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Theory of Space Charge Between Parallel Plane Electrodes

CHRISTOPHER SHERMAN

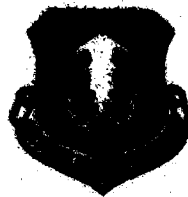12 December 1982

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Chief Scientist

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
The theory of space charge between parallel plane electrodes is treated as a boundary value problem. This treatment leads in a natural way to the organization and classification of the many-valued solutions to this problem.

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Preface

I thank Pat Bench for programming the numerical solutions.

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Theory of Space Charge Between Parallel Plane Electrodes

1. INTRODUCTION

The theory of space charge between parallel plane electrodes has been treated by numerous authors beginning with Child's work in 1911[1] and culminating in a fairly detailed paper in 1938 by Fay, Samuel, and Shockley.[2] *The present report differs from these earlier publications in a major respect; namely that the earlier papers treat an initial value problem, while the current work treats a boundary value problem. From a practical standpoint, if similar ranges of parameters are covered, this difference is of small consequence, since for equivalent families of solutions the information contained is equivalent. From the mathematical point of view, however, the treatment as a boundary value problem differs substantially from the initial value treatment*

The planar space charge problem is one of the simplest naturally occurring problems in electrostatics. Since most problems in electrostatics are formulated by necessity as boundary value problems, it is of interest to solve the planar space charge problem in this manner. The various types of solutions are also more easily organized and classified with this treatment. Finally, such a solution serves as a rare prototype which can be solved analytically to solutions of more difficult problems.

1. Child, C.D. (1911) Phys. Rev. 32:492.

2. Fay, C.E., Samuel, A.L., and Shockley, W. (1938) Bell Systems Technical Journal, 17:49.

2. FORMAL CLASSIFICATION OF THE SOLUTIONS OF THE SPACE CHARGE EQUATION

The space charge equation for a plane, written in MKS units is

$$\epsilon_0 \frac{d^2 v}{dx^2} = n e = J \left(2e/M (v_0 + v) \right)^{-1/2} \tag{1}$$

where

v = electric potential,

n = charged particle number density,

e = electronic charge,

J = current density,

M = particle mass,

x = spatial coordinate,

ϵ_0 = permittivity of free space,

v_0 = initial energy per unit charge of emitted particle.

Let

$$y = x/L, \quad A = 9/4 \; \frac{J L^2 \sqrt{\frac{M}{2e}}}{\epsilon_0 v_0^{3/2}}, \quad \varphi = v/v_0$$

where L = distance between electrodes.

Then

$$\frac{d^2 \varphi}{dy^2} = \frac{4/9 \, A}{\sqrt{1 + \varphi}} . \tag{2}$$

Integrating twice, we have

$$\frac{d\varphi}{dy} = \pm (2 C + 16/9 \, A \sqrt{1 + \varphi})^{1/2} \tag{3}$$

and

$$\frac{4/9 \, A \sqrt{1 + \varphi} - C}{16/27 \, A^2} \; (2 C + 16/9 \, A \sqrt{1 + \varphi})^{1/2} \pm y + C' \tag{4}$$

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where C and C′ are constants of integration. Let $\varphi = 0$ when $y = 0$. Then

$$\frac{4/9\,A\,\sqrt{1+\varphi}-C}{16/27\,A^2}\,(2C+16/9\,A\,\sqrt{1+\varphi})^{1/2} = \pm\; y + \frac{4/9\,A - C}{16/27\,A^2}\,(2C+16/9\,A)^{1/2}.$$

(5)

The common space charge limited solution sets $\frac{d\varphi}{dy} = 0$ when $y = 0$. Here, we treat the general case by letting $\varphi = \varphi_0$ when $y = 1$. There are then three distinct classes of solutions which must be distinguished:

(I)   A potential minimum for $0 \le y \le$ does not exist.

(II)   A potential minimum at $0 \le y_M \le 1$ does exist, the corresponding value of potential φ_M is greater than -1.

(III) A potential minimum at $0 \le y_M \le 1$ exists; the corresponding potential $\varphi_M = -1$; and the current divides at $y = y_M$.

We examine each of these in turn.

For class I, a single solution, with either the + ($\varphi_0 > 0$) or - ($\varphi_0 < 0$) sign is continued from $y = 0$ to $y = 1$. The imposition of $\varphi = \varphi_0$ when $y = 1$ then leads to the following equation for the determination of C

$$\frac{4/9\,A\,\sqrt{1+\varphi_0}-C}{16/27\,A^2}\,(2C+16/9\,A\,\sqrt{1+\varphi_0})^{1/2} = \pm\; 1 + \frac{4/9\,A - C}{16/27\,A^2}\,(2C+16/9\,A)^{1/2}.$$

(6)

If this equation is cleared of fractional powers it results in a cubic in C,

$$2\,C^3 - (1 - B^2)^2\,C^2 - 16/27\,A\,[2\,A + (1 - B^2)\,(B^3 - 1 - A)]\,C$$

$$+ (8/27)^2\,A^2\,[4\,A - (B^3 - 1 - A)^2] = 0 \qquad (7)$$

where

$$B \equiv \sqrt{1+\varphi_0}$$

Eq. (7) contains implicitly roots corresponding to the $\pm$ sign appearing in Eq. (6); it may also contain extraneous roots introduced by the clearing of fractional exponents. Thus any (branch of) solutions for C of Eq. (7) must be tested in Eq. (6) to insure that they actually represent solutions of the original boundary value problem. There are thus both advantages and disadvantages in working with Eq. (7) rather than directly with the original Eq. (6). The advantage is that Eq. (7) is of a well known explicitly solvable form, so that the roots can be systematically found and classified; the disadvantage, noted above, that once a root is found it must be checked further to see if it also satisfies Eq. (6).

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The general explicit solutions of Eq. (7) are too lengthy to be of value and we do not write them down here; we discuss rather their general properties.

The class I solutions may be further divided as follows:

(Ia) $\varphi_O \geq 0$.

The + sign in Eqs. (3) through (6) is chosen and

$$\left.\frac{d\varphi}{dy}\right)_{y=0} = 16/9\ A + 2\ C \geq 0 \ .$$

For any given φ_O there is a maximum value of A, corresponding to C = -8/9 A, and obtained by putting this value of C into Eq. (6)

$$A_M^+ = (B+2)^2\ (B-1) \ . \tag{8}$$

(Ib) $\varphi_O \leq 0$.

The - sign in Eqs. (3) through (6) is chosen and

$$\left.\frac{d\varphi}{dy}\right)_{y=1} = 16/9\ A\ B + 2\ C \leq 0 \ .$$

For any given φ_O there is again a maximum value of A, here corresponding to C = - 8/9 A B and obtained by putting this value of C into Eq. (6)

$$A_M^- = (1 + 2\ B)^2\ (1 - B) \ . \tag{9}$$

The maximum value of A_M^- is obtained for $\varphi_O = -3/4$ (B = 1/2) and is given by

$$\left(A_M^-\right)_{MAX} = 2.$$

Class II, $-1 \leq \varphi_O \leq \infty$.

For this class, the curve of φ vs y is split into two branches and Eq. (6) cannot be used directly to fix C. The separate boundary conditions for the two branches are:

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Left hand branch	Right hand branch

<div align="center">

Left hand branch   Right hand branch

(- sign applies)   (+ sign applies)

</div>

$y = 0$ $\varphi = 0$ $y = y_M$ $\varphi = \varphi_M$ $\left. \dfrac{d\varphi}{dy} \right)_M = 0$

$y = y_M$ $\varphi = \varphi_M, \left. \dfrac{d\varphi}{dy} \right)_M = 0$ $y = 1$ $\varphi = \varphi_0$

For the left hand branch we use Eqs. (3) and (5) with the - sign to obtain

$$2C + 16/9 \, A \sqrt{1 + \varphi_M} = 0 \text{ with } -1 \leq \varphi_M \leq 0 \tag{10a}$$

$$0 = -y_M + \frac{4/9 \, A - C}{16/27 \, A^2} \, (2C + 16/9 \, A)^{1/2}, \quad 0 \leq y_M \leq 1 . \tag{10b}$$

For the right hand branch we use Eq. (4) with the + sign and obtain

$$\frac{4/9 \, AB - C}{16/27 \, A^2} \, (2C + 16/9 \, A B)^{1/2} = 1 - y_M \tag{11a}$$

$$\frac{4/9 \, AB - C}{16/27 \, A^2} \, (2C + 16/9 \, AB)^{1/2} = 1 - \frac{4/9 \, A - C}{16/27 \, A^2} \, (2C + 16/9 \, A)^{1/2} . \tag{11b}$$

The potentials are given by

$$\frac{4/9 \, A \sqrt{1 + \varphi} - C}{16/27 \, A^2} \, (2C + 16/9 \, A \sqrt{1 + \varphi})^{1/2} = \pm \, (y_M - y) \tag{12}$$

where the + sign goes with the left hand branch and the - sign with the right hand branch.

The only difference between Eq. (11b) and Eq. (6) is in the signs on the right hand side. When, however, Eq. (11b) is cleared of fractional powers to form a cubic in C, these differences are eliminated, and once again we obtain Eq. (7). Again, clearing Eq. (11b) of fractional powers may introduce spurious roots, so that solutions of Eq. (7) must be examined carefully to see if they satisfy both Eqs. (10a) and (11b), and are thus proper solutions of the original boundary value problem.

The lower limits of A for class II solutions are the upper limits for class I, namely $A_M^{\pm}$. If all solutions were unique, the upper limits would be found from Eqs. (10a) and (10b), the former with $\varphi_M = -1$ leading to C = 0, the latter yielding $y_M = 1/\sqrt{A}$. Putting these values in Eq. (11a)

<div align="center">11</div>

$$A_L^+ = \left[B^{3/2} + 1 \right]^2 = \left[(1 + \varphi_o)^{3/4} + 1 \right]^2 . \tag{13}$$

In fact, however, not all pairs of values (A, B) do lead to unique solutions, and A_L^+ does not always give the upper limit for class II solutions. It does, however, play a significant role in the classification of solutions.

For some values of φ_o we find $A_L^+ \geq A_M^+$, A_M^- while for others this is not the case. The transition occurs for

$$(B_U + 2)^2 (B_U - 1) = (B_U^{3/2} + 1)^2 \qquad \text{for } \varphi_o > 0 \tag{14}$$

and for

$$(2 B_L + 1)^2 (1 - B_L) = (B_L^{3/2} + 1)^2 \qquad \text{for } \varphi_o < 0 . \tag{15}$$

Eqs. (14) and (15) when cleared of fractional powers are quartics with one root, which clearly does not satisfy Eqs. (14) and (15) at $B_U = B_L = 1$. If these quartics are divided by $B_U - 1$ and $B_L - 1$ respectively, the resulting cubics are

$$9 B_U^3 + 5 B_U^2 - 25 B_U - 25 = 0 \tag{16}$$

$$25 B_L^3 + 25 B_L^2 - 5 B_L - 9 = 0 \tag{17}$$

from which we see directly that $B_L = 1/B_U$.

Solving Eq. (16) we have for the one real, positive root, $B_U = 1.81612$ and hence $B_L = 0.55062$.

The corresponding values of φ_o are

$$\left. \varphi_o \right)_U = 2.2983 \text{ and } \left. \varphi_o \right)_L = -0.69682 .$$

The maximum value of

$$\frac{A_M^+}{A_L^+} = \frac{(B + 2)^2 (B - 1)}{(B^{3/2} + 1)^2}$$

is $4/3$ and occurs for $B = 4$ or $\varphi_o = 15$. The maximum value of

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$$\frac{A_M^-}{A_L^+} = \frac{(1 + 2B)^2 (1 - B)}{(B^{3/2} + 1)^2}$$

is also $4/3$ and occurs for $B = 1/4$ or $c_0 = -15/16$.

The significance of the separation of values of c_0 into those for which $c_0 \gtrless c_0)_U$ and $c_0 \gtrless c_0)_L$ will be clarified in what follows.

3. DISCUSSION OF SOLUTIONS FOR THE INTEGRATION CONSTANT FOR CLASSES I AND II

We commence our examination of the functions $C(A, B)$, implicitly defined by Eq. (7), by an application of Descartes' rule. We note that the coefficients of the first and second terms are always positive and negative respectively. The coefficient of the third term is positive for small A and changes sign when

$$A_K = \frac{(B^2 - 1)(B^3 - 1)}{B^2 + 1}$$

while the coefficient of the last term is negative for small A and changes sign when

$$A_L^{\pm} = (B^{3/2} \pm 1)^2 .$$

Now,

$$(B^{3/2} + 1)^2 > \frac{(B^2 - 1)(B^3 - 1)}{B^2 + 1}$$

for all

$$0 < B < \infty$$

while

$$(B^{3/2} - 1)^2 \leq \frac{(B^2 - 1)(B^3 - 1)}{B^2 + 1}$$

the equality holding only for $B = 1$. We thus have the following sequence of variations in signs of coefficients as A goes from 0 to ∞.

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A	Signs of Coefficients			
$0 < A < A_L^-$	+	–	+	–
$A_L^- < A < A_K$	+	–	+	+
$A_K < A < A_L^+$	+	–	–	+
$A_L^+ < A < \infty$	+	–	–	–

The change in sign from the second to the third row does not change the number of variations in sign and hence we have for the number of ± roots, $N_\pm$ according to DesCartes' rule,

	N_+	N_-
$0 < A < A_L^-$	3 or 1	0
$A_L^- < A < A_L^+$	2 or 0	1
$A_L^+ < A < \infty$	1	2 or 0

The ambiguities in N_+ and N_- are resolved by examination of the discriminant of Eq. (7).

If we use the common notation for the coefficients of cubics

$$C^3 + pC^2 + qC + r = 0 \tag{18}$$

$$a = q - p^2/3$$

$$b = 2/27\, p^3 - 1/3\, pq + r$$

then the discriminant is defined by

$$R = b^2/4 + a^3/27 .$$

The discriminant of Eq. (7) is of eighth order in A, and is factorable into the following form:

$$R = \frac{(16)^2}{(27)^4} A^3 \,[A - (1+B)^3]\, [A - (1-B)^2 (B + \tfrac{1}{2})]^2 \,[A - (1-B)^2 (B/2 + 1)]^2 . \tag{19}$$

Thus, for $A < (1+B)^3$ there are three real roots and for $A > (1+B)^3$ only one. Let $A_R \equiv (1+B)^3$; then the numbers of roots are

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	N_+	N_-
$0 < A < A_L^-$	3	0
$A_L^- < A < A_L^+$	2	1
$A_L^+ < A < A_R$	1	2
$A_R < A < \infty$	1	0

With this table and the preceding material in hand we are in a position to organize, and in what follows, to assign types of solutions in the parameter space. Plotting the four curves A_M^+, A_M^-, A_L^+, and A_R in the A, B (or A, φ_0) plane, we find the quadrant A, B $\geq$ 0 divided into seven regions. Each region will be characterized by the type(s) and numbers of solutions allowed in that region (Figure 1). Although there is a change in sign of one coefficient at $A = A_L^-$, there is no corresponding change in the class type of solution and hence A_L^- does not appear in the figure. A further subdivision of, and final assignment of classes in the regions of the A, B plane is accomplished in Section 4.
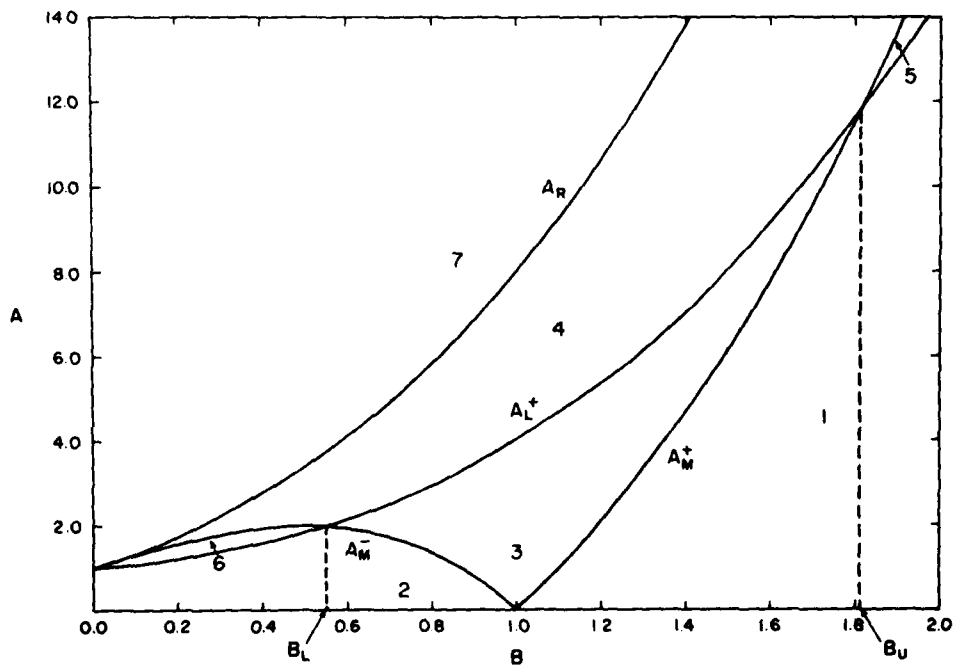

Figure 1. Division of (A, B) Plane Into Separate Regions Characterized by Numbers and Classes of Solutions in Each Region

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We now proceed to examine the solutions of Eq. (7) so that classes of solutions may be assigned to the different regions.

We intend to examine the values of C as functions of A for fixed ω_o. In this, we define C_1, C_2, C_3 so that outside of values for C at the singular points $A_1 = (1 - B)^2 (B + 1/2)$, $A_2 = (1 - B)^2 (B/2 + 1)$ we will have $C_1 \geq C_2 \geq C_3$. The labelling of the roots in this order also identifies them permanently by branch. If we put $C = C' - p/3$ then Eq. (18) becomes

$$(C')^3 + a C' + b = 0 . \tag{20}$$

If

$$\alpha = \left(- b/2 + \sqrt{R} \right)^{1/3}$$

$$\beta = \left(- b/2 - \sqrt{R} \right)^{1/3}$$

our solutions are labelled so that

$$C'_1 = \alpha + \beta \tag{21a}$$

$$C'_2 = - \frac{\alpha + \beta}{2} - \frac{\alpha - \beta}{2} \sqrt{-3} \tag{21b}$$

$$C'_3 = - \frac{\alpha + \beta}{2} + \frac{\alpha - \beta}{2} \sqrt{-3} \tag{21c}$$

3.1 Asymptotic Limits for Small A

For small A the roots of Eq. (7) are given by relatively simple expressions. If the equation is modified so that terms in A only as high as the second power are retained, we have

$$2 C^3 - (B^2 - 1)^2 C^2 - 16/27 \, A[A(1 + B^2) - (B^2 - 1)(B^3 - 1)] \, C + (8/27)^2 A^2 (B^3 - 1)^2 = 0 . \tag{22}$$

If we assume a solution $O[C] \gg O[A]$ where $O[\ ] = $ "order of" we may drop the term in A^2, and the resulting quadratic has as one solution

$$C_1 = \frac{1}{2} (B^2 - 1)^2 - 16/27 \, \frac{B^3 - 1}{B^2 - 1} A . \tag{23}$$

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The other root of the quadratic leads to $O[C] = O[A]$ and is hence contrary to our assumption. If Eq. (6) is expanded out to terms of order A^2 we find $\sqrt{2C} = \pm (B^2 - 1)$. Thus the single solution of Eq. (22), $C_1 = 1/2 (B^2 - 1)^2$ contains the two solutions of Eq. (6) corresponding to the $\pm$ sign.

This does not however exhaust the solutions of Eq. (22). If we now assume that there is a solution of the form $O[C] = O[A]$ the term in C^3 is of higher order in powers of A than the other three terms and solution of the quadratic resulting from neglecting the term in C^3 yields, for $B \neq 1$

$$C_2 = C_3 = 8/27 \, \frac{B^3 - 1}{B^2 - 1} \, A = 8/27 \, \frac{B^2 + B + 1}{B + 1} \, A \; . \tag{24}$$

This is a double root of Eq. (21) and hence the existing three roots have been found.

For $B = 1$, the approximations $A \, f(B) \ll (B^2 - 1)$, $(B^3 - 1)$ with $f(B) \neq 0$ are no longer valid and we must return to Eq. (7). Setting $B = 1$ and then retaining only the lowest power of A in the coefficients we have

$$C^3 - 16/27 \, A^2 \, C + 2 (8/27)^2 \, A^3 = 0 \; .$$

The solutions are

$$C_1 = C_2 = 4/9 \, A \quad \text{(double root)} \tag{25a}$$

$$C_3 = - 8/9 \, A \; . \tag{25b}$$

If Eq. (24) is put into Eq. (6) and the roots expanded to obtain the lowest non-vanishing order in A, we find that C_2 is not a solution of Eq. (6). Although it is a solution of Eq. (11a) it is also not a solution of Eq. (10a).

Thus, of the three solutions of Eq. (22), only C_1 is a solution of the original problem; and, this contains the two solutions corresponding to the $\pm$ sign.

3.2 Other Special Values of C

For $A = A_R = (B + 1)^3$ (and $R = 0$), Eq. (7), when solved by use of standard cubic formulae yields expressions which may be simplified to obtain

$$C_1 = 1/6 \, (B + 1)^2 \, (3 B^2 + 14/3 \, B + 3) \tag{26a}$$

$$C_2 = C_3 = - 8/9 \, B(1 + B)^2 \; . \tag{26b}$$

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Similarly, for $A = A_2 = (1 - B)^2 (B/2 + 1)$

$$C_1 = 1/18 (B - 1)^2 (B^2 + 2B + 9) \tag{27a}$$

$$C_2 = C_3 = 2/9 (B - 1)^2 B(B + 2) \tag{27b}$$

and for $A = A_1 = (1 - B)^2 (B + 1/2)$

$$C_1 = 1/18 (B - 1)^2 (9 B^2 + 2B + 1) \tag{28a}$$

$$C_2 = C_3 = 2/9 (B - 1)^2 (2B + 1) . \tag{28b}$$

For $A = A_M^+ = (B + 2)^2 (B - 1)$ we know one root from the definition of A_M^+; namely

$$\left. \frac{d\varphi}{dy} \right)_{y = 0} = 2C + 16/9 A_M$$

and

$$C = C_3 = - 8/9 (B + 2)^2 (B - 1) . \tag{29a}$$

From this we obtain an equation quadratic in C, and the remaining two roots

$$C_{1, 2} = 1/4 \left\{ (B^2 - 1)^2 + 16/9 (B + 2)^2 (B - 1) \pm (B^2 - 1) \left[(B^2 - 1)^2 + 32/3 (B + 2)^2 (B - 1) \right]^{1/2} \right\} \tag{29b}$$

where the $\pm$ signs are associated with the subscripts 1, 2 of C respectively.

Similarly for $A = A_M^- = (2B + 1)^2 (1 - B)$ one root is

$$C_3 = - 8/9 A B = - 8/9 B(2B + 1)^2 (1 - B) \tag{30a}$$

and the other two roots are found to be

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$$C_{1,2} = 1/4 \left\{ (B^2 - 1)^2 + 16/9\, AB \pm \left[\left[(B^2 - 1)^2 + 16/9\, AB \right]^2 \right.\right.$$

$$\left.\left. - 64/81\, AB(B - 1)(-25\, B^3 - 25\, B^2 + 5\, B + 9) \right]^{1/2} \right\} \tag{30b}$$

where again the $\pm$ signs go respectively with the subscripts 1, 2.

3.3 The Zeros of C

The zeros of C are obtained when

$$4A - (B^3 - 1 - A)^2 = 0$$

or

$$A = A_L^{\pm} = (B^{3/2} \pm 1)^2 . \tag{31}$$

For any B there are thus two values of A for which $C = 0$. It must be determined however which branches, C_1, C_2 or C_3 these zeros belong to. This, as well as the further properties of C require more complete solutions for C as functions of A. We now turn to these solutions.

3.4 Systematic Presentation of Solutions for C

We continue our examination of solutions of Eq. (7) by graphing solutions of C vs A for seven values of φ_o: -0.9; -0.5; 0.0; 1.0; 3.0; 30; and 1000. These are shown in Figures 2 through 8. The first five values are chosen to fall in the separate regions illustrated in Figure 1; the last two to show behavior at large φ_o. In addition, Figures 9, 10 and 11 show the behavior of the roots for small A, for φ_o = -0.5; 1.0; and 3.0.

We note first that the zero at $A = (B^{3/2} - 1)^2$ occurs in the C_3 branch, while that at $A = (B^{3/2} + 1)^2$ occurs in the C_2 branch. Further, again aside from singularities at $A = A_1$ and $A = A_2$ the intent to label the branches so that $C_1 > C_2 > C_3$ is accomplished. We next turn to a discussion of these singularities.
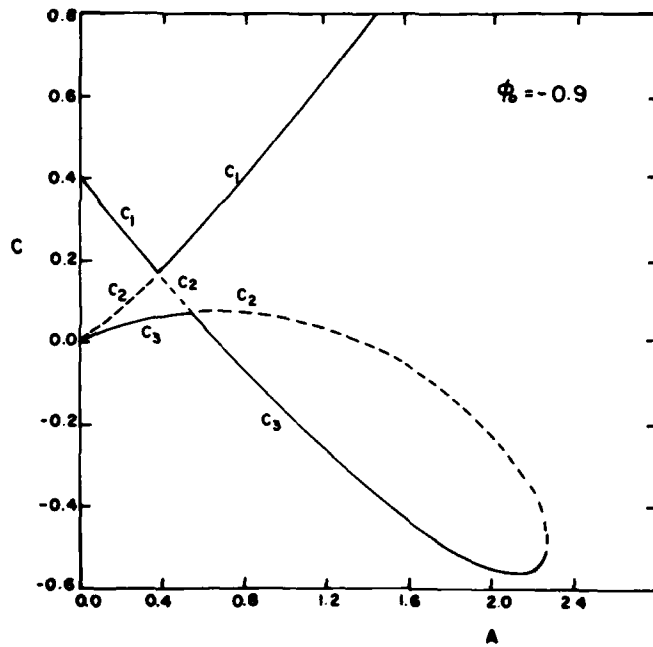
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Figure 2. Graph of Integration Constant C vs
Non-dimensionalized Current Density A, for
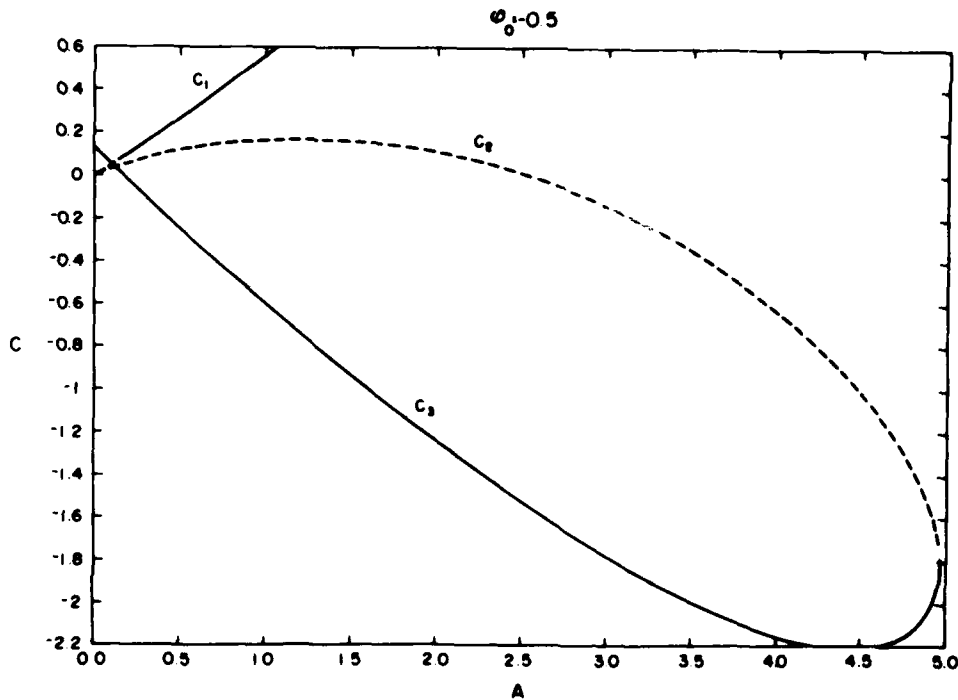Non-dimensionalized Voltage φ_o = -0.9

Figure 3. Graph of Integration Constant C vs Non-dimensionalized Current
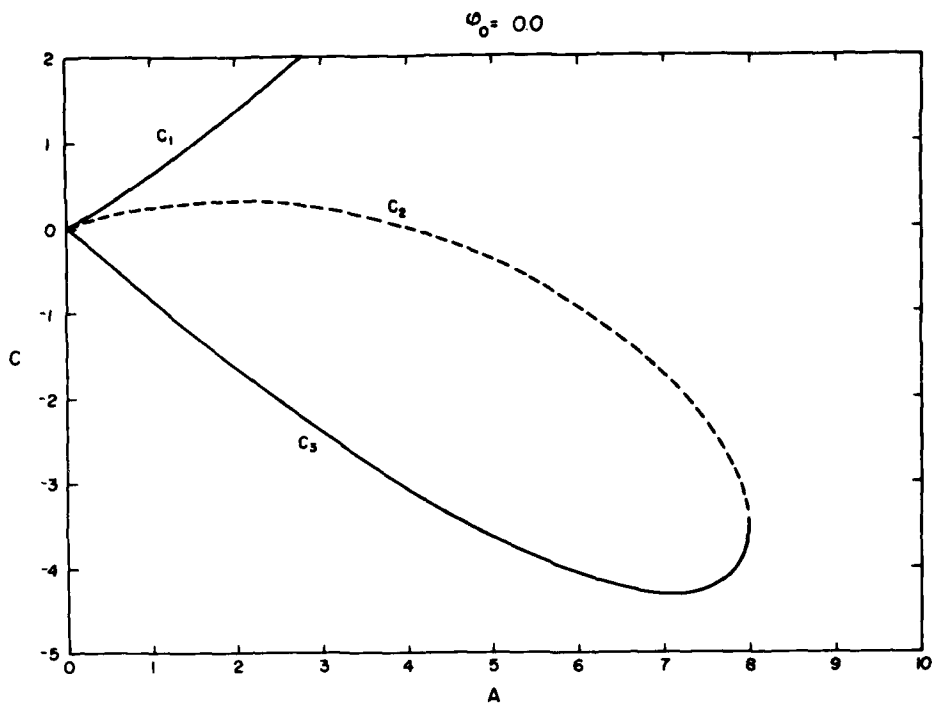Density A for Non-dimensionalized Voltage φ_o = -0.5

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Figure 4. Graph of Integration Constant C vs Non-dimensionalized Current Density A, for Non-dimensionalized Voltage $\varphi_o = 0.0$

Figure 5. Graph of Integration Constant C vs Non-dimensionalized Current Density A, for Non-dimensionalized Voltage $\varphi_o = 1.0$

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$\varphi_0 = 3.0$

Figure 6. Graph of Integration Constant C vs Non-dimensionalized Current
Density A, for Non-dimensionalized Voltage $\varphi_O = 3.0$

$\varphi_0 = 30$

Figure 7. Graph of Integration Constant C vs Non-dimensionalized Current
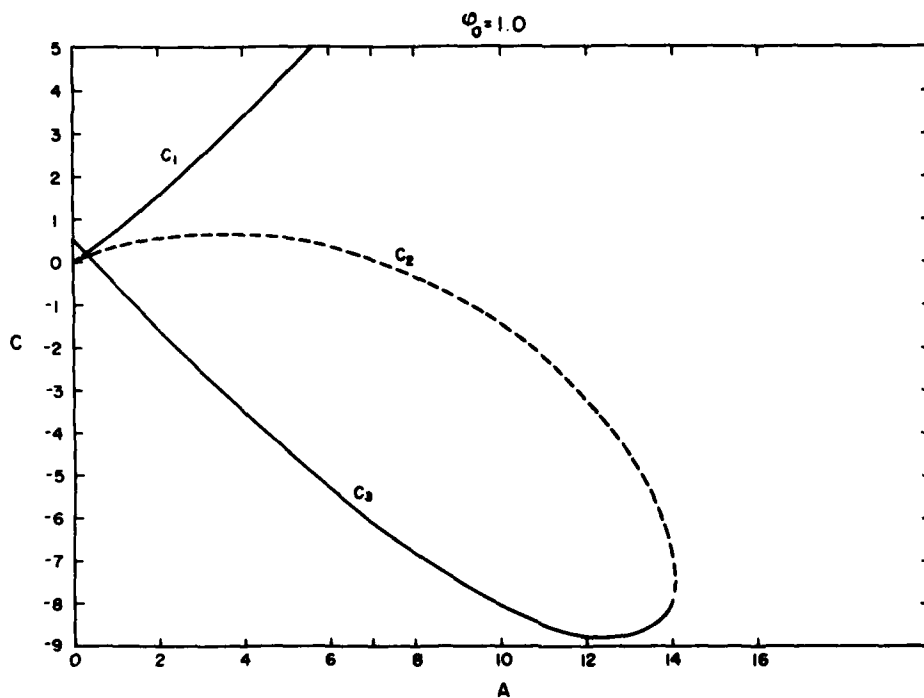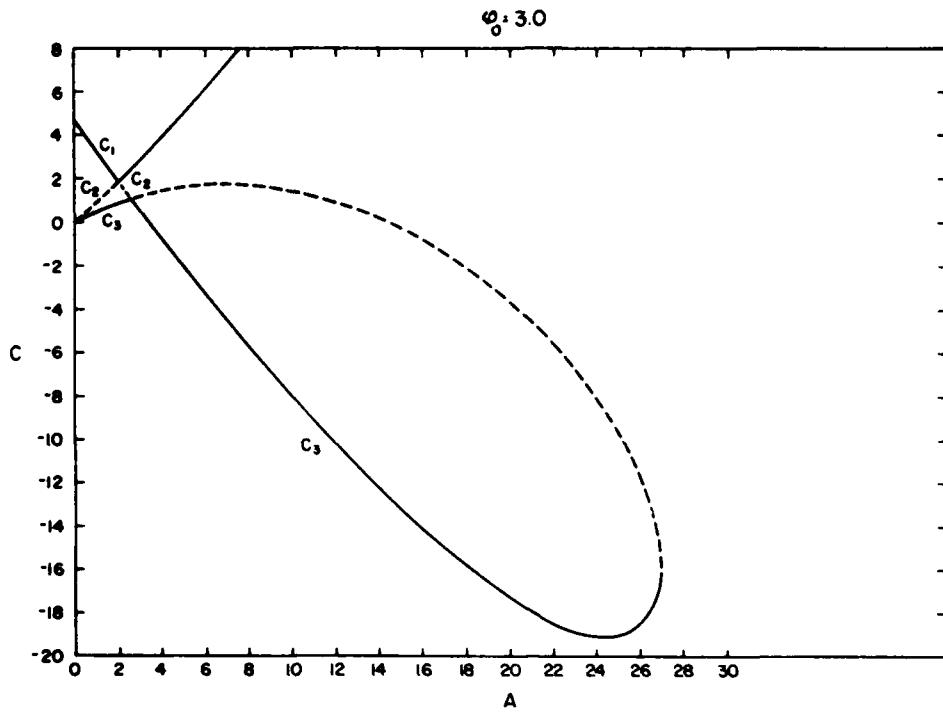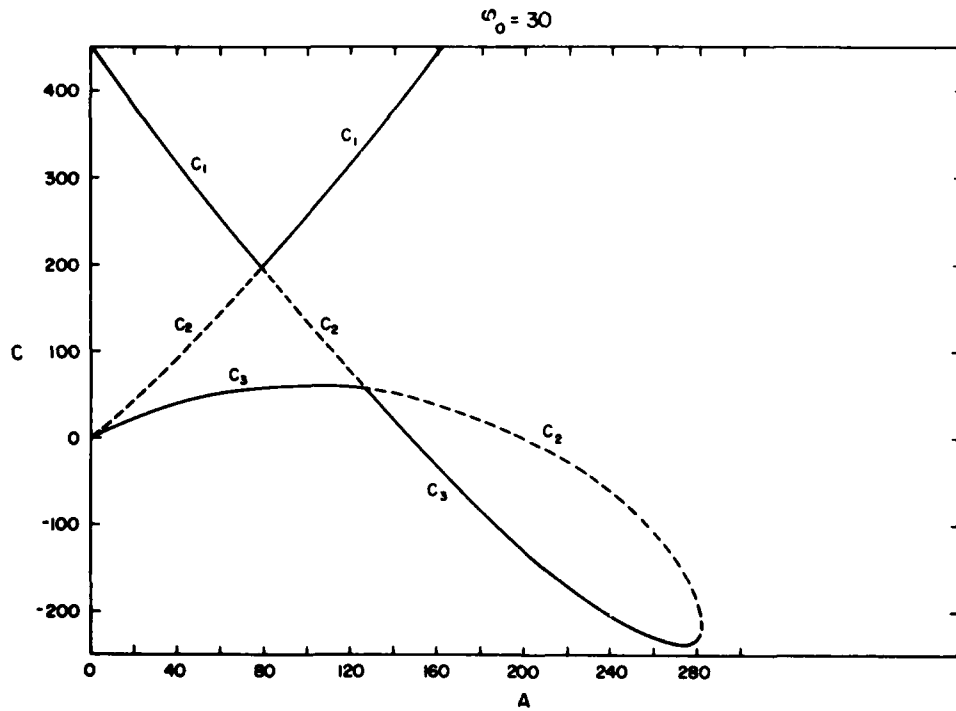Density A, for Non-dimensionalized Voltage $\varphi_O = 30.0$

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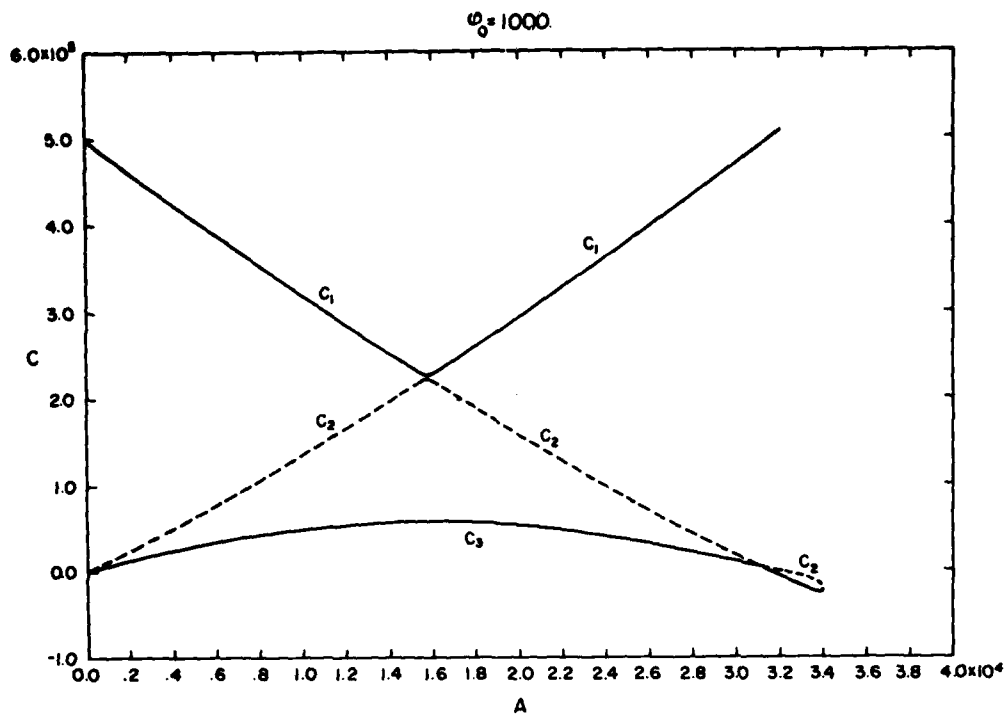
Figure 8. Graph of Integration Constant C vs Non-dimensionalized Current Density A, for Non-dimensionalized Voltage $\varphi_O = 1000$

Figure 9. Graph of Integration Constant C vs Non-dimensionalized Current Density A, for Non-dimensionalized Voltage $\varphi_O = -0.5$. Details for small A

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Figure 10. Grapn of Integration Constant C vs Non-dimensionalized Current Density A, for Non-dimensionalized Voltage φ_0 = 1.0. Details for small A

Figure 11. Graph of Integration Constant C vs Non-dimensionalized Current Density A, for Non-dimensionalized Voltage φ_0 = 3.0. Details for small A
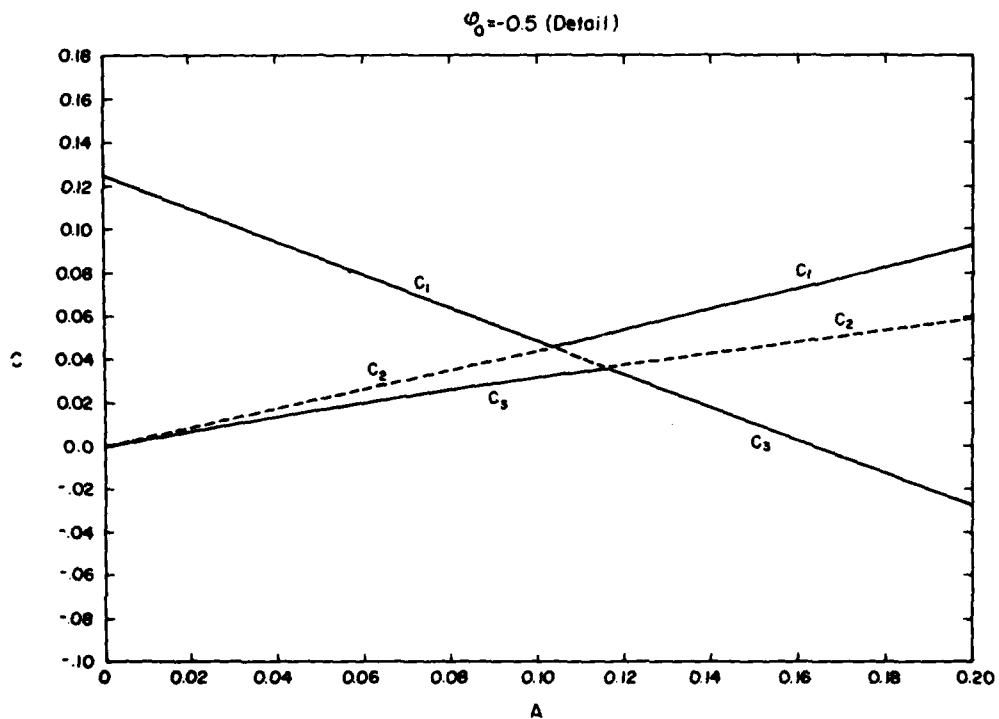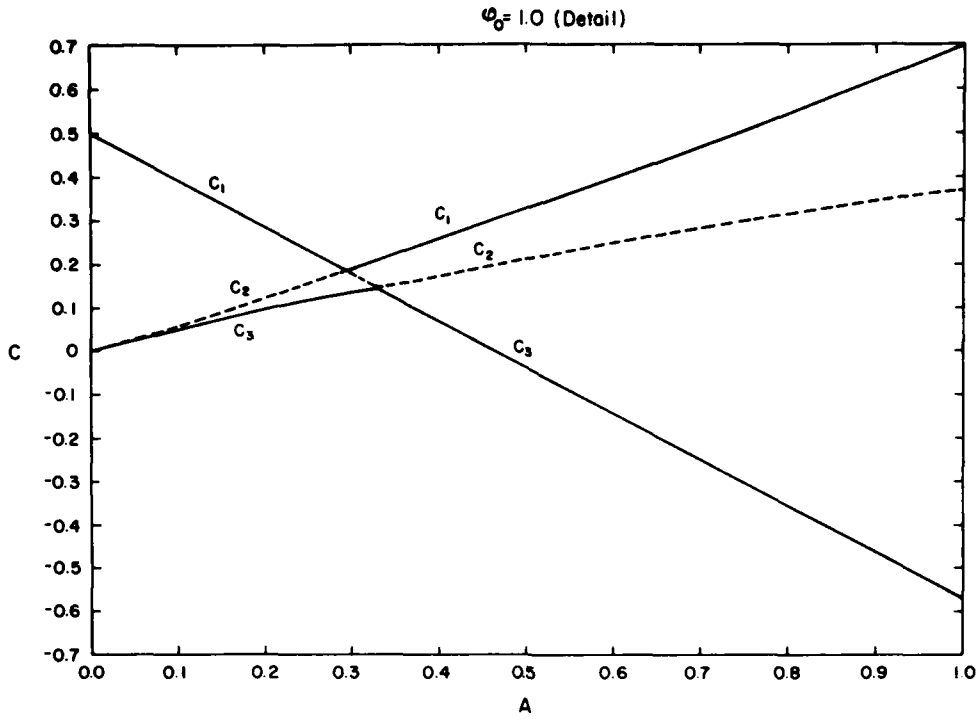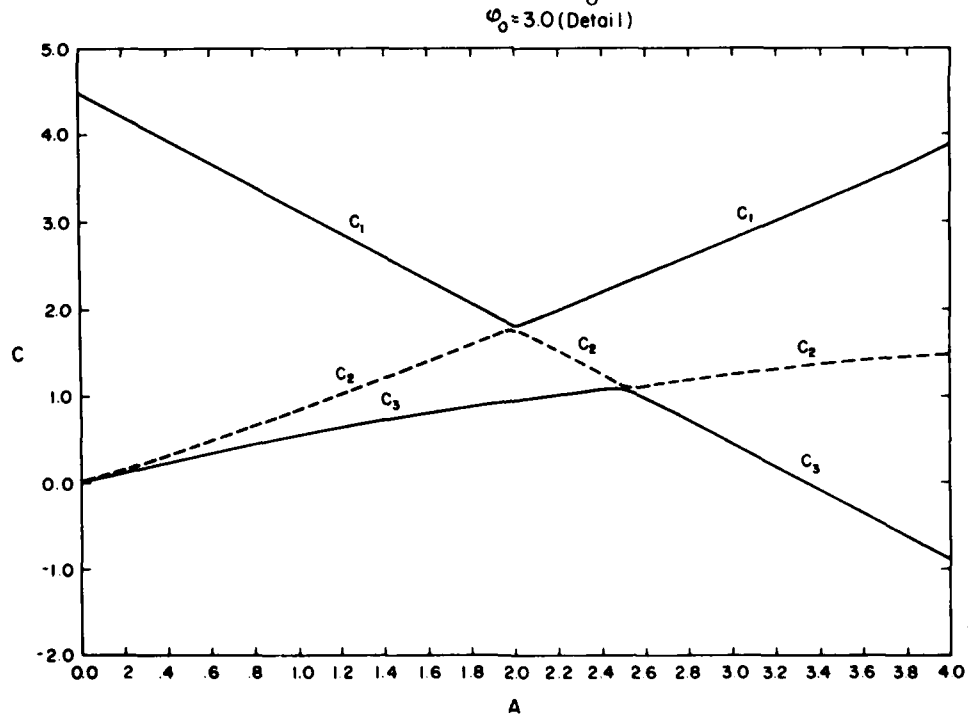
3.5 Singularities in C

The singularities of C occur at the branch points given by $R(A) = 0$. We confine ourselves to some general comments pointing out several of the more obvious features of these singularities.

We note to start with that as far as first derivatives are concerned a visual inspection of Figures 2 through 11 indicates that for $B < 1$ and $A_1 < A_2$, C_1 and C_2 have discontinuities at $A = A_1$, while C_2 and C_3 have discontinuities at $A = A_2$. For $B > 1$, and $A_1 > A_2$, the roles of A_1 and A_2 are interchanged.

For the behavior of the functions themselves Eqs. (27) and (28) yield the following:

B	$A = A_1$	$A = A_2$
$B < 1$	$C_1 < C_2 = C_3$	$C_1 > C_2 = C_3$
$B > 1$	$C_1 > C_2 = C_3$	$C_1 < C_2 = C_3$
$B = 1$	$C_1 = C_2 = C_3$	$C_1 = C_2 = C_3$

At $A =$ the smaller of A_1 and A_2 the evidence of a discontinuity is clear. The figures show at this point that $C_1 = C_2 > C_3$; Eqs. (21) on the other hand have when $R = 0$, in general, $C_2 = C_3 \neq C_1$. There thus exists, for each fixed value of B a singularity of C at $A =$ the smaller of A_1, A_2. This singularity has the effect of exchanging the neighboring values of C_1 and C_3. Numerical solutions are consistent with this supposition. Thus we have for example for $B = 2$ the following:

A	C_1	C_2	C_3
1.99997	1.77782	1.77775	0.944434
1.99998	1.77780	1.77776	0.944437
1.99999	1.77779	1.77777	0.944441
2.00000	0.94444	1.77778	1.777778
2.00001	1.77779	1.77776	0.944448
2.00002	1.77780	1.77775	0.944452
2.00003	1.77781	1.77774	0.944455

At $A =$ the larger of A_1, A_2 there is no obvious requirement that a singularity exist, since graphical values in the figures and the results of Eqs. (21) both lead to $C_1 > C_2 = C_3$.

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3.6 Assignment of Classes to Solutions

Having obtained the solutions of Eq. (7) it now remains to determine which of these satisfy Eq. (6) (class I) and which satisfy Eqs. (10a) and (11b) (class II). This is accomplished by computer by substituting values of C into the appropriate equations. If this is done, changes in numbers and classes are found to occur only for $A = A_M^+$, A_M^-, A_L^+, and A_R. This information is presented in Table 1. The assignment of solutions in class III are discussed in the following section.

Table 1. Numbers and Classes of Solutions in the Various Regions of Parameter Space

Region	Solutions	Class I	Class II	Class III
1	1	1	0	0
2	1	1	0	0
3	1	0	1	0
4	3	0	2	1
5	3	1	1	1
6	3	1	1	1
7	1	0	0	1
8	3	1	0	2
9	3	0	1	2

3.7 Uniqueness

For $A < A_L^+$ (classes I and II) and $A > A_R$ (class III) the solutions for C are unique functions of φ_0 and A. For $A_L^+ < A < A_R$ on the other hand the solutions are many valued. Which solution among the multiple solutions is actually realized, may be determined by; (1) the manner in which the operating conditions are approached, and (2) questions of stability of operation. We do not investigate either of these questions here.

3.8 Determination of Values of C

Value(s) of C for any φ_0, A may be found by use of Table 2. The table is constructed by examination of solutions corresponding to the regions shown in Figure 1 and by observing that

$$A_L^+ \geq A_1, A_2 \qquad -1 \leq \varphi_o < \infty$$

$$A_M^- \geq A_1, A_2 \qquad -1 \leq \varphi_o \leq 0$$

$$A_M^+ \geq A_1, A_2 \qquad 0 \leq \varphi_o < \infty$$

Table 2. Assignment of C to Branches Corresponding to Ranges of A and φ_o

$-1 \leq \varphi_o \leq \varphi_L$ and $\varphi_{U} \leq \varphi_o < \infty$ $(A_L^+ \geq A_1, A_2)$		
Range of A	Branch of C	Class
$A \leq A_1$	C_1	I
$A_1 \leq A \leq A_2$	C_2	I
$A_2 \leq A \leq A_L^+$	C_3	I
$A_L^+ \leq A \leq A_M^\pm$	C_2/C_3	II/I
$A_M^\pm \leq A \leq A_R$	C_2, C_3	II

$\varphi_L \leq \varphi_o \leq 0$ $(A_M^- \geq A_1, A_2)$ and $0 \leq \varphi_o \leq \varphi_U$ $(A_M^+ > A_1, A_2)$		
Range of A	Branch of C	Class
$A \leq A_1$	C_1	I
$A_1 \leq A \leq A_2$	C_2	I
$A_2 \leq A \leq A_M^\pm$	C_3	I
$A_M^\pm \leq A \leq A_L^+$	C_3	II
$A_L^+ \leq A \leq A_R$	C_2, C_3	II

$$A_M^\pm = A_M^+ \quad \text{for } \varphi_o > 0$$

$$A_M^\pm = A_M^- \quad \text{for } \varphi_o < 0$$

4. THE SPACE CHARGE LIMITED (CLASS III) CASE

We introduce the following additional terms (Figure 12):

j = Fraction of emitted current which is transmitted past the potential minimum.

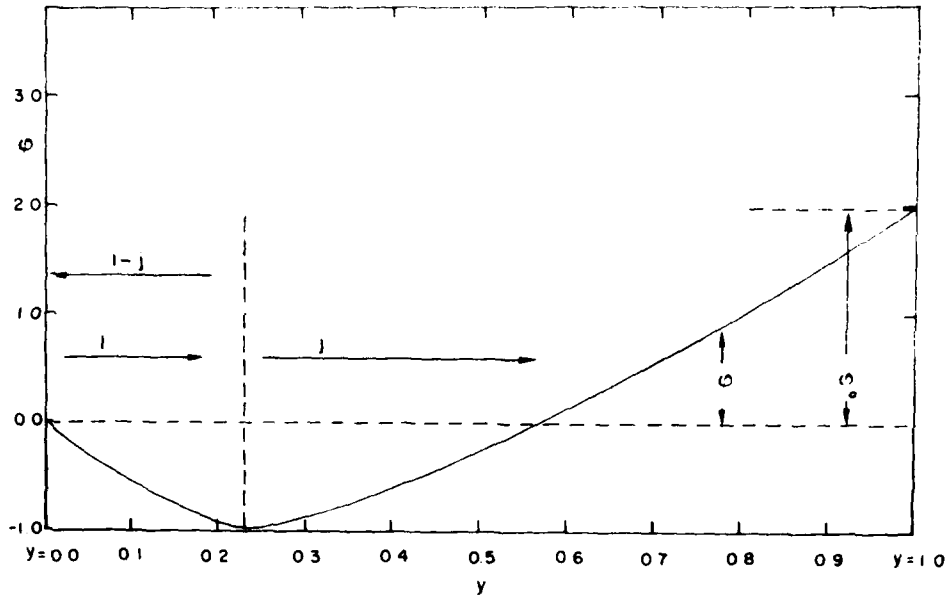y_1 = Value of y at the potential minimum.

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Figure 12. Typical Class III Potential Distribution Showing Fractional Current Flow to Left and Right of Potential Minimum at $\varphi = -1.0$. $B = \sqrt{3}$; $A = 13.98$; $y_1 = 0.2279$; $j = 0.6235$

Eq. (2) is then replaced by

$$\frac{d^2\varphi}{dy^2} = \frac{4/9\,A}{\sqrt{1+\varphi}}\,(2-j) \qquad 0 < y < y_1 \tag{32}$$

$$\frac{d^2\varphi}{dy^2} = \frac{4/9\,A}{\sqrt{1+\varphi}}\,j \qquad y_1 < y < 1\,. \tag{33}$$

Solutions of Eqs. (32) and (33) which satisfy the boundary conditions

$$\varphi = -1 \qquad\qquad \frac{d\varphi}{dy} = 0$$

are

$$\varphi = -1 + [A(2-j)]^{2/3}\,(y_1 - y)^{4/3} \tag{34}$$

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$$\varphi = -1 + (A\,j)^{2/3}\,(y - y_1)^{4/3} \tag{35}$$

respectively. The boundary conditions at $y = 0$ and $y = 1$ then yield

$$1 = A(2 - j)\,y_1^2 \tag{36}$$

$$B^3 = A\,j(1 - y_1)^2\ . \tag{37}$$

Eliminating j,

$$2\,A = 1/y_1^2 + \frac{B^3}{(1 - y_1)^2} \tag{38}$$

and clearing of fractions,

$$F \equiv 2\,A\,y_1^4 - 4\,A\,y_1^3 + (2\,A - 1 - B^3)\,y_1^2 + 2\,y_1 - 1 = 0 \tag{39}$$

an equation of fourth order in y_1. The role of y_1 here replaces that played by the integration constant in the other two cases; once it is known (a function of A and B) the potential distribution is fixed.

Since $F(0) = -1$ and $F(-\infty) = \infty$, Eq. (39) has at least one real root between 0 and $-\infty$. Since $F(1) = -B^3$ and $F(\infty) = \infty$, there is at least one real root between $+1$ and $+\infty$. From Eq. (38) it is clear that the other two roots, when real, lie between $y_1 = 0$ and $y_1 = 1$.

Consider Eq. (38) for a fixed value of B. For $y_1 = 1$ or $y_1 = 0$, $A \to \infty$. For some $0 < y_1 < 1$, A has a (unique) minimum. A must always be larger than this minimum, and finding this minimum supplies a necessary condition in defining the domain of A, B space over which there will be allowable solutions for y_1. We have,

$$2\,\frac{dA}{dy_1} = \frac{-2}{y_1^3} + \frac{2\,B^3}{(1 - y_1)^3} = 0$$

$$(1 - y_1)^3 = B^3\,y_1^3 \tag{40}$$

$$\frac{1}{y_1} = B + 1$$

$$2A \geq 2A_{MIN} = (B + 1)^2 + B^3\left(\frac{B + 1}{B}\right)^2 = (1 + B)^3 = \frac{1}{y_1^3} = A_R\ . \tag{41}$$

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Although $A > A_{MIN}$ guarantees a solution of Eq. (39) for $0 \leq y_1 \leq 1$, there are further requirements on the values of A necessary to insure that solutions of physical interest exist. The value of j can never be greater than j = 1. Putting j = 1 in Eqs. (36) and (37) we obtain

$$\frac{1}{y_1^2} \geq \frac{B^3}{(1 - y_1)^2} . \tag{42}$$

For $B \geq 1$ this limits y_1 to values

$$y_1 \leq \frac{1}{1 + B^{3/2}} \qquad \text{and A to values}$$

$$A \geq \left(1 + B^{3/2}\right)^2 = A_L^+ .$$

For $B = 1$, $A_{MIN} = A_L^+$; for all $B > 1$, $A_L^+ > A_{MIN}$ and hence we must have $A \geq A_L^+$ in order that solutions of physical interest exist.

For $B \leq 1$, Eq. (42) yields

$$y_1 < \frac{1}{1 \pm B^{3/2}}$$

or

$$A \geq 1 \pm B^{3/2} = A_L^{\pm} . \tag{43}$$

Since $A_L^- \leq A_{MIN} \leq A_L^+$ the dual inequalities Eqs. (41) and (43) can be satisfied by $A > A_{MIN}$. Now, when $A = A_{MIN}$, a double root of y_1 is

$$y_1 = \frac{1}{B + 1}$$

and

$$j = 2 \frac{B}{B + 1} \leq 1 .$$

Of the two solutions for y_1 for $A > A_{MIN}$, one increases with increasing A, the other decreases with increasing A. This hold for $A_{MIN} \leq A \leq A_L^+$ with both solutions for $0 \leq y_1 \leq 1$ satisfying $j \leq 1$. At $A = A_L^+$, one value of j equals unity, and for $A > A_L^+$, the solution with the larger value of y_1 has $j > 1$, so that only one solution

of physical significance exists for $A > A_L^+$. These considerations are clarified by graphical display in Figures 13a and 13b.
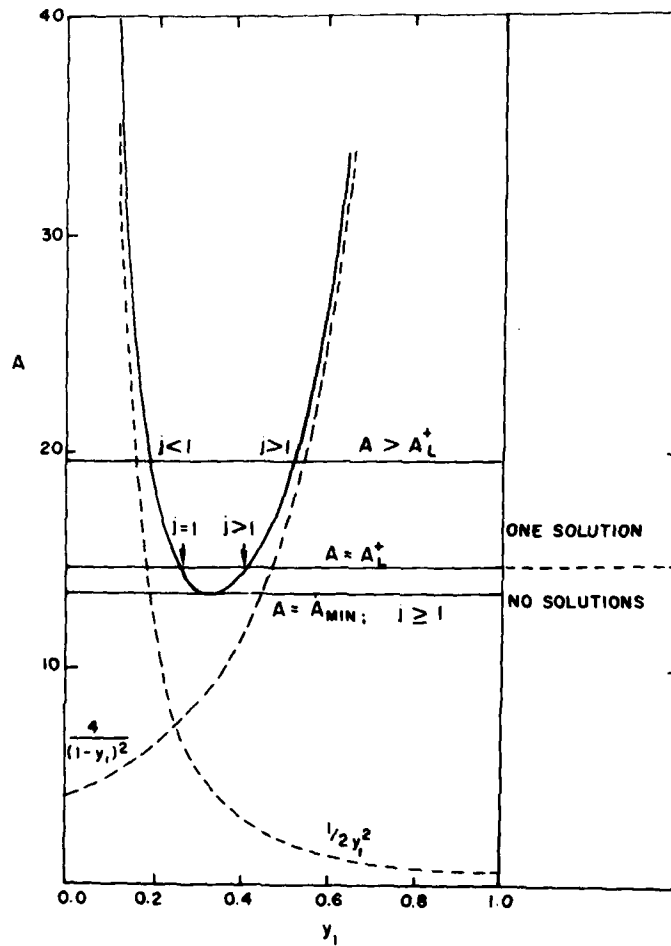

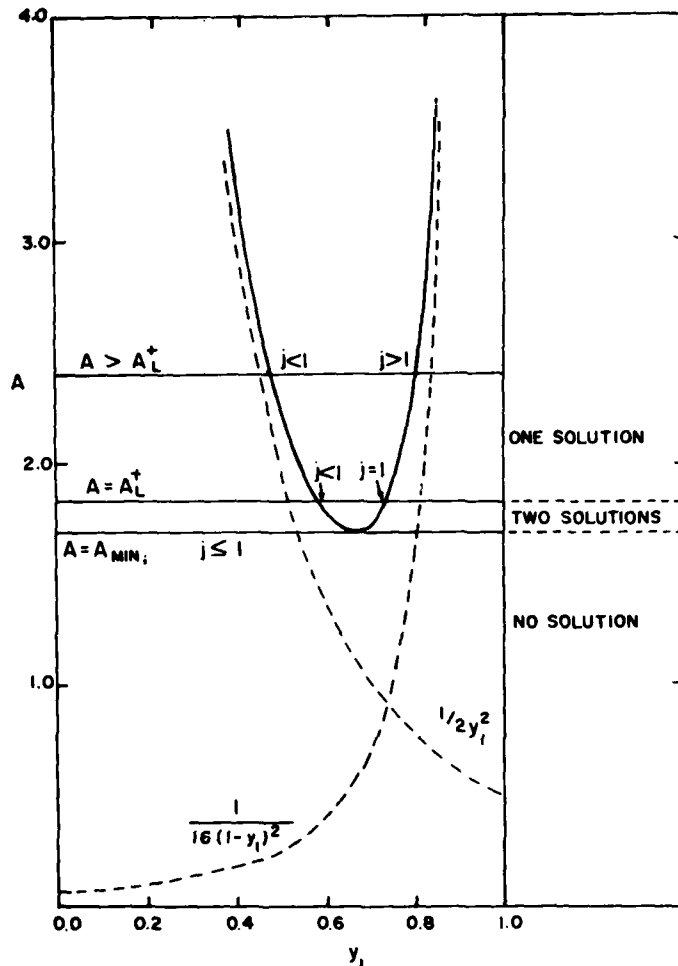Figure 13a. Geometric Illustration of Limits and Numbers of Class III Solutions for $B \geq 1$

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Figure 13b. Geometric Illustration of Limits and
Numbers of Class III Solutions for $B \leqq 1$

We note that A_{MIN} intersects A_M^- at $B = 1/\sqrt{3} > B_L$ and intersects A_L^+ at $B = 1$. Figure 14, a superposition of A_{MIN} onto Figure 1 completes the classification of solutions, dividing the A, B plane into nine distinct regions. The region labelled 2 in Figure 1 is split into two regions, 2 and 8, in Figure 14; and the region labelled 3 in Figure 1 is likewise split into regions 3 and 9 in Figure 14. The numbers of class III solutions in each region has already been given in Table 1.
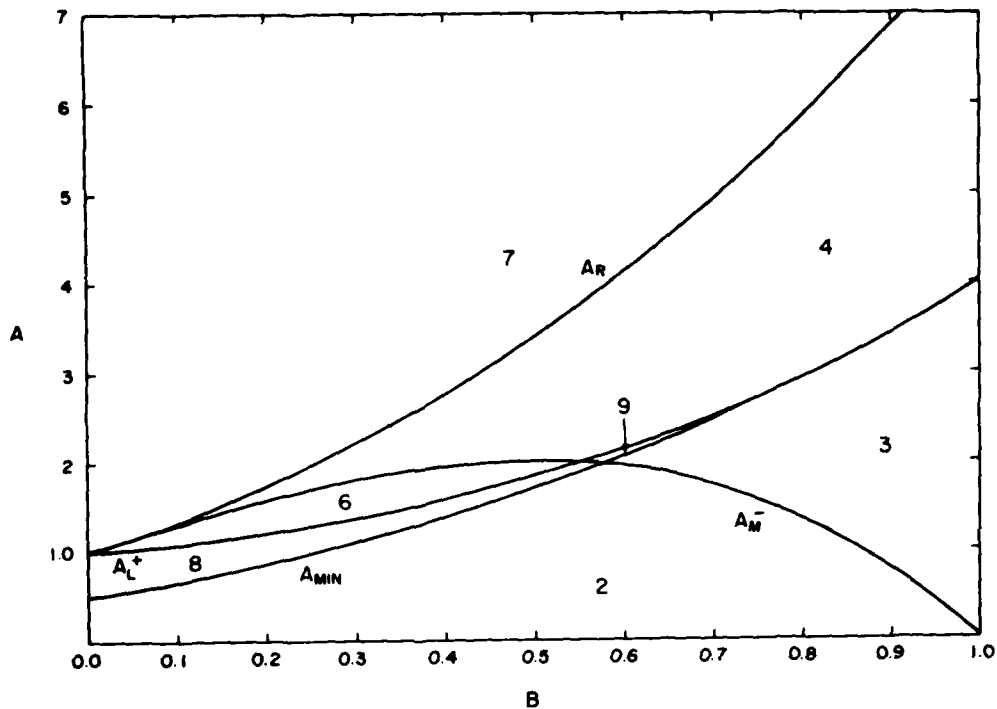
Figure 14. Division of (A, B) Plane Into Separate Regions Characterized by Numbers and Classes of Solutions in Each Region. Detail for B ≤ 1 shows modification of Figure 1 to include class III solutions

4.1 Special Values of y_1

For several particular values of B, the solutions of Eq. (33) are of simple form.

For B = 0, the four roots are

$$y_1 = \pm \frac{1}{\sqrt{2A}}, \ 1 \qquad \text{(the last a double root)}.$$

For B = 1, the four roots are

$$y_1 = \pm \left[\frac{A + 2 \pm 2\sqrt{2A+1}}{4A} \right]^{1/2} + \frac{1}{2}.$$

The two roots $0 \le y_1 \le 1$ are those for which the ± sign inside the radical is taken as −.

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For $A = (B^{3/2} + 1)^2$ one root is

$$y_1 = \frac{1}{\sqrt{A}} \; .$$

The residual cubic is

$$2\sqrt{A}\, y_1^3 + (2 - 4\sqrt{A})\, y_1^2 + (\sqrt{A} - 2)\, y_1 + 1 = 0$$

and the remaining roots do not seem to have simple forms.

For $A = A_{MIN} = \dfrac{(B+1)^3}{2}$, the four roots are

$$y_1 = \frac{1}{1+B} \quad \text{(double root)}$$

$$y_1 = \frac{B \pm \sqrt{B^2 + B + 1}}{B + 1} \; .$$

Figure 15 shows typical values of y_1 as a function of A with B as parameter. Solid lines are solutions of the original boundary value problem with $j \leq 1$. Dashed values continue solutions for all values $0 \leq y_1 \leq 1$. Dotted lines ($1/\sqrt{A}$ for $B \geq 1$ and $1/\sqrt{A}$ and $(2A)^{-1/3}$ for $B \leq 1$) separate regions according to number of class III solutions existing in each region.
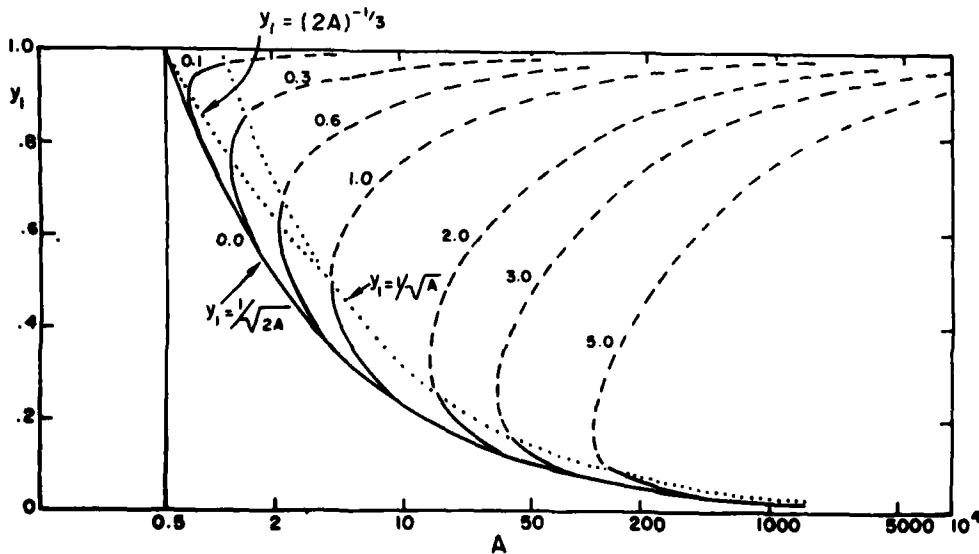

Figure 15. Values of $0 \leq y_1 \leq 1$ From Eq. (33) as Functions of A With B as Parameter. Solid lines; values of y_1 for which $j \leq 1$; dashed lines; values of y_1 for which $j \geq 1$; dotted lines; separate regions according to number of class III solutions existing in each region

5. FORMS OF THE POTENTIAL

The primary purposes of this report are to investigate structure, to classify solutions, and to organize methods of obtaining the latter. We do not, therefore, present here large numbers of graphs of potential vs distance; these are given in works previously referred to. We do, however, for illustrative purposes, give several examples of such graphs and to this end we choose parameters lying in regions 4, 5, 8 and 9, in all of which there are three solutions corresponding to each pair of parameters (A, B). Figures 16, 17, 18, and 19 each show the three potential curves corresponding to the values of parameters shown in Table 3.

Table 3. Parameters For Graphs of Potential. Numerals in Parenthesis Give Class

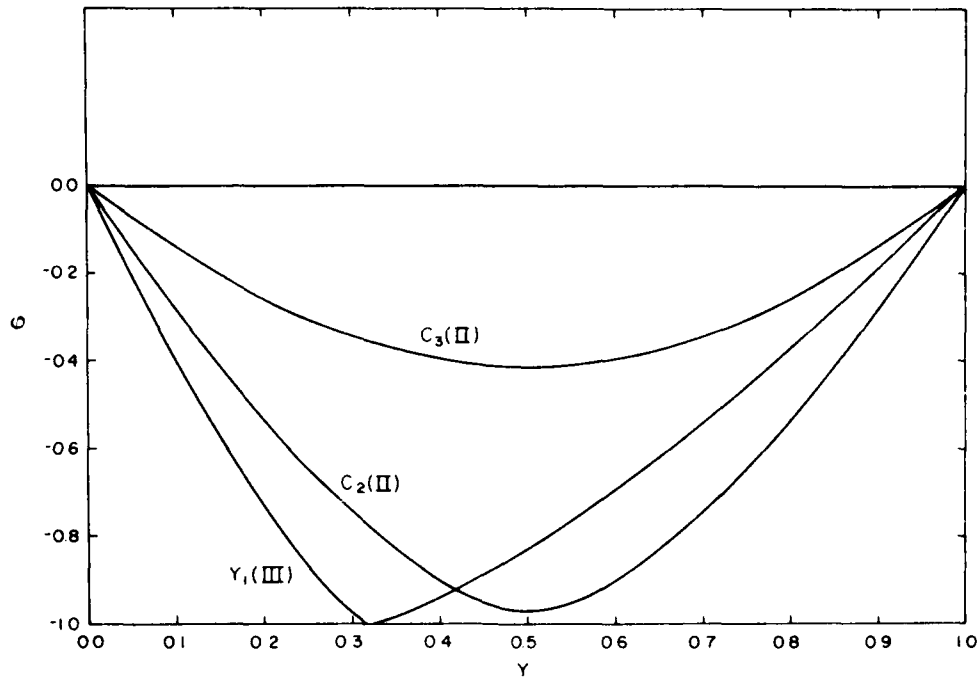| Figure Number	16	17	18	19
Region	4	5	8	9
B	1.0	2.0	0.1	0.6
φ_0	0.0	3.0	-0.99	-0.64
A	6.0	15	1.0	2.1
C_1	---	---	---	---
C_2	-0.9261(II)	-0.1651(II)	---	---
C_3	-4.086 (II)	-13.31 (I)	-0.02545(I)	-1.118(II)
y_1	0.3187	0.2525	0.7092	0.5794
y_1	------	------	0.9672	0.6679
j	0.3591	0.9546	0.0118	0.5813
j	------	------	0.9311	0.9324

Figure 16. Potential Curves Corresponding to Region 4. B = 1.0; $\varphi_0 = 0.0$; A = 6.0
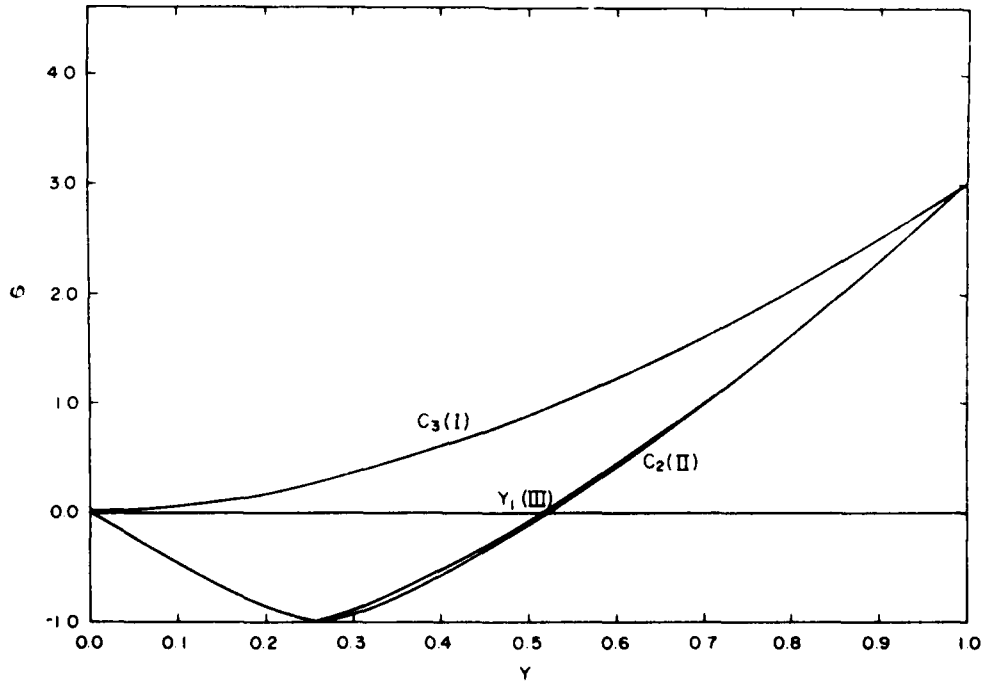

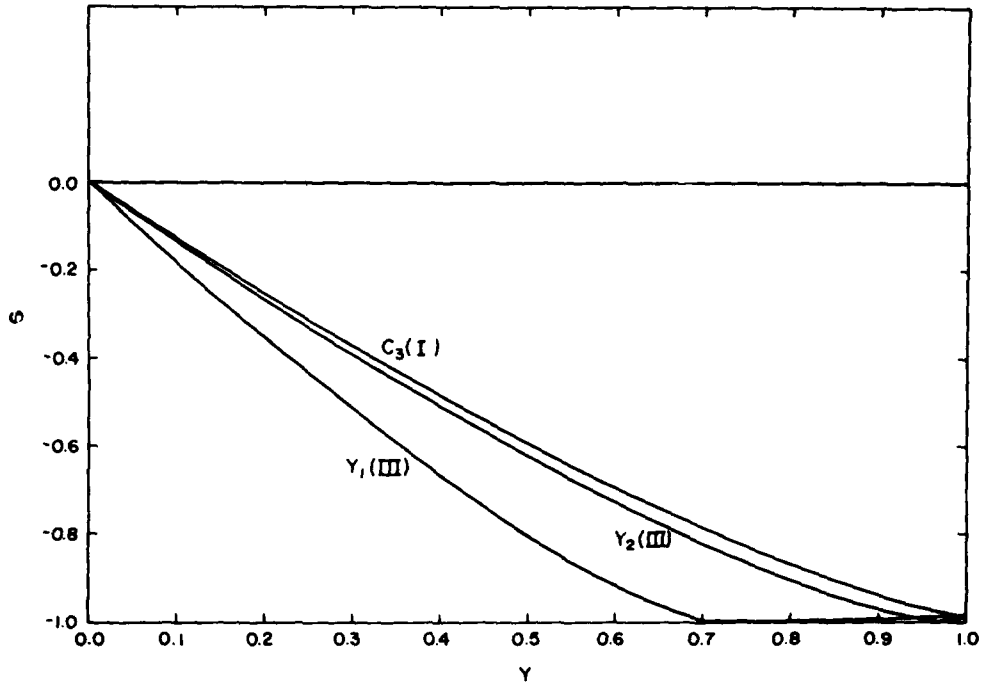Figure 17. Potential Curves Corresponding to Region 5. B = 2.0; $\varphi_0 = 3.0$; A = 15.0

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Figure 18. Potential Curves Corresponding to Region 8. $B = 0.1$; $\varphi_O = -0.99$; $A = 1.0$
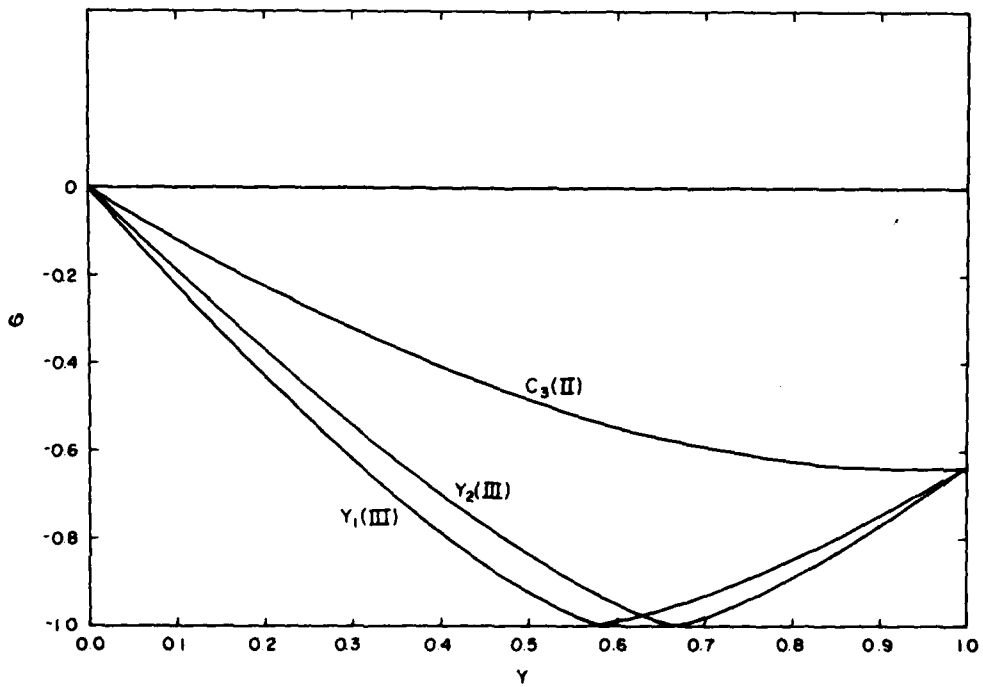

Figure 19. Potential Curves Corresponding to Region 9. $B = 0.6$; $\varphi_O = -0.64$; $A = 2.1$

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Appendix A

Listing of Special Value of A

$$A_K = (B^2 - 1)(B^3 - 1)(B^2 + 1)^{-1}$$

$$A_L^\pm = (B^{3/2} \pm 1)^2$$

$$A_M^+ = (B + 2)^2 (B - 1)$$

$$A_M^- = (2B + 1)^2 (1 - B)$$

$$A_R = (B + 1)^3$$

$$A_{MIN} = \frac{1}{2}(B + 1)^3$$

$$A_1 = (1 - B)^2 (B + 1/2)$$

$$A_2 = (1 - B)^2 (B/2 + 1)$$

$(A_1, A_2, A_R$ are roots of the discriminant of Eq. (7).)