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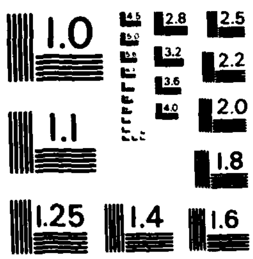

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A LOWER BOUND FOR THE BAYES RISK FOR TESTING SEQUENTIALLY THE SIGN OF THE DRIFT PARAMETER OF A WIENER PROCESS

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ABSTRACT

Let $x(t)$ be a Wiener process with drift μ and variance 1 per unit of time. For testing $H: \mu \leq 0$ vs $A: \mu > 0$ with the loss function $|\mu|$ if the wrong decision is made and 0 otherwise, c cost of observation per unit time and μ has a prior distribution which is normal with mean 0 and variance σ_0^2, we followed an idea of Bickel and Yahav to obtain a lower bound for the Bayes risk and showed that this lower bound is strict as $\sigma_0 \to \infty$ for all c.

Key Words: Sequential tests, S.P.R.T, Bayes, stopping times, lower bound, asymptotic expansion.

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1. Introduction : Let $x(t)$ be a Wiener process with drift μ and variance 1 per unit of time. Chernoff [2] considered the following problem, test

$$H: \mu \leq 0 \quad \text{vs} \quad A: \mu > 0$$

with the loss function $|\mu|$ if the wrong decision is made and 0 otherwise, c cost of observation per unit time and μ has a prior distribution which is normal with mean μ_0 and variance σ_0^2. Chernoff [3] showed that the Bayes risk

$$(1.1) \quad B(\mu_0,\sigma_0^2) = c^{\frac{2}{3}}\left[K\sigma_0^{-1}\phi\left(\frac{\mu_0}{\sigma_0}\right) - 6c^{\frac{1}{3}}\sigma_0^{-2}\ln\sigma_0\left(1+o(1)\right)\right]$$

$$\text{as } \sigma_0 \to \infty$$

Where K is an unknown constant. Throughout this paper ϕ and Φ are the standard normal density and cumulative respectively.

By considering the above testing problem with the additional information of the magnitude of μ, Bickel and Yahav [1] obtained a lower bound for the Bayes risk for the case of μ having the improper prior distribution and conjectured that the lower bound can be attained as $c \downarrow 0$. In this note we assume that μ has a normal prior distribution with mean 0 and variance σ_0^2. By using similar techniques as in Bickel and Yahav [1], we obtained a lower bound for the Bayes risk, then showed that this lower bound is not asymptotically achievable as $\sigma_0^2 \to \infty$ for all $c > 0$.

2. <u>Lower Bound For Bayes Risk:</u> From Chernoff [3], the posterior cost of wrong decision is given by

$$(2.1) \quad Y_t = (t+\sigma_0^{-2})^{-1/2} \left\{\phi(\alpha)-|\alpha|\Phi(-|\alpha|)\right\}$$

where $\alpha = (t+\sigma_0^{-2})^{-1/2} X(t)$. Let the posterior risk at time t be,

$$(2.2) \quad R(c,t) = Y_t + ct$$

We are interested in a stopping rule τ_0 for which

$$E[R(c,\tau_0)] = \inf_{\tau \epsilon T} E[R(c,\tau)]$$

where T is the class of all stopping times.

Using the idea of Bickel and Yahav [1], let us consider the following problem of testing,

$$H: \mu = \mu_0 \quad \text{vs} \quad A: \mu = -\mu_0$$

with $|\mu_0|$ for cost of wrong decision and prior distribution $P(\mu = \mu_0) = P(\mu = -\mu_0) = \frac{1}{2}$. Then the posterior cost of wrong decision is

$$\tilde{Y}_t = |\mu_0| \, P(X(t)\mu < 0 \mid X(t))$$

Let

$$\tilde{R}(c,t) = \tilde{Y}_t + ct$$

To solve the above Bayes problem, we have to find a stopping rule τ^* such that

$$E(\tilde{R}(c,\tau^*)) = \inf_{\tau \epsilon T} E(\tilde{R}(c,\tau))$$

From the property of S.P.R.T we have the following lemma.

<u>Lemma 2.1:</u> <u>The stopping rule</u> τ^*: <u>Stop the first</u> $|X(t)| = a$, <u>where</u> "a" <u>is determined by the minimization of</u>

$$|\nu_0|(1+\exp(2a|\nu_0|))^{-1} + ca|\nu_0|^{-1}(1-2(1+\exp(2a|\nu_0|))^{-1})$$

<u>is the optimal stopping rule for the above problem.</u>

<u>Lemma 2.2:</u>

$$(2\pi\sigma_0^2)^{-1/2} \int_{-\infty}^{\infty} E_\mu[\tilde{R}(c,\tau^*)]\exp(-\mu^2/2\sigma_0^2)d\mu \leq E[R(c,\tau_0)]$$

<u>Proof:</u> τ_0 is a Bayes rule for a symmetric problem and hence is symmetric in μ. Hence

$$E_\mu[R(c,\tau_0)] \geq E_\mu(\tilde{R}(c,\tau^*)] \quad \text{for all } \mu$$

From it the lemma follows.

<u>Theorem:</u>

$$(2\pi\sigma_0^2)^{-1/2} \int_{-\infty}^{\infty} E_\mu[\tilde{R}(c,\tau^*)]\exp(-\mu^2/2\sigma_0^2)d\mu$$

$$= c^{\frac{2}{3}}[K'\sigma_0^{-1} - \frac{3}{2} c^{\frac{1}{3}}\sigma_0^{-2}\ln\sigma_0(1+o(1))]$$

$$\text{as } \sigma_0 \to \infty$$

<u>where</u>

$$K' = (2\pi)^{-\frac{1}{2}} 2^{\frac{1}{3}} 3^{-1} \int_1^{\infty} (z-z^{-1}+2\ln z)^{-4/3}(1+z^{-2}+2z^{-1})$$

$$(1+ \ln z-z^{-1})dz$$

<u>Proof:</u> Let

$$(2.3) \qquad z = e^{2a\mu}$$

where "a" is the solution of the minimization problem in Lemma 2.1. Then z should satisfy the relation

$$(2.4) \qquad 2\mu^3 = c(z-z^{-1} + 2\ln z)$$

We have by using (2.3), (2.4) and Lemma 2.1,

$$\int_{-\infty}^{\infty} E_\mu[\tilde{R}(c,\tau^*)]\exp(-\mu^2/2\sigma_0^2)d\mu$$

$$= 2^{1/3} 3^{-1} c^{2/3} \int_1^{\infty} (z-z^{-1} + 2\ln z)^{-\frac{4}{3}} (1 + \ln z-z^{-1}) \times$$

$$(1 + 2z^{-1} + z^{-2}) \exp[-c^{2/3}(z-z^{-1} + 2\ln z)^{2/3}\sigma_0^{-2} 2^{-5/3}]dz$$

Let

$$\gamma = 2^{-5/3}\, c^{2/3}\, \sigma_0^{-2}$$

$$I(z) = (z - z^{-1} + 2 \ln z)^{-4/3}(1 + \ln z - z^{-1})(1 + 2z^{-1} + z^{-2})$$

We have

$$(2.5) \quad \int_{-\infty}^{\infty} E_\mu \{\tilde{R}(c, \tau^*)\} \exp(-\mu^2/2\sigma_0^2)\, d\mu = 2^{1/3} 3^{-1} c^{2/3} \times$$

$$\int_1^{\infty} I(z) \exp(-\gamma (z - z^{-1} + 2 \ln z)^{2/3})\, dz$$

<u>Lemma 2.3</u>: $\displaystyle\int_1^{1/\gamma} I(z) \exp(-\gamma (z - z^{-1} + 2 \ln z)^{2/3})\, dz$

$$= \int_1^{\infty} I(z)\, dz + 3\gamma^{1/3} \ln \gamma - 12\gamma^{1/3} + O(\gamma^{2/3} \ln \gamma).$$

<u>Proof:</u>

$$\int_1^{1/\gamma} I(z) \exp(-\gamma (z - z^{-1} + 2 \ln z)^{2/3})\, dz$$

$$= \int_1^{\gamma^{-1}} I(z) [1 - \gamma (z - z^{-1} + 2 \ln z)^{2/3} (1 + o(1))]\, dz$$

$$= \int_1^{\gamma^{-1}} I(z)\, dz - \gamma (1 + o(1)) \int_1^{\gamma^{-1}} I(z) (z - z^{-1} + 2 \ln z)^{2/3}\, dz$$

$$= \int_1^{\infty} I(z)\, dz - \int_{\gamma^{-1}}^{\infty} I(z)\, dz - \gamma (1 + o(1)) O(\gamma^{-1/3} \ln \gamma)$$

$$= \int_1^{\infty} I(z)\, dz + 3\gamma^{1/3} \ln \gamma - 12\gamma^{1/3} + O(\gamma^{2/3} \ln \gamma)$$

<u>Lemma 2.4</u>: $\displaystyle\int_{1/\gamma}^{\infty} I(z) \exp(-\gamma (z - z^{-1} + 2 \ln z)^{2/3})\, dz$

$$= 12\gamma^{1/3} - 3\gamma^{1/3} \ln \gamma + 9 \cdot 2^{-1} \pi^{\frac{1}{2}} \gamma^{\frac{1}{2}} \ln \gamma (1 + o(1))$$

<u>Proof:</u> Let $w = \gamma (z - z^{-1} + 2 \ln z)^{2/3}$

$$I(z) \exp(-\gamma (z - z^{-1} + 2 \ln z)^{2/3})\, dz$$

$$= 3 \cdot 2^{-1} \gamma^{1/2} w^{-3/2} (1 + \ln z - z^{-1}) e^{-w}\, dw$$

Let

$$u = z - z^{-1} + 2\,\ell n\, z = (w/\gamma)^{3/2}$$

For $z \geq \gamma^{-1}$

$$1 + \ell n\, z - z^{-1} = 1 + \ell n\, u + O(u^{-1}\,\ell n\, u)$$

$$= 1 + 3\cdot 2^{-1}\,\ell n(w/\gamma) + O((w/\gamma)^{-3/2}\,\ell n\,(w/\gamma))$$

Then

$$\int_{\gamma^{-1}}^{\infty} I(z)\,\exp(-\gamma(z - z^{-1} + 2\,\ell n\, z)^{2/3})dz$$

$$= 3\cdot 2^{-2} \int_{\gamma(\gamma^{-1}-\gamma-2\,\ell n\,\gamma)^{2/3}}^{\infty} \gamma^{1/2} w^{-3/2}(3\,\ell n\, w - 3\ell n\,\gamma + 2)e^{-w}dw +$$

$$O(\gamma^{4/3}\,\ell n\gamma)$$

$$= 12\gamma^{1/3} - 3\gamma^{1/3}\,\ell n\,\gamma + 9\cdot 2^{-1}\,\pi^{1/2}\gamma^{1/2}\,\ell n\,\gamma(1 + o(1))$$

From (2.5), Lemma 2.3 and Lemma 2.4 we get the Theorem.

From (1.1), Lemma 2.2 and the Theorem, we have the following corollary to the Theorem.

Corollary: $K > K'$

Remark: Consider the case of ν having a prior distribution of Lebesgue measure. For any stopping rule τ,

$$\int_{-\infty}^{\infty} R(\nu,\tau)d\nu = \lim_{\sigma_0\to\infty} (2\pi\sigma_0^2)^{\frac{1}{2}} [(2\pi\sigma_0^2)^{-1/2} \int_{-\infty}^{\infty} R(\nu,\tau)e^{-\nu^2/2\sigma_0^2}\,d\nu]$$

$$\geq \lim_{\sigma_0\to\infty} (2\pi\sigma_0^2)^{1/2}\, B(0,\sigma_0^2)$$

$$= K\,c^{2/3}$$

So the Bayes risk with respect to Lebesgue measure

$$\inf_{\tau} \int_{-\infty}^{\infty} R(\nu,\tau)\,d\nu \geq Kc^{2/3} > K'\,c^{2/3}$$

for all $c > 0$.

Here, $K'\,c^{2/3}$ is the lower bound derived in [1]. Therefore, we have shown that Bickel and Yahav's lower bound cannot be attained.

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[3] Chernoff, H. (1965). "Sequential tests for the mean of a normal distribution III (Small t)". Ann. Math. Statist, , 36, 28 - 54.

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Γ the author

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delta sub 0 to the 2nd power

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