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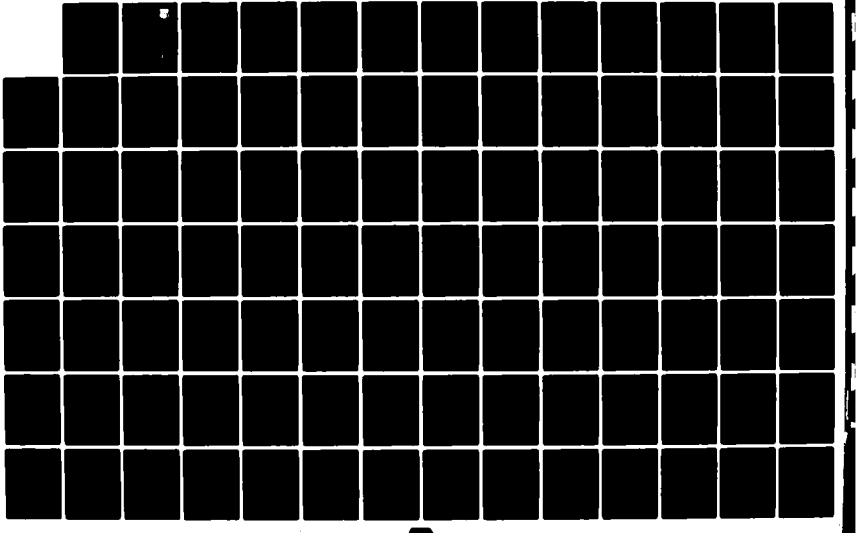

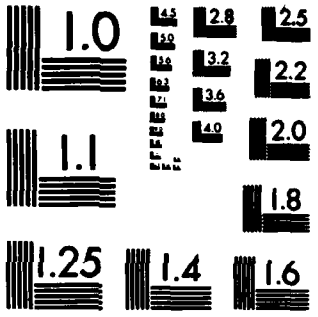

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ASYMPTOTIC FAR FIELD CONDITIONS FOR UNSTEADY SUBSONIC AND TRANSONIC FLOWS

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April 1983
Final Report for period June-December 1982

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This technical report has been reviewed and is approved for publication.

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Far field conditions for unsteady flow at subsonic and transonic Mach numbers are derived under the assumption that the changes that occur in the flow field are not too slow. One then can derive far field conditions from analytic expressions which approximate the behavior of wave fronts and sound rays at large distances. In subsonic flows, one uses for this purpose equations linearized for the vicinity of a parallel

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cont → flow with the free stream Mach number. This leads to the result of Bayliss, Gunzburger, and Turkel, but extended to nonspherical wave fronts. Some theoretical insight from a different point of view is provided by interpreting the wave fronts as characteristic surfaces. The linearization used for subsonic flows fails for free stream Mach number one, because it leads to a piling up of waves in a plane perpendicular to the free stream direction. A linearization is again carried out but with respect to a flow field in which the term of the steady flow, which dominates at a large distance, is taken into account. The analysis requires a considerable amount of analytic work (although a closed analytic representation for the underlying flow field is available), but the resulting formulae for the far field conditions are quite tractable.

PREFACE

This report has been written under Contract F33615-81-K-3216 entitled "Mathematical Questions Related to the Computation of Compressible Flow Field," to the University of Dayton for the Aeroelastic Group, Analysis and Optimization Branch, Structures and Dynamics Division, AFWAL/FIBRC under Project 2304, Task 2304N1, and Program Element 61102F. The work was performed during the period June through December 1982. Dr. Karl G. Guderley of the University of Dayton Research Institute was Principal Investigator. Dr. Charles L. Keller, AFWAL/FIBRC, (513) 255-7384, Wright-Patterson Air Force Base, Ohio 45433 was Program Manager.

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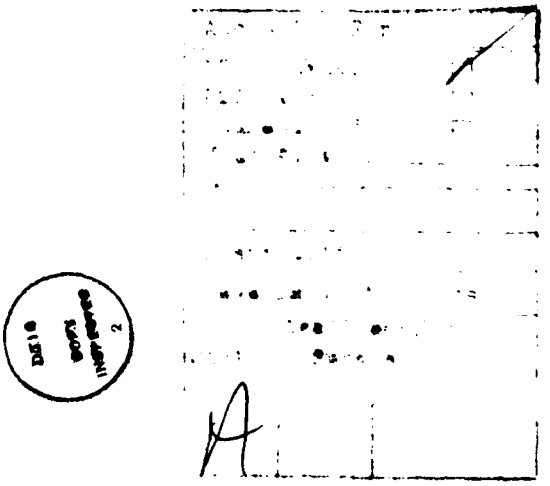

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

SECTION I
INTRODUCTION

In deriving far field conditions for unsteady flows at high subsonic Mach numbers, one usually makes the assumption that in the distant field the deviations from a parallel flow with the given free stream Mach number are small. Two quantities are of importance, the ratio of the velocity perturbations to the free stream velocity of sound and the ratio of the velocity perturbations to the difference between the free stream velocity and the sonic velocity (the velocity for which velocity of sound and velocity of motion are the same). Accordingly, it is not permissible to choose a boundary of the computed part of the flow field for which the local Mach number reaches the value of one. The distance from the profile for which this prerequisite is satisfied increases to infinity as the free stream Mach number approaches one.

The original goal of the research effort reported here was the derivation, by means of asymptotic techniques, of far field conditions for small unsteady perturbations with harmonic time dependence superimposed to a steady flow with the free stream Mach number one. It was planned to illustrate the mathematical technique by applying it to problems with a high subsonic free stream Mach number. This attempt led to some developments beyond those found in the literature. Usually one assumes that the perturbations arriving at the outer edge of the computed flow field can be approximated by cylindrical or spherical waves whose origin is known. (Actually, it must be estimated.) The analysis leads to equations for which the wave fronts may have a more general character, although one must assume that they are rather smooth. The information provided in the standard methods by estimating the location of the origin is then derived from flow field data. Furthermore, it is a rather simple step to proceed from perturbations which are harmonic in time to general unsteady perturbations. Of course, asymptotic results are not always applicable. The requirement that the frequency of harmonic oscillations is fairly

high or that changes of the flow field with time are rather quick is not always satisfied. Moreover, the description of the perturbation field by means of smooth wavefronts is not always possible. Then one must apply far field conditions (of a more complicated nature) in which such assumptions are not made (Ref. 1 and 2). Of theoretical interest is the relation of surfaces of constant phase (which play an important role in the asymptotic considerations) to characteristic surfaces in the space spanned by the space coordinates and the time.

Accordingly, this report consists of two nearly independent parts, one consisting of Sections II to VI which deals with far field conditions at a high subsonic Mach number and the remainder which treats far field conditions for oscillatory flows at a free stream Mach number one.

Regarding the latter we make the following observations. We deal with small unsteady perturbations with harmonic time dependence superimposed to a steady flow with the free stream Mach number one. Because of the so-called freezing of the Mach number distribution in the flow field for free stream Mach numbers in the vicinity of one, the results are also applicable for flow fields with such Mach numbers. In this study the linearization is carried out for the vicinity of a steady flow perturbed by the presence of a body, (while in the derivation for subsonic Mach numbers, one linearizes for the vicinity of a parallel undisturbed flow). It is assumed that the boundary at which the far field conditions are to be applied lies at such a distance that the flow field can be approximated by the dominant term in the development with respect to distance. For these perturbations analytic expressions are available, even for the axisymmetric problem. They have been found, simultaneously by a number of authors (Randall, Mueller and Matschat, and Euvrard, Refs. 3, 4, and 5). We shall use the form given by Randall. The conventional treatment of far field conditions for subsonic flows makes use of analytical expressions for the unsteady linearized far field. For flow fields with a free stream Mach number one, such expressions are not

available. One can dispense with information of this kind by asymptotic considerations. This is a technique used in physics to make the transition from wave to ray optics. The asymptotic expressions are valid if the frequencies are sufficiently high. Such techniques have been applied before for problems of sound radiation (for recent applications see Bayliss, Gunzburger, and Turkel, Ref. 6, 7). The mathematical nature of conventional far field conditions seems to indicate that one comes into the realm of such high frequency approximation as the free stream Mach number approaches one.

SECTION II

THE SUBSONIC PROBLEM, BASIC EQUATIONS, PARTICULAR SOLUTIONS

Let $\bar{x},\bar{y}$ be a system of Cartesian coordinates, $\bar{t}$ the time, and $\bar{\phi}$ the velocity potential which describes the deviations from a parallel subsonic flow. The linearized differential equation which we take as point of departure for the present investigations can be obtained either by linearizing the potential equation for unsteady flow for the vicinity of a steady parallel flow, or by a transformation of the equation for the propagation of sound in air at rest. The differential equation for the latter problem is given by

$$\bar{\phi}_{\bar{x}\bar{x}} + \bar{\phi}_{\bar{y}\bar{y}} - a^{-2}\bar{\phi}_{\bar{t}\bar{t}} = 0 \qquad (1)$$

where "a" is the constant sound velocity. The equation for perturbation in air that moves from right to left with the velocity U is obtained by the following transformation

$$\bar{\phi}(\bar{x},\bar{y},\bar{t}) = \tilde{\phi}(\tilde{x},\tilde{y},\tilde{t})$$

where

$$\begin{aligned} \tilde{x} &= \bar{x} + U\bar{t} \\ \tilde{y} &= \bar{y} \\ \tilde{t} &= \bar{t}. \end{aligned} \qquad (2)$$

One obtains

$$(1-U^2/a^2)\tilde{\phi}_{\tilde{x}\tilde{x}} + \tilde{\phi}_{\tilde{y}\tilde{y}} - 2(U/a^2)\tilde{\phi}_{\tilde{x}\tilde{t}} - (1/a^2)\tilde{\phi}_{\tilde{t}\tilde{t}} = 0. \qquad (3)$$

We introduce the free stream Mach number

$$M = U/a$$

make all length dimensionless with a characteristic length L, for instance the chord of the profile,

$$x = \tilde{x}/L, y = \tilde{y}/L$$

and introduce a dimensionless time

$$t = a\ \tilde{t}/L = a\ \bar{t}/L. \tag{4}$$

Then one has instead of Eq. (2)

$$\bar{x} = L(x - M\ t).$$

With $\hat{\phi}(x,y,t) = \tilde{\phi}(\tilde{x},\tilde{y},\tilde{t})$, one then obtains from Eq. (3)

$$(1-M^2)\hat{\phi}_{xx} + \hat{\phi}_{yy} - 2M\hat{\phi}_{xt} - \hat{\phi}_{tt} = 0 \tag{5}$$

This derivation emphasizes the fact (actually a rather obvious one), that perturbations once introduced behave like perturbations in still air, except that they float downstream with the velocity U. In the last equation, U is replaced by the dimensionless velocity M. We now consider particular solutions which are periodic in time with the circular frequency ν

$$\hat{\phi}(x,y,t) = \phi(x,y)\ \exp(i\nu\bar{t})$$

or

$$\begin{aligned}\tilde{\phi}(\tilde{x},\tilde{y},\tilde{t}) &= \phi(x,y)\ \exp(i(\nu L/U)(Ut/a)) \\ &= \phi(x,y)\ \exp(i\,\mu' Mt) \\ &= \phi(x,y)\ \exp(i\omega t)\end{aligned} \tag{6}$$

In the second version of Eq. (6) the familiar reduced frequency, here denoted by μ', has been introduced

$$\mu' = (\nu L/U) \tag{7}$$

Furthermore, we have set

$$\omega = \mu' M = (\nu L/a) \tag{8}$$

One then obtains

$$(1-M^2)\phi_{xx} + \phi_{yy} - 2i\omega M\phi_x + \omega^2\phi = 0 \tag{9}$$

In Ref. 5 Eq. (9) has been transformed in the following manner. First one carries out the Prandtl Glauert coordinate distortion

$$\begin{aligned} \hat{x} &= x \\ \hat{y} &= (1-M^2)^{1/2} y \end{aligned} \tag{10}$$

One obtains

$$\phi_{\hat{x}\hat{x}} + \phi_{\hat{y}\hat{y}} - 2i\omega M(1-M^2)^{-1}\phi_{\hat{x}} + (1-M^2)^{-1}\omega^2\phi = 0 \tag{11}$$

Next one removes the term with $\phi_{\hat{x}}$ by the transformation

$$\phi(\hat{x},\hat{y}) = \exp(i\omega M(1-M^2)^{-1}\hat{x})\hat{\phi}(\hat{x},\hat{y}) \tag{12}$$

Introducing

$$\mu = (1-M^2)^{-1}\omega = M(1-M^2)^{-1}\mu' \tag{13}$$

One obtains

$$\hat{\phi}_{\hat{x}\hat{x}} + \hat{\phi}_{\hat{y}\hat{y}} + \mu^2\hat{\phi} = 0 \tag{14}$$

This is the Helmholtz equation. Usually the Helmholtz equation arises in the study of periodic perturbations in still air. One then substitutes the following expression into Eq. (1)

$$\bar{\phi}(\bar{x},\bar{y},\bar{t}) = \bar{\bar{\phi}}(\bar{x}/L),(\bar{y}/L))\exp(i\nu t)$$

Using Eq. (8) one obtains

$$\frac{\partial^2\bar{\bar{\phi}}}{\partial(\bar{x}/L)^2} + \frac{\partial^2\bar{\bar{\phi}}}{\partial(\bar{y}/L)^2} + \omega^2\bar{\bar{\phi}} = 0. \tag{15}$$

Notice that Eqs. (14) and (15) have different factors of $\hat{\phi}$ and $\bar{\bar{\phi}}$. The substitute frequency μ tends to infinity as M tends to one. This suggests that, in the vicinity of Mach number one, the high frequency limits of the far field conditions is applicable. The transformations, Eqs. (10) and (12), amount to a rather complicated distortion of the original flow pattern. There is no direct physical relation between the flow fields described by the functions $\hat{\phi}$ and $\bar{\bar{\phi}}$.

Particular solutions of Eq. (14) are given by

$$\hat{\phi}_m(\hat{x},\hat{y}) = H_m^{(2)}(\mu r)\begin{Bmatrix}\cos(m\theta)\\ \sin(m\theta)\end{Bmatrix} \tag{16}$$

Here

$$r = [\hat{x}^2 + \hat{y}^2]^{1/2} = [x^2 + (1-M^2)y^2]^{1/2} \tag{17}$$

$$\left.\begin{aligned}\cos\theta &= \hat{x}/r = x/r\\ \sin\theta &= \hat{y}/r = (1-M^2)^{1/2}y/r\end{aligned}\right\} \quad 0 \leq \theta < 2\pi \tag{18}$$

$H_m^{(2)}$ is the Hankel function of order m, where m is a positive integer or zero.

One has for large values of a complex variable z

$$H_m^{(2)} = \text{const } z^{-1/2} P(z^{-1}) \exp(-iz) \tag{19}$$

where P stands for a semiconverging power series in z^{-1}. The exact form of the constant is unessential in the present context. These particular solutions, partially written in terms of the original coordinates, read

$$\phi = \exp(i\omega M(1-M^2)^{-1}x)$$

$$H_m^{(2)}(\omega(1-M^2)^{-1}(x^2 + (1-M^2)y^2)^{1/2}) \begin{Bmatrix}\cos(m\theta)\\ \sin(m\theta)\end{Bmatrix} \tag{20}$$

and asymptotically

$$\phi = \text{const } r^{-1/2} \exp\{i\omega(1-M^2)^{-1}[Mx - (x^2+(1-M^2)y^2)^{1/2}]\} \begin{Bmatrix}\cos(m\theta)\\ \sin(m\theta)\end{Bmatrix} \tag{21}$$

The factor $(1-M^2)^{-1}$ in the exponential function and in the argument of the Hankel functions makes it obvious that these solutions will fail as the Mach number approaches one.

An intuitive picture can be obtained in the following manner. Assume that one generates perturbations by short pulses spaced at equal time intervals in a flow with Mach number M. We consider the wave front pertaining to each perturbation at some later time t. We mentioned previously that these fronts spread out as in air at rest (that is, with the sound velocity "a") but at the same time move downstream with the dimensionless velocity M. For a Mach number smaller than 1, the system of such wave fronts is drawn in Figure 1. The distance between two adjacent wave fronts is smaller on the upstream side of the point where the perturbations are introduced and larger at the downstream side compared with the wave fronts in air at rest (which would be formed by concentric circles). At a free stream Mach number 1 the upstream distance reduces to zero (Fig. 2). This accumulation of perturbations suggests the difficulties which arises in a flow with the free stream Mach number 1. It is obvious that in such a flow the local

deviations of velocity of sound and of the particle motion from those of the free stream conditions can no longer be disregarded. A discussion where this modification is made will be carried out later.

SECTION III

THE SUBSONIC PROBLEM, HIGH FREQUENCY APPROXIMATION

To obtain a high frequency approximation to solutions of Eq. (9) we make the transformation

$$\phi(x,y) = h(x,y)\ \exp(-i\omega g(x,y)) \tag{22}$$

and satisfy Eq. (9) in the dominant powers of ω.

One has

$$\phi_x=[(h_x/h)-i\omega g_x]\phi$$

$$\phi_{xx} = [(h_{xx}/h)-2(h_x/h)i\omega g_x - \omega^2 g_x^2 - i\omega g_{xx}]\phi$$

$$\phi_{yy} = [(h_{yy}/h)-2(h_y/h)i\omega g_y - \omega^2 g_y^2 - i\omega g_{yy}]\phi.$$

Then from Eq. (9)

$$\begin{aligned} &-\omega^2[(1-M^2)g_x^2 + g_y^2 + 2Mg_x - 1] \\ &-i\omega h^{-1}[(1-M^2)2h_x g_x + 2h_y g_y + 2Mh_x + h((1-M^2)g_{xx} + g_{yy})] \\ &+h^{-1}[(1-M^2)h_{xx} + h_{yy}\ ] = 0. \end{aligned} \tag{23}$$

The dominant terms, (those with the factor ω^2) determine the function g(x,y). One obtains the equation

$$(1-M^2)g_x^2 + g_y^2 + 2Mg_x - 1 = 0. \tag{24}$$

The function h(x,y) depends upon ω. Including one term beyond the lowest order approximation we set

$$h(x,y,\omega) = H_1(x,y) + \omega^{-1}H_2(x,y) \tag{25}$$

The terms of order ω and of order 1 in Eq. (23) give respectively

$$H_{1,x}[(1-M^2)g_x+M] + H_{1,y}g_y + (1/2)H_1[(1-M^2)g_{xx}+g_{yy}] = 0 \qquad (26)$$

$$H_{2,x}[(1-M^2)g_x+M] + H_{2,y}g_y + (1/2)H_2[(1-M^2)g_{xx} + g_{yy}] + (i/2)[(1-M^2)H_{1,xx} + H_{1,yy}] = 0. \qquad (27)$$

Eq. (24) is a first order partial differential equation in one dependent variable. Applying a standard procedure one differentiates Eq.(24) with respect to x and y

$$[(1-M^2)g_x + M]g_{xx} + g_y g_{xy} = 0$$

and (28)

$$[(1-M^2)g_x + M]g_{xy} + g_y g_{yy} = 0.$$

We denote by D differentiation in a direction given by

$$(dy/dx) = g_y/[(1-M^2)g_x + M] \qquad (29)$$

Let $D\ell$ be the line element. Then one has

$$\frac{Dx}{D\ell} = \frac{(1-M^2)g_x + M}{[g_y^2 + ((1-M^2)g_x+M)^2]^{1/2}} \qquad (30)$$

$$\frac{Dy}{D\ell} = \frac{g_y}{[g_y^2 + ((1-M^2)g_x+M)^2]^{1/2}}$$

$$\frac{D}{D\ell} = \frac{Dx}{D\ell}\frac{\partial}{\partial x} + \frac{Dy}{D\ell}\frac{\partial}{\partial y} \qquad (31)$$

Curves with the slope given by Eq. (29) are called characteristics. Then from Eqs. (28)

$$Dg_x/D\ell = 0$$

and

$$Dg_y/D\ell = 0. \tag{32}$$

Furthermore

$$Dg/D\ell = g_x(Dx/D\ell) + g_y(Dy/D\ell) \tag{33}$$

and, from Eqs. (26) and (27)

$$D(\log H_1)/D\ell + (1/2)[(1-M^2)g_{xx} + g_{yy}]/[g_y^2 + ((1-M^2)g_x+M)^2]^{1/2}=0$$

$$DH_2/D\ell+(1/2)H_2[(1-M^2)g_{xx}+g_{yy}]/[g_y^2 +((1-M^2)g_x + M)^2]^{1/2} \tag{34}$$

$$+ (i/2)[(1-M^2)H_{1,xx}+H_{1,yy}]/[g_y^2 + ((1-M^2)g_x + M)^2]^{1/2} = 0$$

It follows from Eqs. (32), that along the characteristics g_x = const and g_y = const and subsequently, from Eq. (29), that these curves are straight lines.

For a preliminary discussion, we consider characteristics which start at the origin. Then

$$Dy/Dx = y/x$$

and because of Eq. (29)

$$g_y = (y/x)[(1-M^2)g_x + M]$$

One obtains, by substituting this expression into Eq. (24) and by solving the resulting quadratic equation,

$$g_x = (1-M^2)^{-1}[-M + x/(x^2 + (1-M^2)y^2)^{1/2}]$$

$$g_y = y/(x^2 + (1-M^2)y^2)^{1/2} \tag{35}$$

Moreover, if g = 0 for x = 0 and y = 0

$$g = g_x x + g_y y$$

$$g = (1-M^2)^{-1}[-Mx + (x^2 + (1-M^2)y^2)^{1/2}] \tag{36}$$

Consider a curve g(x,y) = K (where K is some constant). One finds from the last equation that such a curve satisfies

$$(x - K\,M)^2 + y^2 = K^2. \tag{37}$$

This is a circle with a radius |K| and center at the point (x,y) = (KM,0). For K positive these are the circles shown in Figures 1 and 2.

The limit M = 1 is readily discussed. One obtains

$$g_y = y/|x|$$

$$g_x = (1-M^2)^{-1}[-M + (x/|x|)]. \tag{38}$$

Hence

$$g_x = (1+M)^{-1} = 1/2, \quad x > 0$$

$$\lim_{M\to 1} g_x = \lim_{M\to 1} -(1-M)^{-1} = \infty, \quad x < 0. \tag{39}$$

Actually, for M = 1 no perturbations can arrive at stations x < 0.

Eq. (37) shows that the factor exp (-iωg(x,y)) in Eq. (22) anticipates the waviness of the flow field. The function h(x,y), gives a modulation of the amplitude and since it may be complex also some modification of phase. The phase modulation is expected to be small. The evaluation of H_1 is best carried out on the basis of Eq. (26) rather than Eq. (34). One obtains for the coefficient of H_1

$$(1/2)[(1-M^2)g_{xx} + g_{yy}] = (1/2)r^{-1}$$

where according to the definition in Eq. (17)

$$r = [x^2 + (1-M^2)y^2]^{1/2}.$$

Using the second of Eqs. (35) and Eq. (29), one obtains from Eq. (26), after multiplication by $r\,H_1^{-1}$

$$(H_{1,x}/H_1)x + (H_{1,y}/H_1)y + (1/2) = 0.$$

This equation is solved by

$$H = f(x/y)[x^2+y^2]^{-1/4}. \tag{40}$$

Using Eq. (22) and Eq. (36) one thus obtains the following approximation for ϕ

$$\begin{aligned}\phi &= f(x/y)(x^2+y^2)^{-1/4}\exp(-i\omega g(x,y) \\ &= f(x/y)(x^2+y^2)^{-1/4}\exp[i\omega(1-M^2)^{-1}(Mx - (x^2+(1-M^2)y^2)^{1/2}].\end{aligned} \tag{41}$$

This is the first term in the asymptotic expressions for exact solutions shown in Eq. (21). Comparing Eqs. (41) and (21) one obtains

$$f(x/y) = [(x^2 + y^2)/(x^2+(1-M^2)y^2]^{1/4}\begin{Bmatrix}\cos(m\theta)\\ \sin(m\theta)\end{Bmatrix} \tag{42}$$

where θ is defined in Eq. (18).

The asymptotic expression, Eq. (21), arises from the development of the Hankel functions. The argument of the Hankel functions tends to infinity even as M tends to 1 for finite x and y. This justifies the high frequency approximation. The definition for θ in Eq. (18) shows that for $M \to 1$, y finite and positive and $x > 0$, θ tends to zero, and for $M \to 1$, y finite and $x < 0$, θ tends to π. For M close to 1 the function $f(x/y)$ therefore changes very rapidly in the vicinity of $x = 0$. This corresponds to the accumulation of perturbations shown in Fig. 2. Notice also that the function $f(x/y)$ has a strong peak at $x = 0$ for $M \to 1$ and assumes the value $(1-M^2)^{-1/4}$.

In preparation for a generalization, we rewrite these equations in terms of the direction normal to the wave fronts (given by curves g = const) and in terms of the radius of curvature. It was shown above, that lines g = const are circles which float downstream with the dimensionless velocity M; at time t the circles have the dimensionaless radius R = t. Let α be the angle of the normal to the wave front with the x-axis at some point of the wave front (Fig. 3) and let β be the angle of the ray AB with the x-axis. Then one has

$$
\begin{aligned}
x &= R(M + \cos\alpha) \\
y &= R \sin\alpha \\
R_1 &= (x^2 + y^2)^{1/2} = R(M^2+2M\cos\alpha +1)^{1/2} \\
\operatorname{tg}\beta &= y/x = \sin\alpha/(M+\cos\alpha) \\
\sin\beta &= \sin\alpha\,(M^2+2M\cos\alpha + 1)^{-1/2} \\
\cos\beta &= (M + \cos\alpha)(M^2 + 2M\cos\alpha + 1)^{-1/2}.
\end{aligned}
\tag{43}
$$

In Eqs. (17) and (18) the quantities r and θ have been defined. One obtains

$$
\begin{aligned}
r &= R(1+M\cos\alpha) \\
\operatorname{tg}\theta &= (1-M^2)^{1/2}\operatorname{tg}\beta
\end{aligned}
\tag{44}
$$

The expression g, Eq. (36), reduces to

$$g = R$$

With a modified function f, one then obtains from Eq. (41)

$$\phi = \bar{f}(\alpha) R^{-1/2} \exp(-i\omega R) \tag{45}$$

The direction of the characteristic is given by a line θ = const, or

$$Dx/Dy = \sin\alpha/(M + \cos\alpha) \tag{46}$$

To visualize what happens in a more general case, in which the initial line g = const is not a circle but a general closed curve, one uses the idea that lines g = const at a later time arise from an initial line in the same manner as in air at rest, except that they float downstream with the dimensionless velocity M. We already found that the characteristics are straight lines in any case. For a point on a line g = const with normal given by α, the slope of the characteristic is then given by Eq. (46).

The above description suggests that, along a specified characteristic, ϕ is determined by the curvature of the initial line g = const in the same manner as for perturbations emanating from a circle. This motivates the following representation; let the inital curve g = const be given by

$$x = x_o(s)$$
$$y = y_o(s) \tag{47}$$

where s is the arc length

$$(dx_o/ds)^2 + (dy_o/ds)^2 = 1. \tag{48}$$

Then the angle of the normal to this curve, denoted again by α, is given by

$$\cos\alpha = dy_o/ds$$

$$\sin\alpha = -(dx_o/ds)$$

one has

$$g_x = |\text{grad } g|\cos\alpha$$

$$g_y = |\text{grad } g|\sin\alpha. \tag{49}$$

By substituting the above equations into Eq. (24) one obtains

$$|\text{grad } g| = 1/(1+M\cos\alpha)$$

and therefore

$$g_x = \cos\alpha\ /(1+M\cos\alpha)$$

$$g_y = \sin\alpha\ /(1+M\cos\alpha) \tag{50}$$

The direction of the characteristics is given by Eq. (29). One has for the denominator

$$(1-M^2)g_x + M = (M+\cos\alpha)/(1+M\cos\alpha)$$

Therefore

$$Dy/Dx = \sin\alpha/(M+\cos\alpha) \tag{51}$$

With a parameter p one then obtains the following parametric representation for the characteristic

$$x = x_o + (p-p_o)(M + \cos\alpha)$$
$$y = y_o + (p-p_o)\sin\alpha \qquad (52)$$

We have denoted the arc length of a characteristic by $d\ell$. One obtains from the last equation

$$d\ell/dp = (M^2 + 2M\cos\alpha + 1)^{1/2}. \qquad (53)$$

Furthermore,

$$dg/dp = g_x(dx/dp) + g_y(dy/dp) = 1. \qquad (54)$$

It is shown in Appendix II, that p can be identified with the radius of curvature of the line g = const, to be denoted by R. The function H_1 is determined from Eq. (26). Using Eq. (50) and Eq. (34) one obtains

$$(d(\log H_1)/d\ell) + (1/2)[(1-M^2)g_{xx}+g_{yy}](1+M\cos\alpha)(1+2M\cos\alpha + M^2)^{-1/2}=0 \qquad (55)$$

In Appendix II it is shown that this leads to

$$d(\log H_1)/dp + (1/2)(1/p) = 0 \qquad (56)$$

Hence

$$H_1 = f_1(s)\ p^{-1/2}$$

For circular waves p is given by

$$p = R = R_1/(1 + 2M\cos\alpha + M^2)^{1/2} \qquad (57)$$

in other words, p is proportional to R_1 with a factor of proportionality which is different for different characteristics. It is also shown that

$$H_2 = f_2(s)p^{-1/2}.$$

In the derivation of the expression for H_2 in Appendix II the assumption has been made that p is large (while p_1 is not too large). The last formulae therefore gives a good approximation only at a sufficient distance from the initial curve. The parameter s characterizes the individual characteristics.

We shall use Eq. (22) in the asymptotic form of Eq. (41), but somewhat extended

$$\phi = [f_1(s)R_1^{-1/2} + f_2(s)R_1^{-3/2}]\exp(-i\omega g) \tag{58}$$

With Eq. (57) and modified functions f_1 and f_2 one can also write

$$\phi = [f_1(s)p^{-1/2} + f_2(s)p^{-3/2}]\exp(-i\omega g). \tag{58'}$$

SECTION IV
ASYMPTOTIC FAR FIELD CONDITIONS

Asymptotic far field conditions are obtained from Eq. (58') in the following manner. One forms the derivative in the direction of the characteristic. Then no derivatives of the unknown functions $f_1(s)$ and $f_2(s)$ will appear. Using Eq. (54) one obtains

$$D\phi/D\ell = (Dp/D\ell)\,[f_1(s)\,(-(1/2)p^{-3/2}-i\omega p^{-1/2})+f_2(s)\,(-(3/2)p^{-5/2}-i\omega p^{-3/2})]\exp(-i\omega g) \quad (59)$$

The expression ϕ of Eq. (58') then satisfies

$$[\frac{D}{D\ell} + \frac{Dp}{D\ell}(\frac{5}{2p} + i\omega)][\frac{D}{D\ell} + \frac{Dp}{D\ell}(\frac{1}{2p} + i\omega)]\phi = 0. \quad (60)$$

The second operator (which is applied first) makes the term $f_1(s)p^{-1/2}(-i\omega p)$ zero and changes the expression $f_2(s)p^{-3/2}\exp(i\omega g)$ into $-\,dp/d\ell\; f_2(s)^{-5/2}\exp(-i\omega g)$. The first part of the operator (which is applied afterwards) causes this term to vanish. For large values of p (or of ω) one may disregard the term $p^{-3/2}$ and then one obtains the simple far field condition

$$[\frac{D}{D\ell} + \frac{Dp}{D\ell}(\frac{1}{2p} + i\omega)]\phi = 0. \quad (61)$$

Also, the term $1/(2p)$ may be omitted if ω is sufficiently large

$$[\frac{D}{D\ell} + (\frac{Dp}{D\ell}\, i\omega)]\phi = 0. \quad (62)$$

The last formulation corresponds to the original formulation of Sommerfeld. Eqs. (61) and (62) are different forms of the far field conditions of Bayliss, Gunzberger, and Turkel (Ref. 6,7).

The expression Eq. (60) written in detail gives

$$\frac{D^2\phi}{D\ell^2} + \frac{Dp}{D\ell}(\frac{3}{p} + 2i\omega)\frac{D\phi}{D\ell} + (\frac{Dp}{D\ell})^2[\frac{3}{4p^2} + \frac{3}{p}\, i\omega - \omega^2]\phi = 0. \quad (63)$$

The occurrence of a second derivative in the boundary conditions for a second order differential equation is unusual. In the present problem second derivatives cannot be avoided, but they can be transformed by means of the differential equation into second derivatives along the contour of the computed region. If one uses higher order boundary conditions of the Bayliss, Gunzberger, Turkel type, then even higher derivatives, first in the direction of the characteristics and, after a transformation, along the contour of the computed region are encountered. Exact boundary conditions for the problem have been derived in Ref. 5; they are of a global nature (that is, all boundary points interact with each other). The occurrence of higher derivatives means that the solution in some vicinity of the point under consideration enters the formulation. In this manner the underlying global character of such far field conditions makes itself felt.

Some further details are added. Let β be the angle of the direction of the characteristic with the x axis. Then, according to Eq. (43),

$$\begin{aligned} \sin\beta &= \sin\alpha\,(M^2 + 2M\cos\alpha + 1)^{-1/2} \\ \cos\beta &= (M + \cos\alpha)\,(M^2 + 2M\cos\alpha + 1)^{-1/2}. \end{aligned} \tag{64}$$

(By our definition α is the angle of the normal to the line g = const with the x-axis.) It may be desirable to express various quantities in the terms of the angle β. We introduce an auxiliary angle δ (Fig. 3).

Then

$$\begin{aligned} \sin\delta &= M\sin\beta \\ \alpha &= (\beta + \delta) \\ (M^2 + 2M\cos\alpha + 1)^{1/2} &= M\cos\beta + \cos\delta \\ DR/D\ell = Dp/D\ell &= (M\cos\beta + \cos\delta)^{-1} \end{aligned} \tag{65}$$

Now we express the derivative $d^2\phi/d\ell^2$. We introduce a local system of coordinates ξ, η, where ξ has the direction of the characteristic, η the direction of the tangent to the boundary of the computed region. Then $d/d\ell = d/d\xi$. Let, at the point under consideration, γ be the angle of the η axis with the x axis (Figure 4).

Then

$$x - x_o = \xi \cos\beta + \eta \cos\gamma$$

$$y - y_o = \xi \sin\beta + \eta \sin\gamma$$

and

$$\begin{aligned}
\phi_{\xi\xi} &= \phi_{xx}\cos^2\beta + \phi_{xy}\sin 2\beta + \phi_{yy}\sin^2\beta \\
\phi_{\eta\eta} &= \phi_{xx}\cos^2\gamma + \phi_{xy}\sin 2\gamma + \phi_{yy}\sin^2\gamma \qquad (66) \\
\phi_{\xi\eta} &= \phi_{xx}\cos\beta\cos\gamma + \phi_{xy}\sin(\beta+\gamma) + \phi_{yy}\sin\beta\sin\gamma.
\end{aligned}$$

In addition, one has the differential equation for ϕ (Eq.(9))

$$2i\omega M\phi_x - \omega^2\phi = (1-M^2)\phi_{xx} + \phi_{yy}.$$

Eliminating ϕ_{xy}, ϕ_{yy}, and ϕ_{xy} from these four equations one obtains

$$\begin{aligned}
&\phi_{\xi\xi}[1-M^2\sin^2\gamma] + \phi_{\eta\eta}(1-M^2\sin^2\beta) - \phi_{\xi\eta}[2\cos(\beta-\gamma) + M^2(\cos(\beta+\gamma) - \cos(\beta-\gamma))] \\
&\qquad - (2i\omega M\phi_x - \omega^2\phi)\sin^2(\beta-\gamma) = 0. \qquad (67)
\end{aligned}$$

This equation is verified in Appendix III. It allows one to express $d^2\phi/d\ell^2 = \phi_{\xi\xi}$ in terms of derivatives of ϕ and ϕ_ξ in the η direction (that is in the direction of the contour of the computed part of the flow field).

In practice, one will choose the boundary of the computed part of the flow field so that the η direction coincides with either the x or the y direction. Then considerable simplification will occur.

These results can be applied in two different forms. In a procedure, which corresponds to that of Bayliss, et al., one assumes that all waves originate at the same origin (somewhere in the middle of the profile). The direction of the characteristics is then given by rays through the origin. The simplest form is obtained from Eq. (58)

$$\phi = [f_1(s)R_1^{-1/2} + f_2(s)R_1^{-3/2}]\exp(-i\omega g).$$

where

$$R_1 = (x^2 + y^2)^{1/2} \tag{68}$$

and according to Eq. (36)

$$g = (1-M^2)^{-1}[-Mx + (x^2 + (1-M^2)y^2)^{1/2}]. \tag{69}$$

According to the derivation of Eq. (36) one has

$$Dg/D\ell = g_x(Dx/D\ell) + g_y(Dy/D\ell) = g_x(x/R_1) + g_y(y/R_1)$$

$$Dg/D\ell = g/R_1$$

According to Eq. (54), one has $dg/dp = 1$. Therefore, from Eq. (62)

$$(D\phi/D\ell) + i\omega(g/R_1)\phi = 0 \tag{70}$$

and from Eq. (61)

$$(D\phi/D\ell) + [i\omega(g/R_1) + (1/(2R_1))]\phi = 0$$

The counterpart to Eq. (63) is

$$(D^2\phi/D\ell^2)+[(3/R_1)+(2i\omega g/R_1)](D\phi/D\ell)+[(3/4R_1^2)+(3i\omega g/R_1^2)-(g\omega/R_1)^2]\phi=0. \tag{71}$$

A second form of the far field condition arises, if one determines the likely origin for the waves which arrive at some point of the contour of the computed flow field from the function ϕ and its derivatives along the contour. One proceeds in the following manner. The analysis is based on the assumption that the function h changes only slowly. In principle, h and ϕ are allowed to be complex. Then one has

$$\arg(\phi) = \arg(h) - \omega g$$

and

$$g = \omega^{-1}[\arg(h) - \arg(\phi)]$$

locally arg(h) can be replaced by a constant.

From this expression one obtains g_x and g_y along the contour of the computed region by numerical differentiation, and hence

$$\cos\alpha = g_x/(g_x^2 + g_y^2)^{1/2}$$
$$\sin\alpha = g_y/(g_x^2 + g_y^2)^{1/2} \tag{72}$$

As a check (which may lead to small corrections) one has from the first of Eqs. (50),

$$|\text{grad } g| = (1 + M\cos\alpha)^{-1}$$

The fact that g = p is identical with the radius of curvature of a line g = const, is hard to apply in a practical computation, because one needs interpolations to identify these lines.

In Appendix II the following formulae have been derived

$$g_{xx} = R^{-1}(1 + M\cos\alpha)^{-3}\sin^2\alpha$$

$$g_{yy} = R^{-1}(1 + M\cos\alpha)^{-3}(M + \cos\alpha)^2$$

$$g_{xy} = R^{-1}(1 + M\cos\alpha)^{-3}(-\sin\alpha(M + \cos\alpha)).$$

These equations can be rewritten as

$$g_{xx}/|\text{grad } g| = R^{-1}(1 + M\cos\alpha)^{-2}\sin^2\alpha$$

$$g_{yy}/|\text{grad } g| = R^{-1}(1 + M\cos\alpha)^{-2}(M + \cos\alpha)^2$$

$$g_{xy}/|\text{grad } g| = R^{-1}(1 + M\cos\alpha)^{-2}(-\sin\alpha(M + \cos\alpha)).$$

Assume that the boundary of the computed region is given by a line x = const. Along this line one can determine g_{yy} and g_{xy} (if one allows the use of first derivative in the direction of the normal to this line. Thus, one obtains two expressions for R (which in some way must be reconciled with each other). If one uses only derivatives along the boundary, then one has only g_{yy} at one's disposal.

Equivalent to this formulation is the following approach. One determines along the boundary line the value of α from Eq. (72). Assume again that the boundary is a line x = const. One can then form $\partial\alpha/\partial y$ numerically. (Since α has a direct geometrical meaning, one will be able to judge whether α is sufficiently accurate and smooth.) The radius of curvature is then obtained in the following manner. If, as in Eqs. (47) etc., ds is the line element along the contour g = const, then one has

$$R^{-1} = d\alpha/ds = -\alpha_x\sin\alpha + \alpha_y\cos\alpha$$

Furthermore since α = const along a characteristic

$$\alpha_x(M + \cos\alpha) + \alpha_y \sin\alpha = 0.$$

Hence

$$R^{-1} = \alpha_y(1 + M\cos\alpha)(M + \cos\alpha)^{-1} \qquad (74)$$

or

$$R^{-1} = -\alpha_x(1 + M\cos\alpha)(\sin\alpha)^{-1}$$

In these formulae (R^{-1}) is obtained by numerical differentiation along the contour of the computed region. Equations (73) and (74) are basically the same; this is shown in the following manner

$$\alpha = \text{arctg}\,(g_y/g_x)$$

$$\alpha_y = \frac{g_{yy}g_x - g_{xy}g_y}{(g_x^2 + g_y^2)} = \frac{g_{yy}\cos\alpha - g_{xy}\sin\alpha}{|\text{grad } g|} = (g_{yy}\cos\alpha - g_{xy}\sin\alpha)(1+M\cos\alpha)$$

Then from Eq. (74)

$$R^{-1} = (g_{yy}\cos\alpha - g_{xy}\sin\alpha)(1 + M\cos\alpha)^2(M + \cos\alpha)^{-1}$$

Substituting here g_{yy} and g_{xy} from Eq. (73) one obtains indeed an identity.

In the last equation one can eliminate the mixed derivative. From the fact that along a characteristic g_x and g_y are const, one obtains

$$g_{xx}\cos\beta + g_{xy}\sin\beta = 0$$

$$g_{xy}\cos\beta + g_{yy}\sin\beta = 0$$

hence, with Eq. (64)

$$g_{xy} = -g_{yy} \sin \alpha (M + \cos \alpha)^{-1}$$

and

$$g_{xx} = g_{yy} \sin^2 \alpha (M + \cos \alpha)^{-2}.$$

We make the following observation. The first form of the far field conditions, given by Eqs. (62) through (65), can be derived from the asymptotic development of the Hankel functions, Eq. (19). In practice, one uses only one or two terms of the development with respect to z^{-1}. The representation for fixed $r\omega$ deteriorates with m (whose meaning can be recognized from Eq. (16). The asymptotic far field conditions (in either form) therefore are applicable only at the outer contour where the amplitude of ϕ does not change too rapidly and if $r\omega$ is not too small.

The far field condition in the original form of Bayliss, et. al, require that one first carry out the transformation (Eq. (12)); in the present formulation one uses directly the original form (Eq. (9)) of the partial differential equation. Whether it is worthwhile to carry out the transformation Eq. (12), also in the region close to the profile where the differential equation is more complicated, depends upon practical considerations. The two formulations are equivalent.

In one formulation of the exact far field conditions (which have global character) (Ref. 1) one replaces the effect of periodic perturbations in the computed field by unknown periodic perturbations at the contour of the computed flow field. The local intensity of these perturbations is one of the unknowns of the problems. The field at a fixed point of the outer contour of the computed flow field appears as a superposition of perturbations at all other points of the contour. This is, of course, a much more general approach than the asymptotic form, which requires that the perturbations behave as if they came from one point. The potential at some point of the contour is then expressed by an integral which contains contributions of the

entire contour. The normal derivative is expressed by such an integral but in addition, it contains a term which depends upon the local strength of the perturbations. In this formulation the normal derivative (rather than the derivative in the direction of the characteristics) appears.

For the purpose of determining far field conditions, the idea of a nonreflecting wall has been put forth. If one uses a local formulation of the boundary conditions, a wall can be made nonreflecting only for waves of a selected direction. The condition of Bayliss et al., gives a wall which is nonreflecting for waves that come from the origin. In the author's opinion this, rather than the direction normal to the wall, is the appropriate choice. It is worth noting that in this formulation the direction of the wall enters only if one uses approximations of a higher order.

SECTION V

GENERAL UNSTEADY PERTURBATIONS IN A SUBSONIC PARALLEL FLOW

Equations (60) through (62) can serve to derive asymptotic far field conditions for more general unsteady perturbations in a subsonic flow, if the changes with time are rather rapid. This generalization is desirable for the following reason.

Upstream of the profile where the waves propogate againsts the oncoming flow, they move fairly slowly. Under these circumstances the dependence of the speed of motion and of the local sound speed upon the amplitudes of the waves become important. Portions of the waves where the pressure is higher travel faster than the average. This leads to a distortion. Even if at the point where they are generated (say at the profile) the waves are sinusoidal in time, they lose this property as they travel over some distance (although they remain periodic). This effect is not taken into account in a linearized approach where the unsteady perturbations are considered as small; there the speed of wave propagation is solely determined by the properties of the underlying steady field. The wave amplitudes for which the linearized approach is sufficient become smaller as the Mach number approaches one. This state of affairs is clearly seen, if one studies the propagation of one-dimensional waves in a tube. Accordingly, it may be desirable to take nonlinear effects into account within the computed part of the flow field. It can be assumed that the far field is still governed by the linearized equation for unsteady flows, only the assumption that the waves are sinusoidal in time will be abandoned.

Asymptotic far field conditions for such flows are obtained in the following manner. The expression Eq.(58') combined with Eq. (6) gives

$$\tilde{\phi} = (f_1(s)p^{-1/2} + f_2(s)p^{-3/2})\exp(i\omega(t-p)) \qquad (75)$$

where s denotes a parameter which is constant along a characteristic and p identifies the station along the individual characteristics;

p has been identified with the local radius of curvature R of the lines g = const. At the moment, it is assumed that one knows the angle α which gives the direction of the normal to the wave fronts and therefore also the angle β which gives the direction of the characteristics (Eq. (64)).(Notice that Eq. (64) does not contain the frequency ω). We denoted by $d\ell$ the line element along a characteristic. We have found in Eqs. (54) and (65)

$$\begin{aligned} dp/d\ell &= dR/d\ell = (M^2 + 2M\cos\alpha + 1)^{-1/2} \\ &= (M\cos\beta + \cos\gamma)^{-1} \end{aligned} \tag{76}$$

For simplicity, we omit temporarily in Eq. (75) the contributions of f_2. Differentiating Eq. (75) with respect to t and $d\ell$ one obtains

$$\partial\tilde{\phi}/\partial t = f_1 i\omega p^{-1/2}\exp(i\omega(t-p)) \tag{77}$$

$$\partial\tilde{\phi}/\partial\ell = f_1(\partial p/\partial\ell)[-i\omega p^{-1/2} - (1/2)p^{-3/2}]\exp(i\omega(t-p))$$

Combining the equations with Eq. (75) and writing R instead of p, one obtains the following equation which is free of ω

$$(\partial\tilde{\phi}/\partial\ell) + (\partial R/\partial\ell)[(\partial\tilde{\phi}/\partial t) + (2R)^{-1}\phi] = 0 \tag{78}$$

Equation (78) is obtained directly from Eq. (61), if one remembers that each factor $i\omega$ arises by a differentiation with respect to t. This observation allows one to derive far field conditions which take higher order terms of the asymptotic development into account immediately from Eq. (60)

$$\left[\frac{\partial}{\partial\ell} + \frac{\partial R}{\partial\ell}\left(\frac{5}{2R} + \frac{\partial}{\partial t}\right)\right]\left[\frac{\partial}{\partial\ell} + \frac{\partial R}{\partial\ell}\left(\frac{1}{2R} + \frac{\partial}{\partial t}\right)\right]\phi = 0$$

or in more detail from Eq. (63)

$$\partial^2\check{\phi}/\partial\ell^2 + (\partial R/\partial\ell)[(3/R)(\partial\check{\phi}/\partial\ell) + 2(\partial^2\check{\phi}/\partial\ell\partial t)]$$

$$+ (\partial R/\partial\ell)^2[(3/4R^2)\check{\phi} + (3/R)(\partial\check{\phi}/\partial t) + \partial^2\check{\phi}/\partial t^2] = 0 \tag{79}$$

Again Eq. (67) can be used to express $\partial^2\check{\phi}/\partial\ell^2$, which in the notation used there is identical with $\partial^2\check{\phi}/\partial\xi^2$, in terms of derivatives along the boundary of the computed part of the flow field.

So far, the angle α which gives the direction of the normal to the lines g = const has been assumed to be known. Here assumptions analogous to those of the preceding section must be made. If one assumes that the origin of the waves is known (some point in the vicinity of the profile, then one obtains the necessary formulae by replacing in Eqs. (70), (71), and (72), $i\omega$ by $\partial/\partial t$. One obtains

$$\begin{aligned}
&(\partial\check{\phi}/\partial\ell) + (g/R_1)(\partial\check{\phi}/\partial t) = 0\\
&(\partial\check{\phi}/\partial\ell) + (g/R_1)(\partial\check{\phi}/\partial t) + (1/2R_1)\check{\phi} = 0\\
&(\partial^2\check{\phi}/\partial\ell^2) + (3/R_1)(\partial\check{\phi}/\partial\ell) + 2(g/R_1)(\partial^2\check{\phi}/\partial\ell\partial t)\\
&\quad +(3/4R_1^2)\check{\phi} + (3g/R_1^2)\partial\check{\phi}/\partial t + (g/R_1)^2(\partial^2\check{\phi}/\partial t^2) = 0
\end{aligned} \tag{80}$$

It is possible to derive in this general case, at least approximately, the normal to the wave fronts. In the case that one has sinusoidal perturbations, we assumed that along a line g = const the amplitude of ϕ and also its phase changes only slowly. Under the present circumstances this amounts to the assumption that

$$\text{grad } \phi \sim \text{grad } g.$$

One then obtains

$$\alpha = \text{arctg } (\phi_y/\phi_x) \tag{81}$$

This defines α except if simultaneously $\phi_y = \phi_x = 0$. Excluding the vicinity of such points, one obtains the value of α and therefore also the local values of p in the same manner as before.

One can then apply directly Equations (74) viz

$$R^{-1} = \alpha_y(1 + M \cos \alpha)(M + \cos \alpha)^{-1} \qquad (82)$$

or

$$R^{-1} = - \alpha_x(1 + M \cos \alpha)(\sin \alpha)^{-1}.$$

Alternatively, one can use one of the Eqs. (73) to determine R. The factor of proportionality between g and ϕ cancels under the assumption made here; one can replace, for instance,

$$g_{xx}/|\text{grad } g| \text{ by } \phi_{xx}/|\text{grad } \phi|.$$

SECTION VI
CHARACTERISTIC SURFACES IN THE x,y,t SPACE

The derivation of far field conditions for unsteady flows shown in the preceding section is somewhat indirect. The following discussion is an attempt to throw light from a different direction on these results. As far as the practical aspects are concerned, it adds nothing, but it gives the connection of the present approach with other mathematical concepts.

One observes that the surfaces of the x,y,t space given by

$$t = g(x,y) + \text{const} \tag{83}$$

are characteristic surfaces for the partial differential equation (Eq.(5)). The derivation is found in Appendix IV. If along a noncharacteristic surface the values of ϕ_x, ϕ_y, and ϕ_t are given, then one can determine by means of the differential equation the second (and higher) derivatives of ϕ, in terms of derivatives formed within the surface. In contrast, these derivatives are not uniquely determined for a characteristic surface. As a consequence it is possible that along a characteristic surface, discontinuities of the second and higher derivatives of ϕ will occur. At a fixed point (x,y,t) the orientations of these surfaces are determined by the differential equations for g, namely

$$(1-M^2)g_x^2 + 2Mg_x + g_y^2 = 1 \tag{84}$$

We introduce coordinates ξ,η within one of the characteristic surfaces, ξ and η are identical with the x and y coordinates of a point (x,y,t) of the surface. Let

$$U = \phi_x,\ V = \phi_y,\ W = \phi_t. \tag{85}$$

Then one has, (since $t = g(x,y)$)

$$U_\xi = \phi_{xx} + \phi_{xt} g_x$$

$$V_\eta = \phi_{yy} + \phi_{yt} g_y \qquad (86)$$

$$W_\xi = \phi_{xt} + \phi_{tt} g_x$$

$$W_\eta = \phi_{yt} + \phi_{tt} g_y .$$

In order for second derivatives of ϕ to exist, the values of U, V, and W within the surface must be connected by the compatibility condition

$$-(1-M^2)U_\xi - V_\eta + (2M + (1-M^2)g_x)W_\xi + g_y W_\eta = 0. \qquad (87)$$

This derivation is found in Appendix IV.

A plot of lines g = const in the x,y-plane can be interpreted in two different manners. If one considers t in Eq. (83) as fixed, then these curves give the intersection of characteristic surfaces in the x,y,t space with a plane of constant t. If one considers the constant in Eq. (83) as fixed, then the curves represent a contour map of one characteristic surface where t corresponds to the altitude.

Assume now that for a given time a curve $g(x,y) = c_o$ is given. To construct the characteristic surface one must find the curves g = const for other values of t. For this purpose one can use the results found above. Consider points (x,y) reached from a certain point (x_o, y_o) of the starting curve $g(x,y) = c_o$ by traveling along straight lines given by

$$(x-x_o) = (p-p_o)(M + \cos \alpha)$$

$$(y-y_o) = (p-p_0)\sin \alpha$$

where α is the angle of the normal to the curve with the x-axis. These are the directions given by Eq. (52).

Since, according to Eq. (54)

$$dg/dp = 1 \tag{89}$$

one obtains a point of the curve $g = c_1$ by setting $p-p_o = c_1-c_o$. The same point is reached, if one proceeds from the original point in the x-direction by a distance $M(c_1-c_o)$ and then in the direction of the normal by the distance c_1-c_o. This is in accordance with the description, given in Section II, that a wave front spreads as in still air except that the perturbation floats downstream with the (dimensionless) velocity M.

For a further discussion, we consider all possible values of g_x and g_y (compatible with Eq. (84)), they are obtained by letting α vary in Eq. (88) from 0 to 2π. The locus of the points $g = c_1$ which are reached from point x_o,y_o is then given by

$$\begin{aligned} x &= x_o + (c_1 - c_o)(M + \cos\alpha) \\ y &= y_o + (c_1 - c_o)\sin\alpha. \end{aligned} \tag{90}$$

This is a circle with center $x_o + M(c_1 - c_o)$ and radius $(c_1 - c_o)$. In the x,y,t space, these circles are the contour lines of an oblique cone with tip at point (x_o,y_o) and a generator given by the above circle. At the point of the pertinent characteristic

$$x = x_o + (c_1 - c_o)(M + \cos\alpha)$$

and

$$y = y_o + (c_1 - c_o)\sin\alpha$$

the circle drawn for a fixed point of the line $g = c_1$ and the line $g = c_1$ have the same tangent; consequently, the cone mentioned above is tangent to the characteristic surface in the x,y,t space. The lines of tangency are called bi-characteristics. The characteristics in the x,y-plane with which our considerations started are the projections into the x,y-plane of the bi-characteristics in the

x,y,t plane. The line $g = c_1$ can be obtained by letting the initial line float in the x-direction by the distance $M(c_1 - c_o)$, by drawing circles with radius $c_1 - c_o$ and centers at the curve so shifted, and finally, by determining the envelope of these circles. According to the description given here, each characteristic surface is determined by the initial line g = const. If two such lines are tangent to each other at some point (x_o, y_o), then they will be tangent to each other along the bi-characteristic that starts at this point.

It is natural to introduce in the compatibility condition (Eq. (87)) instead of ξ and η, the derivatives in the direction of the bi-characteristics and of the curve $g = c_o$. We denote by ξ_1, the projection of the bi-characteristic into the x,y-plane, and by η_1, the coordinate in the direction of the tangent to the line g = const. The projection of the bi-characteristic into the x,y-plane is identical with the characteristic lines considered above. The angle of ξ_1 with the ξ-axis (which coincides with the x-axis) is, therefore, given by the angle β, determined by Eq. (43). The angle of the normal to the contour is denoted as before by α. One then has

$$\begin{aligned} \xi &= \xi_1 \cos\beta - \eta_1 \sin\alpha \\ \eta &= \xi_1 \sin\beta + \eta_1 \cos\alpha. \end{aligned} \tag{91}$$

Hence

$$\begin{aligned} \frac{\partial}{\partial \xi_1} &= \cos\beta \frac{\partial}{\partial \xi} + \sin\beta \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \eta_1} &= \sin\alpha \frac{\partial}{\partial \xi} + \cos\alpha \frac{\partial}{\partial \eta} \end{aligned} \tag{92}$$

and

$$\frac{\partial}{\partial \xi} = (\cos\alpha \frac{\partial}{\partial \xi_1} - \sin\beta \frac{\partial}{\partial \eta_1})/\cos(\beta-\alpha)$$

$$\frac{\partial}{\partial \eta} = (\sin\alpha \frac{\partial}{\partial \xi_1} + \cos\beta \frac{\partial}{\partial \eta_1})/\cos(\beta-\alpha). \tag{93}$$

This and the expression for g_x and g_y from Eq. (50) are now substituted into Eq. (87); one obtains

$$-(1-M^2)\cos\alpha(\partial U/\partial \xi_1) - \sin\alpha(\partial V/\partial \xi_1) + (1 + M\cos\alpha)(\partial W/\partial \xi_1)$$

$$+ (M^2+2M\cos\alpha + 1)^{-1/2}[\sin\alpha(1-M^2)(\partial U/\partial \eta_1) - (M + \cos\alpha)(\partial V/\partial \eta_1)$$

$$-M\sin\alpha(\partial W/\partial \eta_1)] = 0 \tag{94}$$

If one changes α by π then one obtains a second bi-characteristic. Let ξ_2 be its direction. The corresponding value of β (denoted by β_2) is then given by

$$\cos\beta_2 = (M - \cos\alpha)(1 - 2M\cos\alpha + M^2)^{-1/2}$$

$$\sin\beta_2 = -\sin\alpha(1 - 2M\cos\alpha + M^2)^{-1/2}. \tag{95}$$

The direction $d\eta_1$ is changed into $-d\eta_1$. One then obtains

$$(1-M^2)\cos\alpha(\partial U/\partial \xi_2) + \sin\alpha(\partial V/\partial \xi_2) + (1-M\cos\alpha)(\partial W/\partial \xi_2)$$

$$+ (M^2-2M\cos\alpha + 1)^{-1/2}[\sin\alpha(1-M^2)(\partial U/\partial \eta_1) + (M-\cos\alpha)(\partial V/\partial \eta_1) \tag{96}$$

$$- M\sin\alpha(\partial W/\partial \eta_1] = 0.$$

In preparation for a discussion of far field conditions, we now discuss the following configuration. We consider a characteristic surface $t = g(x,y)+c_o$ (with c_o fixed), in the vicinity of a point x,y. For time $t > g(x,y)+ c_o$ let ϕ be identical to zero. At the characteristic surface as everywhere else, ϕ is continuous, but its first derivative changes very rapidly as far as this is compatible

with continuous $\tilde{\phi}$. For $t > g(x,y) + c_o + \epsilon$, the first derivatives reach some fixed (or nearly fixed) value. Ultimately, we allow ϵ to tend to zero, then a jump of some first derivative propagates along the characteristic surface. In this situation the derivatives of U, V, and W (ϕ_x, ϕ_y, and ϕ_t) with respect to η (that is within the characteristic surface), remain bounded. The derivatives along the bicharacteristic which belongs to the same normal to the characteristic surface and which passes through this surface becomes very large. We integrate Eq. (96) with respect to ξ_2, the path of integration goes from the undisturbed region through the jump region to a point shortly beyond it. Let [U], [V], and [W] denote the jumps of these quantities; they are actually identical with U, V, and W at the point beyond the jump region. Let the expression containing derivatives with respect to η temporarily be denoted by Q_2. Q_2 remains bounded even in the jump region. One then obtains

$$(1-M^2)\cos\alpha[U] + \sin\alpha[V] + (1-M\cos\alpha)[W] + \int Q_2 d\xi_2 = 0. \qquad (97)$$

In the limit as $\epsilon \to 0$, the length of ξ_2 over which the integration is extended becomes zero. One thus obtains

$$(1-M^2)\cos\alpha\phi_x + \sin\alpha\phi_y + (1-M\cos\alpha)\phi_t = 0. \qquad (98)$$

Because of the continuity of ϕ, and since $\phi \equiv 0$, before the jump the gradient of ϕ is normal to the curve $g(x,y) = t-c_o$, for any fixed time. One remembers that α gives the normal to lines $g(x,y) =$ const. Therefore,

$$\begin{aligned} \phi_x &= |\text{grad}\phi| \cos\alpha \\ \phi_y &= |\text{grad}\phi| \sin\alpha. \end{aligned} \qquad (99)$$

Thus, one obtains from Eq. (98)

$$(1 + M\cos\alpha)\ |\text{grad}\phi| + \phi_t = 0. \qquad (100)$$

We introduce the direction of the characteristics in the x,y-plane; we have found that such characteristics are the projection of the bicharacteristics (of the characteristic surface) into the x,y plane. The angle of the characteristics with the x-axis has been denoted by β; the line element of the characteristic by $d\ell$. Since ϕ = const along the line $g(x,y) = t-c_o$, one has

$$\frac{d\phi}{d\ell} = |\text{grad}\phi| \cos(\beta-\alpha) \tag{101}$$

and with Eq. (64)

$$\frac{d\phi}{d\ell} = |\text{grad}\phi|(1+M\cos\alpha)(M^2+2M\cos\alpha+1)^{-1/2} \tag{102}$$

Equation (100) can then be written in the form

$$(\partial\phi/\partial\ell) + .(M^2 + 2M\cos\alpha + 1)^{-1/2}(\partial\phi/\partial t) = 0 \tag{103}$$

This is Eq. (78) combined with Eq. (77) but with the term $(2R^{-1})$ omitted. For a jump of $|\text{grad}\phi|$ along a characteristic surface the last formula is exact.

In general, the expression $\int R d\xi_2$ which vanishes for a jump will play a role. The line along which one integrates intersects the characteristic surfaces $t = g(x,y)+c$. In an oscillatory flow these are surfaces of constant phase. But for ω large the phase changes rapidly as one passes from one such surface to the next one (this happens if one moves in the direction ξ_2). The derivatives with respect to η_1 (which are derivatives along the different line g = const) remain bounded. One thus obtains an integral with a strongly oscillatory integrand; as ω increases the oscillations become narrower. Hence it follows, that in a low order approximation in ω^{-1}, the integral can be disregarded. It is perhaps possible to estimate this integral (using integration by parts and the second mean value theorem), but actually the derivation of higher order formulae in the manner shown in Section V is simpler.

The last discussions have shown that the far field conditions can be interpreted as an expression for the compatibility condition for the bi-characteristics of a special characteristic surface, defined by the postulate, that for a fixed t it intersects the characteristic surface of constant phase $t = g(x,y) + c_o$, along the curve $g(x,y) = t - c_o$. This special choice is necessary in order to ensure that the factor R in the integral $\int R d\xi_2$ remains bounded.

In Eq. (103) the radius of curvature of the line g(x,y) = const does not appear. The results agree with Eq. (78) only for $R \to \infty$. In other words, one uses a relation for plane waves. The dependence on the parameter p (which is identical with the radius of curvature R) and which would be needed in order to derive higher order approximations can be obtained on the basis of the last formula, if one uses in addition the compatibility conditions for the surface $t = g(x,y) + c_o$.

First we express the fact that at this surface ϕ = const. Since α gives the direction of normal to the line g(x,y) in the x,y-plane and since ϕ = const, one has:

$$\begin{aligned} \phi_x &= |\text{grad}\phi| \cos\alpha \\ \phi_y &= |\text{grad}\phi| \sin\alpha \end{aligned} \tag{104}$$

where $|\text{grad}\phi|$ refers to the function $\phi(x,y,t)$ for constant t. Along the intersection of a plane y = const with $t = g(x,y) + c_o$ one has

$$dt = g_x dx.$$

Therefore, since $\phi = 0$

$$\phi_t + \phi_x \frac{dx}{dt} = \phi_t + g_x^{-1} \phi_x. \tag{105}$$

Hence,

$$\phi_t = -g_x^{-1} \, |\text{grad}\phi| \cos \alpha$$

and with Eq. (50)

$$\phi_t = -|\text{grad}\phi| \, (1+M \cos\alpha) \tag{106}$$

Our next task is the determination of the derivatives with respect to η_1 in Eq. (94). The expression to be evaluated is

$$Q_1 = [\sin\alpha(1-M^2)(\partial U/\partial\eta_1) - (M+\cos\alpha)(\partial V/\partial\eta_1) - M\sin\alpha(\partial W/\partial\eta_1)]$$

$$[M^2 - 2M\cos\alpha + 1]^{1/2} \tag{107}$$

with

$$U = \phi_x = |\text{grad}\phi| \cos \alpha$$

$$V = \phi_y = |\text{grad}\phi| \sin \alpha$$

$$W = \phi_t = |\text{grad}\phi| \, (1+M \cos\alpha).$$

The derivatives of U, V, and W in the expression Q_1, therefore, are expressed in terms of derivatives of $|\text{grad}\phi|$ and of α with respect to η_1. The coefficient of

$$\frac{d}{d\eta_1} \; |\text{grad}\phi|$$

turns out to be zero, for

$$\sin\alpha(1-M^2)\cos\alpha - (M + \cos\alpha)\sin\alpha + M\sin\alpha(1+M\cos\alpha) = 0.$$

Therefore,

$$Q_1 = -|\text{grad}\phi| \frac{d\alpha}{d\eta_1} \, (1+M\cos\alpha)(M^2 + 2M\cos\alpha + 1)^{1/2}.$$

In a similar manner, one obtains for the ξ_1 derivatives in Eq. (94)

$$-(1-M^2)\cos\alpha(\partial U/\partial\xi_1) - \sin\alpha(\partial V/\partial\xi_1) + (1+M\cos\alpha)(\partial W/\partial\xi_1) =$$

$$-2[\partial|\mathrm{grad}\phi|/\partial\xi_1](1+M\cos\alpha)$$

No terms $\partial\alpha/\partial\xi_1$ occur. This is in keeping with Eq. (32), which shows that along a characteristic (here bi-characteristic) g_x, g_y and consequently also α are constant. One also observes that

$$d\alpha/d\eta_1 = R^{-1}$$

(where as before R is the radius of curvature of a line g = const). One thus obtains from the compatibility condition Eq. (94)

$$-2(d|\mathrm{grad}\phi|/d\xi_1) - R^{-1}|\mathrm{grad}\phi|(M^2 + 2M\cos\alpha + 1)^{-1/2} = 0$$

In Appendix II we have identified R with a parameter p, the line element $d\xi_1$ corresponds to $d\ell$. Using Eq. (53) one then obtains

$$d(\log|\mathrm{grad}\phi|)/dp + (1/2p) = 0.$$

Hence,

$$|\mathrm{grad}\phi| = \text{const } p^{-1/2} = \text{const } R^{-1/2}. \tag{108}$$

From Eqs. (104) and (105)

$$\begin{aligned} \phi_x &= \text{const } R^{-1/2}\cos\alpha \\ \phi_y &= \text{const } R^{-1/2}\sin\alpha \\ \phi_t &= \text{const } R^{-1/2}(1 + M\cos\alpha) \end{aligned} \tag{109}$$

The constant in these three expressions is the same but may vary from characteristic to characteristic. According to Eq. (77)

$$dR/d\ell = (M^2 + 2M\cos\alpha + 1)^{-1/2}.$$

Note that these expressions satisfy the Sommerfeldt radiation condition (Eq. (78)) with the term $(2R^{-1})$ omitted. This can be shown with the aid of Eqs. (102) and (77).

The conditions (103) arise from bi-characteristics which cross the characteristic surface along which a jump of $|\text{grad}\phi|$ propagates. Here the contribution of Q_2 vanishes in the limit of a sudden change of $|\text{grad}\phi|$. In this equation, no contributions of R appears. The expressions Eq. (108) and (109) are derived from the relations for bi-characteristics lying directly above the characteristic surface at which the jump occurs. Here the term Q_1, the counterpart of Q_2, is not negligible in comparison to other terms of the compatibility condition. This fact is responsible for the dependence upon the radius of curvature R of the line $g(x,y) = \text{const}$.

SECTION VII
THREE-DIMENSIONAL PROBLEM

The three-dimensional problem has actually more practical significance than the two-dimensional problem. Most of the necessary formulae can be fairly easily derived by using the analogy with the two-dimensional problem. Eq. (22) is now replaced by

$$\phi(x,y,z) = h(x,yz)\ \exp(-i\omega g(x,y,z)). \tag{110}$$

Equation (24) is replaced by

$$(1-M^2)g_x^2 + g_y^2 + g_z^2 + 2Mg_x - 1 = 0. \tag{111}$$

Let D denote differentiation in the direction of a characteristic, and let $D\ell$ be the line element of a characteristic. With unit vectors $\vec{e}_x$, $\vec{e}_y$, and $\vec{e}_z$ respectively in the x,y, and z directions, one obtains

$$\begin{aligned}\vec{e}_x(Dx/D\ell) + \vec{e}_y(Dy/D\ell) + \vec{e}_z(Dz/D\ell) = \\ = \text{const}\ [\vec{e}_x((1-M^2)g_x+M) + \vec{e}_y g_y + \vec{e}_z g_z]\end{aligned} \tag{112}$$

Then one finds, that along the characteristics

$$Dg_x/D\ell = Dg_y/D\ell = Dg_z/D\ell = 0.$$

Therefore, g_x,g_y, and g_z are constants. The characteristics are straight lines. Let (n,x), (n,y), and (n,z) be, respectively, the angles of the normal to the surface g = const with the x, y, and z axes. One has, of course,

$$\cos^2(n,x) + \cos^2(n,y) + \cos^2(n,z) = 1 \tag{113}$$

Equation (50) suggests that

$$
\begin{aligned}
|\text{grad } g| &= (1 + M\cos(n,x))^{-1} \\
g_x &= \cos(n,x)(1 + M\cos(n,x))^{-1} \\
g_y &= \cos(n,y)(1 + M\cos(n,x))^{-1} \\
g_z &= \cos(n,z)(1 + M\cos(n,x))^{-1}.
\end{aligned}
\tag{114}
$$

On readily verifies that this is the solution of Eq. (111). It depends upon the parameters $\cos(n,x)$, $\cos(n,y)$, and $\cos(n,z)$. Because of Eq. (113), there are actually only two free parameters. One also verifies that Eq. (112) is satisfied if one sets

$$
\begin{aligned}
x &= x_o + p\,(\cos(n,x) + M) \\
y &= y_o + p\,\cos(n,y) \\
z &= z_o + p\,\cos(n,z)
\end{aligned}
\tag{115}
$$

where p is a parameter whose meaning so far is not specified (in the two-dimensional case, p has been identified with the radius of curvature of the curve g = const at the point under consideration; in the three-dimensional case, one does not have a single radius of curvature). One then has

$$
D\ell/Dp = (M^2 + 2M\cos(n,x) + 1)^{1/2} \tag{116}
$$

$$
Dg/Dp = g_x(\cos(n,x) + M) + g_y\cos(n,y) + g_z\cos(n,z) = 1 \tag{117}
$$

and hence

$$
(Dg/D\ell) = (Dg/Dp)(Dp/D\ell) = (M^2 + 2M\cos(n,x) + 1)^{-1/2}. \tag{118}
$$

If all the characteristics emanate from the origin, then one has

$$x = p\,(\cos(n,x) + M)$$
$$y = p\cos(n,y) \qquad (119)$$
$$z = p\cos(n,z);$$

$$g = g_x x + g_y y + g_z z = p\,. \qquad (120)$$

A surface g = const is a sphere with radius p and center M p, for one has as a consequence of Eqs. (114) and (119)

$$(x - Mp)^2 + y^2 + z^2) = p^2. \qquad (121)$$

In analogy with Eq. (36), we set,tentatively,for this case

$$g(x,y,z) = (1-M^2)^{-1}[-Mx + (x^2 + (1-M^2)(y^2 + z^2))^{1/2}]. \qquad (122)$$

Then

$$g_x = (1-M^2)^{-1}\{-M + x(x^2 + (1-M^2)(y^2 + z^2))^{-1/2}]\}$$
$$g_y = y[x^2 + (1-M^2)(y^2 + z^2)]^{-1/2}$$
$$g_z = z[x^2 + (1-M^2)(y^2 + z^2)]^{-1/2}.$$

Then one can verify that Eq. (111) is satisfied. Next one must determine H. In principle, one could return to the counterpart of Eq. (26) and evaluate the coefficients of this equation. But one can also refer to the counterpart of the particular solutions, Eq. (16). They are spherical harmonics, and they have the form

$$r^{-1}P(r^{-1})\exp(-i\mu r)$$

with

$$r = (x^2 + y^2 + z^2)^{1/2}. \qquad (123)$$

P is the generic expression for a power series. The analog to Eq. (58) suggests the following approximations for ϕ

$$\phi=[f_1(x/R_1,y/R_1,z/R_1)R_1^{-1} + f_2(x/R_1,y/R_1,z/R_1)R_1^{-2}]\exp(-i\omega g(x,y,z)). \tag{124}$$

Along a characteristic, the functions f_1 and f_2; along with the values of g_x, g_y, and g_z, are constant. This leads to the counterpart of Eq. (60)

$$(\frac{D}{D\ell} + \frac{3}{R_1} + i\omega \frac{Dg}{D\ell})(\frac{D}{D\ell} + \frac{1}{R_1} + i\omega \frac{Dg}{D\ell})\phi = 0 \tag{125}$$

and the simpler (and less accurate forms)

$$[\frac{D}{D\ell} + \frac{1}{R_1} + i\omega \frac{Dg}{D\ell}]\phi = 0 \tag{126}$$

and

$$[\frac{D}{D\ell} + i\omega \frac{Dg}{D\ell}]\phi = 0. \tag{127}$$

Here

$$R_1 = (x^2 + y^2 + z^2)^{1/2}. \tag{128}$$

The direction of the characteristics is given by the rays through the origin. One has

$$\frac{Dg}{D\ell} = \frac{g}{R_1} \tag{129}$$

and g is given in Eq. (122).

For general unsteady flows with waves originating from the origin, one than obtains the far field conditions

$$[\frac{D}{D\ell} + \frac{3}{R_1} + \frac{g}{R_1}\frac{\partial}{\partial t}][\frac{D}{D\ell} + \frac{1}{R_1} + \frac{g}{R_1}\frac{\partial}{\partial t}]\phi = 0. \tag{130}$$

or the simpler forms

$$[\frac{D}{D\ell} + \frac{1}{R_1} + \frac{g}{R_1}\frac{\partial}{\partial t}]\phi = 0 \tag{131}$$

$$[\frac{D}{D\ell} + \frac{g}{R_1}\frac{\partial}{\partial t}]\phi = 0. \tag{132}$$

The second derivative which occurs in the above equation is expressed by

$$D^2/D\ell^2 = (x^2/R_1^2)(\partial^2/\partial x^2) + (y^2/R_1^2)(\partial^2/\partial y^2) + (z^2/R_1^2)(\partial^2/\partial^2)$$
$$+2(xy/R_1^2)(\partial^2/\partial x\partial y) + 2(xz/R_1^2)(\partial^2/\partial x\partial z) + 2(yz/R_1^2)(\partial^2/\partial y\partial z). \tag{133}$$

Assume that one boundary of the computed flow field is a plane $x = \text{const}$. From the values of ϕ and ϕ_x within this plane, one determines ϕ_{yy}, ϕ_{zz}, ϕ_{yz}, ϕ_{xz}, ϕ_{xy}, ϕ_t, and ϕ_{tt}. ϕ_{xx} is expressed in terms of ϕ_{yy}, ϕ_{zz}, ϕ_{xt}, and ϕ_{tt} by the differential equation for ϕ.

Equations (124) through (133) give far field conditions for which one estimates the origin of the waves (in essence, the surfaces $g = \text{const}$). The location of the origin determines the direction of the characteristics (the ray through the origin of the wave), the value of $R_1 = (x^2 + y^2 + z^2)^{1/2}$. One obtains

$$g/R_1 = (1-M^2)^{-1}[-Mx + (x^2+(1-M^2)(y^2+z^2)^{1/2}][x^2+y^2+z^2]^{-1/2}.$$

This approximation may sometimes be unsatisfactory. In three-dimensional problems at an intermediate distance from the airplane, surfaces $g = \text{const}$ are more likely to resemble ellipsoids than spheres. Then one may proceed as follows. We consider sinusoidal perturbations. One identifies surfaces $g = \text{const}$ with surfaces of equal phase of ϕ. One determines by numerical differentiation

$$\cos(n,x) = g_x/|\text{grad } g|$$
$$\cos(n,y) = g_y/|\text{grad } g| \qquad (134)$$
$$\cos(n,z) = g_z/|\text{grad } g|.$$

One ought to have, according to Eq. (114)

$$|\text{grad } g| = (1 + M\cos(n,x))^{-1} \qquad (135)$$

This gives a check (and probably also an adjustment) of the numerical results. The characteristic direction then follows from Eq. (115)

$$Dx/D\ell = (\cos(nx) + M)/(M^2 + 2M\cos(n,x) + 1)^{1/2}$$
$$Dy/D\ell = \cos(ny)/ (M^2 + 2M\cos(n,x) + 1)^{1/2}$$
$$Dz/D\ell = \cos(nz)/ (M^2 + 2M\cos(n,x) + 1)^{1/2}$$

This defines all quantities occurring in the Sommerfeldt far field condition, (the generalization of Eq. (132))

$$D\phi/D\ell + i\omega Dg/D\ell\phi = 0.$$

The next higher approximation which corresponds to Eq. (131) requires the evaluation of the function h in Equation (110). Proceeding in analogy with the two-dimensional case (Eqs. (25) and (26)), one arrives at the analog to Eq. (34)

$$D(\log H_1)/D\ell + (1/2)[1-M^2)g_{xx}+g_{yy}+g_{zz}]/[g_y^2+g_z^2 +((1-M^2)g_x+M)^2]^{1/2}=0.$$

Using Eqs. (134) and (135), one obtains (136)

$$D(\log H_1)/D\ell +(1/2)[1-M^2)g_{xx}+g_{yy}+g_{zz}][1+M\cos(nx)[M^2+2M\cos(nx)+1]^{-1/2}.$$

Then one obtains from the approximate representation for ϕ

$$\phi = H_1 \exp(-i\omega g)$$

$$D\phi/D\ell + [i\omega(Dg/D\ell) - D(\log H_1)/D\ell]\phi = 0.$$

$(\log H_1)D$ is evaluated from Eq. (136). This requires the determination of the factor $(1-M^2)g_{xx} + g_{yy} + g_{zz}$, which can be done numerically. A geometrical interpretation is formed in Eq. (A.31), but it is preferable to express directly that along the characteristics, g_x, g_y, and g_z are constant.

$$g_{xx}(M + \cos(n,x)) + g_{xy}\cos(n,y) + g_{xz}\cos(n,z) = 0$$

$$g_{xy}(M + \cos(n,x)) + g_{yy}\cos(n,y) + g_{yz}\cos(n,z) = 0$$

$$g_{xz}(M + \cos(n,x)) + g_{yz}\cos(n,y) + g_{zz}\cos(n,z) = 0.$$

We describe the procedure for a boundary surface given by a plane x = const. There one finds g_x, g_y, and g_z by numerical differentiation. This allows one to determine cos(n,x), cos(n,y), and cos(n,z). By further differentiation within this plane (that is with respect to y and z) one can express

$$g_{yy},\ g_{yz},\ \text{and}\ g_{zz}.$$

Then one obtains g_{xy} and g_{xz} from the last two equations

$$g_{xy} = -(g_{yy}\cos(n,y) + g_{yz}\cos(n,z))/(M + \cos(n,x))$$

$$g_{xz} = -(g_{yz}\cos(n,y) + g_{zz}\cos(n,z))/(M + \cos(n,x))$$

and from the first equation

(137)

$$g_{xx} = (g_{yy}\cos^2(n,y) + 2g_{yz}\cos(n,y)\cos(n,z) + g_{zz}\cos^2(n,z)/M+\cos(n,x))^2.$$

This allows the evaluation of $d(\log H_1)/d\ell$, Eq. (136).

An interpretation of the expression

$$(1/2)[1-M^2)g_{xx} + g_{yy} + g_{zz}][1 + M\cos(nx)]$$

which occurs on the right hand side of the equation for $D(\log H_1)D\ell$ is given at the end of Appendix II. Using Eq. (135) one recognizes that it represents the average curvature of the surface g = const at the point under consideration. The average curvature is obtained in the following manner. One forms the curvature of two curves which arise from the intersection of the surface g = const with two planes through the normal to this surface which are perpendicular to each other, and then forms the average. One of these planes can be chosen arbitrarily, the other one is then determined. The average curvature is independent of the orientation of these planes. For a certain orientation, the radii of curvature assumes simultaneously extreme values. Let these values be $R_{1,0}$ and $R_{2,0}$ for the surface $g = g_o$, and assume that for this surface p = 0. The dependence upon the parameter p along a fixed characteristic is then given by

$$C_{average} = (1/2)(R_{10}+p)^{-1}+(R_{20}+p)^{-1}.$$

One recognizes that the expression (136) contains certain terms of higher order in p^{-1}.

With Eq. (116) one can now rewrite Eq. (136)

$$D(\log H_1)/dp = -\frac{1}{2}[(R_{10}+p)^{-1}+(R_{20}+p)^{-1}].$$

Hence,

$$H_1 = \text{const}\ [(R_{10} + p)(R_{20} + p)]^{-1/2}. \qquad (138)$$

The expression $[(R_{10}+p)(R_{20}+p)]^{-1/2}$ is the Gaussian curvature of the surface g = const at the point under consideration. The constant may change from characteristic to characteristic. For p large one obtains

$$H_1 = \text{const } p^{-1}.$$

This is the form suggested by the particular solutions (123). The present form of the far field conditions is somewhat more general. In the two-dimensional case one of the principal radii of curvature (and the constant in Eq. (138)) are infinite. Then one obtains the results of the preceding section

$$H_1 = \text{const } p^{-1/2}.$$

The estimation of the H_2 carried out in the two-dimensional case, is valid only for p large. Then one has for the three-dimensional case $R_1 \sim R_2$, and

$$H_2 \sim \text{const } (R_1 R_2)^{-1}.$$

On this basis it would be possible to formulate approximations to the far field conditions of the next order. However, one may have doubts whether in a practical case this refinement is justified. The derivations are based on the assumption that surfaces of constant g can be approximated by surfaces of constant phase of ϕ. This is only an approximation. The application of these ideas to general unsteady flows presupposes again, that the computed flow permits one to recognize wave fronts (which define the surfaces g = const). In critical cases it is probably preferable to use far field conditions which are more complicated but do not require the assumption of high frequency or the identification of wave fronts or wave origins.

SECTION VIII

OSCILLATORY PERTURBATIONS IN A PLANE FLOW FIELD WITH A FREE STREAM MACH NUMBER ONE

Figure 2 shows the wave patterns obtained in the linearized treatment of subsonic flows, which arises in the limit where the free stream Mach number goes to one. The perturbations pile up at the value of x where the waves originate. This happens because the sound velocity which governs the manner in which perturbations spread through the flow field is considered as constant throughout the flow field. In reality, the profile generates, even at a free stream Mach number one, a subsonic region upstream of the profile in which perturbations will travel upstream. A pile-up of waves will not occur, although at a great distance from the profile the wave length pertaining to a certain frequency will be very short. The following analysis takes the modification of the sound velocity by the presence of the profile into account. At a sufficient distance from the body the flow field at a free stream Mach number one can be described by a similarity solution. In the following we study periodic perturbation in this part of the flow field.

The differential equations for the underlying steady flow field is obtained in the familiar manner; one assumes that the deviations from the sonic parallel flow are small, but retains a critical term which allows the change of the type of the differential equation from elliptic in the subsonic region to hyperbolic in supersonic region. The simplified differential equation for the steady flow field is then given by

$$-\frac{\gamma+1}{2}\frac{\partial}{\partial x}\left(\phi_{ox}^2\right)+\phi_{oyy}=0. \tag{139}$$

Here ϕ_o is the potential which gives in the basic steady flow field the deviation of the flow from a parallel flow with the Mach number one. In this field one now studies unsteady perturbations which are small even in comparison to the steady perturbations introduced by the presence of the body. Let $\tilde{\phi}$ be the potential describing these perturbations. They satisfy the following partial differential equation

$$-(\gamma+1)\frac{\partial}{\partial x}(\phi_{ox}\tilde{\phi}_x) + \tilde{\phi}_{yy} - 2\tilde{\phi}_{xt} - \tilde{\phi}_{tt} = 0. \tag{140}$$

Here the space coordinates have been made dimensionless by a characteristic length L and the time by $(La)^{-1}$ (where "a" is the velocity of sound. The basic potential ϕ_o is considered as known. Eq. (140) is the counterpart of Eq. (5). Introducing

$$\tilde{\phi}(x,y,t) = \phi(x,y)\exp(i\omega t) \tag{141}$$

one obtains

$$-(\gamma+1)\frac{\partial}{\partial x}(\phi_{ox}\phi_x)\phi_x + \phi_{yy} - 2i\omega\phi_x + \omega^2\phi = 0 \tag{142}$$

which corresponds to Eq. (9). In analogy to Eq. (22) one sets

$$\phi = h(x,y)\exp(-i\omega g(x,y)). \tag{143}$$

Substituting this equation into Eq. (142) and collecting the terms with powers ω^2 and ω one obtains the counterparts to Eqs. (24) and (26)

$$(\gamma+1)\phi_{ox}g_x^2 - 2g_x - g_y^2 + 1 = 0 \tag{144}$$

and

$$[(\gamma+1)\phi_{ox}2g_x-2](H_x/H) - 2g_y(H_y/H)+(\gamma+1)(\phi_{ox}g_{xx}+\phi_{oxx}g_x)-g_{yy}=0. \tag{145}$$

As in Section II we reduce the integration of Eq. (145) to the integration of ordinary differential equations by introducing the concept of characteristics. Differentiating Eq. (144) with respect to x and y one obtains

$$\begin{aligned}[(\gamma+1)\phi_{ox}g_x-1]g_{xx} - g_y g_{xy} + ((\gamma+1)/2)\phi_{oxx}g_x^2 = 0\\ [(\gamma+1)\phi_{ox}g_x-1[g_{xy} - g_y g_{yy} + ((\gamma+1)/2)\phi_{oxy}g_x^2 = 0.\end{aligned} \tag{146}$$

The direction of the characteristics is given by

$$Dy/Dx = g_y/(1-(\gamma + 1)\phi_{ox}g_x) \tag{147}$$

Notice that Eq. (144) can serve to express g_y by g_x.

It is convenient to introduce the arc length along the characteristics (to be denoted by s) and use it as independent variable, although x and y can be used for the same purposes, at least along part of the characteristics. Then one has

$$\frac{Dx}{Ds} = \frac{1 - (\gamma +1)\phi_{ox}g_x}{[(1-(\gamma+1)\phi_{ox}g_x)^2 + g_y^2]^{1/2}} \tag{148}$$

$$\frac{Dy}{Ds} = \frac{g_y}{[(1-(\gamma+1)\phi_{ox}g_x)^2 + g_y^2]^{1/2}} \tag{149}$$

and from Eq. (146)

$$\frac{Dg_x}{Ds} = \frac{((\gamma+ 1)/2)g_x^2\ \phi_{oxx}}{[(1-(\gamma+1)\phi_{ox}g_x)^2 + g_y^2]^{1/2}} \tag{150}$$

$$\frac{Dg_y}{Ds} = \frac{((\gamma+ 1)/2)g_x^2\phi_{oxy}}{[(1-(\gamma+1)\phi_{ox}g_x)^2 + g_y^2]^{1/2}} \tag{151}$$

Then

$$\frac{Dg}{Ds} = g_x \frac{Dx}{Ds} + g_y \frac{Dy}{Ds}. \tag{152}$$

Finally, from Eq. (145)

$$\frac{D(\log H)}{Ds} = (1/2)\ \frac{(\gamma+ 1)(\phi_{ox}g_{xx} +\phi_{oxx}g_x)-g_{yy}}{[(1-(\gamma+ 1)\phi_{ox}g_x)^2 + g_y^2]^{1/2}} \tag{153}$$

To determine the functions g_{xx} and g_{yy} which occur in the last equation, one differentiates Eqs. (146), the first one with respect to x, the second one with respect to y. One thus obtains equations for Dg_{xx}/Ds and Dg_{yy}/Ds in terms of g_x, g_y, and data pertaining to the basic field. For the developments of this report these details are unnecessary.

Equations (148) through (151) represent a system of ordinary differential equations for x, y, g_x, and g_y. The unknowns g, H, and auxiliary quantities can subsequently be found by quadratures. For a flow field given by ϕ and a starting line in the x, y plane, one then can determine asymptotic solutions in the form Eq. (143).

For the usual formulation of far field conditions at subsonic flows, one assumes that the perturbations originate from a point. There it can be justified by the observation that at a sufficient distance from the origin outgoing perturbations can be expressed by a superposition of particular solutions derived from Eq. (16). In the present context, such particular solutions are not available. Nevertheless, we make a similar assumption; namely, that the line at which the far field conditions are applied lies at such a distance from the profile that the exact point where the perturbations originate does not matter.

Now we assume that the underlying steady flow field (described by $\phi_o(x,y)$ can be approximated in the far field by the dominant term in the development with respect to the distance. For flows with a free stream Mach number one, this term reflects the profile displacement; the unsymmetry of the airfoil (including the effect of an angle of attack) gives rise to higher order corrections. The formulae derived here can probably be extended to include such terms.

The dominant term in $\phi_o(x,y)$ and of such corrections have the form of similarity solutions. A survey is found in Reference 8. One has for two-dimensional flows

$$\phi_o = \tilde{\mu}^3 y^{2/5} \bar{f}(\bar{\zeta}) \tag{154}$$

with

$$\bar{\zeta} = \tilde{\mu}^{-1} \zeta$$

and

$$\zeta = (\gamma + 1)^{-1/3} x\, y^{-4/5} \tag{155}$$

The intensity of the perturbation expresses itself by the constant $\tilde{\mu}$. It is obtained during the computation of the basic field (see Reference 7). Let

$$\mu = (\gamma + 1)^{1/3} \tilde{\mu}. \tag{156}$$

Then one has

$$\begin{aligned} \bar{\zeta} &= \mu^{-1} x\, y^{-4/5} \\ \partial\bar{\zeta}/\partial x &= \mu^{-1} y^{-4/5} \\ \partial\bar{\zeta}/\partial y &= -(4/5) y^{-1} \bar{\zeta}. \end{aligned} \tag{157}$$

The function $\bar{f}$ appears in a closed form if one changes the independent variable

$$\bar{\zeta} = \sigma^{-2/5}((\sigma/2)-1) \tag{158}$$

Then

$$\bar{f} = (1/24)\sigma^{-1/5}(\sigma^2 - 6\sigma + 48) \tag{159}$$

The function g is now determined by a similarity hypothesis (rather than by integrating Eqs. (148) through (152). Details of the following computations are found in Appendix V. Here we give only an overview of the essential steps.

Let g_o be an approximation to g which satisfies Eq. (144) in the highest power of y. One finds that g_o is then determined from

$$(\gamma+1)\phi_{ox}g_{ox}^2 - 2g_{ox} - g_{oy}^2 = 0. \tag{160}$$

This equation arises, if one omits the term ϕ_{tt} in Eq. (140). One usually calls this the low frequency approximation. We set

$$g_o = \mu^{-1}\, y^{6/5}\sigma^{-3/5}\gamma_o(\sigma). \tag{161}$$

Substituting Eq. (161) into Eq. (160), one obtains

$$\mu^{-2}y^{2/5}\sigma^{-1,5}(\sigma+(4/3))^{-1}\Big\{\sigma(\sigma-(16/3))(d\gamma_o/d\sigma)^2-(d\gamma_o/d\sigma)\sigma(2\gamma-(20/3))$$

$$+\gamma_o(-3\gamma_o+4)\Big\} = 0. \tag{162}$$

The expression within the braces can be regarded as a quadratic equation for $d\gamma_o/d\sigma$. One notices that the coefficient of $(d\gamma_o/d\sigma)^2$ vanishes for $\sigma = 16/3$. The curve $\sigma = 16/3$ represents in the xy-plane the so-called limiting characteristic of the far field. (Perturbations originating downstream of the limiting characteristic will never reach the sonic line and therefore have no effect on the subsonic field.) In general, one of the roots of this quadratic equation will be infinite at the limiting characteristic. The solution which is smooth throughout the flow field is immediately found by inspection

$$\gamma_o = 4/3 \tag{163}$$

Thus from Eq. (161)

$$g_o = \mu^{-1}(y^2/\sigma)^{3/5}(4/3). \tag{164}$$

The approximation of the next order is obtained by setting

$$g = g_o + g_1. \tag{165}$$

One assumes that for y sufficiently large, g_1 is sufficiently small so that nonlinear terms can be disregarded. A hypothesis of a similar kind can probably be used to take higher order terms in the development of ϕ_o into account. One obtains from Eq. (144)

$$((\gamma+1)\phi_{ox}g_{ox} - 1)g_{1x} - g_{oy}\, g_{1y} + (1/2) = 0. \tag{166}$$

It follows from Eq. (161) that g_{oy} has a factor $y^{1/5}$. A comparison of the second and third term in the last equation then suggests that g_1 has the form

$$g_1 = (y^2/\sigma)^{2/5}\gamma_1(\sigma). \tag{167}$$

This leads to the equation

$$\sigma(d\gamma_1/d\sigma) + (2/3)\gamma_1 = (1/4)/\sigma + (4/3). \tag{168}$$

The one solution that is smooth at the origin is given by

$$\gamma_1 = (1/2) + (3/20)\sigma. \tag{169}$$

One thus has the following approximation for g

$$g = \mu^{-1}(y^2/\sigma)^{3/5}[(4/3) + A((1/2)+(3/20)\sigma)] \tag{170}$$

with

$$A = \mu^2(\sigma/y^2)^{1/5}. \tag{171}$$

Equation (158) gives a formula for (σ/y^2) which applies in the vicinity of $\sigma = 0$, $y = 0$. In the next step one would determine the function H. This is shown in Appendix V and one finds that H amounts only to a correction of higher order in y.

With an approximation to g known, one can derive far field conditions in the following manner. At a given point of the flow field the characteristic direction can be computed from Eq. (147). Then one obtains from Eq. (143)

$$D\phi/Ds - (D(\log H)/Ds - i\omega Dg/Ds)\phi = 0. \tag{172}$$

In principle, D(logH)/Ds can be computed from Eq. (153), but actually it is small of higher order. Equation (172) is a local condition which relates the derivative in a certain direction with the perturbation potential. Because of the simplification introduced, this relation holds only in the far field. The potential and its derivative at the outer edge of the computed field must satisfy these conditions in order to match with the far field. One still needs formulae in terms of σ and y, for these are the variables which are used in the computations. Here only the final formulae are written down; the derivation is found in Appendix V.

It can be assumed that for a given point (x,y), the value of σ is available. For constant σ, A defined in Eq. (171), decreases with increasing y. Only first order terms in A are taken into account. The following formulae are related to the direction of the characteristics. In practice one will probably use the third one

$$D(\log y)/D(\log\sigma) = (4/3)(1 + (1/8)A(\sigma+ (4/3)) \tag{173}$$

$$D(\log(-x))/D(\log\sigma) =(2/3)[(1-(5/4)\sigma)(1-(\sigma/2))^{-1} + (1/5)A(\sigma+ (4/3))]$$

$$D(\log(-x))/D(\log y)= (1/2)[(1-(5/4)\sigma)(1-(\sigma/2))^{-1} + (1/5)A(\sigma+ (4/3))]$$

$$[1 + (1/8)A(\sigma+ (4/3))]^{-1}$$

At $x = 0, \xi = 0$ and $\sigma = 2$. For this vicinity, Eq. (173) is used in the form

$$D(-x)/D(\log\sigma) = (1/2)[\{(-x)/(1-(\sigma/2)\}\ (1-(5/4)\sigma) + \tag{174}$$

$$(-x)(1/5)A(\sigma+ (4/3))][1+ (1/8)A(\sigma+(4/3))]^{-1}$$

The term $(-x)/(1-(\sigma/2)$, which at $x = 0$ is undetermined, is found from Eq. (157) and (158),

$$(-x)/(1 - '\sigma/2)) = \mu(\sigma/y^2)^{-2/5}. \tag{175}$$

No difficulty arises for $\sigma \to 0$, $y = 0$, if $x \neq 0$. One will, of course, evaluate Dy/Dx rather than Dx/Dy.

The above equations are correct only to the first order in A. Some second order terms have been included, however, so that no jumps occur in the characteristic slope if one passes from the formula for $D(\log y)/D(\log\sigma)$ to that for $D(\log x)/D(\log\sigma)$.

From Eqs. (173) one can find approximate analytic expressions for the characteristics. For the formulation of the far field conditions they are not necessary; but they are useful for a comparison with characteristics found by direct computation. One obtains

$$y = c\,\sigma^{4/3} + c^{3/5}\,\mu^2(\sigma/4)(\sigma- (8/3)). \tag{176}$$

Here c is a parameter which is constant along a characteristic.

One needs, in addition, Dg/Ds and (for completeness) D(logH)/DS. One has

$$Dg/D\sigma = (\partial g/\partial\sigma) + (\partial g/\partial y)(Dy/D\sigma).$$

This leads to

$$Dg/D(\log\sigma) = \mu^{-1}(y^2/\sigma)^{3/5}\{[-(4/5) + A(-(1/5) + (9/100)\sigma] + D(\log y)/D(\log\sigma)[(8/5) + A(4/5)((1/2) + (3/20)\sigma)]\} \quad (177)$$

Here D(logy)/D(logσ) is found from Eq. (173). One obtains in the lowest approximation

$$Dg/D\sigma = \mu^{-1}(4/3)\sigma^{-1}(y^2/\sigma)^{3/5}$$

or

$$Dg/Dy = \mu^{-1}(y^2/\sigma)^{3/5} \quad (178)$$

and by substituting Eq. (176)

$$Dg/Dy \sim y^{-1/4}.$$

In the lowest order approximation of remarkable simplicity

$$D(\log H)/D(\log y) = -(1/2) \quad (179)$$

or

$$D(\log H)/Dy \sim y^{-1}.$$

Accordingly, the contributions of D(logH)/dy in the far field condition is small in comparison to that of Dg/Dy.

Equations (173) serve to determine the direction of the characteristics Dx/Dy and Dσ/Ds.

One has

$$Ds/D\sigma = ((Dx/D\sigma)^2 + (Dy/D\sigma)^2)^{1/2}.$$

This, then, allows one to compute for a given point (x,y) all data needed to evaluate Eqs. (172).

SECTION IX

UNSTEADY PERTURBATIONS IN A THREE-DIMENSIONAL FLOW FIELD WITH A FREE STREAM MACH NUMBER ONE

The treatment of the three-dimensional problem is entirely analogous to that of the plane problem. At great distances the flow field generated by a body of finite dimensions approaches that of an axial symmetric body. The approximation to the basic field used here is therefore the axisymmetric solution. As in the plane case the characteristics are lines; for an axisymmetric flow field these lines happen to lie in the meridian planes. In dealing with an axisymmetric body, the contour shown is the intersection of the surface with a meridian plane. The characteristics in the meridian plane determine, of course, an axisymmetric surface; but this surface has no meaning in its own right.

The following formulae are completely analogous to those of the preceding section. The coordinate y stands for the radius in cylindrical coordinates. The counterpart to Eq. (140) is

$$-(\gamma+1)\frac{\partial}{\partial x}(\phi_{ox}\breve{\phi}_x) + \breve{\phi}_{yy} + \breve{\phi}_y/y - 2\breve{\phi}_{xt} - \breve{\phi}_{tt} = 0. \qquad (180)$$

The hypothesis

$$\breve{\phi}(x,y,t) = \phi(x,y)\exp(i\omega t) \qquad (181)$$

yields

$$-(\gamma+1)\frac{\partial}{\partial x}(\phi_{ox}\phi_x) + \phi_{yy} + \phi_y/y - 2i\omega\phi_x + \omega^2\phi = 0. \qquad (182)$$

One sets

$$\phi = h(x,y)\exp(-i\omega g(x,y)) \qquad (183)$$

The terms with power ω^2 in Eqs. (182) give

$$(\gamma+1)\phi_{ox}g_x^2 - 2g_x - g_y^2 + 1 = 0. \qquad (184)$$

An approximation, $\bar{H}(x,y)$ for $H(x,y)$ is obtained with the power ω

$$[(\gamma+1)\phi_{ox}g_x - 1](H_x/H) - g_y(H_y/H) + (\gamma+1)/2)\,\partial(\phi_{ox}g_x)/\partial x$$

$$- (1/2y)\,\partial(yg_y)/\partial_y = 0.$$

Equation (184) agrees with Eq. (144). Eqs. (147) through (152) therefore can be taken over immediately. The equation corresponding to Eq. (153) assumes the form

$$\frac{D\log\bar{H}}{Ds} = (1/2)\,\frac{(\gamma+1)\partial(\phi_{ox}g_x)/\partial x - (1/y)\,\partial(yg_y)/\partial y}{[(1-(\gamma+1)\phi_{ox}g_x)^2 + g_y^2\,]^{1/2}} \qquad (185)$$

Next, the expression in ϕ_o which dominates at a great distance is introduced

$$\phi_o = \tilde{\mu}^3\, y^{-2/7}\bar{f}(\bar{\zeta}) \qquad (186)$$

with

$$\bar{\zeta} = \tilde{\mu}^{-1}\zeta$$

and

$$\zeta = (\gamma+1)^{-/3}xy^{-4/7}.$$

Again,

$$\begin{aligned} \mu &= (\gamma+1)^{1/3}\tilde{\mu} \\ \bar{\zeta} &= \mu^{-1}\, xy^{-4/7} \end{aligned} \qquad (187)$$

and

$$\begin{aligned} \partial\bar{\zeta}/\partial x &= \mu^{-1}y^{-4/7} \\ \partial\bar{\zeta}/\,y &= -(4/7)y^{-1}\bar{\zeta}. \end{aligned} \qquad (188)$$

The following closed form of f has been given by Randall

$$\bar{\zeta} = \sigma^{-2/7}(2\sigma-1)$$
$$\bar{f} = -(8/9)\sigma^{1/7}(-2\sigma^2 + 3\sigma + 6) \qquad (189)$$

By steps analogous to the case of plane flows, one obtains

$$g = \mu^{-1} (y^2/\sigma)^{5/7}[(3/10) + A((1/2) + (5/7)\sigma)] \qquad (190)$$

with

$$A = \mu^2(\sigma/y^2)^{3/7}. \qquad (191)$$

$$D(\log y)/D(\log\sigma) = (6/5)(1 + (4/3)A(\sigma+ (1/5)] \qquad (192)$$

$$D(\log(-x))/D(\log\sigma) = (2/5)[(1-7\sigma)(1-2\sigma)^{-1}+(16/7)A(\sigma+(1/5))]$$

$$D(\log(-x))/D\log y) = (1/3)[(1-7\sigma)(1-2\sigma)^{-1}+ (16/7)A(\sigma+ (1/5))]$$

$$[1 + (4/3)A(\sigma+ (1/5)]$$

One obtains as approximation to the shape of the characteristics

$$y = c\ \sigma^{6/5} + \mu^2\ c^{1/7}\ \sigma^{3/5}\ (4\sigma - (8/15)$$

$$Dg/D(\log\sigma) = \mu^{-1}(y^2/\sigma)^{5/7}\ \{[-(3/14) + A\mu^2(-(1/7) + (25/24)\sigma)]$$

$$+ D(\log y)/D(\log\sigma)[(3/7) + A\mu^2((2/7) + (20/49)\sigma)]\}$$

One obtains in a first approximation

$$Dg/D\sigma = \mu^{-1}(y^2/\sigma)^{5/7}\ \sigma^{-1}(3/10)$$

$$Dg/Dy = \mu^{-1}(y^2/\sigma)^{5/7}y^{-1}(1/2)$$

Finally, in the lowest order approximation

$$D(\log H)/D(\log y) = -1.$$

SECTION X

VALIDITY OF THE LINEARIZED APPROXIMATION FOR TIME DEPENDENT PERTURBATIONS AT MACH NUMBER ONE

It is conceivable that the velocities caused by the unsteady perturbations decrease more slowly as one goes to infinity than the deviation of the velocities in the basic flow from the free stream velocity. These velocities and the pertinent sound velocity determine the propagation of perturbations. We have assumed that the unsteady perturbations are small and that their effect on the speed of propagation of perturbations can be disregarded. But, if the unsteady contributions do not stay small in comparison to those of the steady flow, then the present analysis is not valid at great distances, although it might be valid in an intermediate range.

We carry out this discussion first in the two-dimensional flow. There one finds from the similarity solution (Eq. (154)) together with the second of Eqs. (157) that along a line σ = const

$$\phi_{xo} \sim y^{-2/5} \tag{193}$$

For the superimposed unsteady flow one obtains as dominant terms, respectively, in $\tilde{\phi}_t$ and $\tilde{\phi}_x$

$$\tilde{\phi}_t = \omega H$$

and

$$\tilde{\phi}_x = H\omega g_x$$

It follows from Eqs. (161) that along a line σ= const

$$g_{ox} \sim y^{2/5} \tag{194}$$

This shows that in a perturbed flow, $\tilde{\phi}_x$ dominates $\tilde{\phi}_t$ at a large distance. One notices that for the basic flow, one has the power $y^{-2/5}$ in the expression ϕ_x, and $y^{+2/5}$ in the expression $\tilde{\phi}_x$ for the superimposed perturbations. This makes the investigations carried out in this section necessary. The question whether at infinity the perturbations can still be regarded as small in comparison to the basic field cannot be dismissed. Decisive is the behavior of the function H. Along a characteristic, H is given by H or $y^{-1/2}$, but characteristics differ from lines σ = const. According to Eq. (171) one has along a characteristic

$$y_\sigma^{-4/3} \sim \text{const.}$$

Accordingly,

$$H = y^{-1/2} f(y\sigma^{-4/3}) \tag{195}$$

where f is used as a generic expression. It has nothing to do with the functions f used to describe the basic field. Some information about f is obtained by considering the function H in the vicinity of the negative x axis. It is reasonable to assume that in the x,y plane, H is a smooth function. If one approaches the negative axis along a line (-x) = const, then by Eq. (175)

$$\sigma y^{-2} = \text{const},$$

but H is likely to behave in the vicinity of the negative x axis as a power of (-x).

Expressing the assumption that on this path H remains smooth, one obtains from Eq. (182)

$$H = y^{-1/2} f(y\sigma^{-3/4}) \sim (\sigma y^{-2})^{\beta} \tag{196}$$

Obviously, the function f must have a singularity at the x axis. We write,

$$f(\sigma y^{-3/4}) = (\sigma y^{-3/4})^{\alpha} \tilde{f}(\sigma y^{-3/4}).$$

If one moves to infinity along a line σ = const, then $\sigma y^{-4/3} \to 0$. In other words, in this limiting process the argument of $\tilde{f}$ goes to zero, throughout the flow field, one can choose α in such a manner that f then assumes a finite limiting value. Thus, one obtains from Eq. (196)

$$y^{-1/2}(\sigma y^{-3/4}) \sim (\sigma y^{-1/2})^{\beta}.$$

The powers of σ and y on the left and right must match

$$-\frac{1}{2} - \frac{3}{4}\alpha = -2\beta$$

$$\alpha = \beta$$

Hence

$$\alpha = 2/5$$

and

$$H = (\sigma/y^2)^{2/5} \tilde{f}(\sigma y^{-3/4})$$

One finds accordingly that for superimposed perturbations

$$\tilde{\phi}_x = \sigma^{2/5} y^{-4/5} g_{ox} \sim y^{-2/5}.$$

Accordingly, the perturbations have the same dependence upon y along a line σ = const as the basic flow.

The same result is found for the three-dimensional flow. One has in the basic steady flow along a line σ = const

$$\phi_x \sim y^{-6/7}$$
$$g_x \sim y^{+6/7}$$

and

$$H = y^{-1} f(y/\sigma^{6/5}) = \text{const.}$$

For lines $-x = \text{const.}$ One has again $(\sigma/y^{-2}) = \text{const.}$

Then one has in a corresponding computation

$$\alpha = \beta = (6/7)$$

and $$H \sim y^{-12/7}$$

along a line $\sigma = \text{const.}$

Consequently, the basic flow and the superimposed unsteady perturbation change with the same power of y along a line $\sigma = \text{const.}$ If the unsteady perturbations are small at an intermediate distance, they will be small also at large distances.

SECTION XI
COMPARISON OF EXACT AND ASYMPTOTIC EXPRESSIONS FOR g

The analytical formulae for g derived in Sections VIII and IX are valid only at a sufficient distance from the origin. We show here a comparison of these analytical results with data obtained by a numerical integration of the characteristic equations. Of course, these computations need not be performed if one merely wants to apply the far field conditions.

For the plane flow with free stream Mach number one, the equations to be integrated are Eqs. (148) to (152). In these equations the underlying steady flow enters via the function ϕ_o. Here the asymptotic expressions for the steady flow are used. One has according to Eqs. (A.50)

$$(\gamma+1)\phi_{ox} = \mu^2(\sigma/y^2)^{1/5}((1/4)\sigma - 1) \tag{204}$$

Hence with Eqs. (A.49)

$$(\gamma+1)\phi_{oxx} = \mu(\sigma/y^2)^{3/5}(\sigma-(2/3))(\sigma + (4/3))^{-1} \tag{205}$$

$$(\gamma+1)\phi_{oxy} = \mu^2 y(\sigma/y^2)^{6/5}(-(1/2)\sigma+ (4/3)(\sigma + (4/3))^{-1}$$

μ is a constant determined by the underlying steady flow.

In these expressions the variable σ is encountered. It is expressed in terms of x and y (which, for the integration along the characteristics, are dependent variables) by Eqs. (157) and (158). To determine σ for given x and y, one first evaluates

$$\bar{\zeta} = \mu^{-1}xy^{-4/5} \tag{206}$$

and then determines σ from Eq. ((159), i.e.,

$$\bar{\zeta} = \sigma^{-2/5}((\sigma/2) - 1). \tag{207}$$

This can be done by a Newton procedure. One obtains a correction $\Delta\sigma$ to an approximation σ_o from

$$\Delta\sigma = \frac{\bar{\zeta} - \bar{\zeta}(\sigma_o)}{(d\bar{\zeta}/d\sigma)/_{\sigma=\sigma_o}} \tag{208}$$

$\bar{\zeta}(\sigma_o)$ and $d\bar{\zeta}/d\sigma$ are found from Eqs. (207) and (A48), respectively.

The method deteriorates in the vicinity of the negative x-axis. There $\bar{\zeta} \to -\infty$. There one iterates as follows

$$\sigma^{(n+1)} = (-\bar{\zeta})^{-5/2}(1 - (\sigma^{(n)}/2)^{5/2}.$$

The superscript n refers to the iteration number. In principle, one must prescribe g along some initial curve. Let this curve be given by

$$x = x(y) \quad \text{or} \quad y = y(x)$$

and consider a fixed point (x_o, y_o) of this curve. Therefore, one has for the system (Eqs. (148) through (152)), initial values for $x = x_o$, $y = y_o$, and $g = g(x_o, y_o)$. One can also assign an initial value for H. To find initial condition for g_x and g_y, one forms the derivative of g along the initial curve

$$dg/dx = g_x + g_y(dy/dx). \tag{209}$$

Here dg/dx and dy/dx are known. Combining this relation with the partial differential equation for g, Eq. (144); namely,

$$(\gamma + 1)\phi_{ox}g_x^2 - 2g_x - g_y^2 + 1 = 0, \tag{210}$$

one can determine g_x and g_y.

This is the general procedure. In our computation, we assumed the initial curve to be a small circle around a point (x_o,y_o), whose radius goes to zero. From this circle, characteristics go out in all directions. In this case one has the same value of ϕ_{xo} for all starting points of the characteristic. Moreover, Eq. (209) can be disregarded. If one assumes that g = const along this small circle, then dg/dx = 0. For any choice of g_x and g_y, one can find a direction (dx/dy) along a circle for which Eq. (209) is satisfied.

As initial condition for the integration of the system (Eq. (148) to (152)),one then has $x = x_o$, $y = y_o$, $g = 0$. The derivatives g_x and g_y must be chosen in such a manner that Eq. (210) is satisfied. To obtain characteristics whose direction is reasonably spaced, we have proceeded in the following manner. Equation (210) is rewritten in the form

$$(-(\gamma+1)\phi_x(x_o,y_o))^{-1}[-(\gamma+1)\phi_x(x_o,y_o)g_x+1]^2+g_y^2 =[-(\gamma+1)\phi_x(x_o,y_o)+1)]$$

$$[-(\gamma+1)\phi_x(x_o,y_o)]^{-1}$$

This equation is satisfied if one sets

$$g_y = r\sin\theta$$

$$[-(\gamma+1)\phi_x(x_o,y_o)g_x+1][-(\gamma+1)\phi_x(x_o,y_o)]^{-1} = r\cos\theta$$

with

$$r = [-(\gamma+1)\phi_x(x_o,y_o) +1]^{1/2}][-(\gamma+1)\phi_x(x_o,y_o)]^{-1/2}.$$

Then,

$$g_x = [r(-(\gamma+1)\phi_x(x_oy_o))\cos\theta-1][-(\gamma+1)\phi_x(x_o,y_o]^{-1}$$

The initial direction of the characteristic is found from Eq. (147)

$$Dy/Dx = (-(\gamma+1)\phi_x(x_o,y_o)^{-1/2}tg\theta$$

The axisymmetric case is treated in an analogous manner. In this case the point (x_o,y_o) is chosen on the negative x axis. If one chooses it away from the axis of symmetry, then one must assume that the initial surface is an infinitely thin ring, whose centerline intersects the meridian plane at the point (x_o,y_o). In this case the characteristics (line) will lie in the meridian plane. With slight modifications, one might also treat the case where the initial surface is a small sphere with center at $(x_o,y_o,z_o=0)$. In this case, one would obtain a two-dimensional family of characteristics, which in general are not confined to a meridian plane.

The necessary formulae are

$$(\gamma+1)\phi_{ox} = \mu^2(\sigma/y^2)^{3/7}(8/3)(\sigma-1)$$

$$(\gamma+1)\phi_{oxx} = \mu(\sigma/y^2)^{5/7}((8/3)\sigma - (4/5))(\sigma+(1/5))^{-1}$$

$$(\gamma+1)\phi_{oxy} = \mu^2(16/3)y(\sigma/y^2)^{10/7}(-\sigma+(4/5))(\sigma+(1/5))^{-1}$$

$$\bar{\zeta} = \mu^{-1}x\,y^{-4/7}$$

$$\bar{\zeta} = \sigma^{-2/7}(2\sigma-1).$$

The initial conditions are chosen in the same manner as in the two-dimensional case.

The computations have been carried out for $\tilde{\mu} = 1$. (See Eqs. (54) and (186).) Then, according to Eq. (156) and (187)

$$\mu = (\gamma+1)^{1/3}.$$

We have chosen $\gamma = 1.4$.

Figure 5 shows in a plane flow with the free stream Mach number one a set of computed characteristics which start at some point of the negative x-axis and also the corresponding one-term approximation. Figure 6 shows the corresponding two-term approximation. The axisymmetric counterparts are shown in Figures 10 and 11. These curves correspond to the straight lines in Figure 2. Some of them start in the upstream direction but eventually all of them are swept downstream. The curves have been terminated at the limiting characteristic belonging to the basic flow field. The approximate curves have been chosen in such a manner that they coincide with the exact curves at the outer points. In practice exact characteristics are not available. In the application of the asymptotic far field conditions the slope of the characteristics is needed. The asymptotic two-term expression gives a satisfactory approximation to these slopes.

Figures 7 (for plane flows) and 12 (for axisymmetric flows) show one- and two-term approximations for the wave fronts given by lines g = const. They are drawn for g = 10, 20, and 30. One sees that the additional term in the approximation makes an essential difference. Figures 8 (for plane flows) and 13 (for axisymmetric flows) show computed characteristics and, marked by asterisk, computed values of g (g = 10, 20, 30) and also the asymptotic two-term approximations for g. The computed values are rather well represented by their asymptotic curves, especially if one takes into account that in the determination of g a constant remains open. The curves correspond to the circles in Figure 2. One no longer encounters a piling up of the wave fronts, although they come closer together at a large distance from the origin. Figures 9 show for plane flow corresponding curves for waves starting at different points. The asymptotic representation of the curves g = const remains the same. One sees that it is not too important where the waves originate.

No counterparts for the three-dimensional case have been drawn because the characteristics for points off the negative x axis will, in general, not be curves that lie in the meridian planes.

One recognizes that the asymptotic expressions developed above can indeed be used to formulate far field conditions, provided, of course, that the boundary of the computed field is not too close to the profile.

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APPENDIX I

SOME REMARKS ABOUT THE HYPOTHESIS EQ. (22)

The hypotheses Eqs. (22) and (12) have a similar effect. In each case a factor is split off from the expression for the potential. The factor $\exp(i\omega M(1-M^2)^{-1}\hat{x})$ in Eq. (12) generates a field with equal waviness in all directions. Because of the interpretation of $g(x,y)$ given by Eq. (36), the factor $\exp(-i\omega g(x,y))$ in Eq. (22) anticipates the waviness of the flow field completely but only for outgoing waves. The original differential equation admits incoming as well as outgoing waves. This holds also for the differential equation for h which one obtains by setting the remaining terms in Eq. (23) equal to zero. To illustrate the effect of the transformation in Eq. (22) we consider the problem for three-dimensional perturbations in air at rest. There the discussion is particularly simple because of the availability of closed solutions. Accordingly, we consider the partial differential equation

$$\phi_{rr} + (2/r)\phi_r - \phi_{tt} = 0$$

The hypothesis

$$\phi = \hat{\phi}(r)\ \exp(i\omega t)$$

then leads to

$$\hat{\phi}_{rr} + (2/r)\hat{\phi}_r + \omega^2\hat{\phi} = 0$$

Particular solutions are given by

$$\hat{\phi} = r^{-1}\ \exp(\pm i\omega r) \qquad \text{(A.1)}$$

(and also by derivatives of this expression with respect to x,y or z)

If one sets in analogy to Eq. (22)

$$\hat{\phi} = h(r)\ \exp(-i\omega g(r)) \tag{A.2}$$

then one has immediately

$$g(r) = r$$

The two particular solutions for h corresponding to those in Eq. (40) are given by

$$h(r) = r^{-1} \tag{A.3}$$

and

$$h(r) = r^{-1}\ \exp(i2\omega r) \tag{A.4}$$

In the expression for outgoing waves (Eq. (A.3)), the waviness has vanished while that for incoming waves (Eq. (A.4)) has waves which possess half of the original wave length.

This has interesting consequences for a numerical approach. If one knows that only outgoing waves are present, then the function h is smoother than the function ϕ. The mesh for computing h can, therefore, be coarser than that for $\hat{\phi}$. But one ought to be aware of the fact that in a coarse mesh incoming waves will be greatly distorted.

In cases where the partial differential equation can be solved by a product hypotheses, so that ultimately one is led to an ordinary differential equation in r, one is tempted to build up the solutions of this ordinary differential equations from particular soltuions which are obtained by solving initial value problems. Roughly speaking, the stiffness of such ordinary differential equations is determined by the largest eigenvalue. If one solves the differential equation by a predictor-corrector method, then the stiffness causes an instability of the procedure which is remedied by a reduction of the step width. This happens even if the particular solution which is being determined does not

contain the particular solution which causes stiffness. The stiffness of the differential equation for $\hat{\phi}$ is determined by the value of ω, that of the differential equation for h by 2ω. Of course, this discussion is rather academic, for in those cases where a product hypotheses can be used it is likely that the ordinary differential equations can be solved in a closed form.

APPENDIX II

GEOMETRIC INTERPRETATION OF CERTAIN ANALYTIC EXPRESSIONS

First we determine the curvature of a curve $g(x,y)=\text{const.}$ At a point x_o,y_o we introduce a local system of Cartesian coordinates ξ, ζ, where ζ is normal to the curve $g(x,y) = g(x_o,y_o)$ and ξ tangential to it (Fig. 14).

Developing g with respect to ξ and ζ but retaining only the terms of the lowest order, one obtains

$$g(x,y) = g(x_o,y_o) + |\text{grad } g|\zeta + g_{\xi\xi}\frac{\xi^2}{2}$$

Here it is assumed that the ζ axis points in the direction of increasing g. The curve $g(x,y) = g(x_o,y_o)$ is then given by

$$\zeta = -|\text{grad } g|^{-1} g_{\xi\xi}\frac{\xi^2}{2}$$

The curvature of a curve $y = f(x)$ is given by the familiar formula

$$R^{-1} = -f''/(1+f'^2)^{3/2}$$

It gives a positive value for R if the region under the curve $y = f(x)$ is convex (Fig. 15). In the present case one, therefore, obtains

$$R^{-1} = |\text{grad } g|^{-1} g_{\xi\xi}$$

Returning to the original x,y system one observes that

$$(x - x_o) = \xi \cos(\xi,x) + \zeta \cos(\zeta,x)$$
$$(y - y_o) = \xi \cos(\xi,y) + \zeta \cos(\zeta,y)$$

where (ξ,x), etc. stands for the angle between the respective axes (here the x and the ξ axes). Since the ξ and ζ axes are

perpendicular to each other one has

$$\cos(\xi,x)\cos(\zeta,x) + \cos(\xi,y)\cos(\zeta,y) = 0$$

One then has

$$g_{\xi\xi} = g_{xx}\cos^2(\xi,x) + 2g_{xy}\cos(\xi,x)\cos(\xi,y) + g_{yy}\cos^2(\xi,y)$$

Let, (Fig. 16)

$$(\zeta,x) = \alpha$$
$$(\zeta,y) = (\pi/2 - \alpha)$$
$$(\xi,x) = (\pi/2 - \alpha)$$
$$(\xi,y) = (\pi - \alpha)$$

(The signs of these angles do not matter because one encounters in the above formulae only the cosines of these angles.) One obtains

$$g_{\xi\xi} = g_{xx}\sin^2\alpha - 2g_{xy}\sin\alpha\cos\alpha + g_{yy}\cos^2\alpha$$

and

$$R^{-1} = (g_{xx}\sin^2\alpha - 2g_{xy}\sin\alpha\cos\alpha + g_{yy}\cos^2\alpha)/(g_x^2 + g_y^2)^{1/2} \qquad (A.5)$$

Notice that in this formula specific properties of the g-field are not taken into account. This is done presently.

One has

$$g_x = |\text{grad } g|\cos\alpha$$
$$g_y = |\text{grad } g|\sin\alpha$$

Substituting these expressions into Eq. (24) one finds

$$|\text{grad } g| = (1 + M\cos\alpha)^{-1} \qquad (A.6)$$

Therefore,

$$g_x = \cos\alpha(1 + M\cos\alpha)^{-1}$$
$$g_y = \sin\alpha(1 + M\cos\alpha)^{-1} \qquad (A.7)$$

The direction of the characteristic is given in Eq. (29). One now obtains

$$Dy/Dx = \sin\alpha/(M + \cos\alpha) \qquad (A.8)$$

It was found in Section II that g_x, g_y and therefore also α and Dy/Dx are constant along a characteristic. Introducing a parameter p which, at the point under consideration assumes the value p_o, one obtains the following parametric representation for a characteristic

$$x = x_o + (p - p_o)(M + \cos\alpha)$$
$$y = y_o + (p - p_o)\sin\alpha \qquad (A.9)$$

Along such a line one has the relation

$$Dg/Dp = g_x(M + \cos\alpha) + g_y \sin\alpha = 1 \qquad (A.10)$$

Equations (A.9) and (A.10) lead to the following geometric interpretation. We consider a point (x_o, y_o) of the curve $g = g_o$. According to Eq. (A.10), the value of p, to be denoted by p_1, for a curve $g = g_1$ is given by

$$p_1 - p_o = g_1 - g_o.$$

One has for the point x_1, y_1, originating from (x_o, p_o)

$$(x_1 - x_o) = (p_1 - p_o)\cos\alpha + (p_1 - p_o)M$$
$$(y_o - y_o) = (p_1 - p_o)\sin\alpha.$$

Accordingly, the line $g = g_1$ arises from the line first $g = g_o$, by proceeding in the direction of the normal by a distance $(p_1-p_o) = (g_1-g_o)$ and afterwards by translating the curve so obtained in the x-direction by a distance $(p_1-p_o)M$. The first step amounts to the determination of a curve equidistant from $g = g_o$ by the distance (p_1-p_o). By this process, the radius of curvature is changed from R_o (at (x_o,y_o) to $R_1 = R_o + p - p_o$ (at the point corresponding to (x_1,y_1)). The subsequent translation leaves the radius of curvature unchanged. Identifying p_o with R_o one finds that at the points of subsequent curves g = const, the parameter p is identical with the radius of curvature.

The analytical derivation of this result is somewhat cumbersome. One can proceed as follows. The fact that g_x and g_y are constant along characteristics is now used to expresss g_{xx}, g_{yy}, and g_{xy} in terms of R^{-1}. It follows from Eq. (A.9) that

$$\begin{aligned} Dg_x/Dp &= g_{xx}(M + \cos\alpha) + g_{xy} \sin\alpha = 0 \\ D_{gy}/Dp &= g_{xy}(M + \cos\alpha) + g_{yy} \sin\alpha = 0. \end{aligned} \tag{A.11}$$

Moreover, from Eq. (A.5) in conjunction with Eq. (A.6)

$$g_{xx} \sin^2\alpha - 2g_{xy} \sin\alpha \cos\alpha + g_{yy}\cos^2\alpha = R^{-1}(1+M\cos\alpha)^{-1} \tag{A.12}$$

Equations (A.11) and (A.12) form a system of linear equations for g_{xx}, g_{xy}, and g_{yy}. One verifies that

$$\begin{aligned} g_{xx} &= R^{-1} \sin^2\alpha/(1 + M \cos\alpha)^3 \\ g_{xy} &= -R^{-1} \sin\alpha(M + \cos\alpha)/(1+M\cos\alpha)^3 \\ g_{yy} &= R^{-1}(M + \cos\alpha)^2/(1 + M\cos\alpha)^3. \end{aligned} \tag{A.13}$$

Hence

$$g_{xx}(1-M^2) + g_{yy} = R^{-1}/(1 + M \cos \alpha) \qquad (A.14)$$

In these expressions R is the local radius of curvature. So far the relation between R and p has not been established. For this purpose we express the derivative of R^{-1} along a characteristic from Eq. (A.14), keeping in mind, that α = const along a characteristic

$$-R^{-2}(dR/d\ell) = (1 + M \cos\alpha)[(1-M^2)(dg_{xx}/D\ell) + (Dg_{yy}/D\ell)] \qquad (A.15)$$

The second derivatives of g encountered here are subject to compatibility conditions (as are all higher derivatives), which are derived from Eq. (28). One obtains by forming derivatives with respect to x and y

$$[(1-M^2)g_x + M]g_{xxx} + g_y g_{xxy} + (1-M^2)g_{xx}^2 + g_{xy}^2 = 0$$

$$[(1-M^2)g_x + M]g_{yyx} + g_y g_{yyy} + (1-M^2)g_{xy}^2 + g_{yy}^2 = 0$$

$$[(1-M^2)g_x + M]g_{xyx} + g_y g_{xyy} + (1-M^2)g_{xy}g_{xx} + g_{yy}g_{xy} = 0.$$

It follows from Eqs. (A.7) that

$$[1-M^2)g_x + M] = (M + \cos\alpha)/(1 + M \cos\alpha).$$

Then with Eqs. (A.7) and (A.9)

$$Dg_{xx}/Dp + (1 + M \cos \alpha)[(1-M^2)g_{xx}^2 + g_{xy}^2] = 0$$

$$Dg_{yy}/Dp + (1 + M \cos \alpha)[(1-M^2)g_{xy}^2 + g_y^2] = 0 \qquad (A.16)$$

$$Dg_{xy}/Dp + (1 + M \cos \alpha)[(1-M^2)g_{xx} + g_{yy}]g_{xy} = 0.$$

This is a system of three nonlinear ordinary differential equations for g_{xx}, g_{yy}, and g_{xy}. It holds along a characteristic. The previous result, that p is identical with R, which in the present context is to be regarded as a conjecture, suggests that one particular solution is given by Eq. (A.13) with R replaced by p. If this is correct, then one has

$$Dg_{xx}/Dp = -p^{-2}\sin^2\alpha/(1 + M\cos\alpha)^3$$

$$Dg_{xy}/Dp = p^{-2}\sin\alpha(M + \cos\alpha)/(1 + M\cos\alpha)^3 \qquad \text{(A.17)}$$

$$Dg_{yy}/Dp = -p^{-2}(M + \cos\alpha)^2/(1 + M\cos\alpha)^3.$$

Equations (A.13) (with R replaced by p) are indeed particular solutions. One obtains, for instance, by substituting into the first of Eqs. (A.16)

$$-p^{-2}\sin^2\alpha(1+M\cos\alpha)^{-3}+p^{-2}(1+M\cos\alpha)^{-5}[(1-M^2)\sin^4\alpha+ \sin^2\alpha(M+\cos\alpha)^2] = 0.$$

Equations (A.13) give only a particular solution, for they do not contain three constants of integration. Actually, one does not need the general solution, for at the initial point ($p = p_o$), Eqs. (A.13) (with R replaced by p) assume, of course, the values of g_{xx}, g_{xy}, and g_{yy} expressed in terms of R_o.

So far we have identified R with p only at the point (x_o, y_o). The identification at other points of the characteristic is still a conjecture. Substituting Dg_{xx}/Dp and Dg_{yy}/Dp (Eq. A.16) into Eq. (A.15) and remembering that according to Eq. (53) $dp/d\ell$=const along a characteristic, one obtains

$$-R^{-2}(dR/dp)=-(1+M\cos\alpha)p^{-2}[(1-M^2)\sin^2\alpha + (M + \cos\alpha)^2]/(1+M\cos\alpha)^3.$$

$$-R^{-2}(dR/dp) = -p^{-2}.$$

Hence

$$R^{-1} = p^{-1} + \text{const.}$$

The constant vanishes, because one can choose at the point (x_o, y_o)

$$p_o = R_o.$$

This fits with the above conjecture. The identification of p with the radius of curvature is therefore justified.

To determine H_1, we rewrite the first of Eqs. (34) using Eqs. (53)

$$D(\log H_1)/Dp = -(1/2)[1-M^2)g_{xx}+g_{yy}][g_y^2 + ((1-M^2)g_x+M)^2]^{-1/2}[M^2+2M\cos\alpha+1]^{1/2}$$

Hence, with Eq. (50)

$$D(\log H_1)/Dp = -(1/2)[1-M^2)g_{xx} + g_{yy}](1 + M\cos\alpha)$$

and with Eqs. (A.13) with R replaced by p

$$D(\log H_1)/Dp = -(1/2)p^{-1}.$$

Hence,

$$H_1 = \text{const } p^{-1/2} \qquad \text{(A.18)}$$

The form of the function H_2 (introduced in Eq. (25)) is suggested by the asymptotic development of the Hankel functions, (Eq. (19)). In the present setting H_2 is found by integrating the second of Eqs. (34). With p as independent variable, Eq. (34) assumes the form

$$DH_2/dp + (1/2)p^{-1}H_2 + (i/2)[(1-M^2)H_{1,xx} + H_{1,yy}](1+M\cos\alpha) = 0 \qquad \text{(A.19)}$$

Crucial is the inhomogeneous term. The constant in Eq. (A.18) may differ from characteristic to characteristic. Accordingly, we write

$$H_1 = f_1(s)p^{-1/2} \qquad \text{(A.20)}$$

where s is the arc length of the initial curve (see Eqs. (47) and (48)). The value of s is then attached to the characteristics that start at the initial curve. To estimate the inhomogeneous terms, we proceed as follows. Introducing s into Eqs. (A.8) we write

$$x = x(p,s) = x_o(s) + (p-p_o(s))(M + \cos\alpha(s))$$
$$y = y(p,s) = y_o(s) + (p-p_o(s)) \sin\alpha(s). \tag{A.21}$$

Inverting this transformation, one obtains

$$p = p(x,y)$$
$$s = s(x,y).$$

One has the following relations

$$\partial p/\partial x = (\partial y/\partial s)/D$$
$$\partial s/\partial x = -(\partial y/\partial p)/D$$
$$\partial p/\partial y = -(\partial x/\partial s)/D \tag{A.22}$$
$$\partial s/\partial y = (\partial x/\partial p)/D$$

with

$$D = (\partial x/\partial p)(\partial y/\partial s) - (\partial y/\partial p)/(\partial x/\partial s) \tag{A.23}$$

Differentiating Eqs. (A.21) one obtains

$$\partial x/\partial p = M + \cos\alpha(s)$$
$$\partial y/\partial p = \sin\alpha(s)$$
$$\partial x/\partial s = -(\partial p_o/\partial s)(M + \cos\alpha) + (\partial x_o/\partial_s) - (p-p_o(s))\sin\alpha(\partial\alpha/\partial s)$$
$$\partial y/\partial s = -(\partial p_o/\partial s)\sin\alpha + (\partial y_o/\partial s) + (p - p_o(s))\cos\alpha(\partial\alpha/\partial s).$$

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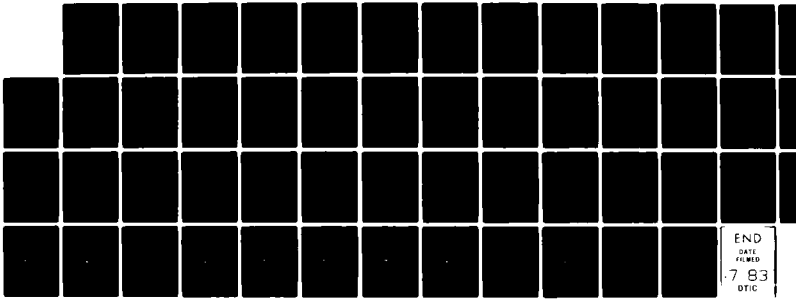

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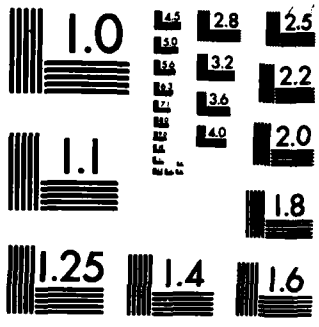

The parameter s is the arc length along the initial curve, α the angle of its normal with the x axis, and p_o the local radius of curvature. Therefore

$$\partial\alpha/\partial s = p_o^{-1}$$

$$\partial x_o/\partial s = -\sin\alpha$$

$$\partial y_o/\partial s = \cos\alpha$$

and

$$\partial x/\partial s = -(\partial p_o/\partial s)(M + \cos\alpha) - (p/p_o)\sin\alpha$$

$$\partial y/\partial s = -(\partial p_o/\partial s)\sin\alpha + (p/p_o)\cos\alpha$$

and from Eq. (A.22)

$$D = (p/p_o)(M\cos\alpha + 1).$$

Then, from Eqs. (A.22)

$$\partial p/\partial x = [-(p_o/p)(\partial p_o/\partial s)\sin\alpha + \cos\alpha](M\cos\alpha + 1)^{-1}$$

$$\partial s/\partial x = -(p_o/p)\sin\alpha\,(M\cos\alpha + 1)^{-1} \qquad \text{(A.24)}$$

$$\partial p/\partial y = [-(p_o/p)(\partial p_o/\partial s)(M + \cos\alpha) + \sin\alpha](M\cos\alpha + 1)^{-1}$$

$$\partial s/\partial y = (p_o/p)(M + \cos\alpha)(M\cos\alpha + 1)^{-1}$$

From these equations one finds the order of magnitude of the expressions for large values of p. $\partial p_o/\partial s$ and p_o are considered as quantities of order 1. One finds

$$\partial p/\partial x = O(1);\ \partial s/\partial x = O(p^{-1});\ \partial p/\partial y = O(1);\ \partial s/\partial y = O(p^{-1}).$$

The orders of magnitude are needed also for derivatives of the next order. One has

$$\frac{\partial^2 p}{\partial x^2} = \left(\frac{\partial}{\partial p}\left(\frac{\partial p}{\partial x}\right)\right)\frac{dp}{dx} + \left(\frac{\partial}{\partial s}\left(\frac{\partial p}{\partial x}\right)\right)\frac{\partial s}{\partial x}$$

and similar formulae for the other derivatives.

In forming derivatives of $(\partial p/\partial x)$, $(\partial p/\partial y)$, $(\partial s/\partial x)$, and $(\partial s/\partial y)$ with respect to s, the order of magnitude of the terms remains unchanged; in forming derivatives with respect to p, the order is lowered by a factor p^{-1}

$$\partial^2 p/\partial x^2 = O(p^{-1})O(1) + O(1)O(p^{-1}) = O(p^{-1})$$

$$\partial^2 p/\partial y^2 = O(p^{-1})O(1) + O(1)O(p^{-1}) = O(p^{-1})$$

$$\partial^2 s/\partial x^2 = O(p^{-2})O(1) + O(p^{-1})O(p^{-1}) = O(p^{-2})$$

$$\partial^2 s/\partial y^2 = O(p^{-2})O(1) + O(p^{-1})O(p^{-1}) = O(p^{-2})$$

Now we can estimate the second derivatives of H_1. One has

$$\partial H_1/\partial x = (\partial H_1/\partial s)(\partial s/\partial x) + (\partial H_1/\partial p(\partial p/\partial x)$$

$$\partial^2 H_1/\partial x^2 = (\partial^2 H_1/\partial_s^2)(\partial s/\partial x)^2 + 2(\partial^2 H_1/\partial s\partial p)(\partial p/\partial x)(\partial s/\partial x) +$$

$$+(\partial^2 H_1/\partial p^2)(\partial p/\partial x)^2 + (\partial H_1/\partial s)(\partial^2 s/\partial x^2) + (\partial H_1/\partial p)(\partial^2 p/\partial x^2)$$

and similarly for $\partial^2 H_1/\partial y^2$.

Now according to Eq. (A.20)

$$H_1 = f(s)p^{-1/2}.$$

Then,

$$\partial^2 H_1/\partial x^2 = O(p^{-1/2})O(p^{-2}) + O(p^{-3/2})O(1)O(p^{-1}) + O(p^{-5/2})O(1)$$

$$+ O(p^{-1/2})O(p^{-2}) + O(p^{-3/2})O(p^{-1})$$

$$\partial^2 H_1/\partial x^2 = O(p^{-5/2}).$$

Similarly,

$$\partial^2 H_1/\partial y^2 = O(p^{-5/2}).$$

The inhomogeneous terms in Eq. (A.19) is therefore $O(p^{-5/2})$. A particular solution of this equation therefore is $O(p^{-3/2})$. The solution for the homogeneous part has $O(p^{-1/2})$, but this contribution can be incorporated in H_1. This is the expected result for H_2.

To arrive at an equation analogous to Eq. (A.14), for the *three-dimensional problem, one can proceed as follows.*

First one derives a formula for the average curvature at a point x_o, y_o, z_o of a surface g = const in the x,y,z space. We introduce a local system of Cartesian coordinates ξ, η, ζ. The ζ axis has the direction of the normal to the surface g = const through the point (x_o, y_o, z_o). The ξ and η axis then lie in the tangential plane. The directional cosines with respect to the x,y,z system are denoted by $\cos(\xi,x)$, $\cos(\xi,y)$, etc. One has

$$\cos^2(\xi,x) + \cos^2(\eta,x) + \cos^2(\zeta x) = 1.$$

Hence

$$\begin{aligned} \cos^2(\xi,x) + \cos^2(\eta,x) &= \sin^2(\zeta,x) \\ \cos^2(\xi,y) + \cos^2(\eta,y) &= \sin^2(\zeta,y) \\ \cos^2(\xi,z) + \cos^2(\eta,z) &= \sin^2(\zeta,z) \end{aligned} \qquad \text{(A.25)}$$

and because the x,y and z axis are perpendicular to each other

$$\begin{aligned}
\cos(\xi,x)\cos(\xi,y)+\cos(\eta,x)\cos(\eta,y) &= -\cos(\zeta,x)\cos(\zeta,y)\\
\cos(\xi,x)\cos(\xi,z)+\cos(\eta,x)\cos(\eta,z) &= -\cos(\zeta,x)\cos(\zeta,z)\\
\cos(\xi,y)\cos(\xi,z)+\cos(\eta,y)\cos(\eta,z) &= -\cos(\zeta,y)\cos(\zeta,z).
\end{aligned} \tag{A.26}$$

The coordinate systems are related by the equation

$$\begin{aligned}
x-x_o &= \xi\cos(\xi,x) + \eta\cos(\eta,x) + \zeta\cos(\zeta,x\\
y-y_o &= \xi\cos(\xi,y) + \eta\cos(\eta,y) + \zeta\cos(\zeta,y)\\
z-z_o &= \xi\cos(\xi,z) + \eta\cos(\eta,z) + \zeta\cos(\zeta,z).
\end{aligned} \tag{A.27}$$

Developing g in the vicinity of the point (x_o,y_o,z_o) up to terms of second order, one obtains

$$\begin{aligned}
g(x,y,z) = g(x_o,y_o,z_o) &+ |\text{grad } g|\xi\\
&+ g_{\xi\xi}(\xi^2/2) + g_{\eta\eta}(\eta^2/2) + g_{\xi\xi}(\zeta^2/2)\\
&+ g_{\xi\eta}\xi\eta + g_{\xi\zeta}\xi\zeta + g_{\eta\zeta}\eta\zeta.
\end{aligned}$$

The intersection of the surface $g = g(x_o,y_o,z_o)$ with the plane $\eta = 0$, is then given by

$$|\text{grad } g|\xi + g_{\xi\xi}(\xi^2/2) + g_{\zeta\zeta}(\zeta^2/2) + g_{\xi\zeta}\xi\zeta = 0.$$

Let R_1 be the radius of curvature of this curve at the origin. One finds

$$R_1^{-1} = g_{\xi\xi}/|\text{grad } g|.$$

Here it is assumed that the region $g < g(x_o,y_o,z_o)$ is convex and the center of curvature lies on the inner normal to this region.

In a corresponding manner, one obtains for the radius of curvature of the intersection of the plane $\xi = 0$ with the surface $g = g(x_o, y_o, z_o)$

$$R_2^{-1} = g_{\eta\eta} / |\text{grad } g|$$

with corresponding rules for the choice of the sign.

The average curvature is then given by

$$C_{\text{average}} = (1/2)(R_1^{-1} + R_2^{-1}) = (1/2)(g_{\xi\xi} + g_{\eta\eta})/|\text{grad } g|. \tag{A.28}$$

We shall see that C_{average} does not depend upon the orientation of the ξ and η axes (as long as they lie in the tangential plane and are perpendicular to each other). We express $(1/2)(g_{\xi\xi} + g_{\eta\eta})$ in the x,y,z system. Using the transformation formulae (Eq. (A.27) one obtains

$$\begin{aligned}(1/2)(g_{\xi\xi} + g_{\eta\eta}) = {} & (1/2)g_{xx}(\cos^2(\xi,x) + \cos^2(\eta,x)) + (1/2)g_{yy}(\cos^2(\xi,y) \\ & + \cos^2(\eta,y)) + (1/2)g_{zz}(\cos^2(\xi,z) + \cos^2(\eta,z)) \\ & + g_{xy}(\cos(\xi,x)\cos(\xi,y) + \cos(\eta,x)\cos(\eta,y) \\ & + g_{xz}(\cos(\xi,x)\cos(\xi,z) + \cos(\eta,x)\cos(\eta,z) \\ & + g_{yz}(\cos(\xi,y)\cos(\xi,z) + \cos(\eta,y)\cos(\eta,z)).\end{aligned}$$

Hence, with the relations (Eqs. (A.25) and (A.26)

$$\begin{aligned}(1/2)g_{\xi\xi} + g_{\eta\eta}) = {} & (1/2)g_{xx}\sin^2(\zeta,x) + (1/2)g_{yy}\sin^2(\zeta,y) + (1/2)g_{zz}\sin^2(\zeta,z) \\ & - g_{xy}\cos(\zeta,x)\cos(\zeta,y) - g_{x,z}\cos(\zeta,x)\cos(\zeta,z) - g_{y,z}\cos(\zeta,y)\cos(\zeta,z).\end{aligned} \tag{A.29}$$

This is the analogon to Eq. (A.14) for the two-dimensional case. The directional angles for the ξ and η axes are no longer present. This shows that this expression does not depend upon the orientation of these axes in the tangential plane.

Again, g_x, g_y, and g_z are constant along the characteristics (see Section VIII) which are given in parametric form in Eq. (115). The ζ direction is the direction of the normal to a surface g = const. One then obtains

$$\begin{array}{l|l} g_{xx}(M+\cos(\zeta,x)) + g_{xy}\cos(\zeta,y) + g_{xz}\cos(\zeta,z) = 0 & (-M+\cos(\zeta,x) \\ g_{xy}(M+\cos(\zeta,x)) + g_{yy}\cos(\zeta,y) + g_{yz}\cos(\zeta,z) = 0 & \cos(\zeta,y) \quad (A.30) \\ g_{xz}(M+\cos(\zeta,x) + g_{yz}\cos(\zeta,y) + g_{zz}\cos(\zeta,z) = 0 & \cos(\zeta,z). \end{array}$$

Multiplying the equations with the factors shown on the right and adding the results to Eq. (A.29) multiplied by 2, one obtains

$$g_{\xi\xi} + g_{\eta\eta} = g_{xx}(1-M^2) + g_{yy} + g_{zz}.$$

The average curvature (Eq. (A.28)) is therefore given by

$$C_{average} = (1/2)(R_1^{-1} + R_2^{-1}) = (1/2)(g_{xx}(1-M^2) + g_{yy} + g_{zz})/|\text{grad } g|. \quad (A.31)$$

In Eq. (115) the characteristics have been represented in terms of a parameter p. To determine the dependence of the average curvature upon p, we use the geometric interpretation of Eq. (115) already applied in this Appendix for the two-dimensional case. Constructing a point of the surface $g = g_1$ = const from point (x_o, y_o, z_o) of a surface $g = g_o$ = const, one first proceeds in the direction of the normal to the original surface by a distance $p_1 - p_o = g_1 - g_o$ (because of Eq. (117), and then shifts the surface in the x-direction by $M(p_1 - p_o)$. In the first step one constructs a surface equidistant to $g = g_o$ at a distance $g_1 - g_o$, in the second the surface as a whole is translated in the x-direction.

The radii of curvature R_1 and R_2 refer to the lines of intersection of a surface g = const with two planes ξ = const and η = const, which are perpendicular to each other and contain

the normal to this surface. Among the possible orientations of these planes, there is one where R_1 and R_2 have their extrema. One then obtains the principal radii of curvature. For these orientations of the planes the normal to the surface g = const remains within the planes even at points of the surface which deviate from (x_o, y_o, z_o) by a first-order distance. One then obtains points of the surface $g = g_1$ by proceeding from a point of the line of intersection of $g = g_o$ with $\xi = 0$, say, to the "corresponding" point of the surface $g = g_1$ by proceeding within the surface $\xi = 0$. In determining the radius of curvature of the intersection of the surface $g = g_1$ with the plane $\xi = 0$, we can confine our attention to this plane. If the radius of curvature of the line of intersection of the surface $g = g_o$ is given by $R_1 = R_{10}$, one finds for the corresponding radius of curvature R_{11}, at $g = g_1$

$$R_{11} = R_{10} + (p_1 - p_o).$$

If one sets $p_o = 0$, then one obtains for the average curvature (Eq. (A.31))

$$C_{average,1} = 2^{-1}((R_{10} + p)^{-1} + (R_{20} + p)^{-1})$$

$$C_{average,1} = \left\{(1/2)[(1-M^2)g_{xx} + g_{yy} + g_{zz}]/|\text{grad } g|\right\}_1$$

$$= (1/2)[(R_{10} + p)^{-1} + (R_{20} + p)^{-1}]$$

where R_{10} and R_{20} are the principal radii of curvature at the point x_o, y_o, z_o. Only for large values of p will this expression behave as p^{-1}.

APPENDIX III
VERIFICATION OF EQUATION (67)

It is practical to rewrite Eqs. (66) and the next equation. Let

$$R_1 = \phi_{\xi\xi},\ R_2 = \phi_{\eta\eta},\ R_3 = \phi_{\xi\eta} \qquad (A.32)$$

$$R_4 = 2i\omega\phi_x - \omega^2\phi.$$

Furthermore

$$X = (1/2)(\phi_{xx} + \phi_{yy})$$

$$Y = (1/2)(\phi_{xx} - \phi_{yy}) \qquad (A.33)$$

$$Z = \phi_{xy}.$$

The following trigonometric identities are used

$$\cos^2\beta = (1/2)(1 + \cos 2\beta)$$
$$\cos^2\gamma = (1/2)(1 + \cos 2\gamma) \qquad (A.34)$$
$$\sin\beta\sin\gamma = (1/2)(\cos(\beta-\gamma) - \cos(\beta+\gamma)).$$

For later use we note some further identities

$$\cos^2\beta+\cos^2\gamma = 1+(1/2)(\cos 2\beta+ \cos 2\gamma)=1+\cos(\beta+\gamma)\cos(\beta-\gamma)$$
$$\cos2\beta+\cos2\gamma = 2\cos(\beta+\gamma)\cos(\beta-\gamma)$$
$$\cos^2\beta-\cos^2\gamma = (1/2)(\cos 2\beta-\cos 2\gamma)= -\sin(\beta+\gamma)\sin(\beta-\gamma) \qquad (A.35)$$
$$\cos2\beta-\cos2\gamma = -2\sin(\beta+\gamma)\sin(\beta-\gamma)$$
$$\sin2\beta-\sin2\gamma = 2\sin(\beta-\gamma)\cos(\beta+\gamma)$$
$$\sin2\beta+\sin2\gamma = 2\sin(\beta+\gamma)\cos(\beta-\gamma)$$

Equations (66) then assume the following form

$$\begin{aligned} X + Y\cos 2\beta + Z\sin 2\beta &= R_1 \\ X + Y\cos 2\gamma + Z\sin 2\gamma &= R_2 \\ X\cos(\beta-\gamma) + Y\cos(\beta+\gamma) + Z\sin(\beta+\gamma) &= R_3 \\ X(2-M^2) - Y M^2 &= R_4 \end{aligned} \tag{A.36}$$

Equation (67), which is to be verified, now appears in the form

$$\begin{aligned} &R_1(1-M^2\sin^2\gamma)+R_2(1-M^2\sin^2\beta)+R_3[-2\cos(\beta-\gamma)-M^2(\cos(\beta+\gamma)-\cos(\beta-\gamma))] \\ &\qquad -R_4\sin^2(\beta-\gamma) = 0 \end{aligned} \tag{A.37}$$

To carry out this verification one must show that the scalar product of the vector formed with the left hand sides of Eqs. (A.36) and the vector formed with the coefficient of R_1 through R_4 in Eqs. (A.37) vanishes. Accordingly, one must evaluate the scalar product

$$\begin{bmatrix} X & +Y\cos 2\beta & +Z\sin 2\beta \\ X & +Y\cos 2\gamma & +Z\sin 2\gamma \\ X\cos(\beta-\gamma) & +Y\cos(\beta+\gamma) & +Z\sin(\beta+\gamma) \\ X(2-M^2) & -YM^2 & \end{bmatrix} \cdot \begin{bmatrix} 1 - M^2\sin^2\gamma \\ 1 - M^2\sin^2\beta \\ \{-2\cos(\beta-\gamma) \\ -M^2(\cos(\beta+\gamma)-\cos(\beta-\gamma))\} \\ -\sin^2(\beta-\gamma) \end{bmatrix}$$

This expression is linear in M^2. The terms not containing M^2 give

$$X[1+1-2\cos^2(\beta-\gamma)-2\sin^2(\beta-\gamma)+Y[\cos 2\beta + \cos 2\gamma - 2\cos(\beta+\gamma)\cos(\beta-\gamma)]$$

$$+Z[\sin 2\beta + \sin 2\gamma - 2\sin(\beta+\gamma)\cos(\beta-\gamma)].$$

The coefficients of Y and Z vanish because of Eqs. (A.35). The vanishing of the coefficient of X is self evident.

Next we evaluate the terms with M^2. It suffices if one sets $M^2 = 1$. The second vector then becomes

$$[\cos^2\gamma,\ \cos^2\beta,\ -[\cos(\beta+\gamma)+\cos(\beta-\gamma)],\ -\sin^2(\beta-\gamma)]^+.$$

One obtains for the coefficient of X

$$\cos^2\gamma+\cos^2\beta-\cos^2(\beta-\gamma)-\cos(\beta-\gamma)\cos(\beta+\gamma)-\sin^2(\beta-\gamma).$$

It vanishes because of the first of Eqs.(A.35).

The term with Y is given by

$$Y[\cos(2\beta)\cos^2\gamma+\cos(2\gamma)\cos^2\beta-\cos^2(\beta+\gamma)-\cos(\beta+\gamma)\cos(\beta-\gamma)+\sin^2(\beta-\gamma)]$$

$$= Y[\cos(2\beta)(1/2)(\cos(2\gamma)+1)+\cos(2\gamma)(1/2)(\cos(2\beta)+1-\cos^2(\beta+\gamma)$$

$$-\cos(\beta+\gamma)\cos(\beta-\gamma)+\sin^2(\beta-\gamma)].$$

Next, with the second of Eqs. (A.35)

$$=Y\ [\cos(2\beta)\cos2\gamma-\cos^2(\beta+\gamma)+\sin^2(\beta-\gamma)]$$

Now $\cos2\beta$ and $\cos2\gamma$ are expressed in terms of $\beta+\gamma$ and $\beta-\gamma$

$$=Y[\cos^2(\beta+\gamma)\cos^2(\beta-\gamma)-\sin^2(\beta+\gamma)\sin^2(\beta-\gamma)-\cos^2(\beta+\gamma)+\sin^2(\beta-\gamma)]$$

$$=Y[-\cos^2(\beta+\gamma)\sin^2(\beta-\gamma)+\sin^2(\beta-\gamma)\cos^2(\beta+\gamma)] = 0.$$

The term with Z is given by

$$Z[\sin(2\beta)\cos^2\gamma+\sin(2\gamma)\cos^2\beta-\sin(\beta+\gamma)(\cos(\beta-\gamma)+\cos(\beta+\gamma))]$$

$$=Z\ [\sin(2\beta)(1/2)(\cos(2\gamma)+1)+\sin(2\gamma)(1/2)(\cos(2\beta)+1)$$

$$-\sin(\beta+\gamma)\cos(\beta-\gamma)-\sin(\beta+\gamma)\cos(\beta+\gamma)].$$

Then, with the fourth of Eqs. (A.35)

$$Z[(1/2)\sin(2\beta)\cos(2\gamma)+\cos(2\beta)\sin(2\gamma)) - \sin(\beta+\gamma)\cos(\beta+\gamma)]$$

$$= Z(1/2\sin(2(\beta+\gamma))- \sin(\beta+\gamma)\cos(\beta+\gamma)] = 0.$$

This gives the desired verification.

APPENDIX IV
CHARACTERISTICS

We compile here the leading ideas of the theory of characteristics applied to the present case. The differential equation under consideration is Eq. (5), with $\hat{\phi}$ replaced by ϕ.

$$(1-M^2)\phi_{xx} + \phi_{yy} - 2M\phi_{xt} - \phi_{tt} = 0 \tag{A.38}$$

We consider some surface in the x, y, t space and assume that it is oriented so that it can be parametrized by x and y. Let this surface be given by

$$t = f(x,y) \tag{A.39}$$

We assume that $\phi_x = U(x,y)$, $\phi_y = V(x,y)$ and $\phi_t = W(x,y)$ are known on this surface. Obviously, these quantities cannot be independent. If the surface S were given by t = const, then in order for a potential to exist, one must have $U_y = V_x$. To derive a corresponding relation for a general surface, we consider the following system of equations

$$\begin{bmatrix} 1 & 0 & 0 & f_x & 0 & 0 \\ 0 & 1 & 0 & 0 & f_y & 0 \\ 0 & 0 & 1 & f_y & 0 & 0 \\ 0 & 0 & 1 & 0 & f_x & 0 \\ 0 & 0 & 0 & 1 & 0 & f_x \\ 0 & 0 & 0 & 0 & 1 & f_y \end{bmatrix} \begin{bmatrix} \phi_{xx} \\ \phi_{yy} \\ \phi_{xy} \\ \phi_{xt} \\ \phi_{yt} \\ \phi_{tt} \end{bmatrix} = \begin{bmatrix} U_x \\ V_y \\ U_y \\ V_x \\ W_x \\ W_y \end{bmatrix} \tag{A.40}$$

The arguments of the derivatives of ϕ are x, y and f(x,y) = t. The arguments of the derivatives of f, U, V, and W are x and y. Premultiplying Eq. (A.34) by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -f_y & 0 \\ 0 & 0 & 0 & 1 & 0 & -f_x \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

one obtains

$$\begin{bmatrix} 1 & 0 & 0 & f_x & 0 & 0 \\ 0 & 0 & 0 & 0 & f_y & 0 \\ 0 & 0 & 1 & 0 & 0 & -f_x f_y \\ 0 & 0 & 1 & 0 & 0 & -f_x f_y \\ 0 & 0 & 0 & 1 & 0 & f_x \\ 0 & 0 & 0 & 0 & 1 & f_y \end{bmatrix} \begin{bmatrix} \phi_{xx} \\ \phi_{yy} \\ \phi_{xy} \\ \phi_{xt} \\ \phi_{yt} \\ \phi_{tt} \end{bmatrix} = \begin{bmatrix} U_x \\ V_y \\ U_y - W_x f_y \\ V_x - W_y f_x \\ W_x \\ W_y \end{bmatrix}$$

The left side of the third and fourth equations in this system are identical. Therefore, the right sides must be identical also.

$$U_y - W_x f_y = V_x - W_y f_x \qquad (A.41)$$

In deriving the characteristics condition, one will omit either the third or fourth equation, but add the partial differential equation for Eq.(A.38), which so far has not bee used. Accordingly, one considers the system

$$\begin{bmatrix} 1 & 0 & 0 & f_x & 0 & 0 \\ 0 & 1 & 0 & 0 & f_y & 0 \\ 0 & 0 & 1 & 0 & 0 & -f_x f_y \\ 0 & 0 & 0 & 1 & 0 & f_x \\ 0 & 0 & 0 & 0 & 1 & f_y \\ (1-M^2) & 1 & 0 & -2M & 0 & -1 \end{bmatrix} \begin{bmatrix} \phi_{xx} \\ \phi_{yy} \\ \phi_{xy} \\ \phi_{xt} \\ \phi_{yt} \\ \phi_{tt} \end{bmatrix} = \begin{bmatrix} U_x \\ V_y \\ U_y - W_x f_y \\ W_x \\ W_y \\ 0 \end{bmatrix}$$

The systematic elimination of the unknows ϕ_{xx}, ϕ_{yy}, etc. amounts to premultiplication of the system by the following 1 by 6 matrix:

$$[-(1-M^2),\ -1,\ \ 0,\ \ 2M+(1-M^2)f_x,\ \ \ f_y,\ \ \ 1].$$

This yields the relation

$$[(2M+(1-M^2)f_x)f_x + f_y^2 - 1]\phi_{tt} = -(1-M^2)U_x - V_y + (2M+(1-M^2)f_x)W_x + f_y W = 0 \tag{A.42}$$

For a characteristic surface, the derivatives (among them ϕ_{tt}) are not uniquely determined. Along characteristic surfaces discontinuities in the second (and higher) derivatives may, therfore, be encountered. Furthermore, derivatives within the surface cannot be chosen independently. This can be recognized in the last equation. One has a characteristic surface, if the factor of ϕ_{tt} vanishes

$$(1-M^2)f_x^2 + 2Mf_x + f_y^2 - 1 = 0 \tag{A.43}$$

But then the right hand side must also vanish and one obtains the compatibility condition

$$-(1-M^2)U_x - V_y + (2M+(1-M^2)f_x)W_x + f_y W_y = 0. \tag{A.44}$$

APPENDIX V

THE TWO DIMENSIONAL UNSTEADY FLOW FIELD AT MACH NUMBER ONE DETAILED COMPUTATIONS

In the main text the general course of the computations is shown. The details are rather cumbersome, although certain expressions which at the beginning, have a rather unwidely form can be considerably simplified. The present appendix gives these computations in sufficient detail to allow a sceptical reader to check the results. The function g is determined from the differential equation (144)

$$(\gamma+1)\phi_{ox}g_x^2 - 2g_x - g_y^2 + 1 = 0 \qquad (A.45)$$

In this equation the term ϕ_{ox} which is determined by the basic field appears; it is expressed by Eqs. (154) through (159). We add the following expression.

It follows from Eqs. (155) and (156)

$$x = \mu y^{4/5}\bar{\xi}$$

Hence, with Eq. (158)

$$-x = \mu(\sigma/y^2)^{-2/5}(1-(\sigma/2) \qquad (A.46)$$

and

$$(\sigma/y^2) = \mu^{5/2}(-x)^{-5/2}(1-(\sigma/2))^{5/2} \qquad (A.47)$$

The following formulae will be needed, because we shall use σ and y (instead of x and y) as independent variables.

One has, from Eq. (158)

$$d\sigma/d\bar{\tau} = (10/3)\sigma^{7/5}(\sigma+(4/3))^{-1}. \qquad (A.48)$$

Then with Eqs. (159)

$$\partial\sigma(x,y)/\partial x = (d\sigma/d\bar{\zeta})(\partial\bar{\zeta}/\partial x) = \mu^{-1}(10/3)y^{-4/5}\sigma^{7/5}(\sigma+(4/3))^{-1}$$
$$\partial\sigma(x,y)/\partial y = (d\sigma/d\bar{\zeta})(\partial\bar{\zeta}/\partial y) = -(4/3)y^{-1}\sigma(\sigma-2)(\sigma+(4/3))^{-1} \tag{A.49}$$

It follows from Eqs. (159) and (A.48) that

$$d\bar{f}/d\bar{\xi} = (d\bar{f}/d\sigma)(d\sigma/d\bar{\xi}) = \sigma^{1/5}(\sigma/4)-1)$$

Then from Eqs. (154), (155), and (156)

$$(\gamma+1)\phi_{ox} = \mu^2(\sigma/y^2)^{1/5}((\sigma/4)-1). \tag{A.50}$$

As a preliminary hypothesis we introduce, instead of Eq. (161)

$$g_o = \mu^{-1}\, y^{6/5}\, \bar{\gamma}_o(\sigma). \tag{A.51}$$

Then

$$g_{ox} = \mu^{-2}(10/3)y^{2/5}\,\sigma^{7/5}(\sigma+(4/3))^{-1}\, d\bar{\gamma}_o/d\sigma \tag{A.52}$$
$$g_{oy} = \mu^{-1}y^{1/5}[(6/5)\bar{\gamma}_o - (4/3)\sigma(\sigma-2)(\sigma+(4/3))^{-1}\, d\bar{\gamma}_o/d\sigma].$$

This expression is substituted into Eq. (160). One obtains with Eq. (A.50)

$$\mu^{-2}y^{2/5}\{\sigma^{1/5}((\sigma/4)-1)(100/9)\sigma^{14/5}(\sigma+(4/3))^{-2}(d\bar{\gamma}_o/d\sigma)^2$$
$$-(20/3)\sigma^{7/5}(\sigma+(4/3))^{-1}(d\bar{\gamma}_o/d\sigma)$$
$$-[(6/5)\bar{\gamma}_o-(4/3)\sigma(\sigma-2)(\sigma+(4/3))^{-1}d\gamma_o/d\sigma]^2\} = 0$$

Next we set

$$\bar{\gamma}_o = \sigma^{-3/5}\gamma_o(\sigma)$$
$$d\bar{\gamma}_o/d\sigma = \sigma^{-8/5}(-(3/5)\gamma_o + \sigma(d\gamma_o/d\sigma)). \tag{A.53}$$

Then, except for a common factor, one obtains integral powers of σ in the last equation

$$\mu^{-2}y^{2/5}\sigma^{-6/5}\{(100/9)\sigma((\sigma/4)-1)(\sigma+(4/3))^{-2}(-(3/5)\gamma_o+ (d\gamma_o/d\sigma))^2$$
$$-(20/3)\sigma(\sigma+ (4/3))^{-1}(-(3/5)\gamma_o + \sigma(d\gamma_o/d\sigma)) \tag{A.54}$$
$$-[(6/5)\gamma_o- (4/3)(\sigma-2)(\sigma+ (4/3))^{-1}(-(3/5)\gamma_o + (d\gamma_o/d\sigma))]^2\} = 0 .$$

The coefficient of γ_o in the last term reduces to

$$(6/5)\gamma_o-(4/3)(\sigma-2)(\sigma+ (4/3))^{-1}(-(3/5)\gamma_o$$
$$= (\sigma+ (4/3))^{-1}2\sigma\gamma_o.$$

One thus obtains for the whole expression (Eq. (A.53))

$$\mu^{-2}y^{2/5}\sigma^{-1/5}(\sigma+ (4/3))^{-2}\{(100/9)((\sigma/4)-1)(-(3/5)\gamma_o+ (d\gamma_o/d\sigma))^2$$
$$-(20/3)(\sigma+ (4/3))(-(3/5)\gamma_o + \sigma(d\gamma_o/d\sigma))$$
$$-\sigma[2\gamma_o - (4/3) (\sigma-2)(d\gamma_o/d\sigma)]^2\} = 0.$$

Next we collect within the braces the quadratic terms with different powers of $d\gamma_o/d\sigma$

Quadratic terms:

$$(d\gamma_o/d\sigma)^2[(100/9)(\sigma/4)-1)\sigma^2 - \sigma(16/9)(\sigma-2)^2$$
$$= (d\gamma_o/d\sigma)^2\sigma[\sigma^2 - 4\sigma - 64/9) = (d\gamma_o/d\sigma)^2\sigma(\sigma + 4/3))(\sigma - (16/3))$$

Linear terms

$$(d\gamma_o/d\sigma)\sigma[(100/9)((\sigma/4)-1)(-6/5)\gamma_o - (20/3)(\sigma+(4/3)+(16/3)\gamma_o(\sigma-2)]$$

$$= (d\gamma_o/d\sigma)\sigma[\gamma_o(8/3+2\sigma)-(20/3)(\sigma+(4/3))]$$

$$= (d\gamma_o/d\sigma)\sigma(\sigma+(4/3))[2\gamma_o-(20/3)].$$

The terms within the braces which do not contain $d\gamma_o/d\sigma$ are

$$(100/9)((\sigma/4)-1)(9/25)\gamma_o^2 - (20/3)(\sigma+(4/3))(-3/5)\gamma_o - 4\sigma\gamma_o^2]$$

$$= \sigma\gamma_o^2 - 4\gamma_o^2 + 4(\sigma+(4/3))\gamma_o - 4\sigma\gamma_o^2$$

$$= (\sigma+(4/3)(-3\gamma_o+4)\gamma_o.$$

One thus obtains

$$\mu^{-2}y^{2/5}\sigma^{-1/5}(\sigma+(4/3))^{-1}\{\sigma(\sigma-(16/3))(d\gamma_o/d\sigma)^2+\sigma(2\gamma_o-(20/3)(d\gamma_o/d\sigma)$$

$$+ (-3\gamma_o+4)\gamma_o\} = 0.$$

The solution which is smooth, even at $\sigma = 16/3$ (the limiting characteristic) is immediately obvious

$$\gamma_o = 4/3 \qquad \text{(A.55)}$$

Combining Eq. (A.51), (A.53), and (A.55), one obtains

$$g_o = (4/3)\mu^{-1}\, y^{6/5}\, \sigma^{-3/5} \qquad \text{(A.56)}$$

A correction g_1 to g_o is computed from Eq. (166)

$$((\gamma+1)\phi_{ox}g_{ox}^{-1})g_{1x} - g_{oy}g_{1y} + (1/2) = 0 \qquad \text{(A.57)}$$

One has from Eqs. (A.56) and (A.49)

$$g_{ox} = -\mu^{-2}(8/3)y^{2/5}\sigma^{-1/5}(\sigma+ (4/3))^{-1} \qquad (A.58)$$

$$g_{oy} = \mu^{-1}(8/3)y^{1/5}\sigma^{2/5}(\sigma+ (4/3))^{-1}$$

and, with Eq. (A.50)

$$(\gamma+1)\phi_{ox}g_{ox} = -(8/3)((\sigma/4)-1)(\sigma+ (4/3))^{-1} \qquad (A.59)$$

$$(\gamma+1)\phi_{ox}g_{ox} -1 = (-(5/3)\sigma + (4/3))(\sigma+ (4/3))^{-1}.$$

First we set

$$g_1 = y^{4/5}\,\bar{\gamma}_1(\sigma) \qquad (A.60)$$

Then one has, with Eqs. (A.49)

$$g_{1x} = \mu^{-1}(10/3)\sigma^{7/5}(\sigma+ (4/3))^{-1}(d\bar{\gamma}_1/d\sigma)$$

$$g_{1y} = y^{-1/5}[(4/5)\bar{\gamma}_1 - (4/3)(\sigma-2)(\sigma+ (4/3))^{-1}\sigma(d\bar{\gamma}_1/d\sigma)].$$

Substituting this into Eq. (A.57), one obtains

$$(-(5/3)\sigma+ (4/3))(\sigma+ (4/3))^{-1}\,\mu^{-1}(10/3)\sigma^{7/5}(\sigma+ (4/3))^{-1}(d\bar{\gamma}_1/d\sigma)$$

$$-\mu^{-1}(8/3)y^{1/5}\sigma^{2/5}(\sigma+ (4/3))^{-1}\,y^{-1/5}[(4/5)\bar{\gamma}_1-(4/3)(\sigma-2)$$

$$(\sigma+(4/3))^{-1}\sigma(d\bar{\gamma}_1/d\sigma)] + (1/2) = 0.$$

This equation simplifies :

$$(\sigma+ (4/3))^{-1}\,\mu^{-1}\,\sigma^{2/5}\left\{-2\,(d\bar{\gamma}_1/d\sigma - (32/15)\bar{\gamma}_1\right\} + (1/2) = 0 \qquad (A.61)$$

This shows that with the hypothesis (Eq.(A.57)), one obtains an ordinary differential equation in which y does not appear.

Next, one introduces

$$\bar{\gamma}_1 = \mu\sigma^{-2/5}\gamma_1. \tag{A.62}$$

Substitution into Eq. (A.61) gives

$$\left\{(4/5)\gamma_1 - 2\sigma(d\gamma_1/d\sigma) - (32/15)\gamma_1\right\} = -(1/2)(\sigma + 4/3))$$

and finally,

$$\sigma(d\gamma_1/d\sigma) + (2/3)\gamma_1 = (1/4)(\sigma + (4/3)).$$

The solution which is smooth at $\sigma = 0$ is given by

$$\gamma_1 = (1/2) + (3/20)\sigma \tag{A.63}$$

Thus,

$$g_1 = \mu\, y^{4/5}\, \sigma^{-2/5}((1/2) + (3/20)\sigma) \tag{A.64}$$

From Eqs. (A.56) and (A.64), the following approximation is obtained

$$g = \mu^{-1}y^{6/5}\sigma^{-3/5}[(4/3)+A((1/2) + (3/20)\sigma)] \tag{A.65}$$

with

$$A = \mu^2(\sigma/y^2)^{1/5}$$

In the vicinity of the negative x-axis, one uses Eq. (A.47) to evaluate (σ/y^2)

Next, formulae for the characteristic directions are developed on the basis of Eq. (147). Parts of the numerator and denominator of the right side are found in Eqs. (A.58) and (A.59). For the missing contributions of g_1, one finds from Eq. (A.64) and (A.49)

$$g_{1y} = \mu y^{-1/5}\sigma^{-2/5}\{(4/5)((1/2) + (3/20)\sigma) - (4/3)(\sigma-2)(\sigma + (4/3))^{-1}[-(2/5)(1/2) + (3/5)(3/20)\sigma]\} \tag{A.66}$$

This simplifies to

$$g_{1y} = \mu(16/15)y^{-1/5\ 3/5}(\sigma + (4/3))^{-1} \tag{A.67}$$

Moreover, from Eqs. (A.64) and (A.49)

$$g_{1x} = \mu y^{4/5}\sigma^{-7/5}(-(1/5)+(9/100)\sigma)\mu^{-1}(10/3)y^{-4/5}\sigma^{7/5}(\sigma+(4/3))^{-1}$$

$$g_{1x} = (-(2/3) + (3/10)\sigma)(\sigma+ (4/3))^{-1}$$

Then with Eq. (A.50)

$$(\gamma+1)\phi_{ox}g_{1x} = \mu^2(\sigma/y^2)^{1/5}((\sigma/4)-1)(-(2/3) + (3/10)\sigma)(\sigma+ (4/3))^{-1} \tag{A.68}$$

Combining Eqs. (A.67) and (A.68) with (A.59) and (A.58) and using the definition of A, Eq. (A.65), one obtains

$$1 -(\gamma+ 1)\phi_{ox}g_x = (\sigma+ (4/3))^{-1}[(5/3)\sigma-(4/3)) + A((\sigma/4)-1)((2/3-(3/10))\sigma]$$

$$g_y = \mu^{-1}\ y^{1/5}\sigma^{2/5}(\sigma+ (4/3))^{-1}(8/3)[1 + (2/5)A]$$

Then, from Eq. (147)

$$Dx/Dy = \mu(\sigma/y^2)^{-2/5}y^{-1}[(5/8)\sigma - (1/2) + A((\sigma/4)-1)(1/4)-(9/80)\sigma][1+(2/5)A]^{-1} \tag{A.69}$$

To obtain $D\sigma/Dy$, we proceed as follows. It follows from Eq. (A.46), that along a curve $\sigma = \sigma(y)$

$$dx/dy = \mu y^{-1}(\sigma/y^2)^{-2/5}[(4/5)((\sigma/2)-1)+(3/10)(\sigma+(4/3))d(\log\sigma)/d(\log y)]$$

One obtains, by substituting into this equation the value of Dx/Dy for the characteristics, Eq. (A.69)

$$\frac{(5/8)\sigma-(1/2)+A((\sigma/4)-1)((1/4)-(9/80)\sigma)}{1+(2/5)A} = (4/5)((\sigma/2)-1)+(3/10)(\sigma+(4/3))D(\log\sigma)/D(\log y)$$

Hence

$$(3/10)(\sigma+(4/3))D(\log\sigma)/D(\log y) =$$

$$= (1+(2/5)A)^{-1}\{(5/8)\sigma-\tfrac{1}{2}+A((\sigma/4)-1)((1/4)-(9/80)\sigma)-(1+(2/5)A)(4/5)((\sigma/2)-1)\}$$

$$= (1+(2/5)A)^{-1}\{(9/40)(\sigma+(4/3))-A(9/320)(\sigma+(4/3)(\sigma-(28/15))\}$$

Hence,

$$D(\log\sigma)/D(\log y) = (1+(2/5)A)^{-1}(3/4)[1-(1/8)A(\sigma-(28/15))]$$

One may develop the denominator with respect to A. Then one obtains

$$D(\log\sigma)/D(\log y) = 3/4[1-(1/8)A(\sigma+(4/3))].$$

Usually, it is convenient to consider σ as independent variable. Again, developing with respect to A, one obtains

$$D(\log y)/D(\log\sigma) = (4/3[1-(1/8)A(\sigma+(4/3))] \quad (A.70)$$

Next, we derive an expression also for $D(\log -x)/D\log\sigma$. One finds from Eq. (A.46)

$$D\log(-x) = (4/5)d(\log y)-(3/10)(\sigma+(4/3))(1-(\sigma)/2))^{-1}d(\log\sigma).$$

Equation (A.70) is used to eliminate d(logy). One then obtains

$$D(\log(-x))/D(\log\sigma) = (2/3)[(1-(5/4)\sigma)(1-(\sigma/2))^{-1}+(1/5)A(\sigma+(4/3)]. \quad (A.71)$$

No development with respect to A has been carried out in deriving Eq. (A.71) from Eq. (A.70). Although both equations neglect terms of higher order in A, they will give the same directions in the x,y plane. The same holds for the quotient of Eqs. (A.71) and (A.70)

$$D(\log(-x))/D(\log y) = (1/2)[(1-5/4)\sigma)(1-(\sigma/2))^{-1}+(1/5)A(\sigma+(4/3))][1+(1/8)A(\sigma+(4/3))]^{-1} \quad (A.72)$$

For the vicinity of the y axis, one writes

$$D(-x)/D\log y)$$

$$=(1/2)[(1-(5/4)\sigma)(-x)(1-\sigma/2))^{-1}-x(1/5)A(\sigma+(4/3))][1+(1/8)A(\sigma+(4/3))]^{-1}$$

and uses Eq. (A.46) to express $(-x)(1-(\sigma/2))^{-1}$, which for x = 0 gives (0/0).

An analytical expression for the characteristics is found from Eq. (A.70). In the lowest approximation in A, one obtains

$$y = c\,\sigma^{4/3} \quad (A.73)$$

where c is the constant of integration. To take first order terms of A, we introduce

$$u = \log y$$
$$v = \log \sigma.$$

This is substituted into Eq. (A.70) and the specific form of A is introduced. One obtains

$$Du/Dv = (4/3) + (1/6)\mu^2\exp(-(2/5)u)[\exp((6/5)v) + (4/3)\exp((1/5v)]=0.$$

In the first approximation, one obtains the above results

$$u_o = (4/3)v + \log c.$$

Next, one sets

$$u = u_o + u_1$$

and neglects higher order terms in u_1. Then

$$Du_1/Dv=(1/6)\mu^2(\exp(-(8/15)-(2/5)\log c)[\exp((6/5)v)+(4/3)\exp((1/5)v)]=0$$

$$Du_1/Dv=(1/6)\mu^2 c^{-2/5}[\exp((2/3)v) + (4/3)\exp(-((1/3)v)] = 0$$

$$u_1 =\mu^2 c^{-2/5}[(1/4)\exp((2/3)v) - (2/3)\exp(-(1/3)v)].$$

Therefore, accurate to the next order

$$u = \log c + (4/3)v + \mu^2 c^{-2/5}[(1/4)\exp((2/3)v)-(2/3)\exp((-1/3)v)]$$

$$y = c\,\sigma^{4/3}\exp[\mu^2 c^{-2/5}(1/4)\sigma^{2/3} - (2/3)\sigma^{-1/3}]$$

For c large (y large) the argument of the exponential function is small. Developing the exponential function, one obtains

$$y = c\,\sigma^{4/3} + \mu^2 c^{3/5}[(1/4)\sigma^2 - (2/3)\sigma] \qquad (A.74)$$

For the application of the far field conditions one needs the derivative of g along the characteristics. The formula which is correct to the first order in A is Eq. (177). The lowest approximation is simply by substituting Eq. (A.74) into Eq. (A.56)

$$g_o = (4/3)\mu^{-1} c^{6/5}\sigma$$

or

$$g_o = (4/3)\mu^{-1} c^{9/20} y^{3/4}.$$

Hence,

$$D\, g_o/D\sigma = (4/3)\mu^{-1}\, c^{6/5}(1 + 0(A)) \qquad (A.75)$$

$$= (4/3)\mu^{-1}\, c^{6/5}(1 + 0(y^{-2/5}))$$

and

$$Dg_o/Dy = \mu^{-1} c^{9/20}\, y^{-1/4}.$$

Also needed is DlogH/Dy. One has from Eqs. (153) and (149)

$$D(\log H)/Dy = (1/2)[(\gamma+1)(\partial(\phi_{ox} g_x)/\partial x) - g_{yy}]/g_y. \qquad (A.76)$$

We evaluate this expression only to the lowest order in A. According to Eq. (A.59)

$$(\gamma+1)\phi_{ox} g_{ox} = -(8/3)((\sigma/4)-1)(\sigma+(4/3))^{-1}.$$

Hence with Eq. (A.49)

$$(\gamma+1)\partial(\phi_{ox} g_{ox})/ = -(8/3)\frac{(1/4)(\sigma+(4/3)) - ((\sigma/4)-1)}{(\sigma+(4/3))^2}\mu^{-1}(10/3)y^{-4/5}\sigma^{7/5}(\sigma+(4/3))^{-1}$$

$$= -\mu^{-1}(320/27)(\sigma/y^2)^{2/5}\sigma(\sigma+(4/3))^{-3}. \qquad (A.77)$$

According to Eq. (A.58)

$$g_{oy} = \mu^{-1}(8/3)y^{1/5}\sigma^{2/5}(\sigma+(4/3))^{-1} = \mu^{-1}(8/3)y(\sigma/y^2)^{2/5}(\sigma+(4/3))^{-1} \qquad (A.78)$$

Then with Eq. (A.49)

$$g_{oyy} = \mu^{-1}(8/3)y^{-4/5}\sigma^{2/5}\{(\sigma+(4/3))^{-1}(1/5) +$$

$$[(2/5)\sigma^{-1}(\sigma+(4/3))^{-1} - (\sigma+(4/3))^{-2}(-(4/3)\sigma(\sigma-2)(\sigma+4/3)^{-1}\}$$

Hence

$$g_{oyy}=\mu^{-1}(8/3)(\sigma/y^2)^{2/5}(\sigma+(4/3))^{-3}\left\{(1/5)\sigma^2+(8/15) + (16/45)\right.$$

$$\left.-(4/3)(\sigma-2)\ [(2/5)(\sigma+ (4/3))-\sigma]\right\} \qquad (A.79)$$

$$g_{oyy} =\mu^{-1}(8/3)(\sigma/y^2)^{2/5}(\sigma +(4/3)^{-3}[\sigma^2-(16/9) + (16/9)]$$

Then from Eqs. (A.77) and (A.79)

$$(\gamma+1)\partial(\phi_{ox}g_{ox})/\partial_x-g_{yy} = \mu^{-1}(\sigma/y^2)^{2/5}(\sigma+(4/3))^{-3}(8/3)[-(40/9)\sigma - \sigma^2$$

$$+ (16/9)\sigma - 16/9] = \mu^{-1}(\sigma/y^2)^{2/5}(8/3)(\sigma+(4/3))^{-1}$$

and from Eq. (A.76) together with (A.78)

$$D(\log H)d\log y = -(1/2) \qquad (A.80)$$

Along a characteristic is the

$$H = \text{const } y^{-1/2}$$

In the expression for the far field conditions, Eq.(80), DlogH/Ds nd Dg/Ds appear in the coefficient of ϕ. For large y,

$$D\log H/Dy \sim y^{-1}$$

and according to Eq. (75), $Dg/Dy \sim = y^{-1/4}$.

The contribution of DlogH/Dy is therefore negligible. Equation (80) is important in the demonstration of Section IX.

APPENDIX VI

THE AXISYMMETRIC UNSTEADY FLOW FIELD AT MACH NUMBER ONE DETAILED COMPUTATIONS

The computations are identical with those for the two-dimensional case, except of course for the specific form of the analytical expressions.

The function g satisfies the same differential equation as before

$$(\gamma+1)\phi_{ox}g_x^2 - 2g_x - g_y^2 + 1 = 0. \tag{A.81}$$

The term for ϕ_{ox}, by which the basic field enters is found from Eqs. (186) through (189). We add, (from Eq. (187))

$$x = \mu y^{4/7}\bar{\zeta}$$

and with Eq. (189)

$$-x = \mu(\sigma/y^2)^{-2/7}(1-2\sigma) \tag{A.82}$$

and

$$(\sigma/y^2) = \mu^{7/2}(-x)^{-7/2}(1-2\sigma)^{7/2} \tag{A.83}$$

One obtains, from Eq. (189)

$$d\sigma/d\bar{\zeta} = (7/10)\sigma^{9/7}(\sigma+(1/5))^{-1} \tag{A.84}$$

and, with Eqs. (188)

$$\begin{aligned} \partial\sigma(x,y)/\partial x &= \mu^{-1}(7/10)y^{-4/7}\,\sigma^{9/7}(\sigma+(1/5))^{-1} \\ \partial\sigma(x,y)/\partial y &= -(2/5)y^{-1}\,\sigma(2\sigma-1)(\sigma+(1/5))^{-1} \end{aligned} \tag{A.85}$$

It follows from Eqs. (189) and (A.84), that

$$d\bar{f}/d\bar{\zeta} = (8/3)\sigma^{3/7}(\sigma-1).$$

Then, from Eqs. (186),(187), and (188)

$$(\gamma+1)\phi_{ox} = \mu^2(\sigma/y^2)^{3/7}(8/3)(\sigma-1). \tag{A.86}$$

We make the preliminary hypothesis

$$g_o = \mu^{-1}y^{10/7}\,\bar{\gamma}_o(\sigma). \tag{A.87}$$

Then,

$$g_{ox} = \mu^{-2}(7/10)y^{6/7}\,\sigma^{9/7}(\sigma+(1/5))^{-1}\,d\bar{\gamma}_o/d\sigma$$

$$g_{oy} = \mu^{-1}y^{3/7}[(10/7)\bar{\gamma}_o - (2/5)\sigma(2\sigma-1)(\sigma+(1/5)^{-1}d\bar{\gamma}_o/d\sigma]. \tag{A.88}$$

The function g_o will satisfy Eq. (A.81) in the highest power in y. One has specifically

$$(\gamma+1)\phi_{ox}g_{ox}^2 - 2g_{ox} - g_{oy}^2 = 0.$$

One obtains, by substituting Eqs. (A.86) and (A.88)

$$\mu^{-2}y^{6/7}\Big\{\sigma^{3/7}(8/3)(\sigma-1)(49/100)\sigma^{18/7}(\sigma+(1/5))^{-2}(d\bar{\gamma}_o/d\sigma)^2$$

$$-(7/5)\sigma^{9/7}(\sigma+(1/5))^{-1}(d\bar{\gamma}_o/d\sigma)$$

$$-[(10/7)\bar{\gamma}_o - (2/5)\sigma(2\sigma-1)(\sigma+(1/5))^{-1}d\bar{\gamma}_o/d\sigma]^2\Big\} = 0.$$

Next we set

$$\bar{\gamma}_o = \gamma^{-5/7}\gamma_o$$
$$d\bar{\gamma}_o/d\sigma = \sigma^{-12/7}[-(5/7)\gamma_o + \sigma(d\gamma_o/d\sigma)] \tag{A.89}$$

Then one obtains

$$\mu^{-2}y^{6/7}\,\sigma^{-10/7}\,\Big\{(98/75)\sigma(\sigma-1)(\sigma+(1/5))^{-2}(-(5/7)\gamma_o+\sigma(d\gamma_o/d\sigma))^2$$

$$-(7/5)\sigma(\sigma+(1/5))^{-1}(-(5/7)\gamma_o+\sigma(d\gamma_o/d\sigma)) \qquad \text{(A.90)}$$

$$-[(10/7)\gamma_o-(2/5)(2\sigma-1)(\sigma+(1/5))^{-1}(-(5/7)\gamma_o+\sigma(d\gamma_o/d\sigma))]^2\Big\}=0.$$

The coefficient of γ_o in the last term reduces to

$$(10/7)\gamma_o-(2/5)(2\sigma-1)(\sigma+(1/5))^{-1}(-5/7)\gamma_o$$

$$=\sigma(\sigma+(1/5))^{-1}2\gamma_o.$$

One then obtains for the whole expression, Eq. (A.90)

$$\mu^{-2}y^{6/7}\,\sigma^{-3/7}(\sigma+(1/5))^{-2}\Big\{(98/75)(\sigma-1)(-(5/7)\gamma_o+\sigma(d\gamma_o/d\sigma))^2$$

$$+(\sigma+(1/5))(\gamma_o-(7/5)\sigma(d\gamma_o/d\sigma))$$

$$-\sigma[2\gamma_o+(2/5)(1-2\sigma)(d\gamma_o/d\sigma)]^2\Big\}=0.$$

Next we collect within the braces the terms with different powers of $d\gamma_o/d\sigma$. Quadratic terms:

$$(d\gamma_o/d\sigma)^2[(98/75)(\sigma-1)\sigma^2-(4/25)\sigma(1-2\sigma)^2]$$

$$=(d\gamma_o/d\sigma)^2\,[(2/3)\sigma^2-(2/3)\sigma-(4/25)]=(d\gamma_o/d\sigma)^2\sigma(2/3)(\sigma+(1/5))(\sigma-(6/5))$$

Linear terms:

$$(d\gamma_o/d\sigma)[(98/75)(\sigma-1)(-10/7)\sigma\gamma_o-(7/5)(\sigma+(1/5)\sigma-(/8/5)\sigma(1-2\sigma)\gamma_o]$$

$$=(d\gamma_o/d\sigma)\sigma(\sigma+(1/5)[(4/3)\gamma_o-(7/5)].$$

Terms without $d\gamma_o/d\sigma$

$$(98/75)(\sigma-1)(25/49)\gamma_o^2 + (\sigma+(1/5))\gamma_o - 4\sigma/\gamma_o^2$$

$$= \gamma_o[-(10/3)\gamma_o + 1](\sigma+(1/5)).$$

One thus obtains, from the terms within the braces

$$(\sigma+(1/5))\left\{(d\gamma_o/d\sigma)^2(2/3)(\sigma-(6/5)) + (d\gamma_o/d\sigma)\sigma((4/3)\gamma_o-(7/5)\right.$$

$$\left. + \gamma_o[-(10/3)\gamma_o + 1)\right\} = 0$$

The solution which is smooth, even at $\sigma = (6/5)$ is given by

$$\gamma_o = 3/10. \tag{A.91}$$

Combining Eqs. (A.87), (A.89), and (A.91), one obtains

$$g_o = (3/10)\mu^{-1} y^{10/7} \sigma^{-5/7}. \tag{A.92}$$

A correction, g_1 to g_o, is obtained from Eq. (A.81) by retaining only first order terms in g_1

$$((\gamma+1)\phi_{ox}g_{ox} - 1)g_{1x} - g_{oy}g_y + (1/2) = 0. \tag{A.93}$$

One has, from Eqs. (A.92) and (A.89)

$$g_{ox} = -\mu^{-2}(3/20)y^{6/7}\sigma^{-3/7}(\sigma+(1/5))^{-1}$$

$$g_{oy} = \mu^{-1}(3/5)y^{3/7}\sigma^{2/7}(\sigma+(1/5))^{-1} \tag{A.94}$$

and with Eq. (A.86)

$$(\gamma+1)\phi_{ox}g_{ox} = -(2/5)(\sigma-1)(\sigma+(1/5))^{-1}$$

$$(\gamma+1)\phi_{ox}g_{ox}-1=(-(7/5)\sigma+(1/5))(\sigma+(1/5))^{-1} \tag{A.95}$$

We set

$$g_1 = y^{4/7}\bar{\gamma}_1(\sigma) \tag{A.96}$$

Then, with Eqs. (A.49)

$$g_{1x} = \mu^{-1}(7/10)\sigma^{9/7}(\sigma+ (1/5))^{-1}d\bar{\gamma}_1/d\sigma$$

$$g_{1y} = y^{-3/7}[(4/7)\bar{\gamma}_1(-2/5)(2\sigma-1)(\sigma+ (1/5))^{-1}\sigma d\bar{\gamma}_1/d\sigma]$$

Substituting this into Eq. (A.93), one obtains

$$(-(7/5)\sigma +(1/5))(\sigma+ (1/5))^{-1}\ \mu^{-1}(7/10)\sigma^{9/7}(\sigma+ (1/5))^{-1}d\bar{\gamma}_1/d\sigma$$

$$-\mu^{-1}(3/5)y^{3/7}\sigma^{2/7}(\sigma+(1/5))^{-1}[(4/7)\bar{\gamma}_1 -(2/5)(2\sigma-1)(\sigma+ (1/5))^{-1}\sigma d\bar{\gamma}_1/d\sigma]$$

$$+ (1/2) = 0.$$

This equation simplifies to

$$\mu^{-1}(1/5)(\sigma+ (1/5))^{-1}\ \sigma^{2/7}[-(5/2)\sigma(d\bar{\gamma}_1/d\sigma) - (12/7)\bar{\gamma}_1]+(1/2) =0. \tag{A.97}$$

Next one introduces

$$\bar{\gamma}_1 = \mu\sigma^{-2/7}\gamma_1$$

Substitution into Eq. (A.97) gives

$$(1/5)[(5/7)\gamma_1 -(5/2)\sigma(d\gamma_1/d\sigma)-(12/7)\gamma_1]] + (1/2)(\sigma+ (1/5)=0$$

and finally

$$\sigma(d\gamma_1/d\sigma) + (2/5) = (\sigma+ (1/5)).$$

The solution which is smooth at $\sigma = 0$ is given by

$$\gamma_1 = (1/2) + (5/7)\sigma \qquad (A.98)$$

Thus

$$g_1 = \mu y^{4/7}\, \sigma^{-2/7}((1/2) + (5/7)\sigma) \qquad (A.99)$$

From Eqs. (A.92) and (A.99), one thus obtains the following approximation

$$g = \mu^{-1}\, y^{10/7}\, \sigma^{-5/7}[(3/10)+A((1/2)+(5/7)\sigma)]$$

with

$$A = \mu^2(\sigma/y^2)^{3/7} \qquad (A.100)$$

In the vicinity of the negative x axis one uses Eq. (A.83) to evaluate (σ/y^2).

Next, formula for the evaluation of the characteristic directions are developed. Since the differential equation for g, Eq. (144), is the same for the plane and the axisymmetric problem, one obtains the same formula for the characteristic directions, viz Eq. (147). Parts of the numerator and the denominator are found in Eqs. (A.94) and (A.95). For the missing contributions of g_1, one finds from Eqs. (A.99) and (A.85)

$$g_{1y} = \mu y^{-3/7}\sigma^{-2/7}\left\{(4/7)(1/2)+(5/7)\sigma) + (-(1/7)+(25/49)\sigma)\right.$$
$$\left. - (2/5)(2\sigma-1)(\sigma+ (1/5))^{-1}[(1/2)(-2/7) + (5/7)^2\sigma\right\}.$$

This simplifies to

$$g_{1y} = \mu(24/35)y^{-3/7}\, \sigma^{5/7}(\sigma+ (1/5))^{-1}. \qquad (A.101)$$

Moreover, from Eqs. (A.99) and (A.85)

$$g_{1x} = \mu y^{4/7} \sigma^{-9/7}(-(1/7) + (25/49)\sigma)\mu^{-1}(7/10)y^{-4/7}\sigma^{9/7}(\sigma + (1/5))^{-1}$$

$$g_{1x} = (-(1/10) + (5/14)\sigma(\sigma + (1/5))^{-1}.$$

Then with Eq. (A.86)

(A.102)

$$(\gamma+1)\phi_{ox}g_{1x}=\mu^2(\sigma/y^2)^{3/7}(8/3)(\sigma-1)(-(1/10)+(5/14)\sigma)(\sigma+(1/5))^{-1}$$

Combining Eqs. (A.101) and (A.102) with (A.94) and (A.95) and using the definitions for A, Eq. (A.100), one obtains

$$1-(\gamma+1)\phi_{ox}g_x = (\sigma+(1/5))^{-1}[(7/5)\sigma-(1/5)+A(\sigma-1)((4/15)-(20/21)\sigma)]$$

$$g_y = \mu^{-1}(3/5)y^{3/7}\sigma^{2/7}(+(1/5))^{-1}(1 + (8/7)A).$$

Then, from Eq. (147)

(A.103)

$$Dx/Dy=\mu(\sigma/y^2)^{-2/7}y^{-1}[(7/3)\sigma-(1/3)+A(\sigma-1)((4/9)-(100/63)\sigma)[1+(8/7)A]^{-1}$$

To obtain $D\sigma/Dy$ we proceed as follows. It follows from Eq. (A.82) that along a curve $\sigma = \sigma(y)$

$$dx/dy =\mu y^{-1}(\sigma/y^2)^{-2/7}[(4/7)(2\sigma-1)+(10/7)(\sigma+(1/5))d(\log\sigma)/d(\log y)]$$

One obtains, by substituting into this equation the expression (A.103) for the value of Dx/Dy along a characteristic

$$\frac{(7/3)\sigma-(1/3)+A(\sigma-1)((4/9)-(100/63)\sigma)}{(1+(8/7)A)} = (4/7)(2\sigma-1)+(10/7)(\sigma+(1/5))\frac{D(\log\sigma)}{D(\log y)}$$

Hence,

$$(10/7)(\sigma+(1/5)D(\log\sigma)/D(\log y)$$

$$= (1+(8/7)A)^{-1}\left\{(7/3)\sigma-(1/3)+A(\sigma-1)((4/9)-(100/63)\sigma)-(1+(8/7)A)(4/7)(2\sigma-1)\right\}$$

$$= (1+(8/7)A)^{-1}\left\{(25/21)(\sigma+(1/5)-A(100/63)(\sigma+(1/5)-(23/35))\right\}$$

Hence,

$$D(\log\sigma)/D(\log y) = (1+(8/7)A)^{-1}(5/6)[1-(4/3)A(\sigma-(23/35))].$$

Developing the denominator with respect to A, one obtains

$$D(\log\sigma)/D(\log y) = (5/6)[1-(4/3)A(\sigma+(1/5))]$$

Usually it is convenient to consider σ as the independent variable. Again, developing the denominator one obtains

$$D(\log y)/D(\log\sigma) = (6/5)[1+(4/3)A(\sigma+(1/5))]. \qquad (A.104)$$

We derive an expression also for $D(\log(-x))/D\sigma$. One finds from Eq. (A.82)

$$d(\log(-x)) = (4/7)d(\log y)+(10/7)(\sigma+(1/5))(2\sigma-1)^{-1}d\log\sigma$$

Equation (A.104) is used to eliminate d(logy). Then

$$D(\log)-x))/D\log\sigma) = (2/5)[(1-7\sigma)(1-2\sigma)^{-1}+(16/7)A(\sigma+(1/5))]. \qquad (A.105)$$

Here no development with respect to A has been carried out. The quotient of Eqs. (A.105) and (A.104) gives

$$D(\log(-x))/D(\log y) = (1/3)[1-7\sigma)(1-2\sigma)^{-1}+(16/7)A(\sigma+(1/5))]$$

$$[1+(4/3)A(\sigma+(1/5))]^{-1}.$$

For the vicinity of the negative y axis one writes

$$D(-x)/D(\log y) = (1/3)[1-7\sigma)(-x)(1-2\sigma)^{-1} + (16/7)(-x)A(\sigma + (1/5))]$$

$$[1+(4/3)A(\sigma+(1/5))]^{-1}$$

and uses Eq. (A.82) to express $(-x)(1-2\sigma)^{-1}$.

To find an analytical expression for the characteristics, we set

$$u = \log y$$
$$v = \log \sigma$$

Then from Eq. (A.104) and (A.100)

$$Du/Dv = (6/5) + (8/5)\mu^2 \exp(-(6/7)u[\exp((10/7)v)+(1/5)\exp((3/7)v)]$$

One obtains in the first approximation

$$u_o = (6/5)v + \log c.$$

Next one sets

$$u = u_o + u_1$$

and neglects higher order terms in u_1. Then,

$$Du_1/Dv=(8/5)\mu^2(\exp(-(36/35)v - (6/7)\log c)[\exp((10/7)v+(1/5)\exp((3/7)v)]$$

$$Du_1/Dv = (8/5)\mu^2 c^{-6/7}[\exp(12/5)v)+(1/5)\exp(-(3/5)v]$$

$$u_1 = \mu^2 c^{-6/7}[4 \exp((2/5)v) - (8/15) \exp(-(3/5)v)].$$

Therefore, correct to the next order,

$$u = \log c + (6/5)v + \mu^2 c^{-6/7}[4\exp((2/5)v) - (8/15)\exp(-3/5)v)$$

$$y = c\sigma^{6/5}\exp\{\mu^2 c^{-6/7}[4\sigma^{2/5} - (8/15)\sigma^{-3/5}]\}$$

For c large (y large) the exponential function is developed. One obtains

$$y = c\,\sigma^{6/5} + \mu^2 c^{1/7}\sigma^{3/5}(4\sigma - (8/15)). \qquad (A.106)$$

We compute in the lowest approximation the derivative of g along a characteristic. This quantity occurs in the application of the far field conditions. One obtains by substituting Eq. (A.106) into Eq. (A.92)

$$g_o = (3/10)\mu^{-1}\, c^{10/7}\sigma$$

or

$$g_o(3/10)\mu^{-1}c^{\,25/42}\, y^{5/6}.$$

Hence

$$Dg_o/D\sigma = (3/10)\mu^{-1}c^{10/7}(1 + O(A))$$

$$= (3/10)\mu^{-1}c^{10/7}(1 + O(y^{-1/2})) \qquad (A.107)$$

and

$$Dg_o/Dy = (1/4)\mu^{-1}c^{25/42}y^{-1/6}(1+O(y^{-1/2})).$$

Also needed in the far field condition (at least until it has been shown that its order can be disregarded) is D(logH)/Dy. The characteristic directions are given by the same general formulae in the axisymmetric and in the plane flows. One obtains from Eqs. (185) and (149)

$$D(\log H)/Dy = (1/2)[(\gamma+1)\partial(\phi_{ox}g_x)/\partial x) - y^{-1}\partial(yg_y)/\partial y]g_y. \qquad (A.108)$$

We evaluate this expression only to the lowest order in A. According to Eq. (A.95)

$$(\gamma+1)\phi_{ox}g_{ox} = -(2/5)(\sigma-1)(\sigma+(1/5))^{-1}.$$

Hence with Eq. (A.85)

$$(\gamma+1)\partial(\phi_{ox}g_{ox})/\partial x = -(2/5)\,\frac{\sigma+(1/5)-(\sigma-1)}{(\sigma+(1/5))^2}\,\mu^{-1}(7/10)y^{-4/7}\sigma^{9/7}(\sigma+(1/5)^{-1}$$

$$= -\mu^{-1}(42/125)(\sigma/y^2)^{2/7}\sigma\,(\sigma+(1/5)^{-3}. \qquad (A.109)$$

According to Eq. (A.94)

$$y\,g_{oy} = \mu^{-1}(3/5)y^{10/7}\sigma^{2/7}(\sigma+(1/5))^{-1} \qquad (A.110)$$

Then with Eq. (A.85)

$$y^{-1}\partial(yg_{oy})/\partial y = \mu^{-1}(3/5)y^{-4/7}\sigma^{2/7}\Big\{(10/7)(\sigma+(1/5))^{-1} +$$

$$+ [(2/7)\sigma^{-1}(\sigma+(1/5))^{-1} - (\sigma+(1/5))^{-2}][-(2/5)\sigma(2\sigma-1)(\sigma+(1/5)^{-1}]\Big\}$$

$$y^{-1}\partial(y\,g_{oy})/\partial y = \mu^{-1}(3/5)(\sigma/y^2)^{2/7}(\sigma+(1/5))^{-3}\Big\{(10/7)(\sigma+(1/5))^2$$

$$+ [(2/7)(\sigma+(1/5)) - \sigma][-(2/5)(2\sigma-1)\Big\}$$

$$y^{-1}\partial(y\,g_{oy})/\partial y = \mu^{-1}(3/5)(\sigma/y^2)^{2/7}(\sigma+(1/5))^{-3}(1/7)\Big\{(10\sigma^2+4\sigma+(2/5)$$

$$+ (2/5)(1-2\sigma)(-5\sigma+(2/5))\Big\}$$

$$y^{-1}\partial(y\,g_{oy})/\partial y = \mu^{-1}(6/5)(\sigma/y^2)^{2/7}(\sigma+(1/5))^{-3}(\sigma^2+(3/25)\sigma+(1/25)) \qquad (A.111)$$

Combining Eqs. (A.109) and (A.111) one obtains

$$(\gamma+1)\partial(\phi_{ox}g_{ox})/\partial x - y^{-1}\sigma(yg_{oy})/\partial y$$

$$= \mu^{-1}(\sigma/y^2)^{2/7}(\sigma+(1/5))^{-3}\left\{-(42/125)\sigma-(6/5)\sigma^2-(18/125)\sigma-(6/25))\right\}$$

$$=-\mu^{-1}(6/5)(\sigma/y^2)^{2/7}(\sigma+(1/5))^{-1}.$$

Hence, from Eqs. (A.108) and (A.110)

$$D(\log H)/Dy = y^{-1} \qquad \text{(A.112)}$$

$$H = \text{const } y^{-1}.$$

Comparing Eqs. A.112) and (A.107), one recognizes that in the evaluation of the far field conditions D(logH)/Dy is negligible in comparison to Dg/Dy. Equation (A.112) is, however, important in the demonstration of Section IX.

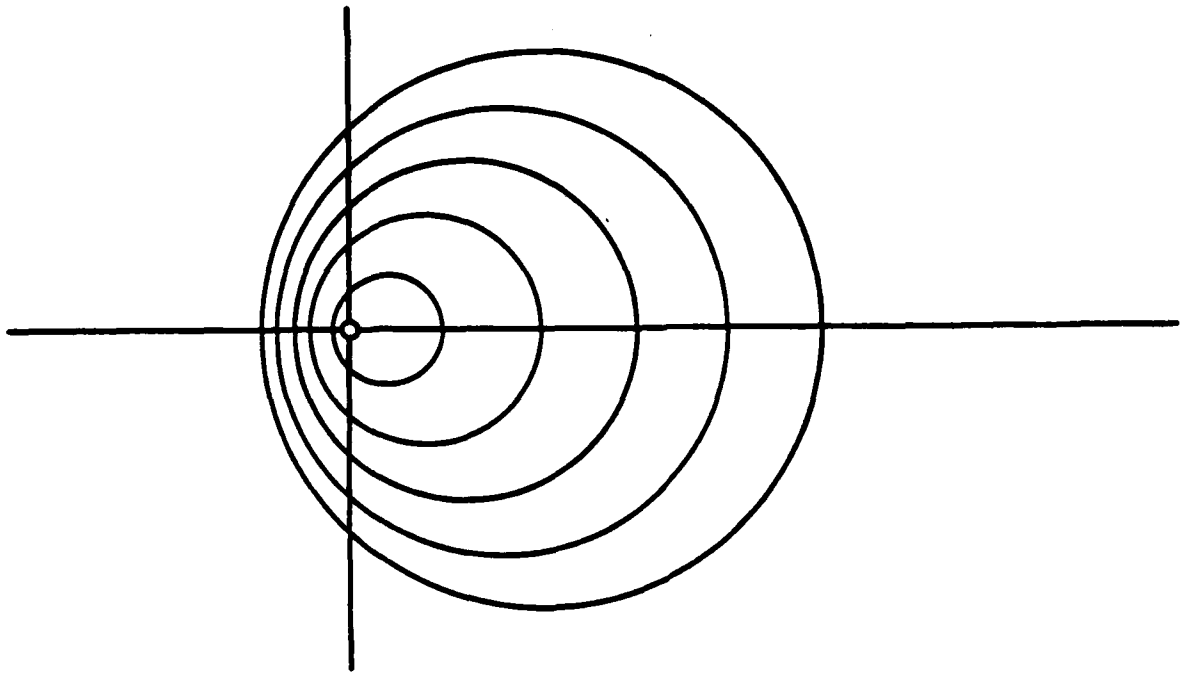

Figure 1. System of Wave Fronts for a Subsonic Free Stream Mach Number in Linearized Theory.

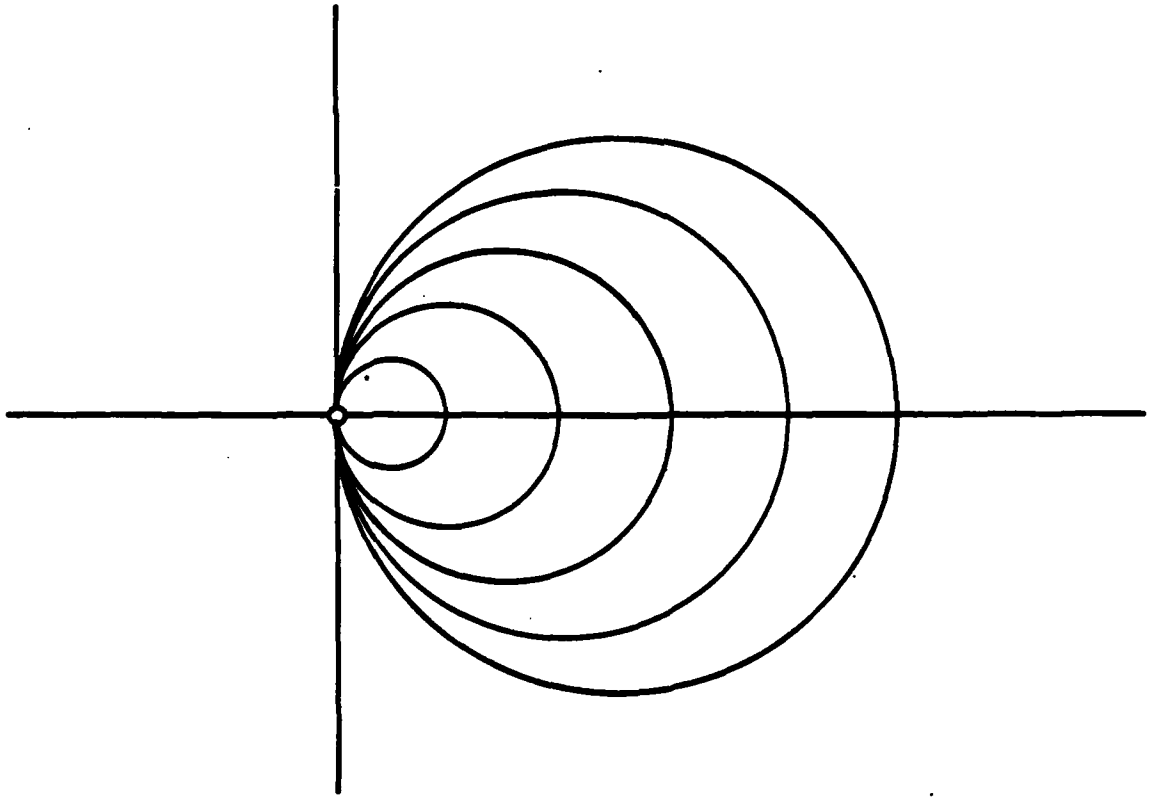

Figure 2. System of Wave Fronts for a Free Stream Mach Number One in Linearized Theory.

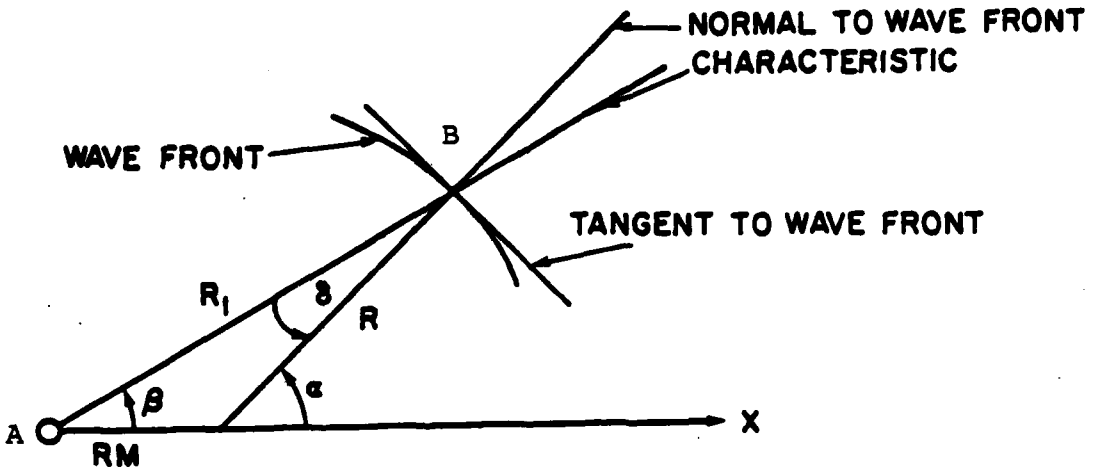

Figure 3. Normal to a Wave Front and Characteristic for a Wave Spreading out from the Origin.

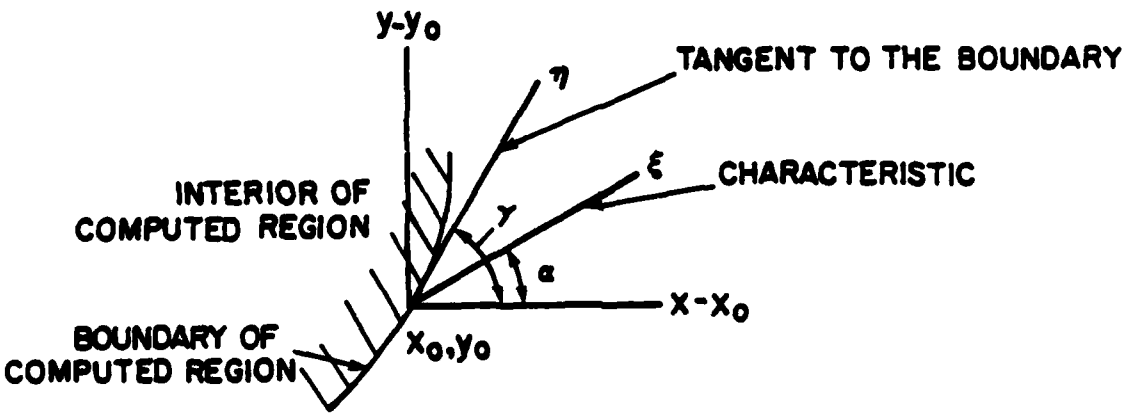

Figure 4. Coordinate System for the Evaluation of Derivatives at the Edge of the Computed Region.

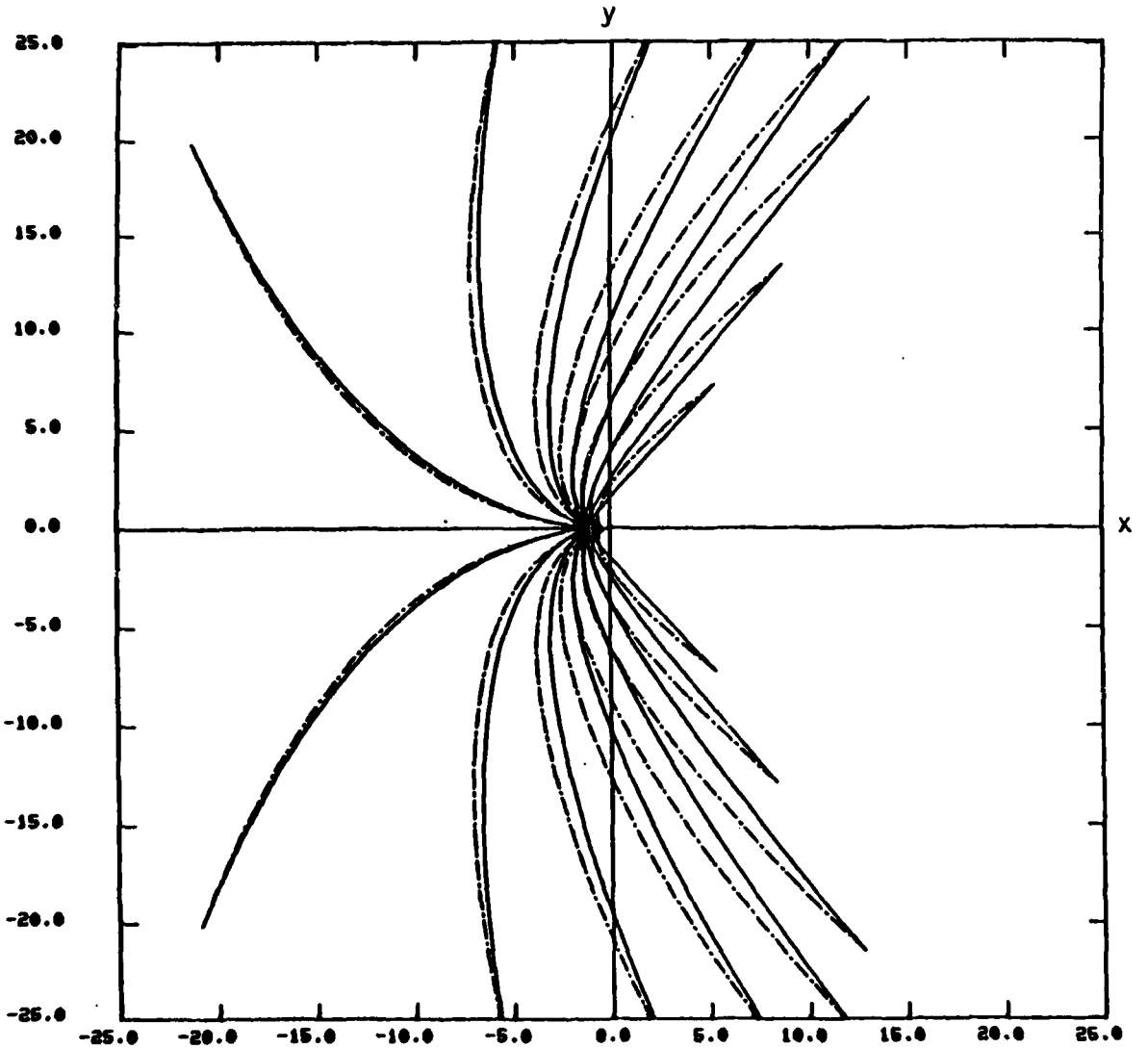

Figure 5. Computed Characteristics (Solid Lines) and Their Analytic One-Term Approximations (Dotted Lines) for Plane Flow with the Free Stream Mach Number One. (The curves have been made to agree at their outer points).

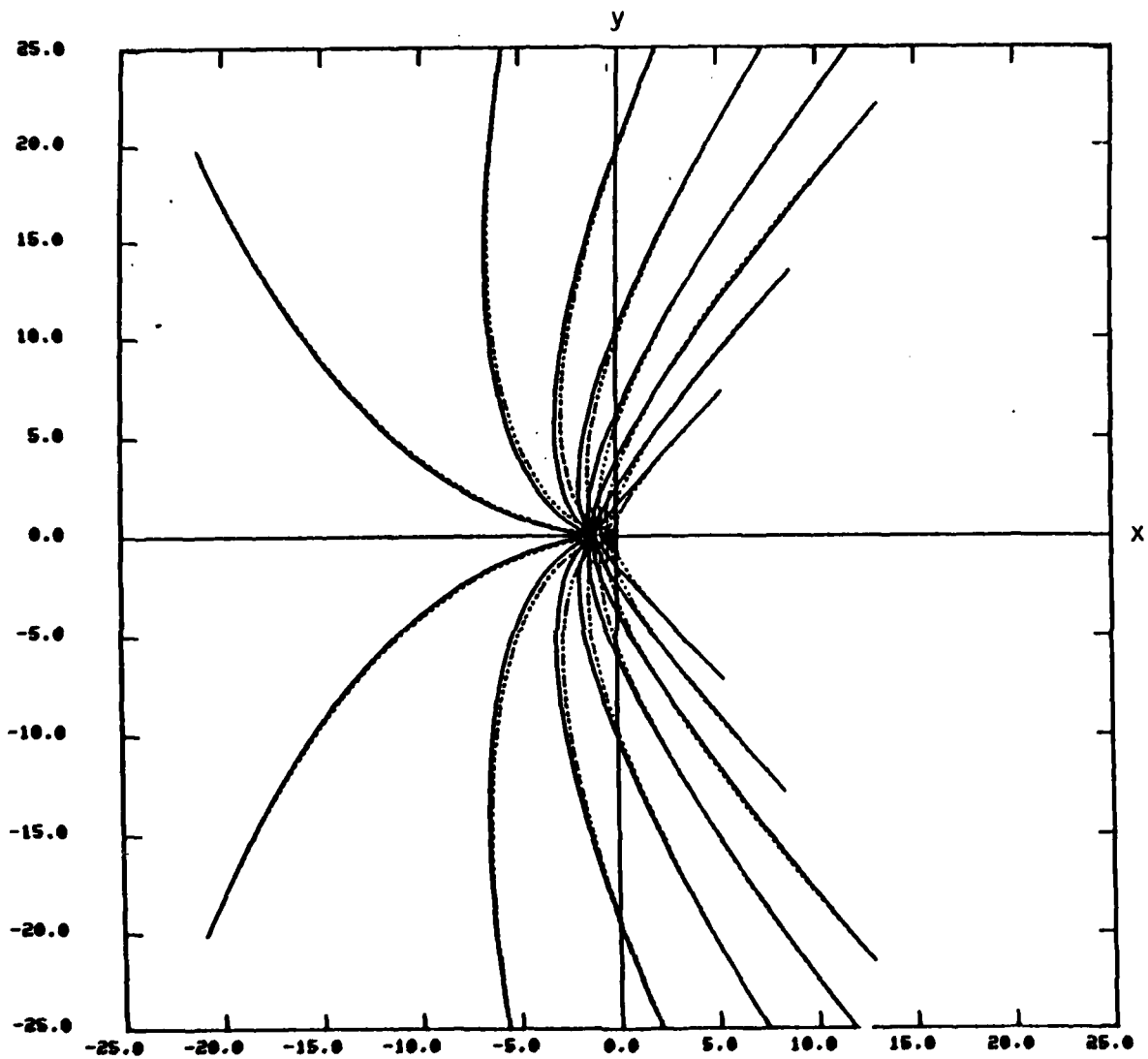

Figure 6. Computed Characteristics (Solid Lines) and Their Analytic Two-Term Approximations (Dotted Lines) for Plane Flow with a Free Stream Mach Number One.

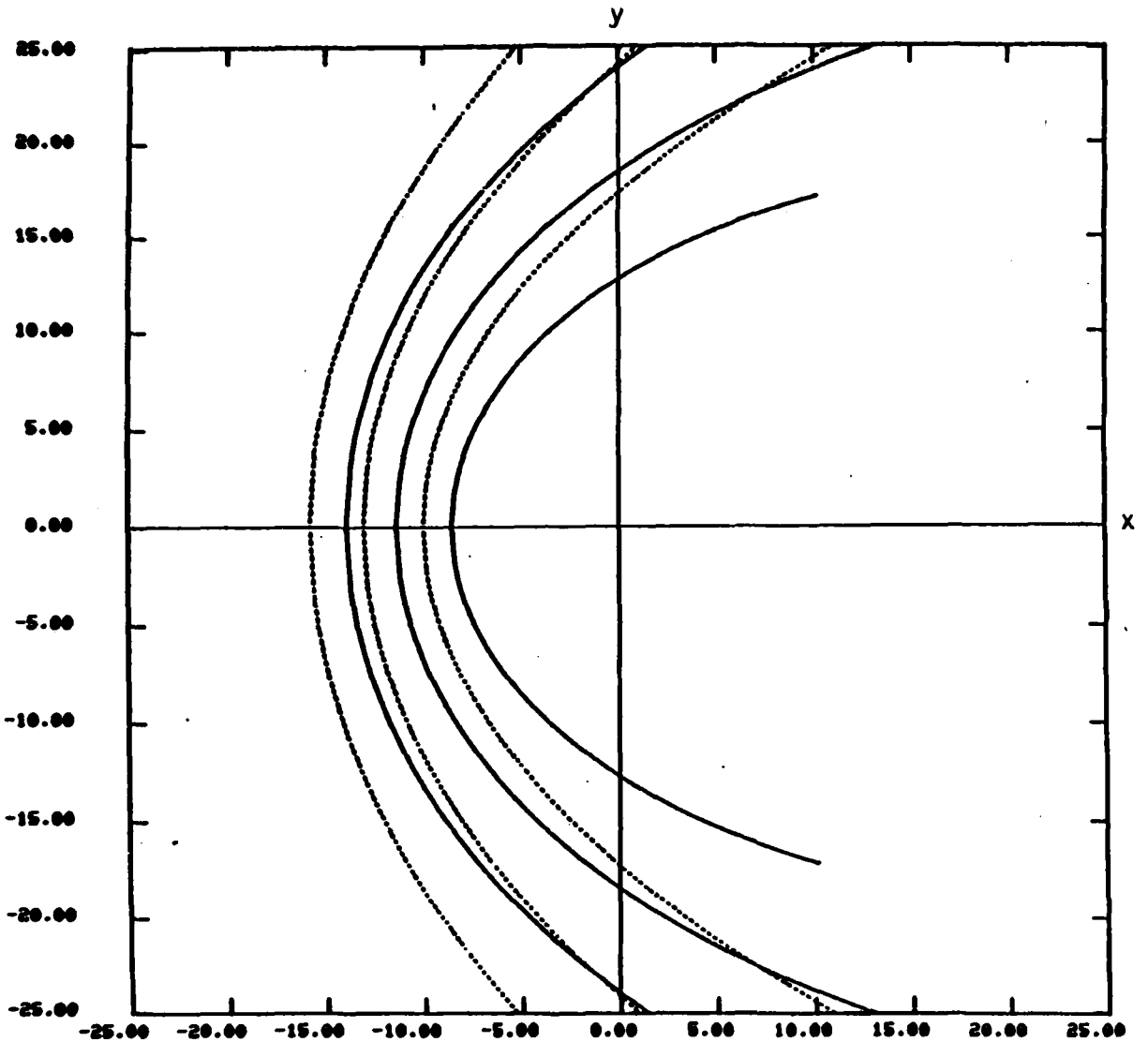

Figure 7. One- and Two-Term Approximations for Wave Fronts (g = 20, 30, 40) in Plane Flow with a Free Stream Mach Number One.

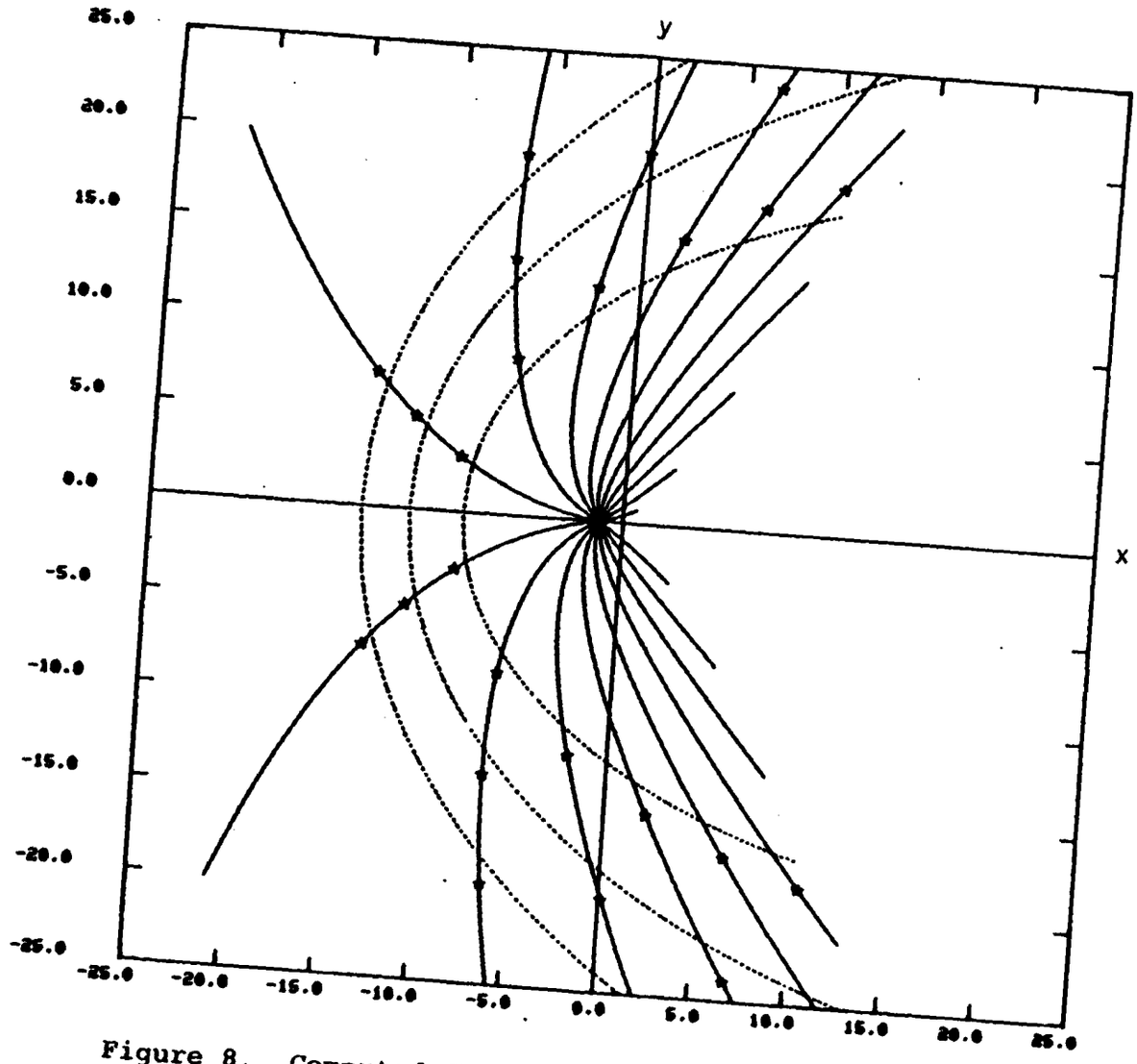

Figure 8. Computed Characteristics with Points Marked by Asterisks for g = 20, 30, and 40 and Two-Term Approximation for the Same Values of g in Plane Flow with Free Stream Mach Number One.

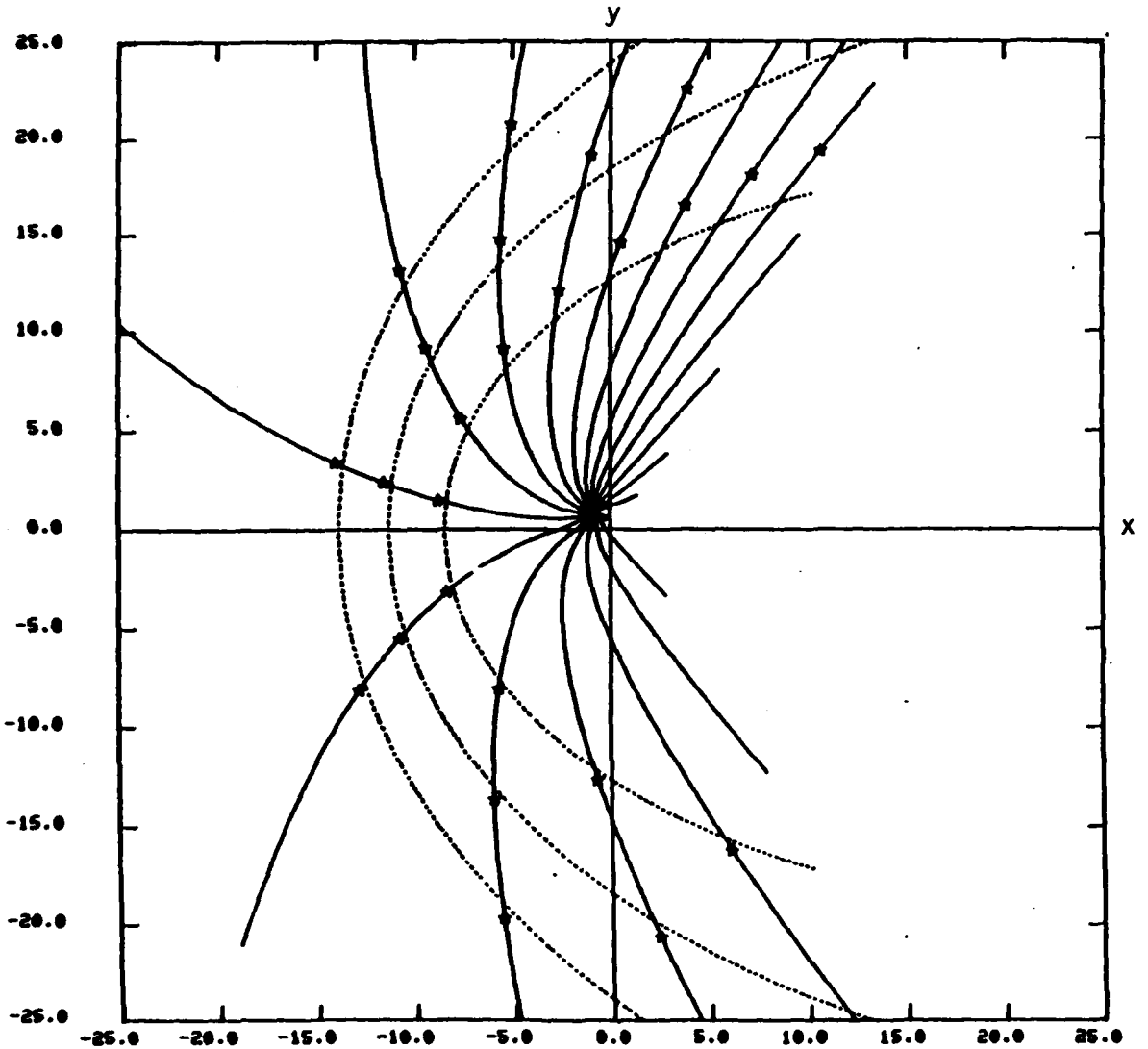

Figure 9a. Same as Figure 8, but with a Different Starting Point of the Computed Characteristics.

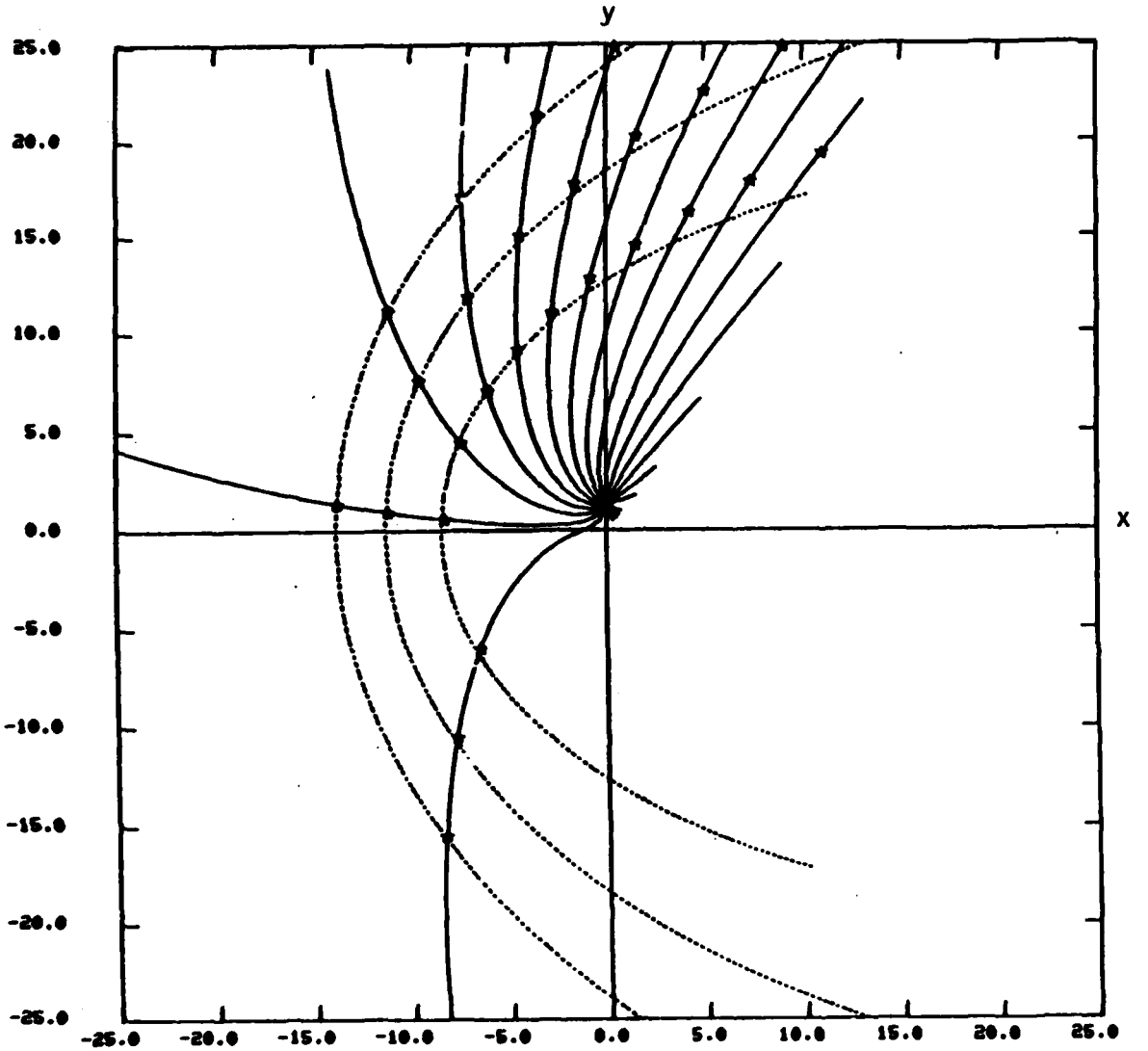

Figure 9b. Same as Figure 8, but with a Different Starting Point of the Computed Characteristics.

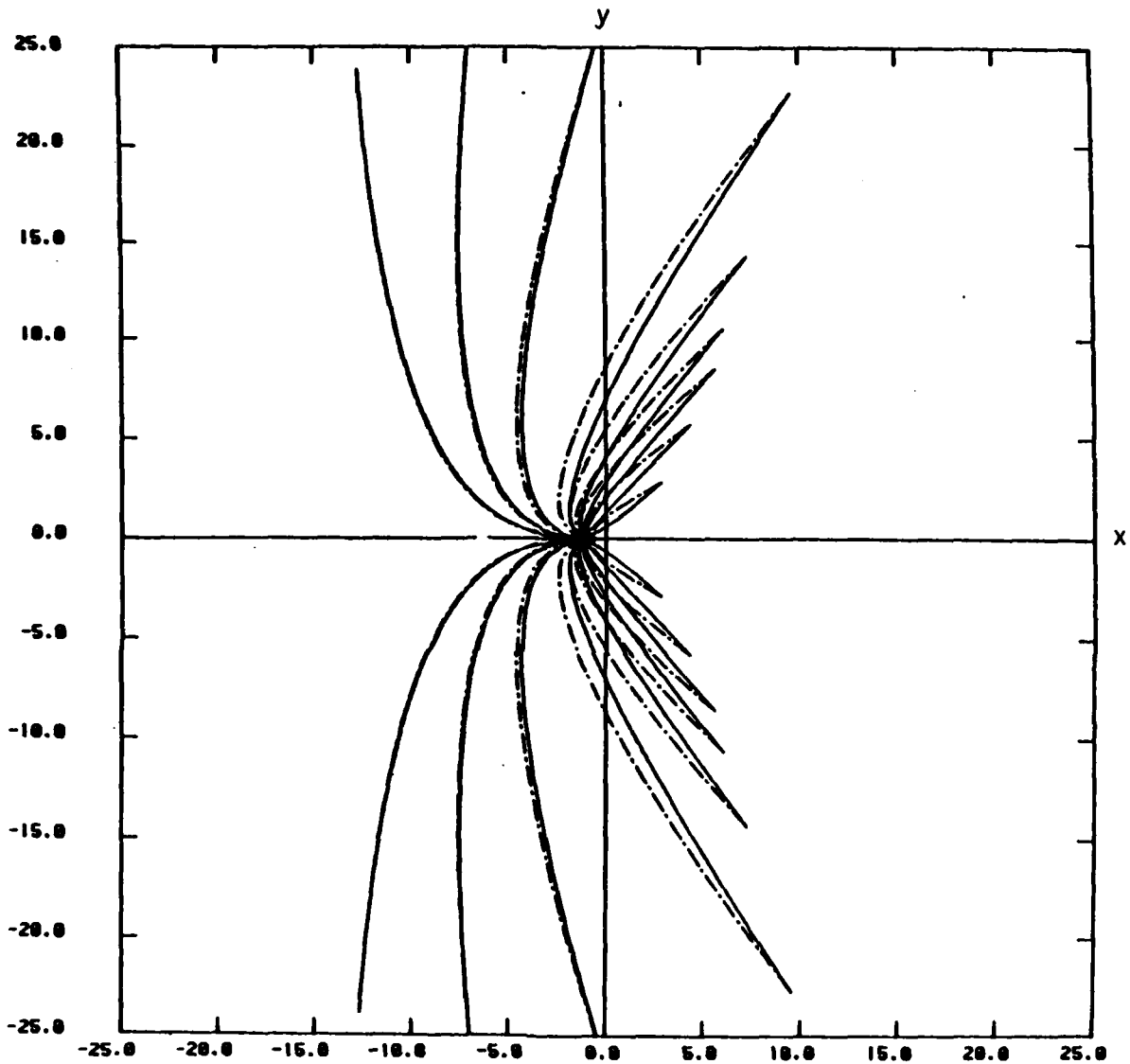

Figure 10. Counterpart of Figure 5 for Axisymmetric Flow.

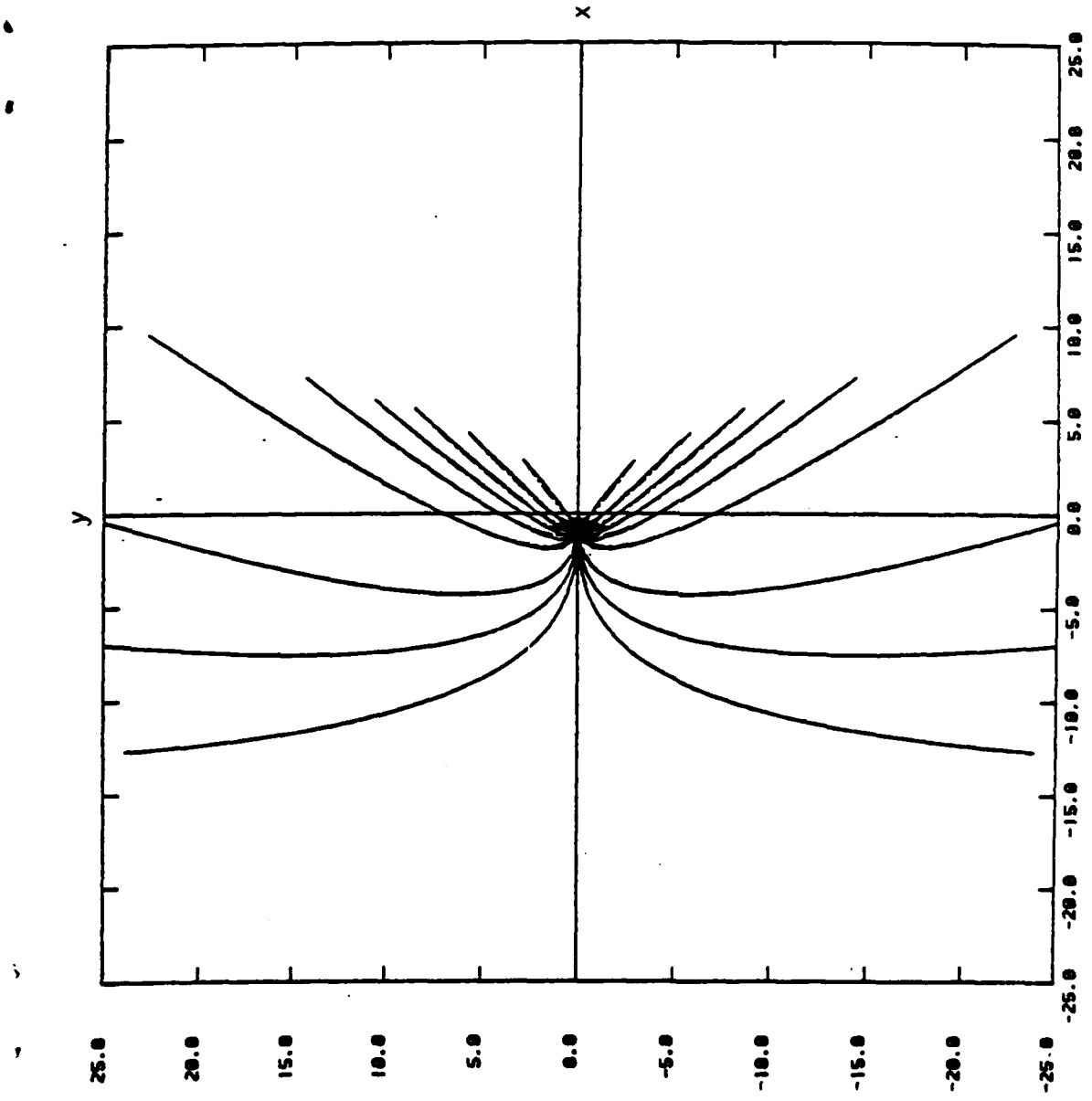

Figure 11. Counterpart of Figure 6 for Axisymmetric Flow.

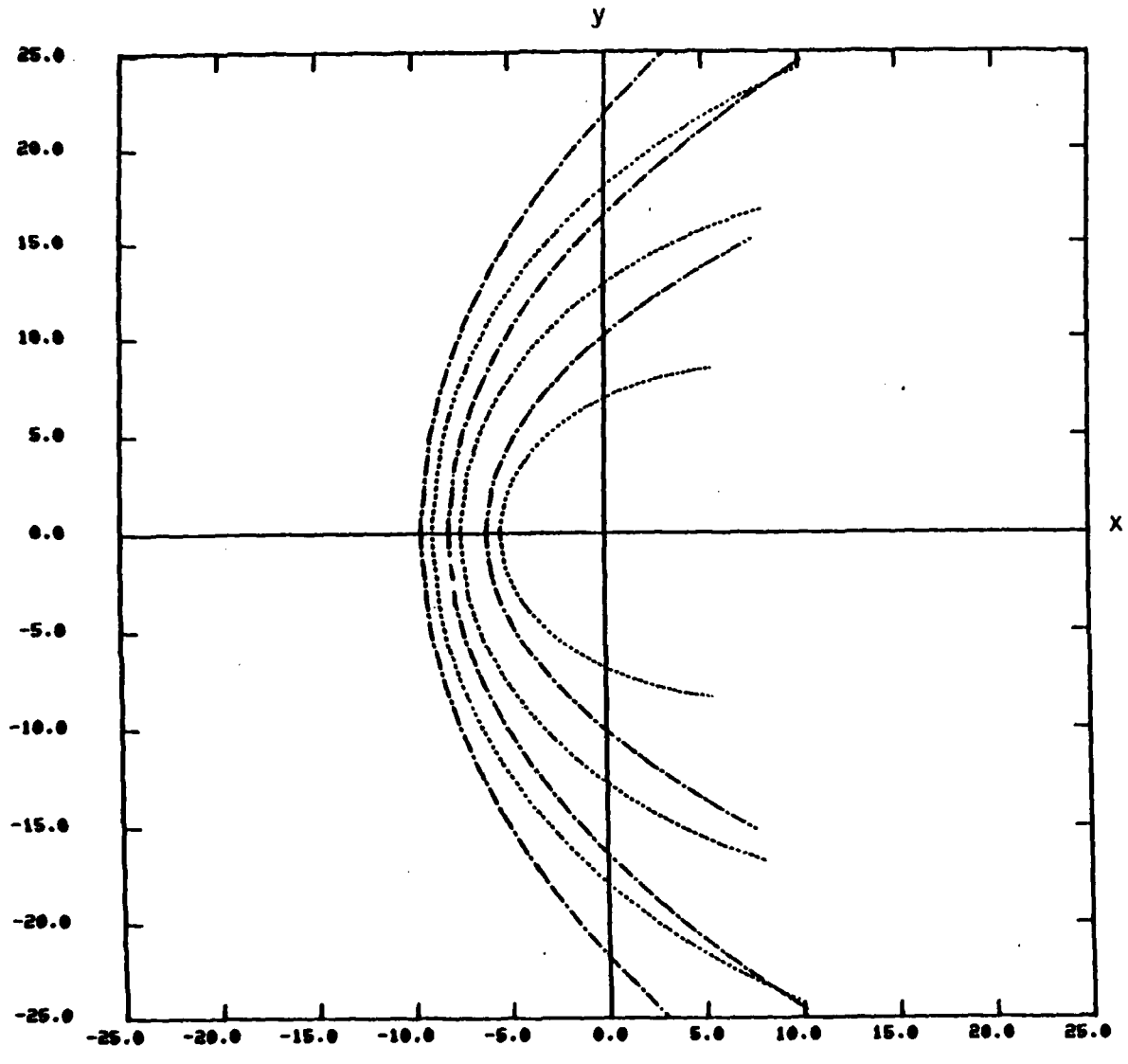

Figure 12. Counterpart of Figure 7 for Axisymmetric Flow.

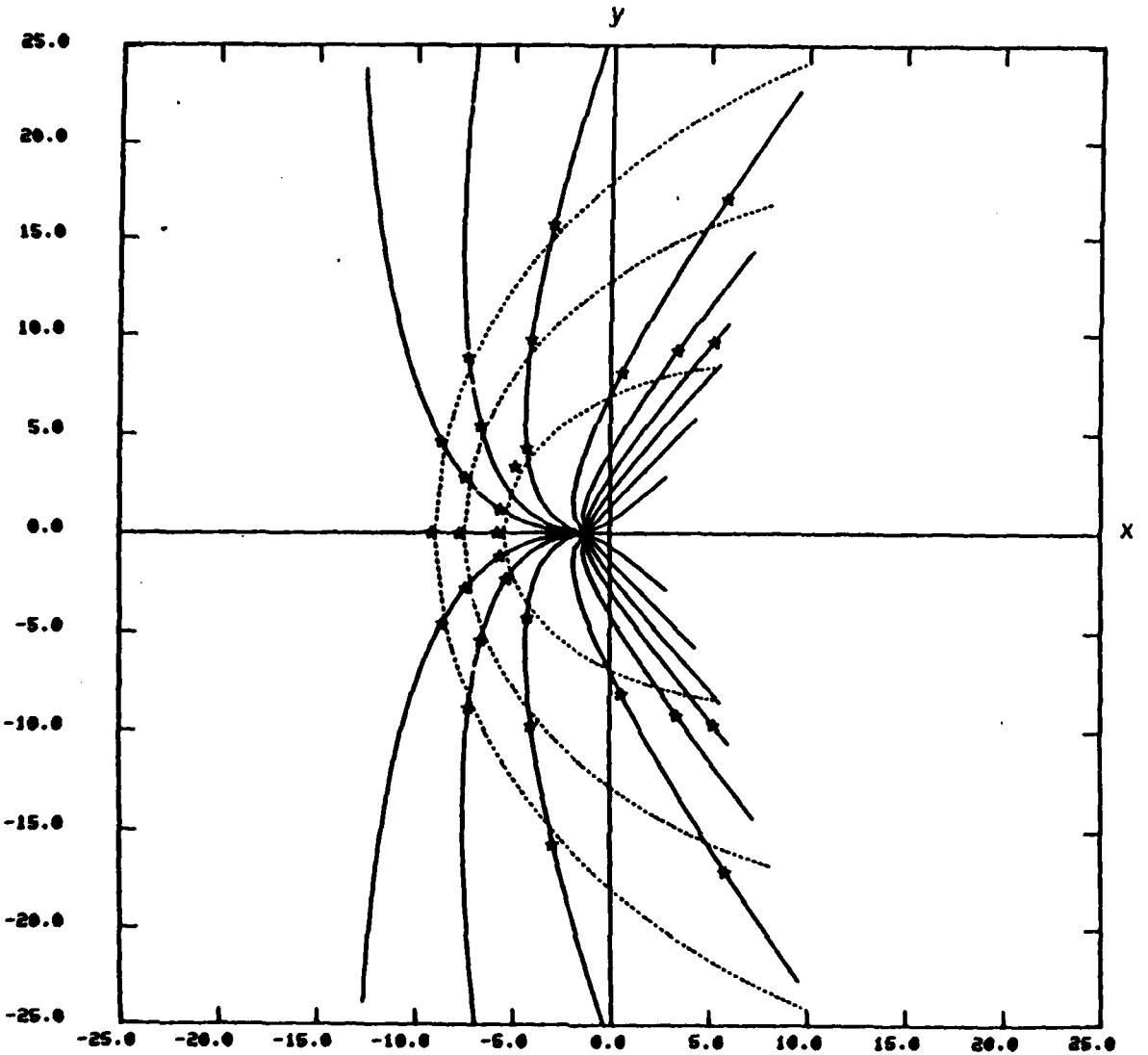

Figure 13. Counterpart of Figure 8 for Axisymmetric Flow.

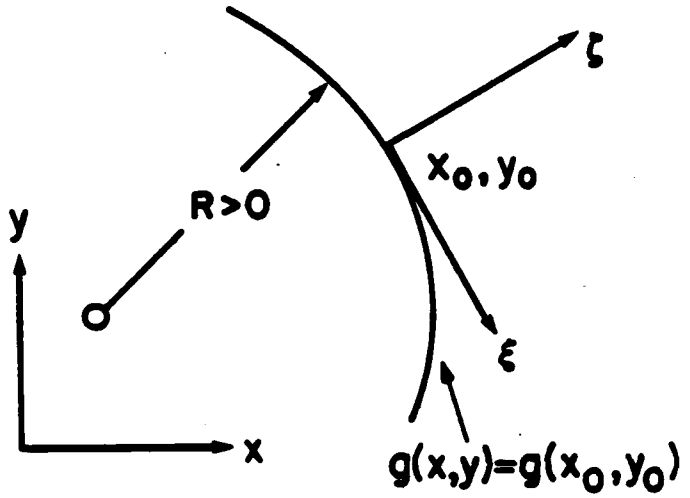

Figure 14. Coordinate System Used to Determine the Radius of Curvature of a Wave Front.

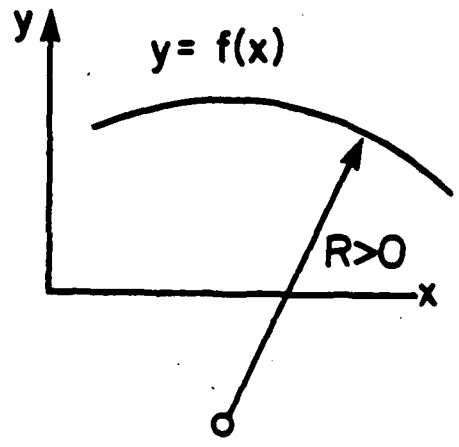

Figure 15. Curve with Positive Radius of Curvature.

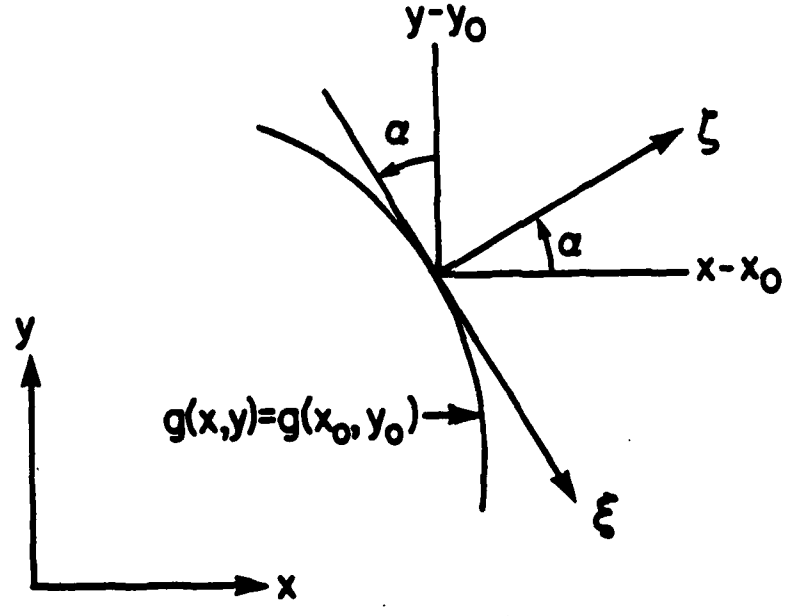

Figure 16. Orientation of the ξ,ζ System with Respect to the x,y System.

*USGPO:1983-659-062-910