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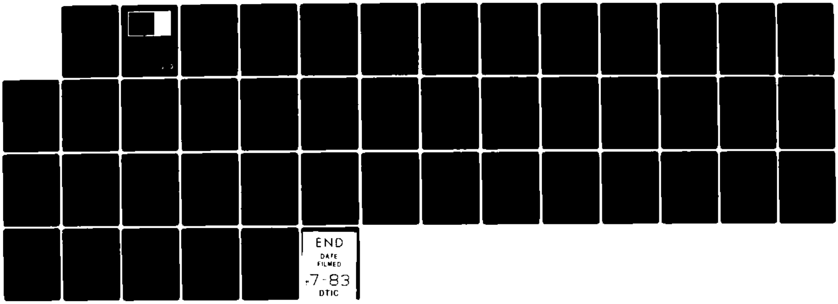

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ON A NONLINEAR DEGENERATE PARABOLIC EQUATION IN INFILTRATION OR EVAPORATION THROUGH A POROUS MEDIUM

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ON A NONLINEAR DEGENERATE PARABOLIC EQUATION IN INFILTRATION OR EVAPORATION THROUGH A POROUS MEDIUM

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ABSTRACT

We prove the existence, continuity and uniqueness of solutions of the Cauchy problem and of the first and mixed boundary value problems for the equation

(E) $$u_t = \phi(u)_{xx} + b(u)_x.$$

ϕ and b are assumed to belong to a large class of functions including the particular cases $\phi(u) = u^m$, $b(u) = u^\lambda$; $m > 1$ and $\lambda > 0$. These results significantly sharpen those currently available in the substantial literature devoted to (E) over the last two decades. In particular, the uniqueness is proved in a generality which allows (E) to model problems invoking the evaporation of a fluid through a porous medium.

AMS (MOS) Subject Classifications: 35K55, 35K65

Key Words: Nonlinear degenerate parabolic equation, infiltration and evaporation problems, continuity and uniqueness of generalized solutions.

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SIGNIFICANCE AND EXPLANATION

During the last two decades a great deal of progress has been made on the mathematical analysis of flows through porous media. Such phenomena led to degenerate nonlinear parabolic equations. The equations obtained are of different nature when the fluid movement takes place in a horizontal column of the medium rather than in a vertical column of the medium. The latter case gives rise to first order nonlinear perturbations of the former case and equations of this more general sort also model the evaporation of a fluid through a porous medium. A significant technical difficulty arises in the evaporation case; the first order nonlinear terms can be singular at the points where the solution vanishes.

In this paper the authors give a mathematical treatment of the Cauchy problem as well as the first and mixed boundary value problems for the relevant equations. Existence, continuity and uniqueness of generalized solutions are proved thereby improving earlier results in the mathematical literature.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

ON A NONLINEAR DEGENERATE PARABOLIC EQUATION IN INFILTRATION OR EVAPORATION THROUGH A POROUS MEDIUM

J. Ildefonso Diaz(1)(*) and Robert Kersner(2)

§1. Introduction.

This paper deals with the nonlinear parabolic equation

$$u_t = \phi(u)_{xx} + b(u)_x \tag{E}$$

where ϕ and b are continuous real functions.

Equation (E), sometimes called the nonlinear Fokker-Planck equation, arises, for example, in the study of the flow of a fluid through a homogeneous isotropic rigid porous medium. If $\theta(t,x,y,z)$ denotes the volumetric moisture content and $\vec{v}\,(t,x,y,z)$ the velocity then the continuity equation is

$$\frac{\partial\theta}{\partial t} + \operatorname{div} \vec{v} = 0$$

the density of the fluid being assumed constant. By the Darcy law

$$\vec{v} = -K(\theta) \cdot \operatorname{grad} \Phi$$

where $K(\theta)$ is the hydraulic conductivity and Φ is the total potential. If absortion and chemical, osmotic and thermal effects are neglected, then, for unsatured flows, Φ may be expressed as the sum of a hydrostatic potential due to capillary suction $\psi(\theta)$ and a gravitational potential ([3],[32]). Thus, if we choose the (x,y,z) coordinate system in such a way that the z-coordinate is vertical and pointing upwards, we may write.

$$\Phi = \psi(\theta) + z \, .$$

Then we obtain

$$\frac{\partial\theta}{\partial t} = \operatorname{div} \{D(\theta) \operatorname{grad} \theta\} + \frac{\partial}{\partial\theta} K(\theta) \tag{1.1}$$

where

$$D(\theta) = K(\theta)\cdot\frac{d\psi}{d\theta}(\theta) \, . \tag{1.2}$$

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If the fluid movement takes place in a vertical column of the medium, equation (1.1) takes the form

(1.3) $$\theta_t = \phi(\theta)_{zz} + b(\theta)_z$$

being

(1.4) $$\phi(s) = \int_o^s D(r)dr, \; b(s) = K(s) \quad \text{for} \quad r \in \mathbb{R} .$$

If the fluid movement takes place in a horizontal column of the medium and x denotes distance along the column, (1.1) reduces to the equation

(1.5) $$\theta_t = \phi(\theta)_{xx} .$$

Equation (1.5) also appears in many other contexts. It is also remarkable that the mathematical theory for this equation is fairly well advanced at the present, in contrast with that of equation (1.3). (See, for instance, the survey article of L.A. Peletier in [30]).

The functions D and K (and then ϕ and b) are usually determined empirically according to the nature of the flow problem, as well as of the nature of the porous medium. In any case a reasonable choice for D and K would be

$$D(u) = D_o u^{m-1} \;, \; K(u) = K_o u^{\lambda}$$

where D_o, k_o, m and λ are positive constants. After a suitable rescaling of the independent variables the equation (1.3) yields (by changing z by x)

$(E_{m,\lambda})$ $$u_t = (u^m)_{xx} + (u^{\lambda})_x .$$

The flow problem which has been treated more frequently in the mathematical literature corresponds to the phenomena of absorption and downward infiltration of a fluid (e.g. water) by the porous medium (e.g. soil). In those cases, some physical experiences show that the corresponding functions K and ψ are such that $\phi \in C^2([o,\infty))$, $\phi(o) = \phi'(o) = 0$, $\phi'(r) > 0$, $\phi''(r) > 0$ if $r > 0$ and $b \in C^2([o,\infty))$, $b(o) = o$, $b'(r) > 0$, $b''(r) > 0$ if $r > 0$. (see [35 p. 220] and [3 p. 511]. In terms of equation $(E_{m,\lambda})$ those cases correspond to the assumptions $m > 1$ and $\lambda > 1$. (Some mathematical papers on such problems are [19], [16], [14], [27], [37], [11], and [38]).

Nevertheless, there are other interesting flow problems that give rise to different elections of the functions K and ψ (and then of ϕ and b); at present, no mathematical literature exists on the corresponding equations.

Thus, the physical problem of evaporation from bare soil when the surface is so dry that water loss is limited by the rate of soil-water movement upwards has been studied for many years (see e.g. [31], [35] and the references therein). In such problems, the hydraulic conductivity function K is a regular <u>concave</u> function (see [24, p.425], [31, p.357] and [35,p. 259] and D is a regular increasing function). An immediate change of variables shows that the value of m and λ for which equation $(E_{m,\lambda})$ governs the evaporation problem are $m > 1$ and $0 < \lambda < 1$.

The main objective of this paper is to consider the equations (E) and $(E_{m,\lambda})$ in a general framework, which includes the corresponding equations of evaporation problems in particular.

To be precise, we shall study the following three problems for equation (E):

$$(CP)\quad \begin{cases} u_t = \phi(u)_{xx} + b(u)_x & \text{on } S = (o,T)\times(-\infty,\infty) \\ u(o,x) = u_o(x) & \text{on } (-\infty,\infty), \end{cases}$$

$$(FBVP)\quad \begin{cases} u_t = \phi(u)_{xx} + b(u)_x & \text{on } R = (o,T)\times(l_1,l_2) \\ u(t,l_1) = \psi-(t),\ \ u(t,l_2) = \psi+(t) & \text{on } (o,T) \\ u(o,x) = u_o(x) & \text{on } (l_1,l_2) \end{cases}$$

and

$$(MBVP)\quad \begin{cases} u_t = \phi(u)_{xx} + b(u)_x & \text{on } H = (o,T)\times(-\infty,l_2) \\ u(t,l_2) = \psi(t) & \text{on } (o,T) \\ u(o,x) = u_o(x) & \text{on } (-\infty,l_2)\ . \end{cases}$$

It is important to remark that the most interesting problem in evaporation (as well as in downward infiltration) corresponds to (MBVP) with $l_2 = 0$. (see [31, p.359] and [35, p. 229]).

Like the porous media equation, (E) is a degenerate parabolic equation. At points (t,x) where $u > 0$ it is parabolic, but at points where $u = 0$ it is not. In consequence, we cannot expect the above problems to have a classical solution (in fact, between a region where $u > 0$ and another one where $u = 0$, u need not be smooth). It is, therefore, necessary to generalize the notion of solutions of these problems. Among the different notion of solutions, we shall follow the one introduced in [19].

Definition 1.1. A function $u(x,t)$ defined on $\bar{S}$ is said to be a generalized solution of the (C.P) problem if

i) u is bounded, continuous and nonnegative[1]

ii) u satisfies the integral identity

$$I(u,\zeta,P) \equiv \int_{t_o}^{t_1}\int_{x_1}^{x_2} [\phi(u)\zeta_{xx} + u\zeta_t - b(u)\zeta_x]dxdt - \int_{x_1}^{x_2} u\zeta dx\Big|_{t_o}^{t_1} - \int_{t_o}^{t_1} \phi(u)\zeta_x dt\Big|_{x_1}^{x_2} = 0$$

for all $P \equiv [t_o t_1] \times [x_1,x_2]$ and for all $\zeta \in C^{1,2}_{t,x}(P)$ such that $\zeta(t,x_1) = \zeta(t,x_2) = 0$ for any $t \in [t_o,t_1]$.

iii) $u(o,x) = u_o(x)$ for all $x \in (-\infty,\infty)$.

Definition 1.2. A function $u(x,t)$ defined on $\bar{R}$ is said to be a generalized solution of the (FBVP) problem if

i) u is bounded, nonnegative and continuous on $\bar{R}$.

ii) u satisfies the integral identity $I(u,\zeta,P) = 0$ for any $P = [t_o,t_1] \times [x_1,x_2] \subset \bar{R}$ and any $\zeta \in C^{1,2}_{t,x}(P)$ such that $\zeta|_{x=x_1} = \zeta|_{x=x_2} = 0$.

iii) $u(t,l_1) = \psi_{-(t)}$, $u(t,l_2) = \psi_+(t)$ for all $t \in [o,T]$ and $u(o,x) = u_o(x)$ for all $x \in [l_1,l_2]$.

Definition 1.3. A function $u(x,t)$ defined on $\bar{H}$ is said to be a generalized solution of the (MBVP) problem if

i) u is bounded, nonnegative and continuous on $\bar{H}$.

ii) u satisfies the integral identity $I(u,\zeta,P) = 0$ for any $P = [t_o,t_1] \times [x_1,x_2] \subset \bar{H}$ and any $\zeta \in C^{1,2}_{t,x}(P)$ such that $\zeta|_{x=x_1} = \zeta|_{x=x_2} = 0$.

iii) $u(t,l_2) = \psi(t)$ for all $t \in [o,T]$ and $u(o,x) = u_o(x)$ for all $x \in (-\infty,l_2)$.

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(1) We shall limit our attention to the physically reasonable case of nonnegative data.

To prove the existence of a generalized solution for each one of the three problems we shall follow the constructive method given initially by O.A. Oleinik, A.S. Kalashnikov and E. Yui Lin in [28] for the case of equation (1.5). To do this, we first obtain a sequence of classical solutions of (E) defined on an expanding sequence of cylinders. We shall show that it tends-pointwise-to a function that we call limit solution. (Such a function satisfies all the properties required *except, perhaps, the continuity*). This will be done in Section 2.

In Section 3 we shall prove that under additional hypotheses the limit solution is continuous (i.e. is a generalized solution). Such results are well known when

$$(1.6) \qquad \begin{cases} b \in C^1([o,\infty)) \quad \text{and} \\ \int_s^1 \{|\phi''(r)| + |b''(r)|\}dr \in L^1(o,1) \end{cases}$$

([14]). In the case of equation $(E_{m,\lambda})$, this corresponds to the assumption $m > 1$ and $\lambda > 1$. The study of the regularity of its solutions is made in [19] and [16]. In both cases, optimal estimates on the modulus of continuity of the solution are given; in fact, such estimates are independent of b and λ. In consequence, the idea that the transport term $b(u)_x$ has not any fundamental importance on the behaviour of the solution is defended in the previous literature. Here we shall show that such conclusion is, in general, erroneous (e.g. $\lambda > 0$) since when $0 < \lambda < 1$ the modulus of continuity of the solution depends on λ. More generally, if the function J defined by

$$J(r) \equiv \int_0^r \frac{ds}{b(\phi^{-1}(s))}$$

is finite for $r > 0$ (this is the case, for instance, of $\phi(u) = u^m$, $b(s) = s^\lambda$ and $m > \lambda$) then we shall prove that the modulus of continuity of the solution of (E) can be estimated in terms of the function $J o \phi$ and the data of each problem.

In Section 4 the uniqueness of the generalized solutions is considered. The problem of uniqueness has been a polemic subject in the existing literature. Indeed, the first uniqueness result seem to be the one obtained in 1975 by A.S. Kalashnikov. In his paper

[19], the uniqueness of a generalized solution of $(E_{m,\lambda})$ is shown under the assumptions $m > 1$ and $\lambda > 1$. In 1976, B.H. Gilding and L.A. Peletier in [16], made a systematic study of equation $(E_{m,\lambda})$ in a way which is totally independent Kalashnikov's work. In fact they introduce a different notion of solution of the problem (CP): they substitute condition ii) and iii) of the Definition 1.1. by

ii)* (u^m) has a bounded generalized derivative with respect to x in S,

iii)* u satisfies the identity

$$\iint_S \{\phi_x[(u^m)_x + u^\lambda] - \phi_t u\}dxdt = \int_{-\infty}^{\infty}\phi(x,o)u_o(x)dx$$

for all $\phi \in C^1(\bar{S})$ which vanish for large $|x|$ and for $t = T$. The uniqueness result of [16] for such class of solutions (called weak solutions) is obtained under the assumption $\lambda > \frac{1}{2}(m+1)$. The important work [16] has been the object of several generalizations in the last years. For instance, B.H. Gilding in [14] proved the uniqueness of weak solutions of (CP), (FBVP) and (MBVP) under the hypothesis

$$(1.7) \qquad (b')^2(s) = O(\phi'(s)) \quad \text{as} \quad s \to 0^+ .$$

More recently, Wu Dequan in [37] has proved the uniqueness of the generalized solution of (FBVP), assuming

$$(1.8) \qquad \begin{cases} (b')(s) = O((\phi')^\alpha(s)) \quad \text{as } s \to 0^+, \ \phi'(s) > Ks^\nu \ \text{ for } s > 0, \text{ and} \\ \alpha > \frac{1}{4} \text{ if } \nu < 2 \text{ and } \alpha > \frac{1}{2} - \frac{1}{2\nu} \text{ if } \nu > 2 . \end{cases}$$

We remark that in terms of equation $(E_{m,\lambda})$ condition (1.7) is equivalent to $\lambda > \frac{1}{2}(m+1)$ and condition (1.8) is equivalent to $\lambda > \frac{1}{4}(m+3)$ if $m < 3$ and $\lambda > \frac{m}{2}$ if $m > 3$. (Other uniqueness results are given in [27], [28] and [11] for some variations of equation (E)). Finally, we point out some recent results obtained in [4] by a different approach.

In this paper we give a general and unified answer to the problem of uniqueness of solutions of (CP), (FBVP) and (MBVP). Our assumptions on ϕ and b are weaker than those of the above papers. In particular, they are fulfilled if in the equation $E_{m,\lambda}$ we assume $m > 1$ and $\lambda > 0$. On the other hand, in Section 3 the equivalence between the generalized and weak solutions is proved. Thus, the uniqueness of a weak solution is also ensured.

Our uniqueness result is a particular consequence of some L^1- estimates. These also show the continuous dependence of solutions with respect to the initial data as well as comparison results. Such estimates also show that the semigroup operator defined by the solution is a nonlinear semigroup of contractions on the space $L^1(-\infty,\infty)$, $L^1(l_1,l_2)$ or $L^1(-\infty,l_2)$ respectively. Some comments are made about the way in which that conclusion is obtained by the theory of accretive operators on Banach spaces.

In order to provide the reader with a summary collecting some of the results of this paper, we shall restrict ourselves the consideration of problem (PC) for equation $(E_{m,\lambda})$. We can state the following result:

Theorem 1.1. Assume $m > 1$ and $\lambda > 0$. Let $u_o \geq 0$ on $(-\infty,\infty)$ be such that u_o^β is Lipschitz continuous for some β such that

$$\max\{(m-1),(m-\lambda)^+\} \leq \beta \leq m \qquad (h^+=\max\{h,o\}).$$

Then there exists an unique generalized solution u of the (CP) problem for the equation $(E_{m,\lambda})$. In addition $u \in C^{\nu,\frac{\nu}{2}}(\bar{S})$ for $\nu = \min\{1, 1/_\beta\}$ (being such exponent ν, in general, optimal), $(u^m)_x \in L^\infty(S)$ and u coincides with the unique weak solution of (CP) (As usual $C^{\nu}, \frac{\nu}{2}(\bar{S})$ denotes the Banach space of the functions $u(t,x)$ which are Hölder continuous with respect to x and t, of exponents ν and $\frac{\nu}{2}$ respectively).

We point out that our results can be easily extended to a more general class of equations of the form

$$u_t = \phi(x,t,u)_{xx} + b(x,t,u)_x + c(x,t,u) ,$$

where $\phi(\cdot,\cdot,u)$ is strictly increasing $\phi(x,t,o) = \phi'_u(x,t,o) = 0$ and $b(\cdot,\cdot,u)$ and $c(\cdot,\cdot,u)$ are allowed to be nonnecessarily Lipschitz continuous at $u = 0$ (some additional hypotheses must be made on b and c, e.g. $c(\cdot,\cdot,u)$ non-increasing in u, and so on).

In a forthcoming article the authors study some qualitative properties, including the propagation of the support of solutions, extending the well-known results for the case where b is Lipschits continuous at $u = 0$ and presenting some new properties of the solution of evaporation type problems associated to $E_{m,\lambda}$ for $m > 1$ and $0 < \lambda < 1$.

§2. Existence of a limit solution.

The basic idea in the study of degenerate equations, like (E), consists in obtaining the solution as the limit of a sequence of functions which are solutions of some adequate non-degenerate parabolic equations approaching equation (E). This idea can be carried out by two different ways: a) by consideration of the equations

$$u_{\varepsilon,t} = ((\phi'(u_\varepsilon) + \varepsilon)(u_\varepsilon)_x)_x + b_\varepsilon(u_\varepsilon)_x \quad ,$$

or, b) by replacing $u_o(x) \geq 0$ by the sequence $u_{o,\varepsilon}(x) \geq \varepsilon > 0$ and then showing (via the maximum principle) that the corresponding solutions u_ε satisfy $u_\varepsilon(t,x) \geq \varepsilon$, so they are solutions of the nondegenerate equations.

Method a) is very useful if the signs of the data (for instance u_o for (CP)) are not "a priori" prescribed. However, the passage to the limit is often a difficult task (see the results of [6] and [34] for the case $b \equiv 0$). Here we shall follow the method b) introduced in [29]. Then we shall obtain a sequence of classical solutions defined on an expanding sequence of cylinders and we shall prove that they converge pointwise to a function that we call limit solution. In the next sections we shall prove that, under some supplementary hypotheses, the limit solution coincides with the unique generalized solution.

Proposition 2.1. Assume that there exists $\alpha \in [0,1]$ such that

(2.1) $\phi \in C^{2+\alpha}((0,+\infty)) \cap C^1([0,\infty))$, $\phi(0) = 0$, and $\phi'(r) > 0$ if $r > 0$

(2.2) $b \in {}^{2+\alpha}((0,\infty))$.

Then:

i) For every $u_o \in C_b(-\infty,\infty)$ (1), $u_o \geq 0$ there exists at least one function u defined on $\bar{S}$ such that $u \geq 0$, $u \in L^\infty(S)$ and u satisfies ii) and iii) of Definition 1.1.

ii) For every $u_o \in C([l_1,l_2])$, $u_o \geq 0$, $\psi_-, \psi_+ \in C([o.t])$, $\psi_-, \psi^+ \geq 0$ and $\psi_-(0) = u_o(l_1)$, $\psi_+(0) = u_o(l_2)$ there exists at least one function u defined on $\bar{R}$ such that $u \geq 0$, $u \in L^\infty(R)$ and u satisfies ii) and iii) Definition 1.2.

(1) $C_b(\Omega)$ denotes the set of all the bounded continuous functions defines on Ω.

iii) <u>For every</u> $u_o \in C_b((-\infty,l_2])$, $u_o > 0$, $\psi \in C([0,t])$, $\psi > 0$ and $\psi(0) = u_o(l_2)$ <u>there exists at least one function u defined on $\tilde{H}$ such that u satisfies ii) and iii) of Definition 1.3.</u>

The proof of Proposition 2.1 is already standard after the deep work [29] and its generalizations (see for instance [19], [16], [27]). Nevertheless, in the next sections we shall need some properties of the function u which are obtained by using the proof of Proposition 2.1. This is the reason for our sketch.

We shall use the following result of the classical theory of quasilinear parabolic equations.

<u>Lemma 2.1.</u> (see e.g. [14]) <u>Let</u> $Q \equiv (\eta_1, \eta_2) \times (0,T]$, ε, $\alpha \in (0,1]$ <u>and</u> $M \in (0,\infty)$. <u>Suppose that</u> $u_o \in C^{2+\alpha}([\eta_1,\eta_2])$, $\psi_1,\psi_2 \in C^{1+\alpha}[0,T]$ <u>and</u>

$$\varepsilon \leq u_o \leq M \; , \; \varepsilon \leq \psi_1, \psi_2 \leq M$$

$$\psi_i(o) = u_o(\eta_i), \; \psi_i'(o) = \phi(u_o)''(\eta_i) + (b(u_o))'(\eta_i) \quad \underline{\text{for } i = 1,2}.$$

<u>Then</u> (<u>under assumptions (2.1) and (2.2)</u>) <u>there exists a unique function $u(t,x)$ such that</u>

$$u \in C^{1,2}_{t,x}(\bar{Q}), \; \phi(u)_x \in C^{1,2}_{x,t}(Q), \; \varepsilon \leq u \leq M \quad \underline{\text{in}} \quad \bar{Q}$$

$$u_t = \phi(u)_{xx} + b(u)_x \quad \underline{\text{on}} \quad Q$$

$$u_{(x,o)} = u_o(x) \quad \underline{\text{on}} \quad [\eta_1 \eta_2]$$

$$u(t,\eta_i) = \psi_i(t) \quad \underline{\text{on}} \quad [o,T], \quad \underline{\text{for } i = 1.2}.$$

■

<u>Proof of Proposition 2.1.</u> We shall prove i). We can always choose $M > 0$, $\{\varepsilon_k\}$, $\{\alpha_k\}$ and $\{u_{o'k}\}$ such that

$$(2.1)\quad \begin{cases} \varepsilon_k, \alpha_k \in (0,1) \; , \; \varepsilon_k \to 0 \text{ as } k \to \infty, \\ u_{o'k} \in C^{2+\alpha_k}(-\infty,\infty), \; \varepsilon_k \leq u_{o'k}(x) \leq M \text{ if } |x|<k \text{ and } u_{o'k}(x) = M \text{ if } |x|>k \\ u_{o'k+1}(x) \leq u_{o'k}(x) \quad \text{for all } x \in (-\infty,\infty) \\ u_{o'k} \to u_o \text{ as } k \to \infty \text{ uniformly on compact subsets of } (-\infty,\infty) \end{cases}$$

Let $Q_k = (-k,k+1) \times (0,T]$. Then, by Lemma 2.1, there exists a unique function $u_k \in C^{1,2}_{t,x}(\bar{Q}_k)$ such that: i) $(u_k)_x \in C^{1,2}_{t,x}(Q_k)$, $\varepsilon_k \leq u_k \leq M$, ii) u_k satisfies (E) in Q_k, iii) $u_k(0,x) = u_{o,k}(x)$ for $|x| < k + 1$,

iv) $u_k(t,\pm(k+1)) = M$ for $t \in [0,T]$. Then, by a standard application of the maximum principle we obtain that $u_{k+1}(t,x) < u_k(t,x)$ for all $(t,x) \in \bar{Q}_k$. Hence, we can define

$$u(t,x) = \lim_{k\to\infty} u_k(t,x) \tag{2.2}$$

for all $(t,x) \in \bar{S}$. The function u is nonnnegative, bounded and satisfies the integral condition ii) in Definition 1.1. The proof of ii) and iii) are similar. The natural modifications now being that $u_{o,k}$ are only defined in $[l_1,l_2]$ (in the proof of part (ii) of Proposition 2.1). Also, there exist $\{\psi_{-,k}\}$ and $\{\psi_{+,k}\}$ (sequences in $C^{1+\alpha_k}([0,T])$) such that $\epsilon_k < \psi_{-,k}, \psi_{+,k} < M$, $\psi_{-,k+1} < \psi_{-,k}$, $\psi_{+,k+1} < \psi_{+,k}$, $\psi_{-,k}(0) = u_{o,k}(l_1)$,
$\psi_{+,k}(0) = u_{o,k}(l_2)$, $(\psi_{-,k})'(0) = (\phi(u_{o,k}))''(l_1), + (b(u_{o,k})'(l_1)$,
$(\psi_{+,k})'(0) = (\phi(u_{o,k}))''(l_2) + (b(u_{o,k}))'(l_2)$.
Finally $\psi_{+,k} \to \psi_+$, $\psi_{-,k} \to \psi_-$ uniformly on $[0,T]$ when $k \to \infty$. ∎

<u>Remark 2.1</u>. Obviously, we can also consider more general quasilinear equations or choose data $u_o, (u_o,\psi_-,\psi_+)$ and (u_o,ψ), *not necessarily continuous* (see [17] and [4]). We remark that the result applies to the equation $(E_{m,\lambda})$ when $0 < m < 1$. When $b \equiv 0$ such equation arise in plasma physic (see the exact references in [30]).

<u>§3 Regularity of the limit solutions. Existence of generalized solutions.</u>

We shall now prove that, under some additional hypotheses, all limit solutions are continuous and, therefore, generalized solutions.

The continuity of the solutions of degenerate parabolic equations is one of the most difficult points in the study of such equations. After the precise estimates on the smoothness of the solution of the porous media equation obtained by Aronson and Kalashnikov in [1] and [8] respectively, the question of the continuity of the solution of the porous media equation in higher dimensions remained an open question for a long time. However, positive answers are well known today, concerning a large class of equations including the porous media equation and some particular formulations of equation (E) when the dimension is equal to one (see [8], [7], [10], [34] and [40]).

Here we shall study the smoothness of the solutions of (E) and we shall try to estimate the modulus of continuity of the solutions. Results of this kind are well known for $(E_{m,\lambda})$ when $m > 1$ and $\lambda > 1$ (see [19]) and more generally for (E) when ϕ and b satisfy (2.1), (2.2) as well as

$$(3.1)\quad \begin{cases} b \in C^1([0,\infty)) \\ \int_s^1 \{|\phi''(r)| + |b''(r)|\}dr \in L^1(0,1) \end{cases}$$

(see [14]). Our special interest is centered on (E) under several assumptions which include the case of equation $(E_{m,\lambda})$ when $m > 1$ and $0 < \lambda < 1$.

An important tool in our study will be the fact that if we define the improper integral

$$J(r) = \int_0^r \frac{ds}{b(\phi^{-1}(s))}$$

for every $r > 0$ (we can suppose, without loss of generality that $b(o) = o$) then, when $\phi(s) = s^m$ and $b(s) = s^\lambda$, $J(r)$ is finite if and only if $m > \lambda$. Then our fundamental hypothesis will be $J(r) < +\infty$ for some $r \in (o,\infty)$.

In order to prove the continuity of the limit solution u constructed in §2, we shall first obtain estimates on the modulus of continuity of u_k which should not depend on k. Afterwards we shall pass to the limit when $k \to +\infty$. We start by studying the general nondegenerate problem given in Lemma 2.1.

Proposition 3.1. Given $\delta \in (0, \frac{1}{2}(n_2-n_1))$ and $\tau \in (o,T)$ let $Q_\delta = (0,T] \times (n_1+\delta, n_2-\delta)$, $Q(\tau) = (\tau,T] \times (n_1,n_2)$, $Q_\delta(\tau) = (\tau,T] \times (n_1+\delta,n_2-\delta)$. Assume (2.1), (2.2) and that for every $r \in (0,\varepsilon)$ the following hypotheses hold

(3.2) $\quad J(r) < +\infty$

(3.3) $\quad b''(r)b(r) \leq -C_1\, \phi'(r)$

(3.4) $\quad |\phi''(r)| \leq C_2|b''(r)|$

for some positive constants C_1 and C_2. Let f be the real function defined by

(3.5) $\quad f(r) = \phi^{-1}(J^{-1}(r))$ for $r > 0$.

Then for any u_o and ψ_1 given as in Lemma 2.1, the solution u satisfies: for any δ and τ there exists a constant C (depending only on τ, δ and M) such that

(3.6) $|(f^{-1}(u))_x| \leq C$ in $Q_\delta(\tau)$.

If in addition, u_o verifies

(3.7) $$\sup_{(\eta_1+\delta,\eta_2-\delta)} |f^{-1}(u_o(x))'| = L < +\infty ,$$

then (3.6) holds in Q_δ.

Before proving the above result, let us explain some facts about the proof. The method we use is due to Bernstein. As is well known, the major difficulty of this method appears in the selection of the function of u to be estimated. The estimate

$$|(g(u))_x| \leq C \text{ in } Q_\delta(t)$$

has been obtained by different authors in the following cases:

a) $g(s) = \phi(s)$ (See [1] for $b \equiv 0$ and [14] if b satisfies (3.1))

b) $g(s) = \int_o^s \frac{\phi'(s)}{s}$, if such integral converge and $b \equiv o$ (see [1] and [18]. The estimate (3.6) is completely new.

For equation $(E_{m,\lambda})$, all the hypotheses of Proposition 3.1 are satisfied if

$$0 < \lambda \leq 1 \leq m .$$

In this case a single computation shows that $f^{-1}(s) = \frac{m}{m-\lambda} s^{m-\lambda}$. *More generally we can* prove (using Proposition 3.1) that

(3.8) $|(u^\beta)_x| \leq C$ in $Q_\delta(t)$

for all $\beta \in \mathbb{R}$ such that

$$\max \{(m-1),(m-\lambda)^+\} \leq \beta \leq m$$

where $h^+ = \max \{h,o\}$. Then, estimate (3.8) includes also the estimates of [1], [19] and [16] for equation $(E_{m,\lambda})$(1).

Proof of Proposition 3.1. Set $w = f^{-1}(u)$. From equation (E) we obtain

(3.9) $$w_t = \frac{1}{f'(w)} [\phi(f(w))]_{ww}(w_x)^2 + \phi'(f(w))w_{xx} - b'(f(w)) w_x .$$

Using the definition of J *and* f *we have*

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(1) Recently Ph. Benilan has introduced in [5] a general method to obtain estimates like (3.6). Estimation (3.8) can be found by this method but this is not the case of the general estimate (3.6).

$$[\phi(f(w))]_{ww} = [J^{-1}(w)]_{ww} = \frac{b'(f(w))\, b(f(w))}{\phi'(f(w))}$$

$$f'(w) = \frac{b(f(w))}{\phi'(f(w))}$$

and

$$\frac{[\phi(f(w))]_{ww}}{f'(w)} = b'(f(w)) \; . \tag{3.10}$$

Then

$$w_t = \phi'(f(w))\, w_{xx} + b'(f(w))w_x^2 - b'(f(w))w_x \; . \tag{3.11}$$

Consider now a smooth function $\zeta\,(t,x)$ such that $\zeta = 1$ on $\bar{Q}_\delta(\tau)$, $\zeta = 0$ on the parabolic boundary of Q and $0 \leq \zeta \leq 1$ in $\bar{Q}$. Define the function $z = \zeta^2 p^2$ where $p = w_x$; at any point $(t_o,x_o) \in Q$ where z attains a positive maximum one has

$$z_x = 0 \quad \text{and} \quad z_t - \phi'(f(w))z_{xx} \geq 0 \; .$$

Hence, at (t_o,x_o) we have $\zeta\, p_x = -\zeta_x p$ and

$$\zeta^2 p(p_t - \phi'(f)p_{xx}) \geq (-\zeta\zeta_t + \phi'(f)\zeta\zeta_{xx} + 2\,\phi'(f)\zeta_x^2)p^2 \; .$$

Differentiating (3.11) with respect to x, multiplying the result by $\zeta^2\, p$ and using the former relations we obtain

$$\begin{aligned} - b''(f)f'\zeta^2\, p^4 \leq \zeta p^3[-\phi''(f)f'\zeta\, \zeta_x - 2b'(f)\zeta_x - b''(f)f'\zeta] \;+ \\ + p^2[\zeta\zeta_t - \phi'(f)\zeta\zeta_{xx} - 2\,\phi'(f)\zeta_x^2 - b'(f)\zeta_x\zeta] \; . \end{aligned} \tag{3.12}$$

Using the hypotheses (3.3) and (3.4) we can find two positive constants K_1 and K_2 depending only on ϕ, b and M, such that

$$2\,\zeta^2\, p^2 \leq K_1\zeta|p| + K_2 \quad \text{at } (t_o,x_o) \; . \tag{3.13}$$

By an elementary argument, (3.13) implies that

$$z(t_o,x_o) \leq K_1 + \tfrac{1}{4}K_2^2 = K_3$$

and hence

$$\sup_{\bar{Q}_\delta(\tau)} |w_x| \leq K_3^{1/2} \; .$$

To prove the second part, we note that $|w_x|$ is now bounded at $t = 0$. Hence we may take a cut-off function $\zeta(t,x) = \zeta(x)$ and allow z to attain its maximum at a point of the lower boundary of Q. Otherwise the proof is the same. ∎

The main result of this section is the following

Theorem 3.1. *Let* ϕ *and* b *satisfying the hypotheses of Proposition 3.1. Then*

(a) *For every* $u_o \in C_b(-\infty,\infty)$, $u_o > 0$ *and for any* $\tau \in (0,T)$ *the limit solutions* u *of (CP) satisfy*

$$|f^{-1}(u(t_1,x_1)) - f^{-1}(u(t_2,x_2))| < K\{|x_1-x_2|^2 + |t_1-t_2|\}^{1/2} \tag{3.14}$$

for some constant K *which depends only on* τ *and* $M = \|u_o\|_{L^\infty(-\infty,\infty)}$, *and for all* $(t_1,x_1),(t_2,x_2) \in [\tau,T] \times (-\infty,\infty)$. *If in addition* $f^{-1}(u_o)$ *is lipschitz continuous i.e.*

$$|f^{-1}(u_o(x_1)) - f^{-1}(u_o(x_2))| < L|x_1-x_2|$$

for some $L > 0$ *and all* $x_1,x_2 \in (-\infty,\infty)$ *then the conclusion (3.14) holds for any* (t_1,x_1), $(t_2,x_2) \in \bar{S}$, *and* K *depends only on* M *and* L.

(b) *For every* $u_o \in C_b([l_1,l_2])$, $u_o > 0$ *and* ψ_-, $\psi_+ \in C([0,T])$, ψ_-, $\psi_+ > 0$ *satisfying* $\psi_-(0) = u_o(l_1)$, $\psi_+(0) = u_o(l_2)$ *and for any* $\tau \in (0,T)$ and $\delta > 0$, $\delta < \frac{(l_2-l_1)}{2}$, *the limit solutions* u *of (FBVP) satisfies (3.14) for every* (t_1,x_1), $(t_2,x_2) \in [\tau,T] \times [l_1+2\delta, l_2-2\delta]$. *In particular* $f^{-1}(u) \in C^o([\tau,T] \times (l_1,l_2))$. *If in addition* $f^{-1}(u_o)$ *is locally Lipschtz continuous on* (l_1, l_2) *then* $u \in C^o([0,T] \times [l_1,l_2])$.

(c) *A similar conclusion to (b) holds for the (MBVP) problem*

Proof of (a). Applying Proposition 3.1 to the sequence u_K constructed in the proof of Proposition 2.1 we obtain

$$|f^{-1}(u_{k+1}(t_1,x_1)) - f^{-1}(u_{k+1}(t_2,x_2))| < C|x_1-x_2|$$

where C depends only on M and τ and for every (t_1,x_1), $(t_2,x_2) \in [\tau,T] \times [-k-1,k+1]$. Now set $w_K(t,x) = f^{-1}(u_K(t,x))$. Then w_K satisfies the equation

$$(w_k)_t = A_k(t,x)\,(w_k)_{xx} + B_k(t,x)(w_k)_x$$

being

$$A_k(t,x) = \phi'(f(w_k))(t,x) \quad \text{and} \quad B_k(t,x) = b'(f(w_k))[(w_k)_x - 1](t,x) .$$

Using Proposition 3.1 we know that

$$0 < A_k(t,x) < M^* \quad \text{and} \quad |B_K(t,x)| < M^*$$

where M^* depends only on M and τ. Then by a well known result (see [13]) there exists a constant K which depends only on M^* (i.e. on τ and M but not on k) such that

(3.15) $$|f^{-1}(u_{k+1}(t_1,x_1)) - f^{-1}(u_{k+1}(t_2,x_2))| \le K\,\{|x_1-x_2|^2 + |t_1-t_2|\}^{1/2}$$

for all $k \ge 1, (t_1,x_1),(t_2,x_2) \in [\tau,T] \times [-k-1,k+1]$. Hence

(3.16) $$|f^{-1}(u(t_1,x_1))-f^{-1}(u(t_2,x_2))| \le K\,\{|x_1-x_2|^2 + |t_1-t_2|\}^{1/2}$$

for all $(t_1,x_1), (t_2,x_2) \in [\tau,T] \times (-\infty,+\infty)$. This proves the first part of (a). The second statement is a direct consequence of the second conclusion of Proposition 3.1, choosing now u_k so that $|(f^{-1}(u_{o,k}))'(x)| \le L$ for some $x \in (-\infty,\infty)$ and $k \ge 1$.

<u>Proof of (b)</u>. Arguing as in part (a) we obtain that (3.16) holds for any $(t_1,x_1),(t_2,x_2) \in [\tau,T] \times [l_1+2\delta,l_2-2\delta]$ and, therefore, it is clear that $u \in C^o([0,T] \times (l_1,l_2))$ when $f^{-1}(u_o)$ is Lipschitz continuous on (l_1,l_2). To prove that u is continuous at the points $[o,T] \times \{l_1\}$ and $[o,T] \times \{l_2\}$, it is enough to prove (for instance, for $[o,T] \times \{l_1\}$) that for any $t_o \in [o,T]$

(3.17) $$\limsup_{(t,x)\to(t_o,l_1)} u(t,x) \le \psi_-(t_o)$$

and

(3.18) $$\liminf_{(t,x)\to(t_o,l_1)} u(t,x) \ge \psi_-(t_o).$$

Proving (3.17) is easy because

$$\limsup_{(t,x)\to(t_o,l_1)} u(t,x) \le \limsup_{(t,x)\to(t_o,l_1)} u_k(t,x) = \psi_{-,k}(t_o)$$

and then letting $k \to \infty$ we conclude the result. The proof of (3.18) is more complex (if $\psi_-(t_o) > 0$). This was shown by B.H. Gilding in [14] (Theorem 5) when ϕ and b satisfy (3.1). His argument is the following: for any $\varepsilon \in (0,\psi_-(t_o))$ he constructs a function $w(t,x)$ on $[0,T] \times [l_1,l_2]$ such that

$$\liminf_{(t,x)\to(t_o,l_1)} w(t,x) = \psi_-(t_o)-\varepsilon$$

and such that

$$u_k(t,x) \ge w(t,x)$$

for all $(t,x) \in [0,T] \times [l_1,l_2]$ and k large enough. We remark that the hypothesis (3.1) is only used by Gilding in the definition of w when the convergence of the integral

$$\rho(c) \equiv \int_0^M \frac{\phi'(r)dr}{c\cdot r+b(r)+\beta}$$

is required (c is any positive constant and $\beta = 1 + \sup_{0<r<M}|b(r)|$). Our conclusion follows by the same argument of Gilding noting that

$$\rho(c) \equiv \int_0^{\phi(M)} \frac{ds}{c\phi^{-1}(s)+b(\phi^{-1}(s))+\beta}.$$

Then, $\rho(c) < \infty$ for every $c \in (0,\infty)$ if the assumption (3.2) is made. The details are extremely technical and hence omitted here. The proof of (c) is analogous to the part (b). ■

Remark 3.1. Arguing as in [18] we can estimate the modulus of continuity of u in terms of function f when $(f^{-1})'' > 0$ on $(0,\delta)$. When $(f^{-1})'' < 0$ then the modulus of continuity is a lineal function. In particular, for the equation $(E_{m,\lambda})$ we obtain that the solution u of the (CP) problem is such that

$$u \in C^{\nu,\nu/2}([0,T] \times (-\infty,\infty))$$

for $\nu = \min\{1,1/\beta\}$ and β and real number such that $\max\{(m-1),(m-\lambda)^+\} < \beta < m$.

A further regularity result is the following:

Theorem 3.2. *Let ϕ and b satisfying the hypotheses of Proposition 3.1 Then*

i) *For any $u_0 > 0$ such that $f^{-1}(u_0)$ is bounded Lipschitz continuous on $(-\infty,\infty)$ there exists at least one generalized solution u of (CP) such that $f^{-1}(u)_x \in L^\infty(S)$ (where the derivative is taken in the sense of distributions). In particular $\phi(u)_x \in L^\infty(S)$ and u satisfies*

(3.19) $$\int_0^T \int_{-\infty}^{\infty} \{\theta_x[\phi(u)_x + b(u)] - \theta_t u\} \, dxdt = \int_{-\infty}^{\infty}\theta(0,x) \, u_0(x)dx$$

for all $\theta \in C^1(\bar{S})$ which vanish for large $|x|$ and $t = T$.

ii) *For any u_0, ψ_- and ψ_+ nonnegative functions such that $\phi(u_0)$ is locally Lipschitz continuous on (l_1,l_2), and such that $\phi(\psi_-)$ and $\phi(\psi_+)$ are absolutely* continuous on [0,T] and $\psi_-(0) = u_o(l_1)$, $\psi_+(0) = u_o(l_2)$, there exists at least one generalized solution u of (FBVP) such that $\phi(u)_x \in L^2(R)$ (distributional derivative) and u satisfies the identity

(3.20) $$\iint_R \{\theta_x[\phi(u)_x+b(u)] - \theta_t u\}dxdt = \int_{l_1}^{l_2} \theta(o,x)u_o(x)dx.$$

for all $\theta \in C^1(\bar{R})$ which vanish for $x = l_1$, $x = l_2$ and $t = T$.

iii) For any u_o, ψ nonnegative functions such that $\phi(u_o)$ is locally Lipschitz continuous and bounded on $(-\infty,l_2)$, $\phi(\psi)$ is absolutely continuous on [o,T], and $\psi(o) = u_o(l_2)$, there exists at least one generalized solution u of (MBVP) such that $\phi(u)_x \in L^2_{loc}(H)$

(distributional derivative) and such that

(3.21) $$\iint_H \{\theta_x[\phi(u)_x+b(u)] - \theta_t u\}dxdt = \int_{-\infty}^{l_2} \theta(o,x)u_o(x)dx .$$

for all $\theta \in C^1(\bar{H})$ which vanish for $x = l_2$, for large $|x|$ and for $t = T$.

The proof of i) is a simple consequence of the fact that

$$|\phi(u_k)_x| = |\phi(f(f^{-1}(u_k)))_x| \le C^* \cdot |f^{-1}(u_k)_x| \le C^* \cdot C$$

where $C^* = \max b(u_k(t,x))$ and $k \ge 1$. Otherwise the proof is standard (see e.g. [14]).

In order to prove ii) we need the following estimate near the boundary.

Lemma 3.1. Assume ϕ and b as in Proposition 3.1. Let u_o and ψ_i satisfy the assumptions of Lemma 2.1,(3.7) and

(3.22) $$\int_0^T |\phi(\psi_i(t))|dt \le L^* \quad \text{for } i = 1,2 \quad (L^* > 0) .$$

Then, for any $\delta > 0$, there exists a constant C^*, which depends only on L, L^*, M, T and δ, such that

(3.23) $$\iint_{Q(\tau)-Q_\delta(\tau)} \{\phi(u)_x\}^2 dxdt \le C^*$$

or any $\tau \in (0,T]$.

Proof. We shall only prove that

$$\int_0^T \int_{\eta_1}^{\eta_1+\delta} \{\phi(u)_x\}^2 dxdt \le \frac{C^*}{2}$$

(the estimate $\int_0^T \int_{\eta_2-\delta}^{\eta_2} \{\phi(u)_x\}^2 dxdt \le \frac{C^*}{2}$ is obtained in a similar way). The key idea is due to Gilding [1[illegible]]. Let $\chi(t,x) = \phi(u(t,x))-\phi(\psi_1(t))$. If we take equation (E), multiply it by χ and integrate by parts we obtain

(3.24) $$\int_0^T \int_{\eta_1}^{\eta_1+\delta} \{\phi(u)_x\}^2 dxdt = \int_0^T \{(\phi(u)_x)(t,\eta_1+\delta) + b(u(t,\eta_1+\delta))\}\chi(t,\eta_1+\delta)\, dt -$$

$$-\int_0^T \int_{\eta_1}^{\eta_1+\delta} b(u)(\phi(u))_x dxdt - \int_0^T \int_{\eta_1}^{\eta_1+\delta} u_t \phi(u) dxdt +$$

$$+ \int_0^T \int_{\eta_1}^{\eta_1+\delta} u_t(t,x)\ \phi(\psi_1(t)) dxdt .$$

We denote the four integrals on the right hand side of (3.24) by I_1, I_2, I_3 and I_4 respectively. The only difficult term to estimate is I_1 (see [14]). But by Proposition 3.1 we know that

$$|f^{-1}(u)_x(t,\eta_1+\delta)| \leq C \quad \text{for any} \quad t \in [0,T] .$$

Then

$$|\phi(u)_x(t,\eta_1+\delta)| = |\phi' f'(f^{-1}(u))_x(t,\eta_1+\delta)| \leq C'$$

for some $C > 0$. So

$$|I_1| \leq 2\,\phi(M)T(C' + \sup_{s \in (0,M]} b(s)) . \blacksquare$$

<u>Proof of ii) of Theorem 3.2.</u> Now the functions $u_{o,k}$, $\psi_{-,k}$ and $\psi_{+,k}$ can be assumed to satisfy the conditions given in the proof of Proposition 2.1 as well as

$$\begin{cases} \text{for any } \delta \in (0,1) \text{ there exists a constant } L(\delta) \text{ such that} \\ |\phi(u_{o,k})'(x)| \leq L(\delta) \quad \text{for all } x \in (l_1+\delta, l_2 - \delta) \end{cases}$$

and

$$\int_0^T |\phi(\psi_{-,k}(t))'|dt, \quad \int_0^T |\phi(\psi_{+,k}(t))'|dt \leq L^* .$$

Then, by Lemma 3.1, there exists a constant C^* (which depends only on $L(\beta)$, L^*, M and $T(\beta = \frac{l_2 - l_1}{4})$) such that

(3.25) $$\int_{l_1}^{l_1+2\beta} \int_0^T \{\phi(u_k)_x\}^2 dxdt + \int_{l_2-2\beta}^{l_2} \int_0^T \{\phi(u_k)_x\}^2\, dxdt \leq C^*$$

for all $k \geq 1$. On the other hand, by Proposition 3.1 there exists a constant C_1 which depends only on $L(\beta)$ and M such that

(3.26) $$|(\phi(u_k))_x(t,x)| \leq C_1 \quad \text{for all} \quad (t,x) \in [0,T] \times [l_1+2\beta, l_2-2\beta] .$$

From (3.25) and (3.24) we obtain that $\|\phi(u_k)_x\|_{L^2(H)}$ is uniformly bounded and by using the fact u_k is a classical solution it is easy to see that the weak limit $v \in L^2(H)$ of $\{\phi(u_k)_x\}$ can only be $\phi(u)_x$. The proof of iii) is analogous. ∎

Remark 3.2. By using a generalization of the Nash Theorem ([25] p.204), it is not difficult to show that, under the assumptions of Theorem 3.2, the generalized solution obtained in the above result is a classical solution of (E) in a neighborhood of any interior point (t_o, x_o) where $u(t_o,x_o) > 0$ (see e.g. [1] or [14]).

Remark 3.3. Suppose, for instance, that $b(s) > 0$ for any $s > 0$. Given $1 \in \mathbb{R}$, we define the stationary function

$$(3.27)\qquad U(t,x) \equiv f((1-x)^+) = \begin{cases} \phi^{-1}(J^{-1}(1-x)) & \text{if } x < 1 \\ 0 & \text{if } x > 1. \end{cases}$$

It is easy to see that u is a generalized solution of (MBVP) and satisfies $u(0,x) = f((1-x)^+)$ for $0 < x < \infty$ and $u(t,0) = f(1)$ for $t \in [0,T]$. Moreover

$$(f^{-1}(u))_x = ((1-x)^+)_x$$

hence $(f^{-1}(u))_x = 0$ if $x > 1$ but $(f^{-1}(u))_x \nearrow -1$ when $x \nearrow 1$. Then the estimate (3.14) is exact and can not be improved. (The function (3.27) will be used in a forthcoming paper of the authors in order to prove the boundedness of the right boundary of the support of the solutions of (E)).

Remark 3.4. In some previous works (see [16], [14]) a different notion of solution of (CP) (respect.(FBVP) and (MBVP)) is introduced by means of the integral equality (3.19) (respect. (3.20) and (3.21)). Thus, following [14], a function u defined on $\bar{S}$ is said to be a weak solution of (CP) if u satisfies i) and iii) of the Definition 1.1 as well as the condition

$$(3.29)\qquad \int_0^T \int_{-\infty}^{\infty} \{\theta_x[\phi(u)_x + b(u)] - \theta_t u\} dxdt = \int_{-\infty}^{\infty} \theta(0,x)u_o(x)dx$$

for every $\theta \in C^1(\bar{S})$ such that θ vanish for large $|x|$ and $t = T$. Analogously, it is defined the notion of weak solutions of (FBVP) and (MBVP) by substituting the integral conditions of Definitions 1.2 and 1.3 by the conditions (3.20) and (3.21) respectively. Theorem 3.2 state that, under some natural assumptions, every generalized solution is also a weak solution. The following result shows the equivalence between both notions of solution.

<u>Theorem 3.3. Assume</u> $\phi \in C^1([0,\infty))$ <u>and</u> $b \in C^0([0,\infty))$. <u>Then every weak solution of (CP) (resp. (FBVP) and (MBVP) is a generalized solution of (CP) (resp. (FBVP) and (MBVP)).</u>

<u>Proof</u>. We shall follow an idea suggested by M.G. Crandall to the first author of this paper. Let u be a weak solution of (CP) and let $P = [t_o,t_1] \times [x_1,x_2]$ and $\zeta \in C^{1,2}_{t,x}(P)$ such that $\zeta(t,x_1) = \zeta(t,x_2) = 0$ for any $t \in [t_o,t_1] \subset [0,T]$. Let $\eta \in C^2(\mathbb{R})$ be such that

a) $\eta(r) = 1$ if $r \leq -1$ and $\eta(r) = 0$ if $r \geq 0$

b) $\eta'(0) = \eta'(-1) = 0$.

For every $\varepsilon > 0$, we define the test function $\theta_\varepsilon(t,x)$ as

$$\theta_\varepsilon(t,x) = \begin{cases} \zeta(t,x)\eta(\frac{t-t_1}{\varepsilon})\eta(\frac{t_o-t}{\varepsilon})\eta(\frac{x-x_2}{\varepsilon})\eta(\frac{x_1-x}{\varepsilon}) & \text{if } (t,x) \in P \\ 0 & \text{otherwise} . \end{cases}$$

From the definition of η, it is immediate that $\theta_\varepsilon \in C^1(\bar{S})$ (and support $\theta_\varepsilon) \subseteq P$. By assumption we have

$$(3.29) \quad 0 = -\int_0^T \int_{-\infty}^{\infty} \theta_{\varepsilon,t} u\,dxdt + \int_0^T \int_{-\infty}^{\infty} \theta_{\varepsilon,x}\phi(u)_x dxdt + \int_0^T \int_{-\infty}^{\infty} \theta_{\varepsilon,x} b(u)dxdt$$
$$= I_{1,\varepsilon} + I_{2,\varepsilon} + I_{3,\varepsilon} .$$

One has

$$-I_{1,\varepsilon} = \iint_P \zeta_x\, u\, \eta(\frac{t-t_1}{\varepsilon})\eta(\frac{t_o-t}{\varepsilon})\eta(\frac{x-x_2}{\varepsilon})\eta(\frac{x_1-x}{\varepsilon})dxdt + \iint_P \frac{\zeta u}{\varepsilon}[\eta'(\frac{t-t_1}{\varepsilon})\eta(\ )\eta(\ )\eta(\ )]dxdt -$$

$$-\iint_P \frac{\zeta u}{\varepsilon}[\eta(\ )\,\eta'(\frac{t_o-t}{\varepsilon})\,\eta(\ )\eta(\ )]dxdt = \iint_P \zeta_t u\eta(\ )\eta(\ )\eta(\ )\eta(\ )dxdt +$$

$$+\int_{x_1}^{x_2}\int_{\frac{t_o-t_1}{\varepsilon}}^{0} \zeta(\varepsilon\tau+t_1,x)u(\varepsilon\tau+t_1,x)\ \eta(\frac{t_o,t_1}{\varepsilon}+\tau)\eta(\frac{x-x_2}{\varepsilon})\eta(\frac{x_1-x}{\varepsilon})\ \eta'(\tau)d\tau -$$

$$-\int_{x_1}^{x_2}\int_{\frac{t_o-t_1}{\varepsilon}}^{o} \zeta(t_o-\varepsilon\tau,x)u(t_o-\varepsilon\tau,x)\ \eta(\frac{t_o-t_1}{\varepsilon}-\tau)\eta(\frac{x-x_2}{\varepsilon})\eta(\frac{x_1-x}{\varepsilon})\eta'(\tau)d\tau .$$

Then when ε converges to zero, we obtain

$$-I_{1,\varepsilon} \dashrightarrow \iint_P \zeta_t u\,dxdt - \int_{x_1}^{x_2} \zeta(t_1,x)u(t_1,x)dx + \int_{x_1}^{x_2}\zeta(t_o,x)u(t_o,x)dx$$

Analogously

$$I_{2,\varepsilon} = \iint_P \theta_{\varepsilon,x}\phi(u)_x dxdt = \int_{t_o}^{t_1} \theta_{\varepsilon,x}\phi(u)\Big|_{x_1}^{x_2} dt - \iint_P \theta_{\varepsilon,xx}\phi(u)dxdt =$$

$$= \int_{t_o}^{t_1} \{(\zeta_x\, \eta(\,)\, \eta(\,)\eta(\,)\eta(\,)\phi(u)) + \zeta\, \eta(\frac{t-t_1}{\varepsilon})\eta(\frac{t_o-t}{\varepsilon})[\frac{1}{\varepsilon}\eta'(\frac{x-x_2}{\varepsilon})\eta(\frac{x_1-x}{\varepsilon}) -$$

$$- \frac{1}{\varepsilon}\,\eta(\frac{x-x_2}{\varepsilon})\eta'(\frac{x_1-x}{\varepsilon})\;]\;\Big|_{x_1}^{x_2} dt - \iint_P \theta_{\varepsilon,xx}\phi(u)dxdt = I^1_{2,\varepsilon} + I^2_{2,\varepsilon}\;.$$

Thanks to the fact that $\zeta\,(t,x_1) = \zeta\,(t,x_2) = 0$, one has

$$I^1_{1,\varepsilon} \longrightarrow \int_{t_o}^{t_1} (\zeta_x(t,x_2)\phi(u(t_1x_2)) - \zeta_x(t,x_1)\phi(u(t,x_1)))dt\;.$$

On the other hand

$$- I^2_{2,\varepsilon} = \iint_P \zeta_{xx}\eta(\,)\eta(\,)\eta(\,)\eta(\,)\phi(u)dxdt + 2\iint_P \zeta_x\eta(\,)\eta(\,)[\frac{1}{\varepsilon}\eta'(\frac{x-x_2}{\varepsilon})\eta(\frac{x_1-x}{\varepsilon})$$

$$- \frac{1}{\varepsilon}\,\eta(\frac{x-x_2}{\varepsilon})\eta'(\frac{x_1-x}{\varepsilon})]\phi(u)dxdt + \iint_P \zeta\eta(\;\;)\eta(\,)[\frac{1}{\varepsilon^2}\eta''(\frac{x-x_2}{\varepsilon})\eta(\frac{x_1-x}{\varepsilon})$$

$$- \frac{2}{\varepsilon^2}\,\eta'(\frac{x-x_2}{\varepsilon})\eta'(\frac{x_1-x}{\varepsilon}) + \frac{1}{\varepsilon^2}\,\eta(\frac{x-s_x}{\varepsilon})\;\eta''(\frac{x_1-x}{\varepsilon})]\,\phi(u))dxdt\;.$$

Arguing similarly as in the integral $I_{1,\varepsilon}$, we obtain

$$\iint \zeta_x\eta(\,)\eta(\,)[\frac{1}{\varepsilon}\eta'(\frac{x-x_2}{\varepsilon})\eta(\frac{x_1-x}{\varepsilon}) - \frac{1}{\varepsilon}\eta'(\frac{x-x_2}{\varepsilon})\eta(\frac{x_1-x}{\varepsilon})]\,\phi(u)dxdt \longrightarrow \int_{t_o}^{t_1}(\zeta_x(t,x_2)\phi(u(t,x_2)) -$$

$$- \zeta_x(t,x_1)\phi(u(t,x_1))dt\;,$$

when $\varepsilon \to 0$. Moreover

$$I^{2,*}_{2,\varepsilon} = \frac{1}{\varepsilon^2}\iint_P \zeta\eta(\frac{t-t_1}{\varepsilon})\eta(\frac{t_o-t}{\varepsilon})\eta(\frac{x_1-x}{\varepsilon})\eta''(\frac{x-x_2}{\varepsilon})\phi(w)\,dxdt = \int_{t_o}^{t_1}\int_{\frac{x_1-x_2}{\varepsilon}}^{0}$$

$$\zeta(\frac{t,x_2+\varepsilon\tau}{\varepsilon\tau})\ \eta(\frac{t_o-t}{\varepsilon})\eta(\frac{x_1,x_2}{\varepsilon}) + \tau) \quad \tau\ \eta''(t)\phi(u(t,x_2+\varepsilon\tau))dxdt$$

and then

$$I^{2,*}_{2,\varepsilon} \xrightarrow{\varepsilon\to 0} \int_{t_o}^{t_1} (\zeta_x(t,x_2)\phi(u(t,x_2))\int_{-1}^{0} \tau\eta''(\tau)d\tau)dt = \int_{t_0}^{t_1}\zeta_x(t,x_2)\phi(u(t,x_2))dt$$

(we recall that $\eta'(0) = \eta'(-1) = 0$). We also remark that

$$1/\varepsilon^2 \iint_P \zeta(t,x)\eta(\frac{t-t_1}{\varepsilon})\eta(\frac{t_o-t}{\varepsilon})\eta'(\frac{x-x_2}{\varepsilon})\eta'(\frac{x_1-x}{\varepsilon})\phi(u(t,x))dxdt = 0$$

for every $\varepsilon > 0$ such that $0 < \varepsilon < \frac{x_2-x_1}{2}$. Then

$$I_{2,\varepsilon} \xrightarrow{\varepsilon\to 0} \int_{t_o}^{t_1} \zeta_x\phi(u)\Big|_{x=x_1}^{x=x_2}\ dt\ - \iint_P \zeta_{xx}\phi(u)dxdt - 2\int_{t_o}^{t_1}\zeta_x\phi(u)\Big|_{x=x_1}^{x=x_2}\ dt$$

$$= \iint_P \zeta_{xx}\phi(u)dxdt - \int_{t_o}^{t_1} \zeta_x\ \phi(u)\Big|_{x=x_1}^{x=x_2} dt .$$

Finally, in a similar way we obtain

$$I_{3,\varepsilon} \xrightarrow{\varepsilon\to 0} \iint_P \zeta_x b(u)dxdt.$$

Then, making $\varepsilon \to 0$ in (3.26) we obtain that $-I(u,\zeta,P) = 0$ and then u is a generalized solution of (CP). The cases of the problems (FBVP) and (MBVP) are similar. ∎

§4. Uniqueness, comparison results and continuous dependence

In this section we prove that the generalized solution obtained in Proposition 2.1 and Theorem 3.1 (i.e. the limit solution) is the unique generalized solution.

Our uniqueness result will be a consequence of some L^1- estimates that also prove the continuous dependence of the solutions on the data. Other important consequences of these L^1-estimates are the comparison results showing the monotone dependence of the solutions with respect to the data.

To formulate general results about the comparison of solutions we introduce the following definition:

Definition 4.1. Let G be a closed set of $\bar{S}$. A function $v(t,x)$ defined on G is a generalized supersolution (resp. subsolution) of the equation (E) in G if

a) v is nonnegative, bounded and continuous

b) v satisfies the integral inequality

$$I(v,\zeta,P) \leq 0 \quad (\text{resp. } \geq 0)$$

(I give in the Definition 1.1) for any rectangle $P = [t_o, t_1] \times [x_1, x_2]$, $P \subset \overset{\circ}{G}$ and for all $\zeta \in C^{1,2}_{t,x}(P)$ such that $\zeta(t,x_1) = \zeta(t,x_2) = 0$ for any $t \in [t_o,t_1]$.

In this section we shall assume the following hypotheses:

$$(H_\phi)\begin{cases} \phi \in C^1([0,\infty))\cap C^2((0,\infty)), \phi(0) = \phi'(0) = 0 \text{ and there exists a convex function} \\ \mu \in C^0([0,\infty))\cap C^2((0,\infty)) \text{ such that } \mu(o) = 0 \text{ and } 0 < \mu'(r) \leq \phi'(r) \text{ for } r > 0 . \end{cases}$$

$$(H_b)\begin{cases} b \in C^0([0,\infty)) \cap C^2((0,\infty)), \liminf\limits_{r\to 0^+} b'(r) > -\infty \text{ and} \\ \limsup\limits_{r\to 0^+} b''(r) < +\infty \text{ if } \limsup\limits_{r\to 0^+} b'(r) = +\infty . \end{cases}$$

We remark that (H_ϕ) obviously holds if ϕ is a convex function and (H_b) is trivially verified if $b \in C^1([0,\infty))$ (no condition on b'' is requested in that case). On the other hand, if $b(s) = s^\lambda$, $\lambda \in \mathbb{R}$, then (H_b) is satisfied if $\lambda \geq 0$.

We start considering the (CP) problem. The main result of this section is the following:

Theorem 4.1 Assume (H_ϕ) and (H_b) or (H_{-b}).

Let u be a limit solution of (CP) continuous on $\bar{S}$ and let $\bar{u}$ (resp. $\underline{u}$) be a generalized supersolution (resp. subsolution) of (E) on $G = \bar{S}$. Then for every $0 \leq t \leq T$ we have

(4.1) $\int_{-\infty}^{\infty}(u(t,x)-\bar{u}(t,x))^+dx \leq \int_{-\infty}^{\infty}(u(0,x)-\bar{u}(0,x))^+dx$

(resp. $\int_{-\infty}^{\infty}(\underline{u}(t,x)-u(t,x))^+dx \leq \int_{-\infty}^{\infty}(\underline{u}(0,x)-u(0,x))^+dx$), where $r^+ = \max\{r,0\}$.

As a first consequence of the above result we can state our main result about uniqueness.

Theorem 4.2. Assume (H_ϕ) and (H_b) or (H_{-b}).

Let $u_o \in C_b(-\infty,\infty)$, $u_0 \geq 0$. Then under one of the following hypotheses there exists an unique generalized solution of (CP):

1) (3.1) is satisfied and $\phi(u_0)$ is Lipschitz continuous.

2) (3.2),(3.3) (3.4) are satisfied and $f^{-1}(u_0)$ is Lipschitz continuous (f defined in (3.5)).

Before giving the proof of the above result let us make some remarks. First of all, we recall that by Theorem 3.3 every "weak solution" (see the definition in Remark 3.4) is a generalized solution. Then, by means of the regularity shown in Theorem 3.2, Theorem 4.2 gives automatically the uniqueness of weak solutions, improving the knowledge in the literature about equation (E) (see the Introduction). Secondly, if we consider the particular case of $\phi(s) = s^m$ and $b(s) = s^\lambda$ (i.e. (E) coincides with $(E_{m,\lambda})$) then, for adequate data, Theorem 4.2 shows the uniqueness of generalized (and weak) solutions under the following restrictions:

$$m \geq 1 \; , \; \lambda > 0 \; .$$

In particular the uniqueness of solutions for the evaporation type problems $(\lambda \in (0,1))$ follows.

Other consequences of Theorem 4.1 will be commented upon later.

Proof of Theorem 4.2. Under the assumptions of the theorem, we know the existence of at least one limit solution of the problem. Moreover, this limit solution is continuous (see [14] and Theorem 3.1). Then, if $\hat{u}$ is another generalized solution of (CP), we can obviously apply the estimate (4.1) and then $u \leq \hat{u}$ on $\bar{S}$. Analogously $\hat{u}$ is also a generalized subsolution of (E) on $\bar{S}$ and the dual estimate of (4.1) implies that $\hat{u} \leq u$ on $\bar{S}$. In conclusion $u = \hat{u}$.

Proof of Theorem 4.1. Let $u = \lim_{k\to\infty} u_k$ be the limit solution of (CP) obtained in Proposition 2.1. i.e. $\{u_k\}$ are classical solutions of (E) on the sets $Q_k = (0,T)x(-k-1,k+1)$. We start approximating u by classical solutions of (E) but defined on full set $S = (0,T) \times (-\infty,+\infty)$. To do this, we construct a sequence of functions $\{u_{o,j}(x)\}$ such that i) $u_{o,j} \in C^\infty(\mathbb{R})$, ii) $u_{o,j}(x) \to u(0,x)$ as $j \to \infty$, uniformly on every bounded interval of $\mathbb{R}$, iii) $u_{o,j+1}(x) \leq u_{o,j}(x)$ for all $j \geq 1$ and $x \in \mathbb{R}$ and iv) $0 < \varepsilon_j \leq u_{o,j}(x) \leq M$ for all $j \geq 1$ and $x \in \mathbb{R}$, being $\varepsilon_j \to 0$ as $j \to +\infty$ and $M = \sup u_o$. Now, by applying Proposition 2.1 to the case of $u_o = u_{o,j}$, we obtain a sequence u_j of classical solutions of (E) on S such that $\varepsilon_j \leq u_j(t,x) \leq M$ for every $j \geq 1$ and u_j is monotone nonincreasing in $\bar{S}$. Finally $u(t,x) = \lim_{j\to\infty} u_j(t,x)$ in $\bar{S}$.

We shall prove estimate (4.1) by showing the inequality

$$(4.2)\quad \int_{-\infty}^{\infty} (u(t,x)-\bar{u}(t,x))^{+}\omega(x)dx \leq \int_{-\infty}^{\infty} (u(0,x)-\bar{u}(0,x))^{+}dx$$

for every $\omega \in C_0^{\infty}(\mathbb{R})$, $0 \leq \omega \leq 1$. To do this, we suppose that (supp ω) $= [-L,L]$. For every $t^* \in (0,T]$, let $P \equiv (0,t^*) \times (-r,r)$ be where $r > L + 1$. Let $\zeta \in C^{1,2}_{t,x}(P)$ such that $\zeta(t,-r) = \zeta(t,r) = 0$ for all $t \in [0,t^*]$. Then $I(u_j,\zeta,P) - I(\bar{u},\zeta,P) \geq 0$, i.e.

$$(4.3)\quad \int_{-r}^{r}(u_j(t^*,x)-\bar{u}(t^*,x))\zeta(t^*,x)dx \leq \int_{-r}^{r}(u_j(0,x)-\bar{u}(0,x))\zeta(0,x)dx +$$

$$-\int_0^{t^*}[\phi(u_j(t,r))-\phi(\bar{u}(t,r))]\zeta_x(t,r)dt + \int_0^{t^*}[\phi(u_j(t,-r))-\phi(\bar{u}(t,-r))]\zeta_x(t,-r)dt$$

$$+ \iint_P (u_j-\bar{u})(\zeta_t+A^j\zeta_{xx}-B^j\zeta_x)dxdt$$

where

$$(4.4)\quad A^j = A^j(t,x) = \int_0^1 \phi'(\theta u_j(t,x) + (1-\theta)\bar{u}(t,x))d\theta$$

and

$$(4.5)\quad B^j = B^j(t,x) = \int_0^1 b'(\theta u_j(t,x) + (1-\theta)\bar{u}(t,x))d\theta\ .$$

By assumption (H_ϕ) and the properties of u_j we have

$$(4.6)\quad 0 < \frac{1}{\varepsilon_j}\,\mu(\varepsilon_j) \leq A^j(t,x) \leq M_1$$

for every $(t,x) \in P$ and for some M_1 independent on j. On the other hand, thanks to hypothesis (H_b) there exist two real numbers M_2 and M_3, (M_2 independent of j), such that

$$(4.7)\quad M_2 \leq B^j(t,x) \leq M_3(j)$$

for every $(t,x) \in P$. Indeed, if $-\infty < \liminf_{s\to 0^+} b'(s) \leq \limsup_{s\to 0^+} b'(s) < +\infty$, there exist M_2 and M_3 (both independent of j) such that $M_2 \leq b'(s) \leq M_3$ for every $s \in [0,M]$ and then (4.9) is obvious. If $\limsup_{s\to 0^+} b'(s) = +\infty$, then by the second part of (H_b), there exist M_2 and M_3^* (both independent of j) such that

$$M_2 < b'(s) \quad \text{and} \quad b''(s) < M_3^*$$

for every $s \in (0,M]$. Therefore

$$M_2 < \int_0^1 b'(\theta u_j + (1-\theta)\bar{u})d\theta < \int_0^1 b'(\theta\varepsilon_j)d\theta + \int_0^1 M_3^*(\theta(u_j-\varepsilon_j) + (1-\theta)\bar{u})d\theta$$

$$< \frac{1}{\varepsilon_j} b(\varepsilon_j) + |M_3^*|,M \ .$$

Then (4.7) holds with $M_3(j) = \frac{1}{\varepsilon_j} b(\varepsilon_j) + |M_3^*|M$.

Analogously, if we suppose (H_{-b}) we can find two real numbers $M_2(j)$ and M_3 (M_3 independent on j) such that $M_2(j) < B^j(t,x) < M_3$ for every $(t,x) \in P$. Hence, in any case, we can assume that $M_2(j) < B^j(t,x) < M_3(j)$ for every $(t,x) \in P$.

Define now, on $P = P_r$, two sequences of smooth functions, $\{A_n^{j,r}\}_{n=1}^\infty$ and $\{B_n^{j,r}\}_{n=1}^\infty$, satisfying

> $\{A_n^{j,r}\}$ is monotonically decreasing on n and converges uniformly to A^j, on P_r (when $n \to +\infty$).
>
> $\{B_n^{j,r}\}$ *is e.g. monotonically* increasing on n and converges uniformly to B^j, on P_r (when $n \to +\infty$).

Then, by (4.6) and (4.7) we have

$$0 < \frac{1}{\varepsilon_j} \mu(\varepsilon_j) < A_n^{j,r} < M_1$$

and

$$M_2 < B_n^{j,r} < M_3(j).$$

On the other hand, inequality (4.3) can be written in the following way:

(4.8) $\int_{-r}^{r} (u_j(t^*,x) - \bar{u}(t^*,x))\zeta(t^*,x)dx < \int_{-r}^{r}(u_j(0,x) - \bar{u}(0,x))\zeta(0,x)dx +$

$$+ \int_0^{t^*} [\phi(u_j(t,-r)) - \phi(\bar{u}(t,-r))]\zeta_x(t,-r)dt - \int_0^{t^*} [\phi(u_j(t,r)) - \phi(\bar{u}(t,r))]\zeta_x(t,r)dt$$

$$+ \iint_{P_r} (A^j-A_n^{j,r})(u_j-\bar{u})\zeta_{xx}dxdt + \iint_{P_r} (B_n^{j,r}- B^j)(u_j-\bar{u})\zeta_x dxdt +$$

$$+ \iint_{P_r} (A_n^{j,r}\zeta_{xx} + \zeta_t - B_n^{j,r}\zeta_x)(u_j-\bar{u})\,dxdt .$$

Now, let $\zeta = \zeta_n^{j,r}$ be the classical solution of the linear parabolic problem

$$(4.9)\qquad \begin{cases} \mathcal{L}\zeta \equiv A_n^{j,r}\zeta_{xx} - B_n^{j,r}\zeta_x + \zeta_t = 0 & \text{on } P_r \\ \zeta(t^*,x) = \omega(x)\,\chi(x) & \text{on } (-r,r) \\ \zeta(t,-r) = \zeta(t,r) = 0 & \text{on } (0,t^*) \end{cases}$$

where χ is a given function such that $\chi \in C_0^\infty(\mathbb{R})$ and $0 \le \chi \le 1$ (The existence and uniqueness of ζ is a well-known result (see [25])). One of the crucial points in the present proof is based on the following estimates of the solution of (4.9).

Lemma 4.1. *Let ζ be the solution of (4.9). Then*

i) $0 \le \zeta(t,x) \le \max|\omega(x)\,\chi(x)| \le 1,$ *for all* $(t,x) \in P_r$

ii) *There exists* $M_4 = M_4(j)$ *such that*

$0 \le \zeta(t,x) \le M_4(j)\, e^{-|x|},$ *for all* $(t,x) \in \overline{P_r}$.

iii) *There exists* $M_4 = M_5(j)$ *such that*

$\max\{|\zeta_x(t,r)|, |\zeta_x(t,-r)|\} \le M_5(j)\, e^{-r},$ *for all* $t \in [0,t^*]$.

iv) *There exists* $M_6 = M_6(j)$ *such that*

$|\zeta_x(t,x)| \le M_6(j)$ *for all* $(t,x) \in P_r$

v) *There exists* $M_7 = M_7(j,r,t^*)$ *such that*

$\int_0^{t^*}\int_{-r}^{r}(\zeta_{xx})^2\,dxdt \le M_7(j,r,t^*)$ *for all* $(t,x) \in P_r$.

Proof of Lemma 4.1. - We shall follow some of the ideas introduced in [28]. i) is a consequence of the maximum principle. To prove ii) let us consider the function $w = z - \zeta$, where

$$z(t,x) = C\exp(-x + \beta(t^*-t)).$$

where C and β will be chosen later. Let $P_r^+ = (0,t^*)\times(0,r)$. then we have

$$\mathcal{L}w \equiv \exp(-x+\beta(t^*-t))(A_n^{j,r} - \beta) \le \exp(-x+\beta(t^*-t))\{M_1 + M_3(j) - \beta\} < 0$$

if $\beta > M_1 + M_3(j)$.

$$w(t^*,x) = C\,e^{-x} - \omega(x)\chi(x) \ge 0, \quad \text{for every } x \in [0,r]$$

if $C\ e^{-L} - 1 \ge 0$ i.e. $C \ge e^L$.

$$w(t,0) = e^{\beta(t^*-t)} \ge 0, \quad \text{for every } t \in [0,t^*],$$

$$w(t,r) = C\exp(-r+\beta(t^*-t)) \ge 0, \quad \text{for every } t \in [0,t^*] .$$

Hence, by using the maximum principle we have

$$0 < \zeta\,(t,x) < e^{\beta(t^*-t)}e^{-x} < M_4^1(j)e^{-x}$$

on P_r^+, being

$$M_4^1(j) = e^L e^{t^*(M_1+M_3(j)+1)} .$$

On the set $P_r^- = [0,t^*]x[-r,0]$ we use the auxiliar function $w = z-\zeta$, where now

$$z(t,x) = C \exp\,(x+\beta(t^*-t)) .$$

Then we obtain

$$0 < \zeta(t,x) < M_4^2(j)e^x$$

on P_r^-, with

$$M_4^2(j) = e^L e^{t^*(M_1-M_2(j)+1)} .$$

This prove ii) for $M_4(j) = \max\,\{M_4^1(j),M_4^2(j)\}$. (We remark that if M_2 and M_3 are independent on j, the same holds for M_4).

In order to prove iii), we define the function

$$w(t,x) = e^{-r+1} \exp \beta(x-r+1) - \zeta(t,x)$$

for some β to be chosen. Consider the cylinder $P(r-1,r) = (0,t) \times (r-1,r)$. Then we have

$$\mathcal{L}w \equiv e^{-r+1} \exp \beta(x-r+1)\{\beta^2 A_n^{j,r} - \beta B_n^{j,r}\} > e^{-r+1} \exp \beta(x-r+1)\{\beta \frac{2\mu(\varepsilon_j)}{\varepsilon_j} -\beta M_3(j)\} > 0$$

if $\beta > \max\{\frac{M_3(j)\varepsilon_j}{\mu(\varepsilon_j)} , 1\}$,

$$w(t,r-1) = e^{-r+1} - \zeta(t,r-1) < C\, e^{-r+1}$$

$$w(t,r) = e^{-r+1}e^{\beta}$$

$$w(t^*,x) = e^{-r+1}e^{\beta(x-r+1)} \text{ (we recall that } r > L) .$$

Then, $w(t,x)$ attains the positive maximum $e^{-r+1}e^{\beta}$ at (t,r). Hence

$$\zeta_x(t,r-0) < e^{-r+1}\beta e^{\beta} = M_5^1\,(j)\; e^{-r} \quad \text{for} \quad M_5^1 = \beta(j)e^{\beta(j)+1} .$$

Now, if we consider the function

$$w(t,x) = e^{-r+1} \exp\,\beta(x-r+1) + \zeta(t,x) ,$$

we have $\zeta_x(t,r-0) > - M_5^1(j)e^{-r}$. Finally, by using the auxiliary functions

$$w(t,x) = e^{-r+1} \exp\,\beta(x+r-1) \pm \zeta(t,x)$$

on the set $P(-r,-r+1) = (0,t^*) \times (-r,-r+1)$, for some suitable β we obtain $|\zeta_x(t,-r+0)| \le M_5^2(j)\, e^{-r}$ for some $M_5^2(j)$. This proves iii) for $M_5(j) = \max\{M_5^1(j), M_5^2(j)\}$.

Part iv) is a consequence of the fact that the coefficents $A_n^{j,r}$ and $B_n^{j,r}$ are bounded independently of n and r. Indeed, in these circumstances we can apply the results of the classic theory of linear parabolic equations (see [25]). Finally, to show v) we multiply the equation in (4.9) by ζ_{xx} and we integrate. Then

$$(4.10)\quad \int_0^{t^*}\int_{-r}^{r} A_n^{jr}\,(\zeta_{xx})^2 dxdt = -\int_0^{t^*}\int_{-r}^{r}\zeta_t\zeta_{xx}dxdt + \int_0^{t^*}\int_{-r}^{r} B_n^{jr}\zeta_x\zeta_{xx}dxdt = I_1 + I_2 .$$

Integrating by parts, it results

$$I_1 = \int_0^{t^*}\int_{-r}^{r}\zeta_{tx}\zeta_x = \tfrac{1}{2}\int_{-r}^{r}\left(\frac{d}{dx}(\omega\ \chi(x))\right)^2 dx - \tfrac{1}{2}\int_{r}^{r}(\zeta_x(0,x))^2 dx \le \tfrac{1}{2}\int_{-L}^{L}\frac{d}{dx}(\omega\ \chi(x))^2 dx$$

$$= M_7^1 .$$

On the other hand,

$$I_2 \le \left[\int_0^{t^*}\int_{-r}^{r}(B_n^{jr}\zeta_x)^2 dxdt\right]^{1/2}\left[\int_0^{t^*}\int_{-r}^{r}(\zeta_{xx})^2 dxdt\right]^{1/2} \le (t^*\ 2r)^{1/2}\max\{|M_2(j)|,|M_3(j)|\}$$

$$M_6(j)\ \left[\int_0^{t^*}\int_{-r}^{r}(\zeta_{xx})^2 dxdt\right]^{1/2} = M_7^2(j,r,t^*)\left[\int_0^{t^*}\int_{-r}^{r}(\zeta_{xx})^2 dxdt\right]^{1/2}.$$

Therefore, from (4.10) we deduce

$$\int_0^{t^*}\int_{-r}^{r}(\zeta_{xx})^2 dxdt \le \frac{\varepsilon_j}{\mu(\varepsilon_j)}\left[M_7^1 + M_7^2(j,r,t^*)\left(\int_0^{t^*}\int_{-r}^{r}(\zeta_{xx})^2 dxdt\right)^{1/2}\right].$$

This ends the proof. ∎

<u>Proof of Theorem 4.1 (continued)</u>. By substituting the solution $\zeta = \zeta_n^{j,r}$, solution of (4.9), in the expression (4.8) and applying Lemma 4.1 we have

$$(4.11)\quad \int_{-r}^{r}(u_j(t^*,x)-\bar{u}(t^*,x))\omega(x)\chi(x)dx \le \int_{-r}^{r}(u_j(0,x)-\bar{u}(0,x))^+ dx\ +$$

$$+ t^* M_6(j)e^{-r}\{ \max_{0<t<t^*} |\phi(u_j(t,r))-\phi(\bar{u}(t,r))| + \max_{0<t<t^*} |\phi(u_j(t,-r))-\phi(\bar{u}(t,-r))|\} +$$

$$+ \max_{\bar{P}r} |A^j-A_n^{j,r}| \max_{\bar{P}_r} |u_j-u| M_7(j,r,t^*) + \max_{\bar{P}r} |B_n^{j,r}-B^j| \max_{\bar{P}_r} |u_j-\bar{u}| \; 2t^*rM_6(j) .$$

By taking limits, first with respect to n $(n \to +\infty)$ and then with respect to r $(r \to \infty)$, we obtain

(4.12) $$\int_{-\infty}^{\infty} (u_j(t^*,x)-\bar{u}(t^*,x))\omega(x)\chi(x)dx \le \int_{-\infty}^{\infty} (u_j(0,x)-\bar{u}(0,x))^+dx$$

(we recall that $|u_j-u| \le M$ and that $|\phi(u_j)-\phi(u)| \le \phi(M)$). Letting, now j diverge to infinity, we have

(4.13) $$\int_{-\infty}^{\infty} (u(t^*,x)-\bar{u}((t^*,x))\omega(x)\chi(x)dx \le \int_{-\infty}^{\infty} (u(0,x)-\bar{u}(0,x))^+dx .$$

Finally, relation (4.13) is also true for the function χ given by $\chi(x) = 1$ on the set $\{x : u(t^*,x) > \bar{u}(t^*,x)\}$ and $\chi(x) = 0$ otherwise. (Indeed: it suffices to approximate the function χ by $\chi_m \in C_0^\infty(\mathbb{R})$ and then passing to the limit on m). This concludes the proof of (4.2). Finally, if $\underline{u}$ is a subsolution of (E) on $\bar{S}$, by an analogous argument we obtain

$$\int_{-\infty}^{\infty} (\underline{u},(t,x)-u(t,x))^+\omega(x)dx \le \int_{-\infty}^{\infty} (\underline{u}(o,x)-u(o,x))^+dx$$

for every $\omega \in C_o^\infty(\mathbb{R})$, $0 \le \omega \le 1$, and the proof of Theorem 4.1 is finished. ∎

For the problems (MBVP) and (FBVP) our answers are similar to theorem 4.1 but the proof is somewhat more delicate.

<u>Theorem 4.3</u>. <u>Assume</u> (H_ϕ) and (H_b) or (H_{-b}).

a) <u>Let u be a limit solution of (FBVP) continuous on $\bar{R}$. Let $\bar{u}$(resp. $\underline{u}$) be a generalized supersolution (resp. subsolution) of (E) on $G = \bar{R}$ such that</u>

$$\psi_-(t) \le \bar{u}(t,l_1) \; , \; \psi_+(t) \le \bar{u}(t,l_2)$$

(<u>resp.</u> $\psi_-(t) \ge \underline{u}(t,l_1)$, $\psi_+(t) \ge \underline{u}(t,l_2)$) <u>for every</u> $t \in [0,T]$. <u>Then</u>

(4.19) $$\int_{l_1}^{l_2} (u(t,x)-\bar{u}(t,x))^+dx \le \int_{l_1}^{l_2} (u(0,x)-\bar{u}(0,x))^+dx$$

(<u>resp.</u> $\int_{l_1}^{l_2} (\underline{u}(t,x)-u(t,x))^+dx \le \int_{l_1}^{l_2} (u(0,x)-u(0,x))^+dx)$.

b) <u>Let u be a generalized solution of (MBVP) continuous on $\bar{H}$. Let $\bar{u}$ (resp. u be</u>

a generalized supersubsolution (resp. subsolution of (E) on $G = \bar{H}$. such that

$$\psi(t) \le \bar{u}(t,l_2)$$

(resp. $\psi(t) \ge \underline{u}(t,l_2)$) for every $t \in [0,T]$. Then

$$(4.15)\quad \int_{-\infty}^{l_2}(u(t,x)-\bar{u}(t,x))^+dx \le \int_{-\infty}^{l_2}((u(0,x)-\bar{u}(0,x))^+dx$$

$$(\text{resp. } \int_{-\infty}^{l_2}(\underline{u}(t,x)-u(t,x))^+dx \le \int_{-\infty}^{l_2}(\underline{u}(0,x)-u(0,x))^+dx) .$$

About the uniqueness question we have

Theorem 4.4. Assume (H_ϕ) and (H_b) or (H_{-b}) .

a) Let $u_o \in C_b([l_1,l_2])$, $u_o \ge 0$ and ψ_-, $\psi_+ \in C([0,T])$, ψ_-, $\psi_+ \ge 0$, satisfy $\psi_-(0) = u_o(l_1)$, $\psi_+(0) = u_o(l_2)$. Then, under one of the following hypotheses there exists an unique generalized solution of (FBVP):

1) (3.1) is satisfied $\phi(u_o)$ is locally Lipschitz continuous on (l_1,l_2) and $\phi(\psi_+),\phi(\psi_-)$ are absolutely continuous on $[0,T)$.

2) (3.2),(3.3) and (3.4) are satisfied and $f^{-1}(u_o)$ is locally Lipschitz continuous on (l_1,l_2).

b) Let $u_o \in C_b((-\infty,l_2])$, $u_o \ge 0$ and $\psi \in C([0,T])$, $\psi \ge 0$. satisfy $\psi(0) = u_o(l_2)$. Then under one of the following assumptions there exists an unique generalized solution of (MBVP):

1) (3.1) is satisfied $\phi(u_o)$ is Lipschitz continuous on $(-\infty,l_2-\delta)$ for every $\delta > 0$ and $\phi(\psi)$ is absolutely continuous on $[o,T]$.

2) (3.2), (3.3) and (3.4) are satisfied and $f^{-1}(u_o)$ is Lipschitz continuous on $(-\infty,l_2-\delta)$ for every $\delta > 0$.

Proof of Theorem 4.3 a) Let u be a limit solution of (FBVP) continuous on $\bar{R}$ and let $\bar{u}$ be a generalized supersolution of (E) on $G = \bar{R}$ such that

$$\psi_-(t) \le \bar{u}(t,l_1) \quad\text{and}\quad \psi_+(t) \le \bar{u}(t,l_2)$$

for every $t \in [0,T]$. Let $P \equiv (0,t^*) \times (l_1,l_2)$. Then if $u = \lim u_j$ we obtain as in (4.8), the following

$$(4.16) \quad \int_{l_1}^{l_2} (u_j(t^*,x)-\bar{u}(t^*,x))\zeta(t^*,x)dx \le \int_{l_1}^{l_2} (u_j(o,x)-\bar{u}(o,x))\zeta(o,x)dx +$$

$$+ \int_o^{t^*} (\phi(u_j(t,l_1))-\phi(\bar{u}(t,l_1))\zeta_x(t,l_1)dt - \int_o^{t^*} (\phi(u_j(t,l_2))-\phi(\bar{u}(t,l_2)))\zeta_x(t,l_2)dt +$$

$$+ \iint_P (A^j-A_n^j\ (u_j-\bar{u})\zeta_{xx}dxdt + \iint_P (B_n^j-B^j)(u_j-\bar{u})\zeta_x dxdt +$$

$$+ \iint_P (A_n^j\zeta_{xx} + \zeta_t-B_n^j\ \zeta_x)(u_j-\bar{u})dxdt,$$

where $\{A_n^j\}$ and $\{B_n^j\}$ are two sequences of smooth functions as in the proof of Theorem 4.1. Now define $\zeta = \zeta_n^j$ to be the classical solution of (4.9) after substituting $A_n^{j,r}$, $B_n^{j,r}$ and P_r by A_n^j, B_n^j and P respectively. Our intention is to pass to the limit in (4.16) first with respect to n and afterwards with respect to j. To do this we need to distinguish two different cases:(1)

a_1) $\bar{u}(t,l_1) > 0$ and $\bar{u}(t,l_2) > 0$ for every $t \in [0,T]$

a_2) $\bar{u}(t_o,l_1) = 0$ or $\bar{u}(t_o,l_2) = 0$ for some $t_o \in [0,T]$.

If a_1) takes place then we can choose $\psi_{-,j}$ and $\psi_{+,j}$ such that

$$(4.17) \quad \varepsilon_j < \psi_{-,j}(t) < \bar{u}(t,l_1) \quad \text{and} \quad \varepsilon_j < \psi_{+,j}(t) < \bar{u}(t,l_2)$$

for every $t \in [0,T]$. Thus, remarking that $\zeta_x(t,l_1) > 0$ and $\zeta_x(t,l_2) < 0$ for every $t \in [0,t^*]$ we obtain the conclusion after passing to the limit in n and j respectively as in the proof of Theorem 4.1.

It is clear that (4.17) cannot be possible in general (for instance if $\bar{u}(t_o,l_1) = 0$ or $\bar{u}(t_o,l_2) = 0$). Now we shall obtain estimates on $\zeta_x(t,l_1)$ and $\zeta_x(t,l_2)$ which are sharper then those stated in Lemma 4.1.

__Lemma 4.2__. __Assume__ (H_ϕ) and (H_b) or (H_{-b}). __Let__ ζ __be the solution of__

$$(4.18) \quad \begin{cases} \mathcal{L}\zeta \equiv A_n^j\ \zeta_{xx}-B_n^j\ \zeta_x + \zeta_t = 0 & \underline{\text{on}}\ P \\ \zeta(t^*,x) = \omega(x)\ \chi(x) & \underline{\text{on}}\ (l_1,l_2) \\ \zeta(t,l_1) = \zeta(t,l_2) = 0 & \underline{\text{on}}\ (0,t^*) \end{cases}$$

__where__ χ __is a given function such that__ $\chi \in C_o^\infty(l_1,l_2)$ __and__ $0 \le \chi \le 1$. __Then there exist two constants__ $M_8(j)$ and $M_9(j)$ __such that__

(1) The authors wish to thank M. Berstch for pointing out some omissions at this point of the proof in a preliminary version of this paper.

(4.19) $0 < \zeta_x(t,l_1) < M_8(j)$,

(4.20) $M_9(j) < \zeta_x(t,l_2) < 0$

for every $t \in (o,t^*)$ Moreover

$$\phi(\varepsilon_j)M_8(j) \to 0 \quad \text{and} \quad \phi(\varepsilon_j)M_9(j) \to 0$$

when $j \to \infty$.

Proof. We shall prove (4.20), (4.19) being obtained in a similar way. To do this, we construct an adequate function $\sigma_j(x)$ in such a way that the function $w(t,x) = \sigma_j(x) + \zeta(t,x)$ has a positive maximum at (t,l_2). Then we shall deduce that $\zeta_x(t,l_2) > -\sigma_j'(l_2)$, that is, (4.20). Consider the cylinder $P(l_2-\delta,l_2) = (o,t) \times (l_2-\delta,l_2)$ for some $\delta > 0$ fixed. Then if $\sigma_j' > 0$ and $\sigma_j'' > 0$ we have

$$\mathcal{L}\,w \equiv A_n^j\,\sigma''_{(x)} - B_n^j\,\sigma'_{(x)} > \frac{\mu(\varepsilon_j)}{\varepsilon_j}\,\sigma''(x) - M_3(j)\sigma'(x) = K_j^2 > 0$$

for some $K_j \in \mathbb{R}$ if we choose

$$\sigma(x) = C_j\,\frac{\mu(\varepsilon_j)}{M_3(j)\varepsilon_j}\,e^{\frac{M_3(j)\varepsilon_j}{\mu(\varepsilon_j)}x} - \frac{K_j^2}{M_3(j)}\,x + L_j$$

for every C_j and L_j satisfying

(4.21) $C_j \exp(\frac{M_3(j)\varepsilon_j}{\mu(\varepsilon_j)}(l_2-\delta)) - \frac{K_j^2}{M_3(j)} > 0$ ($\sigma' > 0$ condition)

and

(4.22) $L_j > \frac{K_j^2}{M_3(j)}\,l_2 - C_j\,\frac{\mu(\varepsilon_j)}{M_3(j)\varepsilon_j}\exp(\frac{M_3(j)\varepsilon_j}{\mu(\varepsilon_j)}\,l_2)$ ($w(t,l_2) > 0$ condition).

(We have used the estimates on A_n^j and B_n^j given in the proof of Theorem 4.1. It is clear that we may suppose $M_3(j) > 0$).

On the other hand, on the parabolic boundary of $P(l_2-\delta,l_2)$ we have

$$w(t,l_2) = \sigma_j(l_2)$$

$$w(t,l_2-\delta) < \sigma_j(l_2-\delta) + 1$$

$w(t^*,x) = \sigma_j(x)$ (if we choose δ such that $\omega(x)\chi(x) = 0$ for $x \in (l_2-\delta,l_2)$).

Then, as $\sigma_j' > 0$, w attains a positive maximum at (t,l_2) if we have $\sigma_j(l_2-\delta)_{+1} < \sigma_j(l_2)$ i.e. if K_j^2 and C_j satisfy

(4.23) $$\frac{K_j^2\delta}{M_3(j)} < C_j\,\frac{\mu(\varepsilon_j)}{M_3(j)\varepsilon_j}\exp(\frac{M_3(j)\varepsilon_j}{\mu(\varepsilon_j)}\,l_2)[1-\exp(-\frac{M_3(j)\varepsilon_j\delta}{\mu(\varepsilon_j)})] - 1 \,.$$

It is easy to see that if we choose

$$C_j = \exp\left(\frac{-M_3(j)\varepsilon_j l_2}{\mu(\varepsilon_j)}\right) \min \left\{1, \frac{1}{\frac{\mu(\varepsilon_j)}{M_3(j)\varepsilon_j} - \left(\delta + \frac{\mu(\varepsilon_j)}{M_3(j)\varepsilon_j}\right) \exp\left(-\frac{M_3(j)\varepsilon_j \delta}{\mu(\varepsilon_j)}\right)}\right\}$$

and

$$K_j^2 = \frac{M_3(j)}{\delta} \left(C_j \frac{\mu(\varepsilon_j)}{M_3(j)\varepsilon_j} \exp\left(\frac{M_3(j)\varepsilon_j}{\mu(\varepsilon_j)} l_2\right) - \frac{K_j^2}{M_3(j)}\right.$$

then (4.21) and (4.23) are satisfied. Thus $\zeta_x(t,l_2) \geq -\sigma_j'(l_2)$ being

$$\sigma_j'(l_2) = C_j \exp\left(\frac{M_3(j)\varepsilon_j}{\mu(\varepsilon_j)} l_2\right) - \frac{K_j^2}{M_3(j)} .$$

Now, the sequences $\left\{C_j \exp\left(\frac{M_3(j)\varepsilon_j l_2}{\mu(\varepsilon_j)}\right)\right\}$ and $\left\{\frac{K_j^2}{M_{3(j)}}\right\}$ are bounded and then $\phi(\varepsilon_j)\sigma_j'(l_2) \to 0$ when $j \to \infty$. ∎

<u>Proof of Theorem 4.3 (continued)</u>. Suppose that a_2 holds. Then if we denote $I = \{t \in (o,t^*) : \phi(\psi_{-,j}(t)) > \phi(\bar{u}(t,l_1))\}$ we have $I = \{t \in (o,t^*) : \bar{u}(t,l_1) = 0\}$ and then

$$\int_o^{t^*} (\phi(\psi_{-,j}(t)) - \phi(\bar{u}(t,l_1))\zeta_x(t,l_1)dt \leq \int_I (\phi(\psi_{-,j}(t)) - \phi(\bar{u}(t,l_1))\zeta_x(t,l_1)dt \leq$$
$$\leq t^* \phi(\varepsilon_j) M_8(j)$$

(because on I we can choose $\psi_{-,j}(t) = \varepsilon_j$). By Lemma 4.2 we have

$$\int_o^{t^*} (\phi(\psi_{-,j}(t)) - \phi(\bar{u}(t,l_1))\zeta_x(t,l_1)dt \to 0$$

when j converges to infinity. Similarly

$$\int_o^{t^*} \phi(\psi_{+,j}(t)) - \phi(\bar{u}(t,l_2))\zeta_x(t,l_2)dt \to 0$$

when j converges to infinity. Then the conclusion follows by passing to the limit in (4.16) in n and then in j.

We remark that in order to prove the conclusion for subsolutions it is not necessary to use Lemma 4.2, because in all cases we may choose $\psi_{-,j}$ and $\psi_{+,j}$ satisfying

$$\underline{u}(t,l_1) \leq \psi_{-,j}(t) \qquad \text{and} \qquad \underline{u}(t,l_2) \leq \psi_{+,j}(t)$$

for every $t \in [0,T]$.

The proof of part b) is an easy modification of the proofs of Theorem 4.1 and the above part a). ∎

The proof of Theorem 4.4 is analogous to that of Theorem 4.2.

Other important consequences of Theorems 4.1 and 4.3 are included in the following theorem, which shows continuous and monotone dependence of generalized solutions with respect to the initial data. (We shall consider only the (CP) problem, analogous statements holding for the others).

Theorem 4.5. Assume the hypotheses of Theorem 4.2.

i) *Let* $u, \hat{u}$ *be generalized solutions of (CP) corresponding to the initial data* u_o *and* $\hat{u}_o$ *respectively. Then*

(4.24) $$u(t,\ )-\hat{u}(t,\ )_{L^1(\mathbb{R})} \le u_o-\hat{u}_o{}_{L^1(\mathbb{R})}$$

for every $t \in [0,T]$.

ii) *Let* u *be a generalized solution of (CP) and* $\bar{u}, \underline{u}$ *generalized super- and subsolutions of (E) on* $G = \bar{S}$. *Then* if $\underline{u}(0,x) \le u_o(x) \le \bar{u}(0,x)$ on $(-\infty,\infty)$ *it follows that*

(4.25) $$\underline{u}(t,x) \le u(t,x) \le \bar{u}(t,x)$$

for every $(t,x) \in \bar{S}$.

Proof. The assertion i) follows from part a) of Theorem 4.1 by applying the estimates to $\bar{u} = \hat{u}$ and $\underline{u} = \hat{u}$. Part ii) is also a trivial consequence of such estimates. ∎

Other estimates giving the continuous dependence on the initial data as well as the numerical treatment of equation (E) for $b \in C^1([0,\infty))$ can be found in [33].

We shall end this section by making several comments on the obtained results.

Remark 4.1. The conclusions of Theorem 4.1 are true even under more general hypotheses. So, for the (CP) problem e.g., it is enough that u, $\bar{u}$, $\underline{u}$ be in the function space $C([0,T] : L^1_{loc}(\mathbb{R}))$. The existence of solutions of (CP) in such a function space is not difficult and some hypotheses on ϕ and b made in Theorem (3.1) can be weakened (See, e.g. the approach made in [2] considering a different nonlinear degenerated parabolic equation).

Remark 4.2. If we denote by $S(t)u_o = u(t,\ )$ the generalized solution of (CP) corresponding to the initial datum u_o it is not difficult to show that $S(t)$ is a semigroup. The estimate (4.24) shows that it is a semigroup of contractions on the space

$X = L^1(\mathbb{R})$. Our conclusion, then, coincides with the one obtained by the abstract theory of accretive operators on Banach spaces and evolution equations. Such an approach has been applied to the concrete case of equation (E) by different authors (see [38], [37] and [5]). We remark that, by means of such an approach, it is possible to prove the existence and uniqueness of a function satisfying (CP) in an adequate sense. This is made under very general assumptions on ϕ, b and u_o (see [5]). Such type of solution is, in fact, a generalized solution of (CP) under hypotheses weaker than that the one in Theorem 4.1. However, the abstract approach does not guarantee the continuity nor the uniqueness (among all the possible generalized solutions) of such a function.

Remark 4.3. There exists a vast literature about the existence and uniqueness of solutions of (CP) when function ϕ is not assumed to be strictly increasing on $\mathbb{R}^+$. It is clear that the approach is very different from ours. Indeed, such an approach includes the case $\phi \equiv 0$ and then equation (E) reduces to the "conservation law" equation

$$u_t - b(u)_x = 0$$

for which it is well known the existence of discontinuous solutions. The uniqueness of solutions is then found by introducing a different notion of generalized solutions of (CP) (see, e.g. [36], [23], [12], [38], [39] and [26]).

Remark 4.4. - Comparison results like the one in part ii) of Theorem 4.5 are of a great utility in the study of the qualitative properties of solutions (see e.g. [19], [15], [9], [21] and [22]). In a forthcoming paper by the authors, Theorem 4.5 will be systematically used to derive some qualitative properties of the solutions of the evaporation equation ($E_{m,\lambda}$, $m > 1$, $0 < \lambda < 1$).

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We prove the existence, continuity and uniqueness of solutions of the Cauchy problem and of the first and mixed boundary value problems for the equation

$$u_t = \phi(u)_{xx} + b(u)_x. \qquad (E)$$

ABSTRACT (Continued)

ϕ and b are assumed to belong to a large class of functions including the particular cases $\phi(u) = u^m$, $b(u) = u^\lambda$; $m > 1$ and $\lambda > 0$. These results significantly sharpen those currently available in the substantial literature devoted to (E) over the last two decades. In particular, the uniqueness is proved in a generality which allows (E) to model problems invoking the evaporation of a fluid through a porous medium.

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