998 | -2.96921 | .467483 | -2.94548 | .467300 | -2.93750 |
| .59375 | | | | | .419387 | -2.36741 | .419206 | -2.35938 |
| .62500 | .374055 | -1.87763 | .371971 | -1.78068 | .371494 | -1.75762 | .371338 | -1.75000 |
| .65625 | | | | | .324264 | -1.11629 | .324143 | -1.10938 |
| .68750 | | | .278453 | -.461889 | .278175 | -.443470 | .278091 | -.437500 |
| .71875 | | | | | .233721 | .260799 | .233674 | .265625 |
| .75000 | .192019 | .931318 | .191492 | .985424 | .191421 | .996522 | .191406 | 1.00000 |
| .78125 | | | | | .151813 | 1.76373 | .151826 | 1.76563 |
| .81250 | | | .115372 | 2.56157 | .115460 | 2.56247 | .115494 | 2.56250 |
| .84375 | | | | | .0829456 | 3.39288 | .0829935 | 3.39063 |
| .87500 | .0540231 | 4.29448 | .0547048 | 4.26842 | .0548768 | 4.25515 | .0549316 | 4.25000 |
| .90625 | | | | | .0318831 | 5.14975 | .031975 | 5.14063 |
| .93750 | | | .0144603 | 6.11049 | .0146170 | 6.07757 | .0146637 | 6.06250 |
| .96875 | | | | | .00375493 | 7.04051 | .00378513 | 7.01563 |
| 1.0000 | 0 | 8.23284 | 0 | 8.09769 | 0 | 8.04265 | 0 | 8.00000 |

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TABLE 3 - EXAMPLE 2: UNIFORM MESH

| N | 64 | | 256 | | 1024 | |
|---|---|---|---|---|---|---|
| $x_1$ | $u_1(x_1,0)$ | $u_2(x_1,0)$ | $u_1(x_1,0)$ | $u_2(x_1,0)$ | $u_1(x_1,0)$ | $u_1(x_1,0)$ |
| 0 | 1.35644 | -9.61960 | 1.31121 | -9.17025 | 1.29959 | -9.05834 |
| .0625 | | | | | 1.29076 | -8.99565 |
| .1250 | | | 1.27571 | -8.91722 | 1.26440 | -8.80644 |
| .1875 | | | | | 1.22086 | -8.48732 |
| .2500 | 1.21137 | -8.57152 | 1.17114 | -8.13994 | 1.16078 | -8.03261 |
| .3125 | | | | | 1.08506 | -7.43436 |
| .3750 | | | 1.00384 | -6.78387 | .994969 | -6.68236 |
| .4375 | | | | | .892184 | -5.76413 |
| .5000 | .812358 | -5.13491 | .785777 | -4.75807 | .778854 | -4.66495 |
| .5625 | | | | | .657709 | -3.36783 |
| .6250 | | | .536873 | -1.93536 | .532167 | -1.85355 |
| .6875 | | | | | .406453 | -1.00715 |
| .7500 | .297489 | 1.57309 | .288256 | 1.84719 | .285755 | 1.91421 |
| .8125 | | | | | .176378 | 4.21680 |
| .8750 | | | .0866609 | 6.78649 | .0859307 | 6.83454 |
| .9375 | | | | | .0235262 | 9.79657 |
| 1,0000 | 0 | 13.0037 | 0 | 13.1088 | 0 | 13.1334 |

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TABLE 4 - EXAMPLE 3: UNIFORM MESH

| N | 64 | | 256 | | 1024 | |
|---|---|---|---|---|---|---|
| $x_1$ | $u_1(x_1,0)$ | $u_2(x_1,0)$ | $u_1(x_1,0)$ | $u_2(x_1,0)$ | $u_1(x_1,0)$ | $u_2(x_1,0)$ |
| 0 | 0 | 14.0294 | 0 | 20.5651 | 0 | 29.0652 |
| .0625 | | | | | .0366752 | 9.02679 |
| .1250 | | | .097985 | 3. 21982 | .0856739 | 4.39889 |
| .1875 | | | | | .133943 | 1.45009 |
| .2500 | .199010 | -1.94006 | .185872 | -.756482 | .176257 | -.427232 |
| .3125 | | | | | .209127 | -1.66725 |
| .3750 | | | .239824 | -2.66616 | .230487 | -2.44152 |
| .4375 | | | | | .239298 | -2.84153 |
| .5000 | .259715 | -3.50934 | .243199 | -3.06997 | .235401 | -2.91838 |
| .5625 | | | | | .219446 | -2.70160 |
| .6250 | | | .198364 | -2.30707 | .192882 | -2.20784 |
| .6875 | | | | | .157971 | -1.44573 |
| .7500 | .127328 | -.718524 | .120795 | -.468820 | .117826 | -.418792 |
| .8125 | | | | | .0764653 | .872537 |
| .8750 | | | .0397549 | 2.43183 | .0388672 | 2.42922 |
| .9375 | | | | | .0110272 | 4.25203 |
| 1 | 0 | 6.41809 | 0 | 6.40323 | 0 | 6.33992 |

## TABLE 5 - EXAMPLE 3: THE ADAPTIVE MESH

| N | 64 | | 208 | | 634 | |
|---|---|---|---|---|---|---|
| $x_1$ | $u(x_1,0)$ | $u_2(x_1,0)$ | $u_1(x_1,0)$ | $u_2(x_1,0)$ | $u_1(x_1,0)$ | $u_2(x_1,0)$ |
| 0 | 0 | 14.0294 | 0 | 20.5741 | 0 | 29.0727 |
| .0625 | | | | | .0366943 | 9.02608 |
| .1250 | | | .0918232 | 3.21720 | .0857160 | 4.39572 |
| .1875 | | | | | .134002 | 1.44502 |
| .2500 | .199010 | -1.94006 | .185894 | -.760749 | .176321 | -.433832 |
| .3125 | | | | | .209179 | -1.67517 |
| .3750 | | | .239801 | -2.67162 | .230509 | -2.45064 |
| .4375 | | | | | .239270 | -2.85175 |
| .5000 | .259715 | -3.50934 | .243121 | -3.07583 | .235305 | -2.92978 |
| .5625 | | | | | .219247 | -2.71465 |
| .6250 | | | .198263 | -2.31303 | .192620 | -2.21778 |
| .7500 | .127328 | -.718524 | .120724 | -.475214 | .117593 | -.429559 |
| .8125 | | | | | .0762835 | .858996 |
| .8750 | | | .039362 | 2.42498 | .0387926 | 2.41697 |
| .9375 | | | | | .0110087 | 4.24102 |
| 1 | 0 | 6.41809 | 0 | 6.40290 | 0 | 6.33114 |

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(4) The exact value of $u_1(x_1,0)$ (when available)

(5) The exact value of $u_2(x_1,0)$ (when available)

(6) Number of elements N on the full domain (without using symmetry)

Table 2 shows the results for Example 1. In this case the exact solution is known. Good results have been obtained especially with respect to the value of $u_1$ (the displacement). The accuracy of the function $u_2$ is also good, but not quite as good as that of $u_1$. Because the solution is smooth, the adaptive mesh is uniform.

Table 3 shows the results for Example 2. The solution is smooth so the adaptively constructed mesh is uniform. The exact solution is not available, but the results improve as the number of elements increases. As before, the function $u_1$ is more accurate than the function $u_2$. This is to be expected since $u_2 = \Delta u_1$.

Table 4 shows the results for Example 3 when a uniform non-adaptive mesh is used. The exact solution has a strong singularity at the origin and $u_2(x_1,0) \to \infty$ as $x_1 \to 0$. The exact solution is not available. Once more the results are good especially with respect to $u_1$.

Table 5 gives the results for Example 3 when an adaptive mesh is used. Again the method provides good results. The accuracy for an adaptive mesh of 634 elements is comparable to the accuracy provided by a uniform mesh of 1024 elements.

The results displayed in these tables are typical. Other test computations were performed which tend to confirm the general behavior shown in these tables.

## CONCLUSIONS

Although convergence of the splitting method is theoretically guaranteed at present only under fairly stringent assumptions, it seems to work very well also when these assumptions are not satisfied. The method is a mixed one of the "saddle point" type. Its performance is very good with respect to the displacement $u_1$ and slightly worse with respect to $u_2$ or $u_1$. It is apparent that the method is reliable, particularly if it is controlled by an error estimator. Accordingly it is recommended for use.

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A reliable extraction of the moments and shear forces from the computed results would appear to require postprocessing. This problem should be addressed. It is hoped that the convergence of the method will be theoretically established under less stringent assumptions that those imposed at present and will apply to the examples tested in this report. The splitting method is preferable to the reduction to the Lame' elasticity equations.

The method based on the reduction to the Lame' elasticity equations is very well founded theoretically, but practically has some essential disadvantages which makes it less attractive. The main disadvantages of this method are:

- The need for a particular solution.
- The loss of accuracy and increased error due to the subtraction of the particular solution.
- The necessity of assuming that the domain is simply connected. (This restriction simplifies the computational effort considerably.)
- The method does not directly provide the displacements.

It would appear that the splitting method is the preferred method for the FEARS user.

## REFERENCES

1. Gignac, D., I. Babuska, and C. Mesztenyi, "An Introduction to the FEARS Program," Report DTNSRDC/CMLD-83/04 (Feb 83).

2. Mesztenyi, C.K., "The FEARS User's Manual for the CYBER 7400 at DTNSRDC," Report DTNSRDC/CMLD- . (to appear)

3. Reissner, E., "The Effect of Transverse Shear Deformation on the Bending of Elastic Plates," Journal of Applied Mechanics, Vol. 12, A-69-A-77, (June, 1945).

4. Reissner, E., "On the Theory of Bending of Elastic Plates," Journal of Mathematics and Physics, Vol. 23, pp. 184-191, (1944).

5. Hermann, L., "Finite Element Bending Analyses for Plates," Journal of Engineering Mechanics Division, ASCE, Vol. 93, EM 5, pp. 49-83 (1964).

6. Hermann, L., "A Bending Analysis for Plates," Matrix Methods in Structural Mechanics, AFFDL-TR-66-80, Wright-Patterson Air Force Base, Ohio, pp. 577-604 (1966).

7. Meoshi, T., "A Finite Element Method for the Solution of Fourth Order Partial Differential Equations," Kunomoto Journal of Science (Math.), Vol. 9, pp. 87-116 (1973).

8. Babuska, I., J. Osborn, and J. Pitkaranta, "Analyses of Mixed Methods Using Mesh Dependent Norms," Mathematics of Computation , Vol. 35, pp. 1039-1062 (1980).

9. Folk, R. and J. Osborn, "Error Estimates for Mixed Methods," Recherche Operationelle/Operations Research (Revue Francois d'Automatique Informatique Recherche Operationelle), Vol. 14, pp. 249-277 (1980).

10. Johnson, C., "On the Convergence of Mixed Finite Element Methods for Plate Bending Problems," Numerische Mathematik, Vol. 21, pp. 397-431, (1976).

11. Scholz, R., "A Mixed Method for 4th Order Problems Using Linear Finite Elements," Recherche Operationelle/Operations Research (Revue Francois d'Automatique Informatique Recherche Operationelle), Vol. 12, pp. 85-90 (1978).

12.   Glowinski, R., "Approximations externes par elementes finis de Lagrange d' order un et deux, du probleme de Dirichlet pour l'operateur biharmonique.  Methodes iteratives de resolution des problems approches," Topics in Numerical Analyses, T.T. H. Miller, Editor, pp. 123-171, Academic Press, New York, 1973.

13. Mercier, B., "Numerical Solution of the Biharmonic Problem by Mixed Finite Elements of Class $C^0$," Bollettino Unione Matematica Italiana, Bologna, Vol. 10, pp. 133-140 (1974).

14.   Ciarlet, P. and P. Raviart, "A Mixed Finite Element Method for the Biharmonic Equation," Symposium on Mathematical Aspects of Finite Elements in Partial Differential Equations, C.  deBoor, Editor, pp. 125-143.  Academic Press, New York, 1974.

15.   Bramble, J.H. and R.S. Folk, "Two Mixed Element Methods for the Simply Supported Plate Problem," Lectures on the Numerical Solution of Partial Differential Equations, Lecture Notes No. 20, pp. 31-46.  Department of Mathematics, University of Maryland (College Park), 1981.

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