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PERIODIC SOLUTIONS OF HAMILTONIAN SYSTEMS OF PRESCRIBED PERIOD

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ABSTRACT

This paper is divided in two parts. In the first part some abstract critical point theorems are proved using minimax arguments. The second part is devoted to applications. We study the existence of periodic solutions of the Hamiltonian systems

$$\dot{p} = -\frac{\partial H}{\partial q}(p,q)$$

(1)

$$\dot{q} = \frac{\partial H}{\partial p}(p,q)$$

where $p,q \in \mathbb{R}^n$ and $H \in C^1(\mathbb{R}^{2n},\mathbb{R})$. First we consider Hamiltonian function having the following form:

(2) $$H(p,q) = \sum_{ij} a_{ij}(q)p_i p_j + \sum_i b_i(q)p_i + V(q)$$

where the matrix $a_{ij}(q)$ is positive definite and $V(q)$ grows more rapidly than quadratically as $|q| \to +\infty$. We prove that (1) has infinitely many periodic solutions of any period $T > 0$ under suitable assumptions on the Hamiltonian (2). Then we consider asymptotically linear Hamiltonians:

(3) $$H_z(z) = H_{zz}(\infty)\, z + o(|z|) \quad \text{for} \quad |z| \to +\infty$$

where $z = (p,q)$ and $H_{zz}(\infty)$ is a symmetric operator in $\mathbb{R}^n$. We also give an estimate for the periodic solutions of (1) when the Hamiltonian satisfies (3). Time-dependent Hamiltonians also are considered.

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SIGNIFICANCE AND EXPLANATION

The existence and the number of periodic solutions of Hamiltonian systems is a problem as old as Hamiltonian mechanics itself; early mathematical results were obtained by Liapounov, Poincare, and Birkhoff. Recent remarkable results of Rabinowitz [R2] gave new interest to this classical field; in fact, his work has shown that the techniques and methods of critical point theory, developed in the contest of partial differential equations, may be successfully applied in this field. One of the main results of Rabinowitz states that a Hamiltonian system has infinitely many periodic solutions of any period provided that the Hamiltonian function $H(p,q)$ $(p,q \in \mathbb{R}^n)$ is superquadratic, i.e., it grows more rapidly than quadratically in both of its variables in a suitable way. Unfortunately Hamiltonians arising from physical problems have the form

$$(1) \qquad H(p,q) = \sum_{ij} a_{ij}(q)p_i + \sum_i b_i(q)p_i + V(q) \ .$$

Such Hamiltonians are not superquadratic in the variable p.

In this paper we generalize some abstract critical point theorems in order to include Hamiltonians of the form (1), and we obtain existence of infinitely many periodic solutions of every period provided that $V(q)$ is superquadratic (plus technical assumptions). Asymptotically quadratic Hamiltonians are also considered; these are Hamiltonians such that

$$(2) \qquad H'(z) = H''(\infty)z + o(|z|) \quad \text{for} \quad \|z\| \to +\infty \ ,$$

where $z = (p,q) \in \mathbb{R}^{2n}$, and $H''(\infty):\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a symmetric operator. If $H'(z) = 0$ and H is twice differentiable at $z = 0$, then it is possible to define an index

$\mathcal{O}(\omega H''(0),\omega H''(\infty))$ where $\omega = (2\pi)^{-1}$ times the period of the solution. Under suitable assumptions on H, we know that the Hamiltonian system has at least

$$\tfrac{1}{2}\,|\mathcal{O}(\omega H''(0),\ \omega H''(\infty))|$$

nonlinear $2\pi\omega$-periodic solutions. This result generalizes a result of Amann and Zehnder (who considered strictly convex Hamiltonians [AZ2]) and a previous result of the first author of this paper (which applies when $\mathcal{O} < 0$ [B2]). Time-dependent Hamiltonian are also studied.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

PERIODIC SOLUTIONS OF HAMILTONIAN SYSTEMS OF PRESCRIBED PERIOD

Vieri Benci*, Alberto Capozzi*, and Donato Fortunato**

0. INTRODUCTION AND STATEMENTS OF THE MAIN RESULTS.

Consider the Hamiltonian system of $2n$ ordinary differential equations

$$\dot{p} = -H_q(t,p,q) \qquad \dot{q} = H_p(t,p,q) \quad p,q \in \mathbb{R}^n,\ t \in \mathbb{R}\ , \tag{0.1}$$

where $H \in C^1(\mathbb{R}^{2n+1},\mathbb{R})$, $\cdot$ denotes $\frac{d}{dt}$, $H_q = \frac{\partial H}{\partial q}$, $H_p = \frac{\partial H}{\partial p}$. The system (0.1) can be represented more concisely as

$$-J\dot{z} = H_z(t,z)\ , \tag{0.2}$$

where $z = (p,q)$, $H_z = \frac{\partial H}{\partial z}$ and J is the simplectic matrix in $\mathbb{R}^{2n}$, i.e.

$$J = \begin{bmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{bmatrix}$$

Id being the identity matrix in $\mathbb{R}^n$.

There are many types of questions, both local and global, in the study of periodic solutions of *(0.2) (cf. e.g. the review article of* Rabinowitz [R3] and its references). We suppose in the sequel that $H(t,z)$ is T-periodic in t.

Here we are concerned about the existence of T-periodic solutions of (0.2). Rabinowitz, in a pioneering work [R2], has proved that if $H(t,p,q)$ is "superquadratic" in both the variables p and q, i.e.

$$\text{(0.3)}\qquad \begin{aligned} &\text{there exist } r > 0 \text{ and } \mu > 2 \text{ s.t.} \\ &(H_z(t,z)|z)_{\mathbb{R}^{2n}} > \mu\, H(t,z) > 0 \quad \text{for } |z| > r \text{ and } t \in [0,T] \end{aligned}$$

and it satisfies other assumptions, then (0.2) has a T-periodic solution. If $\frac{\partial H}{\partial t} \equiv 0$ and $H(t,z)$ satisfies (0.3), then Rabinowitz has proved that (0.2) has a nonconstant T-

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periodic solution for every prescribed period T [R4]. Later many other papers appeared dealing with (0.2) when $H(t,z)$ is "superquadratic" [AM, B2, BF2, BR, ClE, E, BB, PT].

Unfortunately the above results on superquadratic Hamiltonians do not cover the classical mechanical problems. In fact, consider a mechanical system with holonomous constraints imbedded in a conservative field of forces. The Hamiltonian of such a system has the form

$$(0.4) \qquad H(t,p,q) = \sum_{i,j=1}^{n} a_{ij}(t,q)p_i p_j + \sum_{i=1}^{n} b_i(t,q)p_i + V(t,q) \ ,$$

where $\{a_{ij}(t,q)\}$ is a positive definite matrix for every t and q. The Hamiltonian (0.4) is quadratic in p, then it does not satisfy (0.3).

If

$$(5) \qquad \begin{array}{ll} a_{ij} \text{ do not depend on } q & (i,j = 1,\dots,n) \\ b_i = 0 & (i = 1,\dots,n) \end{array}$$

(0.1) can be reduced to a second order system of n equations of the form

$$(0.6) \qquad \ddot{x} = -\frac{\partial U}{\partial x} \qquad U = U(t,x) \qquad x \in \mathbb{R}^n$$

which is more easy to study then (0.1) (cf. discussion in [BF3]). In this case, for example, it is known that if $\frac{\partial U}{\partial t} = 0$ and U grows more than quadratically at infinity, in the sense of (0.3), then (0.5) has a non-constant T-periodic solution for each fixed $T > 0$ (cf. [R1,BF1] and references in [R3]).

In this paper first we consider Hamiltonians with the form (0.4) without the restriction (0.5) and with "superquadratic" growth in q. We make the following assumptions on the Hamiltonian (0.4):

<u>Assumptions (Ho)</u>:

(V_1) There exist constants $R > 0$, $\alpha > 2$ s.t.

$$0 < \alpha\, V(t,q) \leq (V_q(q,t),q) \quad \text{for} \quad |q| > R \quad \text{and every} \quad t \in \mathbb{R} \ .$$

(V_2) There exist $C_1, C_2, s, R > 0$ s.t.

$$|V_q(q,t)| \leq C_1 V(q,t) \leq C_2 |q|^s \quad \text{for} \quad |q| > R \quad \text{and every} \quad t \in \mathbb{R} \ .$$

(A_1) There exists a real, continuous function $\nu(q) > 0$ s.t.

$$\sum_{ij} a_{ij}(q,t)p_i p_j > \nu(q)|p|^2 \quad \text{for every } p,q \in \mathbb{R}^n \text{ and } t \in \mathbb{R}.$$

(A_2) There are constants $\beta \in]0,\alpha-2[$ and $\mu > 0$ such that

$$\sum_{ij} M_{ij}(q,t)p_i p_j > \mu|p|^2 \quad \text{where } \{M_{ij}(q,t)\} = \left\{\beta a_{ij} + \sum_k \frac{\partial a_{ij}}{\partial q_k} q_k\right\}.$$

(A_3) There exists a constant C_3 s.t.

$$\left|\sum_{ij} \frac{\partial a_{ij}}{\partial q_k}(q,t)p_i p_j\right| > C_3 \sum_{ij} a_{ij}(q,t)p_i p_j \quad \text{for every } k = 1,\dots,n;\ q \in \mathbb{R}^n,\ t \in \mathbb{R}.$$

(A_4) There exists $C_4 > 0$ s.t.

$$|a_{ij}(q,t)| \le C_4 V(q,t) \quad \text{for } |q| \text{ large and every } t \in \mathbb{R}.$$

$$(B_1) \quad \lim_{|q|\to\infty} \frac{b_i(q,t)^2}{\nu(q)V(q,t)} = 0 \quad \text{for every } i = 1,\dots,n$$

$$(B_2) \quad \lim_{|q|\to\infty} \frac{\left|\frac{\partial b_i}{\partial q_k}(q,t)q_k\right|^2}{\nu(q)V(q,t)} = 0 \quad \text{for every } i,k = 1,\dots,n.$$

Remark. Assumptions (V_1) implies that V grows more than $|q|^\alpha$ at infinity. It replaces assumption (0.3) of other papers.

(A_1) is a physical assumption which depends on the fact that the "kinetic energy" is positive. Observe that it is allowed that $\nu(q) \to 0$ as $|q| \to \infty$.

(A_2) is a technical assumption which is deeply related to the nature of our results. Probably it has some meaning which we have not fully understood.

(V_2), (A_3), (A_4), (B_1), (B_2) are growth conditions on the coefficients of (0.4). Probably they can be weakened using a cut-off technique as in [R1, BR or R4]. We have the following results for Hamiltonians of the form (0.4).

__Theorem 0.1__. Suppose that H satisfies the assumptions (H_o) and

(H_1) the system is atonomous i.e. $\frac{\partial H}{\partial t} = 0$.

Then (0.2) has infinitely many nonconstant T-periodic solutions for every prescribed period T.

(*)

__Theorem 0.2__. Suppose that H satisfies the assumptions (H_o) and

(H_2) $H(t,z)$ is T-periodic in t

(H_3) $H(t,z)$ is even in z.

Then (0.2) has infinitely many nonconstant T-periodic solutions.

__Theorem 0.3__. Suppose that H satisfies (H_o), (H_2) and

(H_4) $z = 0$ is the minimum point of H for every $t \in \mathbb{R}$

(H_5) H is twice differentiable for $z = 0$

(H_6) there exists a constant $\gamma \in]0,1[$ such that

$$\sum_{i,j} \frac{\partial^2 H(t,0)}{\partial z_i \partial z_j} \zeta_i \zeta_j < \frac{2\pi}{T} \gamma |\zeta|^2 \quad \text{for every} \quad t \in \mathbb{R} \quad \text{and} \quad \zeta \notin \mathbb{R}^{2n} .$$

Then (0.2) has at least a nonconstant T-periodic solution.

__Remark 0.4__. If H does not depend on t and it is twice differentiable for $z = 0$, Theorem 0.1 can be deduced from Theorem 0.3. In fact by virtue of the assumptions (H_o), H has a minimum in $\mathbb{R}^{2n}$. It is not restrictive to suppose that the minimum point is $z = 0$. Given any period T, there is a period $T_1 = T/_{k_1}$ $(k_1 \in \mathbb{N})$ such that (H_6) is satisfied. Since a T_1-periodic solution is also a T-periodic solution, we can deduce from Theorem 0.3 that for any period $T > 0$ we have a nonconstant T-periodic solution $z_1(t)$. Also there exists a number h_1 such that z_1 has the minimal period equal to $T/_{h_1 k_1}$. If we take $k_2 > h_1 k_1$ we can find, using Theorem 0.3 a $(T/_{k_2})$ - periodic solution z_2 which is of course a T-periodic solution and $z_2 \neq z_1$. In this way we can find infinitely many

*Warning: Theorem 0.1 just states the existence of periodic solutions but not of prime periodic solutions i.e. solution for which T is the minimal period.

nonconstant T-periodic solutions. We finally observe that, if $b_i = 0$ $(i = 1,\dots,n)$, and $\frac{\partial H}{\partial t} = 0$, variants of Theorem 0.1 can be found in [BCF, G].

Next we consider the case in which H is asymptotically quadratic, i.e. there exists a linear operator $H_{zz}(\infty): \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ s.t.

(0.8) $$H_z(Z) = H_{zz}(\infty)z + o(z) ,$$

where $\frac{o(z)}{|z|} \to 0$ as $|z| \to \infty$. Moreover we suppose that

(0.9) $H(z)$ is twice differentiable for $z = 0$.

The aim is to give a lower bound for the number of $2\pi\omega$-periodic solutions by the comparison between the operators $H_{zz}(0)$ and $H_{zz}(\infty)$. We define as in [B2] an even integer number $\theta(\omega H_{zz}(0), \omega H_{zz}(\infty))$, which will provide such a bound. Given two Hermitian operators $A,B : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$, we set

$$N(A) = \{\text{number of negative eigenvalues of } A\}$$

$$\bar{N}(A) = \{\text{number of nonpositive eigenvalues of } A\} ,$$

and

$$\theta(A,B) = \sum_{k\in\mathbb{Z}} N(ikJ + A) - \bar{N}(ikJ + B) .$$

Observe that $\theta(A,B)$ is a finite number. In fact for k big enough $N(ikJ + A) = \bar{N}(ikJ + B) = n$. Let $\sigma(A)$ denote the spectrum of an Hermitian matrix A. If

(0.10) $$\sigma(i\omega JH_{zz}(\alpha)) \cap \mathbb{Z} = \phi ,$$

and

(0.11) $$\sigma(i\omega JH_{zz}(0)) \cap \mathbb{Z} = \phi ,$$

then $\theta(\omega H_{zz}(\infty), \omega H_{zz}(0)) = -\theta(\omega H_{zz}(0), \omega H_{zz}(\infty))$.

We prove the following theorem:

<u>Theorem 0.5</u>. Suppose that H satisfies (0.8), (0.9), (0.10) and

(0.12) $H_{zz}(\infty)$ is positive definite

(0.13) $H(z) > 0$ for every $z \in \mathbb{R}^{2n}$ s.t. $H_z(z) = 0$,

then (0.1) has at least $\frac{1}{2}\theta(\omega H_{zz}(\infty), \omega H_{zz}(0))$ non-constant $2\pi\omega$-periodic solutions whenever $\theta(\omega H_{zz}(\infty), \omega H_{zz}(0)) > 0$.

If the assumptions (0.12) and (0.13) are replaced by the following ones

(0.12a) $$H_{zz}(0) \text{ is positive definite}$$

(0.13a) $$H(z) < 0 \quad \text{for every} \quad z \in \mathbb{R}^{2n} \text{ s.t. } H_z(z) = 0 \; ,$$

then (0.1) has at least $\frac{1}{2}\,\theta(\omega H_{zz}(0), \omega H_{zz}(\infty))$ non-constant $2\pi\omega$-periodic solutions whenever $\theta(\omega H_{zz}(0), \omega H_{zz}(\infty)) > 0$.

<u>Remark 0.6</u>. The first part of Theorem 0.5 is contained in Theorem 5.1 in [B2]. So Theorem 0.5 can be considered as a natural complement to the results of [B2]. Conditions (0.12a) and (0.13a) are dual to (0.12) and (0.13). However the proof of the second part is much more technical in nature.

<u>Remark 0.7</u>. The assumption (0.10) is a non-resonance condition. If (0.10) does not hold the same conclusion of theorem (0.5) holds if we replace (0.10) by the following assumptions

(0.14) $$H(z) - \tfrac{1}{2}(H_z(z)|z) > c_1|z|^{\alpha} - c_2$$

(0.15) $$|H_z(z)| < c_3 + c_4|z|^{\beta}$$

where $\alpha > \beta > 0$.

From Theorem 0.5 the following corollary easily follows:

<u>Corollary 0.8</u>. If $H(z)$ satisfies (0.8), (0.9), (0.10), (0.12), (0.12a) and

(0.16) $$H_z(z) \neq 0 \quad \text{for every} \quad z \in \mathbb{R}^{2n} - \{0\} \; ,$$

then the system (0.1) has at least

$$\frac{1}{2}\,|\theta(\omega H_{zz}(\infty), \omega H_{zz}(0))|$$

$2\pi\omega$-periodic solutions.

Amman and Zehnder in [AZ2] have obtained a similar result using, instead of (0.12) and (0.12a), the stronger assumption of uniform convexity of $H(z)$.

This paper is divided in two sections. In the first section we have some abstract theorems. In the second one we apply these theorems to obtain the results which we have just stated.

I. SOME ABSTRACT CRITICAL POINTS THEOREMS.

1. Statements of the Abstract Results.

Before stating the main results of this section we shall introduce some notations and definitions. We denote by E a real Hilbert space, by $(\cdot,\cdot)$ the scalar product in E, by $\|\cdot\|$ the norm in E. By $C^1(E,\mathbb{R})$ we denote the space of Frechét differentiable maps from E to $\mathbb{R}$ and, if $f \in C^1(E,\mathbb{R})$ by $f'(u)$ its derivative at $u \in E$. We shall identify E with its dual E' so that $f' \in C^0(E,E)$. For $u \in E$ and $R > 0$ we set $B(u,R) = \{v \in E | \|v - u\| < R\}$, $B_R = B(0,R)$, $S_R = \partial B_R$. Let G be a compact Lie group and let $r : G \to U(E)$ be a representation of G on the group of the unitary linear transformations on E. We set $\mathcal{G} = r(G)$.

<u>Definition 1.1</u>. A functional f on E is called $\mathcal{G}$-<u>invariant</u> if $f \circ T = f$ for every $T \in \mathcal{G}$.

<u>Definition 1.2</u>. A map h from E to E is called $\mathcal{G}$-<u>equivariant</u> if $h \circ T = T \circ h$ for every $T \in \mathcal{G}$.

<u>Definition 1.3</u>. A subset $A \subset E$ is called $\mathcal{G}$-<u>invariant</u> if $T(A) = A$ for every $T \in \mathcal{G}$.

Sometimes, when no ambiguity is possible, we will write "G-invariant", and "G-equivariant" etc. instead of "$\mathcal{G}$-invariant" etc. We set $\mathrm{Fix}\,\mathcal{G} = \{u \in E | T(u) = u$ for every $T \in \mathcal{G}\}$. If $u \in E$ the "<u>orbit</u>" of u is the set $\{T(u) : T \in \mathcal{G}\}$. In the sequel we shall consider $G = \mathbb{Z}_2$ or $G = S^1 = \{z \in \mathbb{C} | \ |z| = 1\}$. Moreover if L is a linear operator on E we denote by $\sigma(L)$ the spectrum of L.

In the sequel we will be concerned with functionals $f \in C^1(E,\mathbb{R})$ satisfying the following assumptions:

(f_1) $f(u) = \frac{1}{2}(Lu,u) - \psi(u)$, where

(i) L is a continuous self-adjoint operator on E

(ii) $\psi \in C^1(E,\mathbb{R})$, $\psi(0) = 0$ and ψ' is a compact operator.

(f_2) (i) $E = \oplus M_\lambda$ where the M_λ's are eigenspaces of L (which might be infinite dimensional).

(ii) 0 is a regular value for L or it is an isolated eigenvalue of finite multiplicity of L.

(f_3) given $c \in]0, +\infty[$, every sequence $\{u_r\}$, for which $\{f(u_r)\} \to c$ and $\|f'(u_r)\| \cdot \|u_r\| \to 0$, possesses a bounded subsequence.

We set

$$E^+ = \overline{\oplus_{\lambda>0} M_\lambda}\ , \ E^- = \overline{\oplus_{\lambda<0} M_\lambda}\ , \quad E^0 = \ker L$$

and let P_+, P_- and P_0 be the relative orthogonal projections. Then

(1.1) $$E = E^- \oplus E^0 \oplus E^- \ .$$

In the case in which E^+ (resp. E^-) is finite-dimensional f is bounded from above (resp. from below) modulo weakly continuous perturbations. In fact we can write $f(u) = \frac{1}{2}(LP_+u, P_+u) - \frac{1}{2}(LP_-u, P_-u) - \psi(u)$ and if, for example, $\dim E^- < +\infty$ then $\Phi(u) = \frac{1}{2}(LP_-u, P_-u) + \psi(u)$ has compact derivative. We shall consider the case in which f can be "strongly indefinite", i.e. E^+ and E^- are both infinite-dimensional, as it occurs in the study of periodic solutions of Hamiltonian systems.

<u>Theorem 1.4.</u> Let $f \in C^1(E, \mathbb{R})$ be a functional satisfying (f_1), (f_2) and (f_3). Moreover we suppose that a unitary representation of the group S^1 acts on E such that

(f_4) L and ψ' are S^1-equivariant

(f_5) there exist two closed linear subspaces $V, W \subset E$ such that

(i) V and W are S^1 invariant.

(ii) $\dim(V \cap W) < +\infty$, $\operatorname{codim}(V + W) < +\infty$

(iii) $\operatorname{Fix}(S^1) \subset V$ or $\operatorname{Fix}(S^1) \subset W$

(iv) there exists positive constants C_0 and ρ such that

$$f(u) > C_0 \quad \text{for every} \quad u \in V \cap S_\rho$$

(v) there exists $C_\infty \in \mathbb{R}$ such that $f(u) < C_\infty$ for every $u \in W$

(vi) $f(u) < C_0$ for $u \in \operatorname{Fix}(S^1)$ s.t. $f'(u) = 0$. Under the above assumptions there exist at least

$$\tfrac{1}{2}(\dim(V \cap W) - \operatorname{codim}(V + W))$$

orbits of critical points, with critical values in $[C_o, C_\infty]$.

We have another theorem for even functional, i.e. for functionals invariant for a $\mathbb{Z}_2$-action.

Theorem 1.5. Let $f \in C^1(E,\mathbb{R})$ be a functional satisfying (f_1), (f_2) and (f_3). Moreover, we suppose that

(f_4') ψ' is odd

(f_5') there exist two closed linear subspaces $V,W \subset E$ which satisfy (f_5)(ii), (f_5)(iii), (f_5)(iv), (f_5)(v).

Then there exists at least

$$\dim(V \cap W) - \operatorname{codim}(V + W)$$

pairs of nonzero critical points with critical values greater or equal than C_o.

Remark 1.6. In the Theorems 1.4 and 1.5 the assumptions (f_2) and (f_3) replace the well known conditions (c) of Palais and Smale (P.S.) used in similar theorems. They do not imply (P.S.), but a weaker condition (i.e. (i) and (ii)) of Lemma 3.4), which has been introduced by G. Cerami (cf. [Ce]; cf. also [BBF]). The conditions (f_5) (resp. (f_5')) are geometrical assumptions, which allow us to give a lower bound to the number of orbits (resp. pairs) of critical points of the functional f.

Remark 1.7. Theorem 1.4 generalizes Theorem 4.1 of [B2] in two points. The assumptions (f_2) and (f_3) are easier to verify than (P.S.). This fact allows to treat Hamiltonians of the form (0.4). Moreover in [B2] the assumption (f_5)(iii) is replaced by the stronger assumption

$$\operatorname{Fix} S^1 \subset W .$$

This generalization permits us to obtain the second part of the Theorem 0.5.

Remark 1.8. If in Theorem 1.5 (f2) and (f3) are replaced by (P.S.) and V (resp W) is finite-dimensional, then we get a variant of a theorem of Clark [Cl1] (resp. Ambrosetti - Rabinowitz [AR]).

In the case in which the functional f does not exibit any symmetry, we have the following theorem:

Theorem 1.9. Let $f \in C^1(E,\mathbb{R})$ be a functional satisfying (f_1) (f_2) and (f_3). Moreover suppose that there exists a L-invariant subspace $V \subset E$, an eigenvector $e \in V$ of L, and positive constants R_1, R_2, C_o, C_∞ with $0 < C_o < C_\infty$ and $\rho < R_1$ such that

(i) $\sup f(Q) = C_\infty$

(ii) $\inf f(S_\rho \cap V) = C_o$

(ii) $\sup f(\partial Q) < 0$

where $Q = \{m + v | m \in V^\perp \cap B_{R_2}, v \in T\}$, $T = \{te | t \in [0,R_1]\}$.

Under the above assumptions f has at least one critical value $c \in [C_o, C_\infty]$.

Remark 1.10. Theorem 1.9 generalizes Theorem 0.1 of Benci-Rabinowitz [BR], because (f1), (f2) and (f3) are weaker assumptions than the respective assumptions in [BR]. This fact allows us to obtain the Theorem 0.3, which applies to Hamiltonian of the form (0.4).

Remark 1.11. Using the techniques developed in this paper it is possible to generalize also Theorem 4.11 of [BR] (cf. [Ca]).

Remark 1.12. The assumption (f_2)(i) is not necessary. In fact, if it does not hold, we can replace the inner product of E with a new inner product such that (f_2)(i) is satisfied.

The new inner product is defined as follows $(u,v)_N = (LP^+u,v) - (LP^-u,v) + (P_ou,v)$.

We observe that every $T \in G$ is a unitary transformation also with respect to the new inner product. If we define a linear operator $\bar{L}: E \to E$ as follows:

$$\bar{L}u = u \quad \text{if} \quad u \in E^+$$

$$\bar{L}u = -u \quad \text{if} \quad u \in E^-$$

$$\bar{L}u = 0 \quad \text{if} \quad u \in E^o$$

then we have

$$(\bar{L}u,v)_N = (Lu,v)$$

and

$$f(u) = \tfrac{1}{2}(\bar{L}\, u,u)_N + \psi(u) \; .$$

So the function f satisfies (f_1), (f_2) and (f_4) or (f_4') in E equipped with the new inner product. Since (f_3) and (f_5) essentially are topological properties, they are as well satisfied (of course minor changes are necessary). Then Theorems 1.4 and 1.5 hold without assumptions (f_2)(ii). A similar remark can be done about Theorem 1.9. However, in the applications which we consider in this paper, assumption (f_2)(ii) is satisfied.

2. Index and Pseudoindex Theory.

In this section we recall some notion (as the notion of index theory) and some theorems which are often used in the critical point theory.

First, some notation is necessary. We get

$$N_\delta(A) = \{u \in E \mid \text{dist}\,(u,A) < \delta\}$$

where dist (u,A) denotes the distance from u to A. For $f \in C^1(E,\mathbb{R})$ and $c \in \mathbb{R}$, we set

$$K_c = \{u \in E \mid f'(u) = 0,\ f(u) = c\}$$

$$A_c = \{u \in E \mid f(u) < c\}\ .$$

Definition 2.1. Let E be a Hilbert space on which a representation $r: G \to r(G) \subset U(E)$ of a compact Lie group G acts. An index theory is a triplet $\{\Sigma, H, i\}$ where

Σ is the family of G-invariant closed subsets of E

H is the set of G-equivariant continuous mappings

$i : \Sigma \to \mathbb{N} \cup \{+\infty\}$ is a mapping, which satisfies the following properties:

(a) $i(A) = 0$ if and only if $A = \phi$

(b) if $A \subset B$ then $i(A) < i(B)$ for all $A,B \in \Sigma$

(2.1) (c) $i(A \cup B) < i(A) + i(B)$ for all $A,B \in \Sigma$

(d) if $A \in \Sigma$ is a compact set, then there exists $\delta > 0$ such that

$$i(N_\delta(A)) = i(A)$$

(e) $i(A) < i(\overline{h(A)})$ for every $A \in \Sigma$ and for every $h \in H$.

Definition 2.2. We say that an index theory satisfies the d-dimension property $(d \in \mathbb{N})$ if

$$i(\partial\Omega \cap V) = \frac{\dim V}{d}$$

where V is a finite dimensional, G-invariant subspace of E such that $V \cap \text{Fix}(G) = \{0\}$ and Ω is a bounded invariant neighborhood of the origin.

The Definition 2.2 makes sense, because, in the examples which we know, if V is as before, then the dimension of V is a multiple of some integer number d.

In the applications we shall use the following index theories:

Example 2.3. The Krasnoselski genus can be considered an index theory which satisfies the 1-dimension property related to the group $Z_2 = \{0,1\}$, where the representation is given by T_o = identity and T_1 = antipodal mapping (cf. e.g. [K], $[R_3]$, $[B_2]$).

Example 2.4. If $G = S^1 = \{w \in \mathbb{C} \mid |w| = 1\}$, then the homological index defined in [F.R.] or the geometrical index defined in $[B_1]$ satisfy the 2-dimension property for any representation $r : G \to U(E)$.

We refer to [Ba] for an abstract construction of an index theory.

In the following theorem we shall list some property of the index which will be used in this paper.

Theorem 2.5. Let $\{\Sigma, H, i\}$ be an index theory which satisfies the dimension property. Then we have

(i) if $[\mathrm{Fix}(G)]^{\perp}$ is infinite dimensional, and $A \cap \mathrm{Fix}(G) \neq \emptyset$, then $i(A) = +\infty$

(ii) if $V \in \Sigma$ is a finite dimensional space and $A \subset V - \mathrm{Fix}(G)$ then $i(A) \leq \frac{\dim V}{d}$

(iii) if $A \cap \mathrm{Fix}(G) = \emptyset$ and $i(A) > 2$ then A contains infinitely many distinct G-orbits

(iv) if $h \in H$ is a homeomorphism, then $i(h(A)) = i(A)$.

For the proof of this theorem we refer to $[B_1]$ and $[B_2]$.

Definition 2.6. Given an index theory $\{\Sigma, H, i\}$ and a group of homeomorphisms $H^* \subset H$, for every $A, B \in \Sigma$ we set

$$i^*(A,B,H^*) = \min_{h \in H^*} i(h(A) \cap B) .$$

The triple $\{\Sigma, H^*, i^*\}$ will be called pseudoindex theory (cf. $[B_2]$ or [BBF]). When no ambiguity is possible we shall write $i^*(\cdot,\cdot)$ instead of $i^*(\cdot,\cdot,H^*)$.

Definition 2.7. Given a G-invariant functional $f \in C^1(E,\mathbb{R})$ and a group of G-equivariant homeomorphism H^*, we say that f satisfies the condition (B) in $]\alpha,\beta[$ $(-\infty \leq \alpha < \beta \leq +\infty)$ with respect to H^* if for every $c \in]\alpha,\beta[$

(i) K_c is compact

(ii) for every $N = N_\delta(K_c)$ there exists $\eta \in H^*$ and a constant $\varepsilon > 0$ such that

(a) $[c-\varepsilon, c+\varepsilon] \subset]\alpha,\beta[$

(b) $\eta(A_{c+\epsilon} - N) \subset A_{c-\epsilon}$.

The concept of pseudoindex and the property (B) are related to the critical point theory by means of the following theorem.

Theorem 2.8. Let $f \in C^1(E,\mathbb{R})$ be a G-invariant functional satisfying the condition (B) in $]\alpha,\beta[$ with respect to $\mathbb{H}^*$. Given $D,F \in \Sigma$, we suppose that

$$
\begin{aligned}
&(i) \quad \sup f(D) = c_\infty < \beta \\
(2.3) \quad &(ii) \quad \inf f(F) = c_0 > \alpha \\
&(iii) \quad i^*(D,F,\mathbb{H}^*) = \bar{k} .
\end{aligned}
$$

If we set

$$\Gamma_k = \{A \in \Sigma \mid i^*(A,F,H^*) \geq k\}$$

then, for $k = 1,\ldots,\bar{k}$, the numbers

$$c_k = \inf_{A\in\Gamma_k} \sup_{u\in A} f(u)$$

are well defined, are critical values of f and

$$c_0 \leq c_1 \leq \ldots\ldots \leq c_{\bar{k}} \leq c_\infty .$$

Moreover if $c = c_k = \ldots\ldots = c_{k+r}$ $(k\geq 1; k+r\leq\bar{k})$, then $i(K_c) \geq r + 1$.

The proof of this theorem follows standard arguments of the critical point theory and it will not be given here (see e.g. [B.B.F.]).

Remark 2.9. If Theorem 2.8 holds we cannot deduce that f has at least $\bar{k}$ distinct orbits of critical points. In fact it might happen that

$$c_1 = \ldots\ldots = c_{\bar{k}} = c$$

and $K_c = \{\bar{u}\}$ where $\bar{u} \in \mathrm{Fix}(G)$.

Then in this case, by Theorem 2.5(i), we have $i(K_c) = +\infty$, but we have only one orbit of critical points i.e. $\{\bar{u}\}$. However if $i(K_c) \geq 2$ and $K_c \cap \mathrm{Fix}(G) = \phi$, by Theorem 2.5(iii) deduce that K_c contains infinitely many distinct orbits. Therefore if the assumptions of Theorem 2.8 hold, we can deduce that one of the following alternatives follows

(a) there exists at least one critical point $\bar{u} \in \mathrm{Fix}(G)$

(b) there exist at least $\bar{k}$ distinct orbits of critical points.

Now we shall enunce the analogous of Theorem 2.8 in the case in which the functional has no symmetry. In this case we can suppose that the function is G-equivariant with respect to the trivial group $G = \{Id\}$. Then the property (B) makes sense.

Definition 2.10. Given two sets D and F and a group of homomorphisms K we say that D and F, K-intersect if

$$h(D) \cap F \neq \phi \quad \text{for every} \quad h \in K .$$

Theorem 2.11. Let $f \in C^1(E,\mathbb{R})$ be a functional satisfying the property (B) in $]\alpha,\beta[$ with respect to K and let $C_0, C_\infty \in R$ be two constants such that

$$\text{(i)} \quad \sup f(D) = C_\infty < \beta$$

$$\text{(2.4)} \qquad \text{(ii)} \quad \inf f(F) = C_0 > \alpha$$

$$\text{(iii)} \quad F \text{ and } D \text{ } K\text{-intersect} .$$

Then f has at least a critical value $c \in [C_0, C_\infty]$. The proof follows standard arguments and it will not be given here (cf. e.g. [B.B.F.]).

3. A Deformation Theorem.

In order to prove Theorems 1.4 and 1.5 we want to use Theorem 2.8. The crucial point is to determine a class of equivariant homeomorphisms $\mathbb{H}^*$ such that

(i) if (f_1), (f_2), (f_3) and (f_4) (or(f_4')) hold, f satisfies the property (B) with respect to $\mathbb{H}^*$

(ii) if (f_5) (or (f_5') hold), then the pseudoindex $i(\cdot,\cdot,\mathbb{H}^*)$ can be estimated by means of $\dim(V \cap W)$ and $\operatorname{codim}(V + W)$.

In order to define $\mathbb{H}^*$ we need the following lemma:

Lemma 3.1. Suppose that L satisfies $(f_2)(i)$ and $(f_2)(ii)$. Moreover suppose that L is G-invariant, where G is a unitary representation of a compact Lie group G. Then

$$E = \overline{\bigoplus_{j \in Z} E_j} \tag{3.1}$$

where the E_j's are G-invariant and L-invariant finite dimensional subspaces, orthogonal with each other.

Proof. If $u \in M_\lambda$, then $LTu = TLu = T\lambda u = \lambda Tu$ for every $T \in G$. So every eigenspace of L is G-invariant.

Then by Peter-Weyl theorem M_λ can be decomposed in finite dimensional G-invariant subspaces orthogonal with each other

$$M_\lambda = \bigoplus_j E_j .$$

Of course, the spaces E_j's constructed in this way, are L-invariant because they are subspaces of an eigenspace of L. $\square$

Now we define the class $\mathbb{H}^*$ as follows:

Definition 3.1'. Let $\mathcal{U}$ be a class of continuous maps $U : E \to E$ such that

(V_1) U is bounded

(V_2) $U(u) = e^{\alpha(u)L}[u]$ where $\alpha : E \to R$ is a G-invariant functional.

Clearly every $U \in \mathcal{U}$ is G-equivariant.

Let $\mathcal{B}$ be a class of continuous maps $b : E \to E$ such that

(b_1) b is G-equivariant and bounded

(b_2) for every $R > 0$, there exists a finite set of indexes $I(R) \subset \mathbb{Z}$ such that

$$b(B_R) \subset \bigoplus_{j \in I(R)} E_j .$$

Finally we define $\mathcal{H}^*$ as the class of all maps h such that

($\mathcal{H}_1^*$) h is an homeomorphism

($\mathcal{H}_2^*$) $h = U_0 + b_0$ where $U_0 \in \mathcal{U}$, $b_0 \in \mathcal{B}$

($\mathcal{H}_3^*$) $h^{-1} = U_1 + b_1$ where $U_1 \in \mathcal{U}$, $b_1 \in \mathcal{B}$

($\mathcal{H}_4^*$) $h(0) = 0$.

Obviously $\mathcal{H}^*$ is a nonempty class of bounded $\mathcal{G}$-equivariant homeomorphisms. It is not difficult to prove the following fact.

<u>Proposition 3.2</u>. $\mathcal{H}^*$ is a group of homeomorphisms.

<u>Proof</u>. By the definition of $\mathcal{H}^*$, it is sufficient to prove that it is closed under composition. Given

$$h_1, h_2 \in \mathcal{H}^*, \quad \text{we set} \quad h_i = U_i + b_i = e^{\alpha_i(\cdot)L}[\cdot] + b_i(\cdot), \qquad (i = 1,2) .$$

Then

$$h_1(h_2(u)) = U_1(h_2(u)) + b_1(h_2(u)) =$$

$$= e^{\alpha_1(h_2(u))L}[h_2(u)] + b_1(h_2(u)) = \tag{3.2}$$

$$= e^{\gamma(u)L}[h_2(u)] + \bar{b}_1(u) ,$$

where $\gamma(u) = \alpha_1[h_2(u)]$ is a $\mathcal{G}$-invariant functional and $\bar{b}_1(\cdot) = b_1(h_2(\cdot)) \in \mathcal{B}$. Then by (3.2), we have

$$h_1(h_2(u)) = e^{\gamma(u)L}\left[e^{\alpha_2(u)L}[u] + b_2(u)\right] + \bar{b}_1(u) =$$

$$= e^{(\gamma(u)+\alpha_2(u))L}[u] + e^{\gamma(u)L}[b_2(u)] + \bar{b}_1(u) =$$

$$= = e^{\beta(u)L}[u] + \bar{b}_2(u) + \bar{b}_1(u) ,$$

where $\beta(u) = \gamma(u) + \alpha_2(u)$ is a $\mathcal{G}$-invariant functional and $\bar{b}_2(\cdot) = e^{\gamma(\cdot)L}[b_2(\cdot)] \in \mathcal{B}$. □

From now on $\mathbb{H}^*$ will denote the class of homeomorphisms just defined and $i^*(\cdot,\cdot) = i^*(\cdot,\cdot,\mathbb{H}^*)$.

The rest of this section is devoted to prove the following theorem:

<u>Theorem 3.3</u>. Suppose that $f \in C^1(E,\mathbb{R})$ satisfies (f_1), (f_2) and (f_3) and that it is G-invariant. Given $c > 0$ and a neighborhood N of K_c, there exists constants $\bar{\varepsilon} > \varepsilon > 0$ (with $\bar{\varepsilon} < c$) and an operator $\eta : E \to E$ such that

(a) $\eta(A_{c+\varepsilon} - N) \subset A_{c-\varepsilon}$

(b) $\eta = U + B \in \mathbb{H}^*$

(c) $U(u) = u$, $B(u) = 0$ for every $u \notin f^{-1}([c-\bar{\varepsilon}, c+\bar{\varepsilon}])$.

In particular f satisfies the condition (B) in $]0,+\infty[$ with respect to $\mathbb{H}^*$ (cf. Definition 2.7).

The proof of Theorem 3.3 is based on the following lemmas:

<u>Lemma 3.4</u>. If f satisfies (f_1) (f_2) and (f_3) then we have:

(i) every bounded sequence $\{u_k\} \subset f^{-1}(]0,\infty[)$ such that $f'(u_k) \to 0$, admits a convergent subsequence

(ii) for every $c > 0$, there exist constants $\bar{\varepsilon}$, $\bar{R}$, b, $\mu > 0$ such that

(a) $[c-\bar{\varepsilon}, c+\bar{\varepsilon}] \subset]0,+\infty[$

(b) $\|f'(u)\| \cdot \|u\| > \mu$ for every $u \in f^{-1}([c-\bar{\varepsilon}, c+\bar{\varepsilon}]) \cap (E - B_{\bar{R}})$

(iii) for every $c > 0$, K_c is compact

(iv) for every c and $R > 0$ and for every neighborhood N of K_c, there exist positive constants $\bar{\varepsilon}, b$ such that

$$\|f'(u)\| > b \quad \text{for every} \quad u \in (A_{c+\bar{\varepsilon}} - A_{c-\bar{\varepsilon}}) \cap (B_R - N) .$$

<u>Proof</u>. (i) We put

$$S = L + \lambda P_o$$

where $\lambda \neq o$ and P_o is the orthogonal projector on $\ker L$. Clearly S is a bounded invertible operator. Now let u_k be a bounded sequence such that $f'(u_k) \to 0$. Then we can write

$$L\, u_k - \psi'(u_k) = v_k$$

with $v_k \to 0$. Then we have

$$S u_k - \lambda P_o u_k - \psi'(u_k) = v_k$$

or

$$S u_k = \lambda P_o u_k + \psi'(u_k) + v_k .$$

Since P_o and ψ' are compact operators, there is a subsequence u'_k such that Pu'_k and $\psi'(u'_k)$ converge. Thus Su'_k converges. Since S is invertible u'_k converges.

(ii) We argue indirectly and we suppose that there exists $c \in]o,+\infty[$ such that for every $n \in \mathbb{N}$ there exists $u_n \in E$ for which

$$\|f'(u_n)\| \cdot \|u_n\| < \frac{1}{n}$$

$$u_n \in f^{-1}([c-\frac{1}{n}, c+\frac{1}{n}]) \cap (E-B_n).$$

Then, for $n \to +\infty$, we have

$$\|f'(u_n)\| \cdot \| u_n\| \to 0$$

$$\|u_n\| \to +\infty$$

$$f(u_n) \to c$$

and this contradicts (f_3).

(iii) From (ii) it follows that K_c is bounded. Because of the continuity of f and f', K_c is closed, and by (i) it follows that it is compact.

(iv) It follows from (i) and standard arguments. □

The conditions (i) and (ii) of the above lemma can be considered as a weakened version of the well known condition (c) of Palais and Smale (cf. Remark 1.6).

<u>Lemma 3.5</u>. Let $k : E \to E$ be a compact operator. For every $\varepsilon > 0$ there exists a compact operator $\tilde{k} : E \to E$ such that:

(a) $\tilde{k}$ is locally Lipschitz continuous

(b) $\|k(u) - \tilde{k}(u)\| \cdot (1 + \|u\|) < \varepsilon$ for every $u \in E$.

Moreover, if k is G-equivariant, $\tilde{k}$ can be chosen G-equivariant.

<u>Proof</u>. The proof follows the same argument as lemma 3.2 in $[B_2]$.

<u>Lemma 3.6</u>. Let $\tilde{k} : E \to E$ be a locally Lipschitz continuous, G-equivariant, compact operator. For every $R > 0$ and $\varepsilon > 0$ there exists an operator $b \in B$ such that

(a) $\|\bar{k}(u) - \bar{b}(u)\| < \varepsilon$ for every $u \in B_R$

(b) $\bar{b}$ is locally Lipschitz continuous.

Proof. Since $\bar{k}(B_R)$ is relatively compact, for every $\varepsilon > 0$ there exist a finite set of points $y_1,\dots,y_s$ such that $\bar{k}(B_R) \subset \bigcup_{i=1}^{s} B(y_i,\frac{\varepsilon}{2})$. Let $n \in \mathbb{N}$ and set P_n the projector on $\bigoplus_{i=-n}^{n} E_i$. If n is big enough, we have

$$\|y_i - P_n y_i\| < \frac{\varepsilon}{2} \qquad \forall i \in \{1,\dots,s\} .$$

Consider now the operator

$$\tilde{b} : B_R \to \bigoplus_{i=-n}^{n} E_i, \qquad \bar{b}\,(u) = \frac{\sum_{i=1}^{s} \mu_i(u) P_n y_i}{\sum_{i=1}^{s} \mu_i(u)} ,$$

where $\mu_i(u) = \operatorname{dist}(\bar{k}(u), E - (B(y_i, \frac{\varepsilon}{2}))$. It is easy to check that $\bar{b}$ is a bounded, Lip. continuous operator and that for every $u \in B_R, \|\bar{k}(u) - \bar{b}(u)\| < \varepsilon$. To prove that $\bar{b}$ can be chosen G-equivariant it is sufficient to repeat the arguments of Lemma 3.2 in $[B_2]$. □

Lemma 3.7. Let $\bar{k} : E \to E$ be as in Lemma 3.6; given $\varepsilon > 0$ there exists an operator $b \in B$ such that

(a) $\|\bar{k}(u) - b(u)\|\cdot(1+\|u\|) < \varepsilon$ for every $u \in E$.

(b) b is locally Lipschitz continuous.

Proof. Given $\varepsilon > 0$, by Lemma 3.6 for every $n \in \mathbb{N}$ there exists a locally Lipschitz continuous operator $\bar{b}_n : B_{n+1} \to V_{n+1}$ such that

(3.3) $$V_{n+1} = \bigoplus_{i\in I(n)} E_i \quad \text{for a finite set } I(n) \subset \mathbb{Z}$$

(3.4) $$\|\bar{k}(u) - \bar{b}_n(u)\| < \frac{\varepsilon}{2(n+1)} \quad \text{for every } u \in B_{n+1} .$$

For every $n \in \mathbb{N}$ we consider a non-increasing map $X_n(t) \in C^1(\mathbb{R},[0,1])$ such that

$$X_n(t) = \begin{cases} 1 & \text{if } t \in [0,n] \\ 0 & \text{if } t \in [n+1/2, +\infty[. \end{cases}$$

we set

$$b_n(u) = \begin{cases} \tilde{b}_n(u) & \text{if } u \in B_{n+1} \\ 0 & \text{if } u \notin B_{n+1} . \end{cases}$$

We define a sequence of operators $c_n : E \to E$ as follows:

$$c_1(u) = b_1(u)$$

(3.5) $$c_2(u) = X_1(\|u\|)\, c_1(u) + (1-X_1(\|u\|))b_2(u)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$c_{n+1}(u) = X_n(\|u\|)c_n(u) + (1-X_n(\|u\|))b_{n+1}(u) .$$

We observe that if $u \in B_n$, $c_n(u) = c_{n+1}(u) = \dots$. We set for $u \in E$

(3.6) $$b(u) = \lim_{n\to\infty} c_n(u) .$$

Clearly $b \in B$ and satisfies (b). Let us prove (a) If $u \in B_{n+1}$ we have

$$\|b(u) - \tilde{k}(u)\| = \|c_{n+1}(u) - \tilde{k}(u)\| =$$

(3.7)
$$= \|X_n(\|u\|)c_n(u) + (1-X_n(\|u\|))b_{n+1}(u) - \tilde{k}(u)\| =$$

$$= \|X_n(\|u\|)(c_n(u) - \tilde{k}(u)) + (1-X_n(\|u\|))(b_{n+1}(u) - \tilde{k}(u))\| <$$

$$< X_n(\|u\|)\|c_n(u) - \tilde{k}(u)\| + (1-X_n(\|u\|))\ \|b_{n+1}(u) - \tilde{k}(u)\| .$$

Since if $u \in B_{n+1}$, $\tilde{b}_{n+1}(u) = b_{n+1}(u)$, then by (3.4) we have

(3.8) $$\|b_{n+1}(u) - \tilde{k}(u)\| < \frac{\varepsilon}{2(n+2)} \quad \text{if} \quad u \in B_{n+1} .$$

To prove (a) it is sufficient to prove that, for every $n \in N$, if $u \in B_n$

(3.9) $$\|b(u) - \tilde{k}(u)\| < \frac{\varepsilon}{1+\|u\|} .$$

In order to prove (3.9) we argue by induction:

if $n = 1$ by (3.5), (3.7) and (3.8) we get

$$\|b(u) - \tilde{k}(u)\| = \|c_1(u) - \tilde{k}(u)\| = \|b_1(u) - \hat{k}(u)\| < \frac{\varepsilon}{4} < \frac{\varepsilon}{1 + \|u\|} .$$

Now suppose that

(3.10) $$\|b(u) - \tilde{k}(u)\| < \frac{\varepsilon}{1+\|u\|} \quad \text{for every } u \in B_n .$$

We have to verify (3.10) for $u \in B_{n+1} - B_n$.

We observe that for $u \in B_{n+1} - B_n$, $c_n(u) = b_n(u)$. Then by (3.4)

(3.11) $$\|c_n(u) - \tilde{k}(u)\| = \|b_n(u) - \tilde{k}(u)\| = \|\tilde{b}_n(u) - \tilde{k}(u)\| < \frac{\varepsilon}{2(n+1)} .$$

Then for $u \in B_{n+1} - B_n$ by (3.7), (3.8) and (3.11) we get

(3.12) $$\|b(u)-\tilde{k}(u)\| < X_n(\|u\|)\frac{\varepsilon}{2(n+1)}+(1-X_n(\|u\|))\frac{\varepsilon}{2(n+2)} < \frac{\varepsilon}{2(n+1)} < \frac{\varepsilon}{1+(n+1)} < \frac{\varepsilon}{1+\|u\|} .$$

Finally by (3.10) and (3.12) we have that

(3.13) $$\|b(u) - \tilde{k}(u)\| < \frac{\varepsilon}{1+\|u\|} \quad \text{for every } u \in B_{n+1}$$ and (3.3) is proved. □

By Lemma 3.5 and 3.7, we get the following lemma:

<u>Lemma 3.8</u>. Let $k : E \to E$ be a G-equivariant, compact operator. Given $\varepsilon > 0$ there exists a bounded operator $b \in \mathcal{B}$ such that

(a) $\|k(u) - b(u)\|\cdot(1+\|u\|) < \varepsilon$ for every $u \in E$

(b) b is locally Lipschitz continuous.

Now we can prove the Theorem 3.3.

<u>Proof</u>. Given $c \in]\alpha,\beta[$, by Lemma 3.4(iii), K_c is compact, hence there exists $\delta > 0$ such that $N \supset M_\delta \supset K_c$, where $M_\delta = N_\delta(K_c)$. Moreover, by Lemma 3.4 (iv) there exist $\bar{\varepsilon} > 0$, and $b > 0$ such that

(3.14) $$\|f'(u)\| > b \quad \forall\, u \in (A_{c+\bar{\varepsilon}} - A_{c-\bar{\varepsilon}}) \cap (B_{\bar{R}} - M_{\delta/8}) .$$

We can assume that $\bar{R}$ is big enough such that $B_{\bar{R}} > M_\delta$. Also we can assume that

(3.15) $$\bar{\varepsilon} < \frac{\delta b}{12} .$$

Let $\gamma > 0$ be such that

(3.16) $$\gamma < \min\left\{\frac{\bar{\varepsilon}}{4} , \frac{b}{4}\right\} .$$

By Lemma (3.8) there exists a locally Lipschitz continuous operator $b \in \mathcal{B}$ such that

(3.17) $$\|k(u) - b(u)\| < \frac{\gamma}{1+\|u\|} \quad \text{for every } u \in E .$$

We set $S = (A_{c+\bar{\varepsilon}} - A_{c-\bar{\varepsilon}}) \cap M_{\delta/8}$, $S_1 = S \cap B_R$, $S_2 = S - B_R$. By (3.16) and (3.14) we have

(3.18) $$\frac{\gamma}{1+\|u\|} < \frac{b}{4} < \frac{\|f'(u)\|}{4} \quad \text{for every } u \in S_1 ,$$

and by (1.16) and Lemma 3.4(ii) we have

(3.19) $$\frac{\gamma}{1+\|u\|} < \frac{\|f'(u)\|}{4} \quad \text{for every} \quad u \in S_2 .$$

Thus, by (3.12), (3.18) and (3.19),

(3.20) $$\|k(u) - b(u)\| < \tfrac{1}{4}\|f'(u)\| \quad \text{for every} \quad u \in S .$$

We observe that if $u \in S$

$$\|Lu + b(u)\| = \|f'(u) - (k(u) - b(u))\| >$$

$$> \|f'(u)\| - \|k(u) - b(u)\| ,$$

then by the above inequality and (3.20)

(3.21) $$\|Lu + b(u)\| > \frac{3}{4}\|f'(u)\| > 0 \quad \text{for every} \quad u \in S .$$

Now we set

(3.22) $$V(u) = 2\,\frac{Lu+b(u)}{\|Lu+b(u)\|^2} \quad \text{for every} \quad u \in S .$$

By (3.21) we have

(3.23) $$\|V(u)\| < \frac{8}{3}\,\frac{1}{\|f'(u)\|} \quad \text{for every} \quad u \in S ,$$

then by Lemma 3.4(ii), (3.14) and (3.23)

(3.24) $$\|V(u)\| < K_1 + K_2\|u\| \quad \text{for every} \quad u \in S ,$$

where K_1 and K_2 are positive constants.

Now we observe that if $u \in S$, by virtue of (1.23),

$$\|k(u) - b(u)\| < \tfrac{1}{4}\|f'(u)\| = \tfrac{1}{4}\|Lu + k(u)\| <$$

$$< \tfrac{1}{4}\|Lu + b(u)\| + \tfrac{1}{4}\|k(u) - b(u)\| ,$$

then

$$\|k(u) - b(u)\| < \frac{1}{3}\|Lu + b(u)\| .$$

From the above inequality, we get

$$\langle V(u), f'(u)\rangle = 2\left\langle \frac{Lu+b(u)}{\|Lu+b(u)\|^2}, Lu+k(u)\right\rangle = \frac{2}{\|Lu+b(u)\|^2}\langle Lu+b(u), Lu+b(u)-b(u)+k(u)\rangle =$$

(3.25)

$$= \frac{2}{\|Lu+b(u)\|^2}\left[\|Lu+b(u)\|^2 + \langle Lu+b(u), k(u)-b(u)\rangle\right] >$$

$$> 2 - 2\,\frac{\|Lu+b(u)\|\cdot\|k(u)-b(u)\|}{\|Lu+b(u)\|^2} > 2 - \frac{2}{3} > 1 \quad \text{for every} \quad u \in S .$$

Now we consider a Lipschitz continuous, functional $\phi : E \to R$ such that

$$(3.26) \qquad \phi(u) = \begin{cases} 0 & \text{if } u \notin f^{-1}([c-\bar{\varepsilon},c+\bar{\varepsilon}]) \text{ or } u \in M_{\delta/8} \\ 1 & \text{if } u \in f^{-1}([c-\varepsilon,c+\varepsilon]) - M_{\delta/4} \end{cases}$$

where $\varepsilon = \frac{\bar{\varepsilon}}{2}$. We can assume that ϕ is G-invariant. We set

$$(3.27) \qquad \tilde{V}(u) = \begin{cases} -\phi(u)V(u) & \text{if } u \in S \\ 0 & \text{if } u \notin S . \end{cases}$$

Consider now the following initial value problem

$$(3.28) \qquad \begin{cases} \frac{d\eta}{dt} = \tilde{V}(\eta) \\ \eta(o) = u \end{cases} \qquad u \in E .$$

Since $\tilde{V}$ is loc. Lipschitz continuous, by (3.24) and standard arguments, it follows that for every $u \in E$, (3.28) has a unique solution $\eta : R \to E$ and if we denote by $\eta(t,u)$ the flow relative to problem (3.28), then $\eta(\bar{t},u) : E \to E$ is a bounded homeomorphism.

In order to prove the part (a) of the theorem, we observe that for $u \in E$, $f(\eta(t,u)) : R \to R$ is not increasing. In fact we have

$$(3.29) \qquad \begin{aligned} \frac{d}{dt} f(\eta(t,u)) &= \langle f'(\eta(t,u)), \frac{d}{dt}\eta(t,u)\rangle = \\ &= -\phi'(\eta(t,u)) \langle f'(\eta(t,u)), V(\eta(t,u)) \rangle . \end{aligned}$$

We set $Q = (A_{c+\varepsilon} - A_{c-\varepsilon}) - M_{\delta/4}$.

By (3.25), (3.26) and (3.29) we have

$$(3.30) \qquad \frac{d}{dt} f(\eta(t,u)) \begin{cases} \leq -1 & \text{for } u \in Q \\ \leq 0 & \text{for } u \in S \cap Q \\ = 0 & \text{for } u \notin S . \end{cases}$$

If $\bar{u} \in Q$ and $t' \in R^+$ is such that $\eta(t,\bar{u}) \in Q \quad \forall t \in [0,t']$ then by (3.30)

$$(3.31) \qquad 2\,\varepsilon > f(\eta(o,\bar{u})) - f(\eta(t',\bar{u})) = -\int_o^{t'} \frac{d}{dt} f(\eta(t,\bar{u}))dt \geq t' .$$

Moreover if $t'' > t'$ is such that $\eta(t,\bar{u}) \in Q \cap B_{\bar{R}}$ for $t \in [t',t'']$, then by (3.14), (3.23) and (3.27)

$$|\eta(t'',\bar u) - \eta(t',\bar u)| = |\int_{t'}^{t''} \bar V(\eta(t,\bar u))dt| <$$

(3.32)

$$< \frac{8}{3}\int_{t'}^{t''} \frac{1}{\|f'(\eta(t,u))\|}\,dt < \frac{8}{3b}(t''-t') < \frac{8t''}{3b} .$$

Finally we set $\eta(u) = \eta(\bar\varepsilon,u) = \eta(2\varepsilon,u)$ and $Y = (A_{c+\varepsilon}-A_{c-\varepsilon}) - M_\delta$. Since $Y \subset Q$ if $u \in Y$ by (3.31) there exists $\bar t \in (0,\bar\varepsilon)$ such that either $\eta(\bar t,u) \in A_{c-\varepsilon}$ or $\eta(\bar t,u) \in M_{\delta/4} - A_{c-\varepsilon}$. The second of these alternatives is not possible, in fact if $\eta(\bar t,u) \in M_{\delta/4} - A_{c-\varepsilon}$ then there exist $t',t'' \in (o,\bar\varepsilon)$, with $t' < t''$, such that $\eta(t,u) \in Q \cap B_{\bar R}$ for $t \in [t',t'')$ and $\eta(t'',u) \in \partial Q$. Then by (3.32) we should have

(3.33) $$t'' > \frac{9}{32} b\delta > \bar\varepsilon$$

and this contradicts the fact that $t'' < \bar t < \bar\varepsilon$. Hence $\eta(\bar t,u) \in A_{c-\varepsilon}$. Then by (3.30) $\eta(\bar\varepsilon,u) \in A_{c-\varepsilon}$.

Thus the part (a) of Theorem 1.24 is proved.

In order to prove (b), we set

$$\bar\phi(u) = \frac{-2\phi(u)}{\|Lu + b(u)\|^2}$$

so the Equation (3.28) becomes

(3.34)
$$\frac{d\eta}{dt} = \bar\phi(\eta)[L\eta+b(\eta)]$$
$$\eta(o) = u .$$

Following an idea of Hofer [H] we set:

(3.35) $$\alpha(t,s,u) = \int_o^{t-s} \bar\phi(\eta(t+s,u))dt .$$

Easy computations show that the Cauchy problem (3.34) is equivalent to the following integral equation:

$$\eta(t,u) = e^{\alpha(t,o,u)L}[u] + \int_o^t \alpha^{\alpha(t,s,u)L}[\bar\phi(\eta(s,u)b(\eta(s,u))]ds .$$

In fact $\eta(0,u) = u$ and

$$\frac{d\eta(t,u)}{dt} = \frac{d}{dt}\alpha(t,o,u)Le^{\alpha(t,o,u)L}[u] + e^{\alpha(t,t,u)L}[\bar{\phi}(\eta(t,u))b(\eta(t,u))]$$

$$+ \int_o^t \frac{d}{dt}\alpha(t,s,u)Le^{\alpha(t,s,u)L}[\bar{\phi}(\eta(s,u))b(\eta(s,u))]ds$$

$$= \bar{\phi}(\eta(t,u))Le^{\alpha(t,o,u)L}[u] + \bar{\phi}(\eta(t,u))b(\eta(t,u))$$

$$+ \int_o^t \bar{\phi}(\eta(t,u))Le^{\alpha(t,s,u)}[\bar{\phi}(\eta(s,u))b(\eta(s,u))]ds$$

$$= \bar{\phi}(\eta(t,u))L\{e^{\alpha(t,o,u)L} + \int_o^t e^{\alpha(t,s,u)}[\bar{\phi}(\eta(s,u))b(\eta(s,u))]ds\}$$

$$+ \bar{\phi}(\eta(t,u))b(\eta(t,u))$$

$$= \bar{\phi}(\eta(t,u))L\eta(t,u) + \bar{\phi}(\eta(t,u))b(\eta(t,u)) =$$

$$= \bar{\phi}(\eta)[L\eta+b\eta] .$$

Observe that, since the operators of G are unitary, $\bar{\phi}$ is G-invariant and by (3.34) and (3.35), η is G-equivariant and $\alpha\ (t,u,\cdot)$ is G-invariant. Then if we set

$$U(u) = e^{\alpha(t,o,u)L}[u]$$

$$B(u) = \int_o^t e^{\alpha(t,s,u)L}[\bar{\phi}(\eta(t,u))b(\eta(t,u))]ds$$

it results that $U \in \mathbf{U}$ and $B \in \mathbf{B}$, moreover $\eta^{-1}(u) = \eta(-\bar{t},u)$, then (b) is proved. By (3.26) and (3.27) it results that $\eta(t,u) = u$ for every $u \notin f^{-1}([c-\bar{\varepsilon},c+\bar{\varepsilon}])$ and every $t \in \mathbb{R}$. Then from (3.26) and (3.35), it follows that $\alpha(t,s,u) = 0$ for every $u \in f^{-1}([c-\bar{\varepsilon},c+\bar{\varepsilon}])$ and every $t,s \in \mathbb{R}$. Therefore, by the definition of U and B, (c) follows.

4. Pseudoindex Evaluation.

In the previous section we have shown that f satisfies the property (B) with respect to the class $\mathbb{H}^*$. In this section we shall compute the pseudoindex of some subsets of E with respect to the class $\mathbb{H}^*$ provided that G satisfies the dimension property. More precisely, we will be concerned in proving the following theorem:

Theorem 4.1. Consider two G-invariant closed linear subspaces $V,W \subset E$ and a bounded G-invariant neighborhood of the origin Ω. Suppose that

(i) $\mathrm{Fix}\, G \subset W$ (or $\mathrm{Fix}\, G \subset V$)

(4.1) (ii) $\dim(V \cap W) < +\infty$, $\mathrm{codim}(V + W) < +\infty$

(iii) the index theory i satisfies the d-dimension property (cf. Definition 2.2).

Then

$$i^*(S \cap V,W) \geq \frac{\dim(V \cap W) - \mathrm{codim}(V+W)}{d}, \tag{4.2}$$

The proof of Theorem 4.1 is based on two lemmas.

Lemma 4.2. Let $V,W,Z \subset E$ be G-invariant, finite dimensional subspaces $(V,W \subset Z)$, and Ω be a bounded G-invariant neighborhood of 0. Given a G-equivariant bounded continuous map $h : E \to E$, we suppose that

(i) $\mathrm{Fix}\, G \subset W$

(ii) the index theory i satisfies the d-dimension property.

(iii) $h(\partial\Omega \cap V) \subset Z$

then

$$i(h(\partial\Omega \cap V) \cap W) \geq \frac{\dim(V \cap W) - \mathrm{codim}_Z(V+W)}{d}. \tag{4.3}$$

Proof. We set $S = \partial\Omega$. We distinguish two cases

Case I $\quad V \cap \mathrm{Fix}\, G \supsetneq \{0\}$

Case II $\quad V \cap \mathrm{Fix}\, G = \{0\}$.

In the Case I we have that

$$V \cap S \cap \mathrm{Fix}\, G \neq \emptyset .$$

Since $h(\mathrm{Fix}\, G) \subset \mathrm{Fix}\, G$,

$$h(S \cap V) \cap \mathrm{Fix}\, G \supset h(V \cap S \cap \mathrm{Fix}\, G) \cap \mathrm{Fix}\, G \neq \emptyset .$$

Using assumption (i) and the above formula we have

$$h(S \cap V) \cap \text{Fix } G \cap W \neq \emptyset .$$

Then by Theorem 2.5(i), it follows that

$$i(h(V \cap S) \cap W) = +\infty .$$

Therefore, in the Case I, (4.3) holds.

We now consider the Case II. Since W is finite dimensional, $h(S \cap V) \cap W \in \sum$ is compact. Then, by (2.1)(d), there exists $N = N_\varepsilon(h(S \cap V) \cap W)$ such that

$$i(N) = i(h(S \cap V) \cap W) . \tag{4.4}$$

We set

$$\begin{aligned} A_1 &= h(S \cap V) \cap N \\ A_2 &= \overline{h(S \cap V) - N} . \end{aligned} \tag{4.5}$$

Obviously $A_1, A_2 \in \sum$ and

$$h(S \cap V) = A_1 \cup A_2 . \tag{4.6}$$

Since $V \cap \text{Fix}(G) = \{0\}$, then

$$\begin{aligned} \frac{\dim V}{d} = i(S \cap V) & \quad \text{(by the dimension property, cf. Def. 2.2)} \\ \leq i(h(S \cap V)) & \quad \text{(by (2.1)(e))} \\ \leq i(A_1 \cup A_2) & \quad \text{(by 4.8 and (2.1)(b))} \\ \leq i(A_1) + i(A_2) & \quad \text{(by (2.1)(c))} . \end{aligned} \tag{4.7}$$

By (4.5), (2.1)(b) and (4.4) we have

$$i(A_1) \leq i(N) = i(h(S \cap V) \cap W) . \tag{4.8}$$

Let $W^\perp$ denote the orthogonal complement of W in Z and let $P_W^\perp$ denote the relative orthogonal projection. $P_W^\perp$ is a G-equivariant map, then, by (2.1)(c)

$$i(A_2) \leq i(P_W^\perp A_2) . \tag{4.9}$$

By the construction of N, $(P_W^\perp A_2) \subset W^\perp - \{0\}$, then since $\text{Fix } G \subset W$,

$$(P_W^\perp A_2) \subset W^\perp - \{0\} = W^\perp - \text{Fix}(G) .$$

Therefore, by Theorem 2.5 (ii)

$$i(P_W^\perp A_2) \leq \frac{\dim W^\perp}{d} . \tag{4.10}$$

By (4.7), (4.8), (4.9) and (4.10), we get

$$\frac{\dim V}{d} \leq i(h(S \cap V) \cap W) + \frac{\dim W^\perp}{d} .$$

By the above formula we have:

$$i(h(S \cap V) \cap W) \geq \frac{\dim V - \dim W^{\perp}}{d} = \frac{\dim V - \operatorname{cod}_Z W}{d} . \qquad \square$$

<u>Lemma 4.3</u>. Let the hypotheses of Lemma 4.2 be satisfied with (i) and (iii) replaced by

(i') $\operatorname{Fix} G \subset V \oplus Z^{\perp}$

(iii') (a) h is a bounded homeomorphism

(b) $h(\Omega \cap Z) \subset Z$

(c) $h(0) = 0$.

Then

$$(4.11) \qquad i(h(\partial\Omega \cap V) \cap W) \geq \frac{\dim(V \cap W) - \operatorname{codim}_Z(V+W)}{d} .$$

<u>Proof</u>. To shorten the notation, we set $S = \partial\Omega$. Since $h(S \cap V) \cap W \in \Sigma$ is compact, by (2.1)(d) there exists $N = N_{\varepsilon_1}(h(S \cap V) \cap W)$ such that

$$(4.12) \qquad i(N) = i(h(S \cap V) \cap W) .$$

There exist constants $\varepsilon_2, \varepsilon_3, \varepsilon > 0$ such that

$$(4.13) \quad N \supset N_{\varepsilon_2}(h(S \cap V) \cap W) \supset h(N_{\varepsilon_3}(S \cap V)) \cap W \supset h(S \cap V_{\varepsilon}) \cap W \supset h(S \cap V) \cap W$$

where $V_{\varepsilon} = N_{\varepsilon}(V) \cap Z$. By the above formula and (2.1)(b) it follows that

$$i(N) \geq i(h(S \cap V_{\varepsilon}) \cap W) \geq i(h(S \cap V) \cap W) .$$

Then, by (4.12),

$$(4.14) \qquad i(h(S \cap V_{\varepsilon}) \cap W) = i(h(S \cap V) \cap W) .$$

We now set

$$R = \overline{Z - V_{\varepsilon}} .$$

Then $Z = V_{\varepsilon} \cup R$ and

$$h(S \cap Z) \cap W = [h(S \cap V_{\varepsilon}) \cap W] \cup [h(S \cap R) \cap W] .$$

By the above formula and (2.1)(c), we have:

$$i(h(S \cap Z) \cap W) \leq i(h(S \cap V_{\varepsilon}) \cap W) + i(h(S \cap R) \cap W) .$$

Comparing this inequality with (4.14), we get

$$(4.15) \qquad i(h(S \cap V) \cap W) \geq i(h(S \cap Z) \cap W) - i(h(S \cap R) \cap W) .$$

Now we shall give an estimate to the terms on the right hand side of (4.15). Let $V^{\perp}$ denote the orthogonal complement of V in Z and $P_V^{\perp}$ the relative projection. Obviously $P_V^{\perp}$ is equivariant. Moreover, by (i'), $P_V^{\perp} R \subset V^{\perp} - \operatorname{Fix}(G)$. Then by (2.1)(e) and Theorem 2.5(ii), we have

$$i(R) \leq i(P_V^\perp R) \leq \frac{\dim V^\perp}{d} \ . \tag{4.16}$$

Now

$$\begin{aligned} i(h(S \cap R) \cap W) &\leq i(h(S \cap R)) \quad \text{(by (2.1)(b))} \\ &= i(S \cap R) \quad \text{(by Theorem 2.5(iv) and (iii')(a))} \\ &\leq i(R) \quad \text{(by (2.1)(b))} \\ &\leq \frac{\dim V^\perp}{d} \quad \text{(by (4.16))} \ . \end{aligned} \tag{4.17}$$

By (iii')(b) and (c), $h(\Omega \cap Z)$ is a bounded neighborhood of 0 in Z. Then the set

$$\tilde{\Omega} = \{z + \tilde{z} | z \in h(\Omega \cap Z), \ \tilde{z} \in Z^\perp, \ |\tilde{z}| < 1\}$$

is a neighborhood of 0 in E. It is easy to check that

$$h(\partial\Omega \cap Z) = \partial\tilde{\Omega} \cap Z \ .$$

Then

$$h(S \cap Z) \cap W = h(\partial\Omega \cap Z) \cap W = \partial\tilde{\Omega} \cap Z \cap W = \partial\tilde{\Omega} \cap W \ .$$

So, by the above inequality and the dimension property it follows that

$$i(h(S \cap Z) \cap W) = i(\partial\tilde{\Omega} \cap W) \geq \frac{\dim W}{d} \ . \tag{4.18}$$

(In the above formula we have to use the inequality because it might happen that $\partial\tilde{\Omega} \cap W \cap \operatorname{Fix} G \neq \phi$; cf. Theorem 2.5(ii)).

Finally, by (4.15), (4.18) and (4.17) we conclude the proof:

$$i(h(S \cap Z) \cap W) \geq \frac{\dim W}{d} - \frac{\dim V^\perp}{d} = \frac{\dim W}{d} - \frac{\operatorname{cod}_Z V}{d} \ . \qquad \square$$

<u>Proof of Theorem 4.1</u>. We set $S = \partial\Omega$ and

$$\begin{aligned} E_2 &= V \cap W \\ E_1 &= \text{orthogonal complement of } E_2 \text{ in } V \\ E_3 &= \text{orthogonal complement of } E_2 \text{ in } W \\ E_4 &= \text{orthogonal complement of } E_1 \oplus E_2 \oplus E_3 \text{ in } E \ . \end{aligned} \tag{4.19}$$

We have, obviously, that $V = E_1 \oplus E_2$, $W = E_2 \oplus E_3$, $E = E_1 \oplus E_2 \oplus E_3 \oplus E_4$. We observe, also, that the subspaces E_1, E_2, E_3, E_4, defined by (4.19) are G-invariant. Let $h = U + b \in H^*$ and $Z \subset E$ be a G-invariant, finite-dimensional subspace such that

$$E_2 \subset Z, \ E_4 \subset Z, \ b(\Omega) \subset Z \ .$$

Then

(4.20) $$h(\Omega \cap Z) \subset Z \; .$$

If we set $Z_1 = E_1 \cap Z$, $Z_3 = E_3 \cap Z$, we have that

(4.21) $$h(S \cap V) \cap W \supset h(S \cap V \cap Z) \cap W \cap Z = h(S \cap (Z_1 \oplus E_2)) \cap (E_2 \oplus Z_3) \; .$$

If we set $\tilde{V} = Z_1 \oplus E_2$, $\tilde{W} = E_2 \oplus Z_3$, we have that $\tilde{V}$ and $\tilde{W}$ satisfy the assumption of *Lemma 4.2 or Lemma 4.3 depending on the fact that* Fix $G \subset V$ or Fix $G \subset W$. Then by (4.20), (4.21), Lemma 4.2 and Lemma 4.3 we have that

$$i(h(S \cap V) \cap W) > \frac{\dim E_2 - \dim E_4}{d} = \frac{\dim(V \cap W) - \operatorname{codim}(V+W)}{d} \; .$$

By the above formula it easily follows that

$$i^*(S \cap V, W) > \frac{\dim (V \cap W) - \operatorname{codim}(V+W)}{d} \; . \qquad \square$$

5. Proof of the Abstract Theorems.

Proof of Theorem 1.4. The proof is based on Theorem 2.8. We have to check that all the assumptions of Theorem 2.8 are fulfilled.

We choose $G = S^1$ and $\mathbb{G} = r(G)$ where r is a unitary representation of S^1. By virtue of Lemma 3.3, f satisfies the condition (B) in $]0,+\infty[$. We set $D = W$ and $F = S_\rho \cap V$. Then (2.3)(i) and (ii) follow from (f_5)(iv) and (v).

By virtue of (f_5)(i), (ii), (iii), the assumptions of Theorem 4.1 are satisfied. Moreover, $\mathbb{G} = r(S^1)$ satisfies the 2-dimension property (cf. example 2.4). Then

$$\tilde{k} = \tfrac{1}{2}\,[\dim(V \cap W) - \operatorname{codim}(V + W)] \ .$$

Therefore $c_1,\dots,c_{\tilde{k}}$ are critical values of f.

By (f_5)(vi), it follows that $K_{c_k} \cap \operatorname{Fix}(S^1) = \emptyset$; then the second alternative of Remark 2.9(b) holds. $\square$

Proof of Theorem 1.5. We argue in the same way as in the proof of Theorem 1.4 except the following changes:

$$G = \mathbb{Z}_2 \quad \text{and} \quad \mathbb{G} = \{\mathrm{Id},\ \text{antipodal map}\} \ .$$

The index theory which we use in this case is the genus, (cf. example 2.3). Then $d = 1$.

Moreover, since $\operatorname{Fix}(\mathbb{G}) = \{0\}$, $K_c \cap \operatorname{Fix}(\mathbb{G}) = \emptyset$ for every $c > 0$. So the second alternative of Remark 2.9(b) holds. $\square$

In order to prove Theorem 1.9, we shall apply Theorem 2.11.

First, we define the class of homeomorphism $\mathbb{K}$ as follows: Set

$$(5.1) \qquad \mathbb{K} = \{h = U + b \in \mathbb{H}^* \mid h(u) = u \ \text{ for every } \ u \in f^{-1}(]-\infty,0])\} \ .$$

In this case $\mathbb{H}^*$ is given by the Definition 3.1' with $\mathbb{G} = \{\mathrm{Id}\}$ i.e. no invariancy property is required for $h \in \mathbb{H}^*$.

Now we need a lemma which is a variant of other similar results (cf. e.g. [BR], [BBF]).

Lemma 5.1. Q and $S_\rho \cap V$, as defined in Theorem 1.9, $\mathbb{K}$-intersect (cf. definition 2.10).

Proof. We have to show that

$$h(Q) \cap (S_\rho \cap V) \neq \emptyset \qquad \forall h \in \mathbb{K} \ .$$

The above formula holds provided that for each $h \in \mathbb{K}$ the following equations have at least one solution:

$$(5.2)\qquad \begin{aligned} &s \in [0,R_1]; \quad u \in B_{R_2} \cap V^{\perp} \\ &\|P_V \circ h(u + se)\| = \rho \\ &P_{V^{\perp}} \circ h(u + se) = 0 \end{aligned}$$

where P_V and $P_{V^{\perp}}$ denote the projections on V and $V^{\perp}$ respectively. Let $h = U + b \in K$, $U = e^{\alpha(\cdot)L}[\cdot]$, then the second equation in (5.2) can be written

$$(5.3)\qquad P_{V^{\perp}}[e^{\alpha(u+se)L}(u + se)] + P_{V^{\perp}}b(u + se) = 0 .$$

Since $se \in V$, we have

$$e^{\alpha(u+se)L}(se) \in V .$$

Then (5.3) can be written as follows

$$(5.4)\qquad P_{V^{\perp}}[e^{\alpha(u+se)L}(u)] + P_{V^{\perp}}b(u + se) = 0 .$$

Moreover, since $u \in V^{\perp}$, we have

$$e^{\alpha(u+se)L}(u) \in V^{\perp} .$$

Then (5.4) can be written

$$(5.5)\qquad e^{\alpha(u+se)L}u + P_{V^{\perp}}b(u + se) = 0 .$$

(5.5) is equivalent to the following equation

$$(5.6)\qquad u + e^{-\alpha(u+se)L}[P_{V^{\perp}}b(u + se)] = 0 .$$

Then (5.2) can be written as follows

$$(5.7)\qquad \begin{aligned} &s \in [0,R_1], \; u \in B_{R_2} \cap V^{\perp} \\ &\|P_V \circ h(u + se)\| = \rho \\ &u + e^{-\alpha(u+se)L}[P_{V^{\perp}}b(u + se)] = 0 . \end{aligned}$$

Using a Leray-Schauder degree argument as in [BR] (cf. also [BBF] and [BF1]) it can be proved that equation (5.7) has at least one solution. $\square$

Proof of Theorem 1.9. If K is the class of homeomorphisms (5.1), then by virtue of Theorem 3.3, f satisfies the property (B) in $]0,+\infty[$. We now set $D = Q$ and $F = S_\rho \cap V$. Then by virtue of Lemma 5.1, F and D K-intersect.

Therefore the conclusion follows from Theorem 2.11. $\square$

II. APPLICATIONS TO HAMILTONIAN SYSTEMS.

6. Some Estimates for the Action Functional.

We initially introduce some functional spaces we shall need in the following. If $m \in \mathbb{N}$ and $t > 1$ we set

$$L^t = L^t(S^1, \mathbb{R}^m) .$$

If $s \in \mathbb{R}$ we set

$$W^s = \{u \in L^2(S^1, \mathbb{R}^{2n}) \mid \sum_{\substack{j \in \mathbb{Z} \\ k=1,\ldots,2n}} (1 + |j|^2)^s |u_{jk}|^2 < +\infty\}$$

where u_{jk}($j \in \mathbb{Z}$, $k = 1,\ldots,2n$) are the Fourier components of u with respect to the basis (in $L^2(S^1, \mathbb{R}^{2n})$)

$$\psi_{jk} = e^{jtJ}\phi_k = \cos(jt)\phi_k + J\mathrm{sen}(jt)\phi_k \tag{6.1}$$

where $\{\phi_k\}$ ($k = 1,\ldots,2n$) is the standard basis in $\mathbb{R}^{2n}$. W^s equipped with the inner product

$$(u|v)_{W^s} = \sum_{j,k} (1 + |j|^2)^s u_{jk} v_{jk} \tag{6.2}$$

is an Hilbert space. We recall that the embedding $W^s \to L^t$ is compact if $\frac{1}{t} > \frac{1}{2} - s$. So in particular $W^{1/2}$ is compactly embedded in L^t for any $t > 1$.

Now consider the Hamiltonian system (0.2) where $H(t,z)$ is T-periodic in t. Making the change of variable $t \to \frac{2\pi t}{T}$, (0.2) becomes

$$-J\dot{z} = \omega H_z(\omega t, z) \quad \text{where} \quad \omega = T/2\pi . \tag{6.3}$$

Obviously the 2π-periodic solutions of (6.3) correspond to the T-periodic solutions of (0.2).

In order to construct the action functional whose critical points are the 2π-periodic solutions of (6.3) we introduce the following bilinear form

$$a(u,v) = \sum_{j \in \mathbb{Z}} \sum_{k=1}^{2n} j \; u_{jk} v_{jk} \qquad u,v \in W^{1/2}$$

where u_{jk}, v_{jk} are the Fourier-components of u,v with respect to the basis (6.1). The bilinear form $a(\cdot,\cdot)$ is symmetric and continuous in $W^{1/2}$. Let $L : W^{1/2} \to W^{1/2}$ be the self-adjoint, continuous operator defined by

(6.4) $$(Lu|v)_{W^{1/2}} = a(u,v) \qquad u,v \in W^{1/2}.$$

Observe that if $u,v \in C^1(S^1,\mathbb{R}^{2n})$

$$(Lu|v)_{W^{1/2}} = \int_0^{2\pi} (-J\dot{u},v)dt .$$

Suppose now that there are positive constants c_1,c_2,s such that

(6.5) $$|H_z(t,z)| \le c_1 + c_2|z|^s \quad \text{for any } t \text{ and } z .$$

Standard arguments show that the functional

(6.6) $$f(z) = \frac{1}{2}(Lz|z)_{W^{1/2}} - \omega\int_0^{2\pi} H(\omega t,z)dt \qquad z \in W^{1/2}$$

is Frechét-differentiable and that its critical points correspond to the 2π-periodic solutions of (6.3). For simplicity in the sequel we shall take $\omega = 1$ and suppose $H(t,z)$ 2π-periodic in t, so (6.6) becomes

(6.7) $$f(z) = \frac{1}{2}(Lz|z)_{W^{1/2}} - \psi(z)$$

where $\psi(z) = \int_0^{2\pi} H(t,z)dt.$

Since $W^{1/2}$ is compactly embedded in L^t for any $t > 1$, by (6.5) we have that the map $z \to H_z(t,z)$ is compact from $W^{1/2}$ on $W^{-1/2}$, then ψ' is compact.

Now it is easy to verify (cf. [BF2] sec. 3) that the spectrum of L consists of the limit points $-1,1$ and of the eigenvalues

$$\lambda_j = \frac{j}{(1+j^2)^{1/2}} \qquad j \in \mathbb{Z} ,$$

and that each eigenvalue λ_j has multiplicity $2n$. Then the functional (6.7) is "strongly indefinite" in the sense used in Section 1, moreover it satisfies the assumptions (f_1) and (f_2) of §1, because we can suppose $H(t,0) = 0$.

Let M_{λ_j} denote the eigenspace corresponding to the eigenvalue λ_j. We set

$$W^+ = \overline{\bigoplus_{j>0} M_{\lambda_j}}, \qquad W^- = \overline{\bigoplus_{j<0} M_{\lambda_j}}, \qquad W^0 = \ker L .$$

Every $z \in W^{1/2}$ can be decomposed as follows

$$z = z^+ + z^- + z^0 .$$

So we have

$$\text{(6.8)} \qquad \begin{aligned} &\text{(a)} \quad \langle Lz,z\rangle = \langle Lz^+,z^+\rangle + \langle Lz^-,z^-\rangle \\ &\text{(b)} \quad \tfrac{1}{2}\|z^+\|^2 \le \langle Lz^+,z^+\rangle \le \|z^+\|^2 \\ &\text{(c)} \quad \tfrac{1}{2}\|z^-\|^2 \le -\langle Lz^-,z^-\rangle \le \|z^-\|^2 . \end{aligned}$$

Now our aim is to find conditions on the Hamiltonian H which guarantee that also the assumption (f_3) is satisfied. We consider a sequence $\{z_n\} \subset W^{1/2}$, $z_n = (p_n,q_n)$ such that

$$\text{(6.9)} \qquad f(z_n) \to c \in]0,+\infty[$$

$$\text{(6.10)} \qquad \|f'(z_n)\| \cdot \|z_n\| \to 0 .$$

Let us initially prove the following lemma.

__Lemma 6.1__. Let $\{z_n\} \subset W^{1/2}$, $z_n = (p_n,q_n)$, be a sequence satisfying (6.9) and (6.10), then the following sequences

$$\text{(6.11)} \qquad \int_0^{2\pi} (H(t,z_n) - (H_p(t,z_n)|p_n))dt$$

$$\text{(6.12)} \qquad \int_0^{2\pi} (H(t,z_n) - (H_q(t,z_n)|q_n))dt$$

are bounded.

__Proof__. Easy computations show that

$$\text{(6.13)} \qquad \begin{aligned} &\text{(a)} \quad \langle f'(z_n),(p_n,0)\rangle = \int_0^{2\pi} ((\dot q_n|p_n) - (H_p(t,z_n)|p_n))dt \\ &\text{(b)} \quad \langle f'(z_n),(0,q_n)\rangle = \int_0^{2\pi} ((\dot q_n|p_n) - (H_q(t,z_n)|q_n))dt \\ &\text{(c)} \quad f(z_n) = \int_0^{2\pi} ((\dot q_n|p_n) - H(t,z_n))dt . \end{aligned}$$

By (6.9) and (6.10) the sequences

$$\langle f'(z_n),(p_n,0)\rangle, \quad \langle f'(z_n),(0,q_n)\rangle, \quad f(z_n)$$

are bounded. Then also right hand sides of the (6.13)'s are bounded. Subtracting (6.13)(c) from (6.13)(a) we get that (6.11) is bounded. Subtracting (6.13)(c) from (6.13)(b) we get that (6.12) is bounded. $\square$

The following lemma will be useful if the Hamiltonian H is asymptotically quadratic (cf. (0.8) and (0.9)) or if it grows more than quadratically in both the variables p and q but does not satisfy the growth condition (0.3) (e.g. $H(z) = |z|^2 \cdot \ln(1 + |z|^2)$).

Lemma 6.2. Suppose that H satisfies (6.5) and that there are positive contants c_3, c_4, α with $\alpha > s$ such that

(6.14) $$|H(t,z) - \tfrac{1}{2}\,(H_z(t,z)|z)| > c_3|z|^\alpha - c_4$$

for any $z \in \mathbb{R}^{2n}$ and $t \in \mathbb{R}$. Then the functional (6.7) satisfies the assumption (f_3).

Proof. Let $\{z_n\}$ be a sequence in $W^{1/2}$ satisfying (6.9) and (6.10). By Lemma 6.1 the sequence

(6.15) $$\int_0^{2\pi} (H(t,z_n) - \tfrac{1}{2}\,(H_z(t,z_n)|z_n))dt$$

is bounded. Then by (6.14), the sequence

(6.16) $$\|z_n\|_{L^\alpha} \text{ is bounded .}$$

Using the decomposition

(6.17) $$W^{1/2} = W^+ \oplus W^- \oplus W^0$$

we set

(6.18) $$z = z^+ + z^- + z^0 \text{ with } z^+ \in W^+,\ z^- \in W^-,\ z^0 \in W^0 .$$

From (6.10) we deduce that for a subsequence, which we continue to call $\{z_n\}$, we have

(6.19) $$\langle Lz_n, z_n^+\rangle - \int_0^{2\pi} (H_z(t,z_n)|z_n^+)dt \to 0 \quad \text{as} \quad n \to \infty .$$

Set $\gamma = \frac{\alpha}{s}$ and $\gamma' = \frac{\alpha}{\alpha - s}$. By (6.19) and (6.5) we have that

$$(6.20)\qquad \|z_n^+\|^2_{W^{1/2}} \le c_5 + c_6 \int_0^{2\pi} (H_z(t,z_n)|z_n^+)dt \le$$

$$\le c_5 + c_6 \left(\int_0^{2\pi} |H_z(t,z_n)|^{\gamma}dt\right)^{1/\gamma} \cdot \left(\int_0^{2\pi} |z_n^+|^{\gamma'}dt\right)^{1/\gamma'} \le$$

$$\le c_7 + c_8 \left(\int_0^{2\pi} |z_n|^{\alpha}dt\right)^{1/\gamma} \cdot \|z_n^+\|_{W^{1/2}}$$

where c_5, c_6, c_7, c_8 are positive constants. By (6.16) and (6.20) we have that

$$(6.21)\qquad \|z_n^+\|_{W^{1/2}} \text{ is bounded .}$$

Analogously it can be proved that

$$(6.22)\qquad \|z_n^-\|_{W^{1/2}} \text{ is bounded.}$$

It remains to prove that also $\|z_n^0\|_{W^{1/2}}$ is bounded. Consider $\phi(z) \in C^1(\mathbb{R}^{2n},\mathbb{R})$ such that

$$\phi(z) = c_9|z|^{\alpha} \quad \text{for} \quad |z| > c_{10}$$

where c_9, c_{10} are suitable positive constants.

Suppose first $\alpha < 1$, then ϕ' is bounded. So by (6.16) and by the mean value theorem we deduce that

$$c_{11} > \int_0^{2\pi} \phi(z_n)dt = \int_0^{2\pi} (\phi(z_n) - \phi(z_n^0))dt + \int_0^{2\pi} \phi(z_n^0)dt >$$

$$(6.23)\qquad > -c_{12} \int_0^{2\pi} |z_n - z_n^0|dt + \int_0^{2\pi} \phi(z_n^0)dt =$$

$$= -c_{12} \int_0^{2\pi} |z_n^+ + z_n^-|dt + \int_0^{2\pi} \phi(z_n^0)dt$$

where c_{11}, c_{12} are positive constants. By (6.21), (6.22) and (6.23) we have that $\|z_n^0\|_{L^\alpha}$ is bounded, then, since $\ker L$ is finite-dimensional, also $\|z_n^0\|_{W^{1/2}}$ is bounded.

Suppose now $\alpha > 1$, then

$$(6.24) \qquad \int_0^{2\pi} |z_n|^\alpha dt + c_{13} \geq \int_0^{2\pi} |z_n| dt \geq \int_0^{2\pi} (|z_n^0| - |z - z_n^0|) dt =$$

$$= \int_0^{2\pi} (|z_n^0| - |z_n^+ + z_n^-|) dt .$$

Finally by (6.24), (6.21) and (6.22) we deduce, also in this case, that $\|z_n^0\|_{W^{1/2}}$ is bounded. □

Now we consider the case in which H has the form (0.4) with a_{ij}, b_i and V of class C^1.

In the sequel we shall use the following shortened notation:

$a(q), A(q), a^k(q)$ $(k = 1,\dots,n)$ will denote respectively the matrices

$$(6.25) \qquad \{a_{ij}(t,q)\},\ \{(\text{grad } a_{ij}(t,q)|q)\},\ \{\frac{\partial a_{ij}}{\partial q_k}(t,q)\} \quad (k = 1,\dots,n) .$$

Moreover

$b(q), B(q), b^k(q)$ $(k = 1,\dots,n)$ will denote respectively the vectors in $\mathbb{R}^n$

$$(6.26) \qquad \{b_i(t,q)\},\ \{(\text{grad } b_i(t,q)|q)\},\ \{\frac{\partial b_i}{\partial q_k}(t,q)\} \quad (k = 1,\dots,n) .$$

Moreover, if v is a vector in $\mathbb{R}^n$ or $\mathbb{R}^{2n}$, $|v|$ will denote its norm.

Lemma 6.3. Assume that the Hamiltonian H has the form (0.4) with a_{ij}, b_i $(i,j = 1,\dots,n)$ and V of class C^1. Assume moreover that $(V_1),(A_1),(A_2),(B_1),(B_2)$ hold. Then, if $\{z_n\}$ $(z_n = (p_n,q_n))$ is a sequence in $W^{1/2}$ satisfying (6.9) and (6.10), the following sequences

$$\int_0^{2\pi} V(t,q_n) dt \qquad \int_0^{2\pi} (a(q_n)p_n|p_n) dt$$

are bounded.

Proof. Let $\delta > 0$ be a constant such that

(6.26') $$\alpha - \beta - 2\delta = 2 .$$

($\alpha - \beta$ are the constants of assumptions (V_1) and (A_2)).

By Lemma 6.1 we have that the sequences

(6.27) $$(1 + \beta + \delta) \int_0^{2\pi} [(a(q_n)p_n|p_n) - V(t,q_n)]dt \quad \text{and}$$

(6.28) $$\int_0^{2\pi} [(A(q_n)p_n|p_n) + (B(q_n)|p_n) + (V_q(t,q_n)|q_n) - H(t,z_n)]dt$$

are bounded.

Adding (6.27) to (6.28) we obtain that the sequence

(6.29) $$\int_0^{2\pi} [\delta(a(q_n)p_n|p_n) + (A(q_n)p_n|p_n) + \beta(a(q_n)p_n|p_n) + \\ + (V_q(t,q_n)|q_n) + (-\beta - 2 - \delta)V(t,q_n) + (B(q_n)|p_n) - (b(q_n)|p_n)]dt$$

is bounded .

By (V_1), (A_2), (6.26') and (6.29) there exists $M_1 > 0$ such that

(6.30) $$M_1 > \int_0^{2\pi} [\delta(a(q_n)p_n|p_n) + \delta V(t,q_n) + (B(q_n)|p_n) - (b(q_n)|p_n)]dt$$

for every $n \in \mathbb{N}$.

Now, by (B_1) and (B_2)

(6.31) $$\frac{|B(q)|^2 + |b(q)|^2}{\delta\nu(q)} < \frac{\delta}{2} V(t,q) + M_2 \quad \text{for every } t \in \mathbb{R} \text{ and } q \in \mathbb{R}^n$$

where M_2 is a positive constant. Then, using (6.31), we get

$$(6.32)\qquad \int_0^{2\pi} [(B(q_n)|p_n) - (b(q_n)|p_n)]dt \le \int_0^{2\pi} [|B(q_n)||p_n| + |b(q_n)||p_n|]dt \le$$

$$\le \int_0^{2\pi} \Big[\frac{|B(q_n)|^2}{\delta\nu(q_n)} + |p_n|^2 \cdot \frac{\delta}{4}\nu(q_n) + \frac{|b(q_n)|^2}{\delta\nu(q_n)} + \frac{\delta}{4}\nu(q_n)|p_n|^2\Big]dt \le$$

$$\le \int_0^{2\pi} \Big[\frac{\delta}{2} V(t,q_n) + \frac{\delta}{2}\nu(q_n)|p_n|^2\Big]dt + M_3 \quad \text{for every } n \in \mathbb{N}$$

where M_3 is a positive constant. By (6.30), (6.32) and (A_1) we deduce that

$$M_1 \ge \int_0^{2\pi} [\delta(a(q_n)p_n|p_n) + \delta V(t,q_n) - \frac{\delta}{2} V(t,q_n) - \frac{\delta}{2}\nu(q_n)|p_n|^2]dt - M_3$$

$$\ge \int_0^{2\pi} \Big[\frac{\delta}{2}(a(a_n)p_n|p_n) + \frac{\delta}{2} V(t,q_n)\Big]dt - M_3 \quad \text{for every } n \in \mathbb{N} .$$

From the above inequality, the conclusion follows. □

__Lemma 6.4__. Let the assumptions of Lemma 6.3 hold. Moreover assume that (V_2), (A_3) and (A_4) hold. Then, if $\{z_n\}$, $(z_n = (p_n,q_n))$, is a sequence in $W^{1/2}$ satisfying (6.9) and (6.10), the sequence

$$\int_0^{2\pi} |H_z(t,z_n)|dt$$

is bounded.

__Proof__. Just computing $H_z(t,z)$, we get

$$(6.33)\qquad |H_z(t,z_n)| \le 2|a(q_n)p_n| + |b(q_n)| + \sum_k |(a^k(q_n)p_n|p_n)| +$$

$$+ \sum_k |(b^k(q_n)|p_n)| + |V_q(t,q_n)| \quad \text{for every } n \in \mathbb{N} .$$

Observe that

$$(6.34)\qquad \text{for every } q,p \in \mathbb{R}^n \ |a(q)p| \le \|a(q)\| + (a(q)p|p) .$$

By (6.34), (A_4) and Lemma 6.3, it follows that

$$(6.35)\qquad \text{for every } n \in \mathbb{N} \int_0^{2\pi} |a(q_n)p_n|dt \le \int_0^{2\pi} [\|a(q_n)\| + (a(q_n)p_n|p_n)]dt \le M_4$$

where M_4 is a positive constant. By (A_1), we get that

(6.36) $$\|a(q)\| \geq \nu(q) \quad \text{for every } q \in \mathbb{R}^n .$$

Then, from (B_1), the above formula and (A_4) we get:

$$\int_0^{2\pi} |b(q_n)|dt \leq \int_0^{2\pi} \nu(q_n)^{1/2} \cdot |V(t,q_n)|^{1/2} dt + M_5 \leq$$

$$\leq (\int_0^{2\pi} \nu(q_n)dt)^{1/2} \cdot (\int_0^{2\pi} |V(t,q_n)|^{1/2} dt) + M_5 \leq$$

$$\leq (\int_0^{2\pi} \|a(q_n)\|dt)^{1/2} \cdot (\int_0^{2\pi} |V(t,q_n)|dt)^{1/2} + M_5 \leq$$

$$\leq M_6 \int_0^{2\pi} |V(t,q_n)|dt + M_7 \quad \text{for every } n \in \mathbb{N} .$$

Then, by Lemma 6.3 and the above inequaltiy, it follows that

(6.37) $$\forall n \in \mathbb{N} \quad \int_0^{2\pi} |b(q_n)|dt \leq M_8 .$$

Now, by (A_3) and Lemma 6.3, we have

(6.38) $$\forall n \in \mathbb{N} \quad \sum_k \int_0^{2\pi} |(a^k(q_n)p_n|p_n)|dt \leq M_9 \int_0^{2\pi} (a(q_n)p_n|p_n)dt \leq M_{10} .$$

Moreover, using (B_2) and (6.36), we have

$$\forall n \in \mathbb{N} \sum_k \int_0^{2\pi} |(b^k(q_n)|p_n)|dt \leq \sum_k (\int_0^{2\pi} \frac{|b^k(q_n)|^2}{\nu(q_n)})^{1/2} \cdot (\int_0^{2\pi} \nu(q_n)|p_n|^2 dt)^{1/2} \leq$$

$$\leq (M_{11} + M_{12} \int_0^{2\pi} V(t,q_n)dt)^{1/2} \cdot (\int_0^{2\pi} (a(q_n)p_n|p_n)dt)^{1/2} .$$

Then, from Lemma 6.3, we get

(6.39) $$\forall n \in \mathbb{N} \quad \sum_k \int_0^{2\pi} |(b^k(q_n)|p_n)|dt \leq M_{13} .$$

At last we observe that by Lemma 6.3 and (V_2)

$$(6.40) \qquad \forall n \in \mathbb{N} \quad \int_0^{2\pi} |V(t,q_n)|dt < M_{14} .$$

So, by (6.33), (6.35), (6.37), (6.38), (6.39) and (6.40), we deduce that the sequence $\int_0^{2\pi} |H_z(t,z_n)|dt$ is bounded. □

<u>Lemma 6.5</u>. Let the assumption of Lemma 6.4 hold. Let $\{z_n\} \subset W^{1/2}$ be a sequence which satisfies (6.9) and (6.10). Then we can select from $\{z_n\}$ a subsequence which is bounded in $W^{1/2}$.

<u>Proof</u>. Suppose that $\{z_n\} \subset W^{1/2}$ satisfies (6.9) and (6.10). Then by Lemma 6.4 $\{H_z(t,z_n)\}$ is bounded in L^1. L^1 is continuously embedded into $W^{-1/2-\eta/2}$, for any $\eta > 0$. Then

$$(6.41) \qquad \|H_z(t,z_n)\|_{W^{-1/2-\eta/2}} \text{ is bounded .}$$

By (6.10) we have:

$$(6.42) \qquad Lz_n - H_z(t,z_n) \to 0 \text{ in } W^{-1/2} .$$

So by (6.41) and (6.42) we have

$$(6.43) \qquad Lz_n \text{ is bounded in } W^{-1/2-\eta/2} .$$

By the definition of the spces W^1 and easy computation, we get

$$(6.44) \qquad \text{for each } z \in W^{1/2} \quad \|\tilde{z}\|_{W^{1/2-\eta/2}} < \text{const.}\|Lz\|_{W^{-1/2-\eta/2}}$$

where $\tilde{z} = z - z^0 = z^+ + z^-$ (cf. (6.18)). By (6.43) and (6.44) we have that

$$(6.45) \qquad \|\tilde{z}_n\|_{W^{1/2-\eta/2}} \text{ is bounded .}$$

Then, since $\eta > 0$ is arbitrary, by the Sobolev embedding theorems,

$$(6.46) \qquad \|\tilde{z}_n\|_{L^t} \text{ is bounded for any } t > 1 .$$

The next step is to prove that

$$(6.47) \qquad \{z_n^0\} \text{ is bounded in } L^1 .$$

We set

$$(p_n^0, q_n^0) = z_n^0 \qquad \forall n \in \mathbb{N} .$$

By (V_1) we have

$$(6.48) \qquad \int_0^{2\pi} |q_n^0|^\alpha dt \le c_1 \int_0^{2\pi} V(t,q_n^0)dt + c_2 \qquad \forall n \in \mathbb{N}$$

where c_1, c_2 are positive constants.

Then, by (6.48) and Lemma 6.3,

(6.49) $\{q_n^0\}$ is bounded in L^α and then in L^1.

Now we have to show that also $\{p_n^0\}$ is bounded in L^1.

To this end we initially show that there exists $\mu > 0$ s.t.

$$(6.50) \qquad \forall n \in \mathbb{N} \quad \int_0^{2\pi} \nu(q_n) \ge \mu .$$

By (6.46) and (6.49) there exists $M > 0$ s.t.

$$(6.51) \qquad \forall n \in \mathbb{N} \quad \|q_n\|_{L^1} \le M .$$

We now set

$$\nu_0 = \inf_{|q| \le M/\pi} \nu(q) \quad \text{and} \quad \Omega_n = \{t \in [0,2\pi) \mid |q_n(t)| \le M/\pi\} .$$

Then

$$\forall n \in \mathbb{N} \qquad M \ge \|q_n\|_{L^1} \ge \int_{[0,2\pi] \setminus \Omega_n} |q_n| dt \ge M/\pi(2\pi - \operatorname{meas} \Omega_n) .$$

From which we get

$$\forall n \in \mathbb{N} \quad \operatorname{meas} \Omega_n \ge \pi .$$

Therefore we have

$$\forall n \in \mathbb{N} \quad \int_0^{2\pi} \nu(q_n)dt \ge \int_{\Omega_n} \nu(q_n)dt \ge \nu_0 \cdot \operatorname{meas} \Omega_n \ge \nu_0 \pi .$$

Then (6.50) holds with $\mu = \nu_0 \pi$.

Now, by Lemma 6.3 and (A_1) there exists $c > 0$ s.t.

$$(6.52) \qquad \forall n \in \mathbb{N} \quad c \geq \int_0^{2\pi} (a(q_n)p_n|p_n)dt \geq \int_0^{2\pi} \nu(q_n)|p_n|^2 = \int_0^{2\pi} \nu(q_n)|p_n^0 + \tilde{p}_n|^2 dt \geq$$

$$= |p_n^0|^2 \int_0^{2\pi} \nu(q_n)dt - 2|p_n^0| \int_0^{2\pi} \nu(q_n)|\tilde{p}_n|dt + \int_0^{2\pi} \nu(q_n)|\tilde{p}_n|^2 dt$$

$$\geq |p_n^0|^2 \int_0^{2\pi} \nu(q_n)dt - 2|p_0| \int_0^{2\pi} \nu(q_n)|\tilde{p}_n|dt \ .$$

Now

$$(6.53) \qquad \int_0^{2\pi} \nu(q_n)|\tilde{p}_n|dt \leq \|\nu(q_n)\|_{L^2} \cdot \|\tilde{p}_n\|_{L^2} \ .$$

By (A_4) and (V_2) we get

$$(6.54) \qquad \forall n \in \mathbb{N} \quad \|\nu(q_n)\|_{L^2}^2 \leq c_1 \int_0^{2\pi} V(t,q_n)^2 dt + c_2 \leq c_3 \int_0^{2\pi} |q_n|^{2s} dt + c_4$$

where c_1, c_2, c_3, c_4 are positive constants.

Moreover, because $\ker L$ is finite dimensional, from (6.49) and (6.46) we deduce that

$$(6.55) \qquad \|q_n\|_{L^{2s}} \text{ is bounded.}$$

Then from (6.53), (6.54), (6.55) it follows that

$$(6.56) \qquad \forall n \in \mathbb{N} \quad \int_0^{2\pi} \nu(q_n)|\tilde{p}_n|dt \leq \|\nu(q)\|_{L^2} \|\tilde{p}_n\|_{L^2} \leq c_6 \|\tilde{p}_n\|_{L^2} \ .$$

Using (6.46) and (6.56) we get

$$(6.57) \qquad \forall n \in \mathbb{N} \quad \int_0^{2\pi} \nu(q_n)|\tilde{p}_n|dt \leq c_7$$

where c_7 is a positive constant. So from (6.52), (6.50) and (6.57) we get

$$(6.58) \qquad \forall n \in \mathbb{N} \quad c \geq \mu|p_n^0|^2 - c_7|p_n^0| \ .$$

Then

(6.59) $\quad |p_n^0|$ is bounded .

Finally, because dim ker L < +∞, from (6.49), (6.59) and (6.46) we deduce that

(6.60) $\quad$ for any $t > 1$ $\quad \|z_n\|_{L^t}$ is bounded .

Let us now show that $\|z_n\|_{W^{1/2}}$ is bounded.

By (6.19) we have

$$\forall n \in \mathbb{N} \quad \|z_n^+\|^2_{W^{1/2}} \leq c_8\Big(1 + \int_0^{2\pi} |H_z(t,z_n)||z_n^+|dt\Big) \tag{6.61}$$

where c_8 is a positive constant .

By (6.33) and the assumptions (H_0) there exists $\gamma > 0$ s.t.

$$\forall z \in \mathbb{R}^{2n}, \ \forall t \in \mathbb{R} \quad |H_z(t,z)| \leq \text{const.}(1 + |z|^\gamma) .$$

Then from (6.61) we get

$$\forall n \in \mathbb{N} \quad \|z_n^+\|^2_{W^{1/2}} \leq \text{const.}(1 + \|z_n\|^\gamma_{L^{2\gamma}} \cdot \|z_n^+\|_{W^{1/2}}) . \tag{6.62}$$

Then from (6.60) and (6.62) it follows that

$$\|z_n^+\| \text{ is bounded} .$$

Analogously it can be proved that

$$\|z_n^-\|_{W^{1/2}} \text{ is bounded} .$$

Finally, because ker L is finite dimensional, we deduce that also

$$\|z_n^0\|_{W^{1/2}} \text{ is bounded} .$$

□

We conclude this section with the following lemma.

<u>Lemma 6.6</u>. If (H_0) hold, the functional (6.7) satisfies (f_1), (f_2) and (f_3) in the space $W^{1/2}$.

<u>Proof</u>. (f_1)(i) and (ii) follow from the construction of L.

By assumptions (V_2), (A_3), (A_4), (B_1), (B_2) and standard majorizations, it follows that f satisfies (6.5). Then (f_1)(ii) is satisfied. (f_3) follows from Lemma 6.5. □

7. Superquadratic Hamiltonians.

In this section we shall prove Theorems 0.1, 0.2 and 0.3. It will be useful to introduce the following notation

$$(7.1)\qquad W_j^+ = \overline{\bigoplus_{k>j} M_{\lambda_k}}\,; \qquad\qquad W_j^- = \overline{\bigoplus_{k<j} M_{\lambda_j}}\,.$$

If $j > 0$, then W_j^+ W^+ so that, for every $z \in W_j^+$, (6.8)(b) holds. The following lemmas provide estimates which shall be used in the proof of the theorems.

<u>Lemma 7.1</u>. For every $c_0 > 0$, there exist $j \in \mathbb{Z}$ and $R > 0$ such that

$$f(z) > c_0 \quad \text{for every} \quad z \in W_j^+,\ \|z\| = R$$

where f is the functional defined by (6.7).

<u>Proof</u>. Since H grows polynomially, there are constants $r, c_1, c_2 > 0$ such that

$$|H(t,z)| < c_1 + c_2|z|^r\,.$$

Then

$$(7.2)\qquad |\psi(z)| < 2\pi c_1 + c_2\|z\|^r_{L^r}\,.$$

Now, by the Sobolev embedding theorem, there are constant $c_3, s > 0$ such that

$$(7.3)\qquad \|z\|_{L^r} < c_3\|z\|_{W^{1/2-s}}\,.$$

If $z \in W_j^+$, $j > 1$, we have

$$\|z\|^2_{W^{1/2-s}} = \sum_{k>j} (1+k^2)^{1/2-s}|z_k|^2 < (1+j^2)^{-s} \sum_{k>j} (1+k^2)^{1/2}|z_k|^2 =$$

$$= (1+j^2)^{-s}\|z\|^2 < j^{-2s}\|z\|^2\,.$$

Then by the above formula (7.2) and (7.3) we get

$$|\psi(z)| < c_4 j^{-\rho}\|z\|^r + c_5 \quad \text{for every} \quad z \in W_j^+$$

where c_4 and c_5 are suitable positive constants and $\rho = sr > 0$.

Then, by (6.8) and the above formula, for $z \in W_j^+$, $\|z\| = R$ we have

$$f(z) = \frac{1}{2}\langle Lz,z\rangle - \psi(z) > \frac{1}{4}R^2 - c_4 j^{-\rho}R^r - c_5 = [\tfrac{1}{4} - c_4 j^{-\rho}R^{r-2}]R^2 - c_5\,.$$

The above formula proves the lemma, in fact, it is sufficient to choose R such that

$\frac{1}{8} R^2 > c_5 + c_0$ and j such that

$$c_4 j^{-\rho} R^{r-2} < \frac{1}{8} . \qquad \square$$

Lemma 7.2. Suppose that H satisfies assumptions (H_0). Then there exist constants a_1 and $a_2 > 0$ such that

$$H(z,t) > a_1 |q|^\alpha - a_2 \tag{7.4}$$

and

$$\beta H(z,t) + (H_z(z,t)|z) > a_1 |q|^\alpha + \mu |p|^2 - a_2 \tag{7.5}$$

where $z = (p,q)$ and μ is the constant in (A_2).

Proof. We prove (7.5).

We shall use the notations introduced in Section 6 (cf. 6.25, 6.26), moreover $c_1, \ldots$ will denote positive constants.

By (A_1), (A_2) and (V_1) we have

$$\begin{aligned} \beta H(z,t) + (H_z(z,t)|z) &= ([\beta a(q) + 2a(q) + A(q)]p|p) + \\ &+ ((\beta + 1)b(q) + B(q)|p) + \beta V(q,t) + (V_q(q,t)|q) > \\ &> \mu |p|^2 + 2\nu(q)|p|^2 - |(\beta + 1)b(q) + B(q)||p| + \beta V(q,t) - c_1 . \end{aligned} \tag{7.6}$$

Using (B_1), (B_2) we have

$$\begin{aligned} |(\beta + 1)b(q) + B(q)||p| &< \frac{|(\beta + 1)b(q) + B(q)|^2}{2\nu(q)} + \frac{\nu(q)}{2} |p|^2 < \\ &< \frac{\beta}{2} V(q,t) + \nu(q)|p|^2 + c_2 . \end{aligned} \tag{7.7}$$

Then, by (7.6), (7.7) we have

$$\beta H(z,t) + (H_z(z,t)|z) > \mu |p|^2 + \nu(q)|p|^2 + \frac{\beta}{2} V(q,t) - c_3 .$$

Then, using again assumption (V_1), we get (7.5). Similar arguments can be used to prove (7.4). $\square$

Lemma 7.2'. Let ϕ a Frechêt differentiable functional on an Hilbert space E, with $\phi(0) = 0$. Suppose that ϕ satisfies the following assumption:

there exist $R, M, \lambda > 0$ s.t.

$$\lambda \phi(x) + \langle \phi'(x), x \rangle < \begin{cases} M & \text{if } \|x\| < R \\ \\ -1 & \text{if } \|x\| > R . \end{cases} \tag{7.8}$$

Then there exist $\bar{R} > 0$ s.t.

$$\phi(x) < 0 \quad \text{for} \quad \|x\| > \bar{R} \ .$$

<u>Proof</u>. Let $v_0 \in H$, $\|v_0\| = 1$ and set

$$g(t) = \lambda\phi(tv_0) \qquad t > 0 \ .$$

We shall initially prove that

(7.9) $$g(t) \leq M \quad \text{for any} \quad t > 0 \ .$$

We argue by contradiction and suppose that there exists $t_1 > 0$ s.t.

$$g(t_1) > M \ .$$

Then, since $g(0) = 0$, there exists $t_0 < t_1$ such that

$$g(t) > M \quad \forall t \in]t_0,t_1[\quad \text{and} \quad g(t_0) = M \ .$$

Obviously there is $\bar{t} \in]t_0,t_1[$ s.t.

$$g'(\bar{t}) > 0 \ .$$

Then

$$g(\bar{t}) + \frac{\bar{t}}{\lambda}\, g'(\bar{t}) > M$$

which means that

$$\lambda\phi(\bar{t}v_0) + \langle\phi'(\bar{t}v_0),\bar{t}v_0\rangle > M$$

and this contradicts (7.8).

Now consider

$$\bar{R} > 0 \quad \text{s.t.} \quad M - \lambda \ln \bar{R}/R < 0 \ .$$

Let us now show that

(7.10) $$\text{there exists} \quad t_2 \in [R,\bar{R}] \quad \text{s.t.} \quad g(t_2) < 0 \ .$$

By (7.8) we have

(7.11) $$g(t) + \frac{1}{\lambda}\, g'(t) + < -1 \quad \text{if} \quad t > R \ .$$

Then, since $g(R) \leq M$ (cf. 7.9), we have:

$$g(\bar{R}) \leq \int_R^{\bar{R}} g'(s) + M < -\int_R^{\bar{R}} \frac{\lambda}{s}\, ds - \int_R^{\bar{R}} \frac{g(s)}{s}\, ds + M = M - \lambda \ln \bar{R}/R - \int_R^{\bar{R}} \frac{g(s)}{s}\, ds$$

$$< -\int_R^{\bar{R}} \frac{g(s)}{s}\, ds \ .$$

From this inequality it is easy to deduce that (7.10) holds.

Now we prove that

$$\phi(x) < 0 \quad \text{for} \quad \|x\| > \bar{R} .$$

Obviously it is sufficient to show that

$$g(t) < 0 \quad \text{for} \quad t > t_2 . \tag{7.12}$$

Arguing by contradiction suppose that there exists $t_4 > t_2$ s.t. $g(t_4) > 0$. Then obviously there exists $t_3 \in]t_2,t_4[$ such that

$$g(t_3) = 0 \quad \text{and} \quad g'(t_3) > 0 . \tag{7.13}$$

Since $t_3 > R$, by (7.8) we get

$$g(t_3) + \frac{g'(t_3)}{\lambda} t_3 < -1 . \tag{7.14}$$

Obviously (7.14) contradicts (7.13). □

<u>Lemma 7.3</u>. Suppose that H satisfies (H_0). Then for any $j \in \mathbb{Z}$. There exists $R > 0$ s.t.

$$f(z) < 0 \quad \text{for} \quad \|z\| > R \qquad z \in W_j^- = \overline{\bigoplus_{k<j} M_{\lambda_k}} .$$

<u>Proof</u>. The interesting case occurs when $j > 0$, otherwise it is trivial.

By virtue of Lemma 7.2' it is enough to prove that

$$\beta f(z) + \langle f'(z),z\rangle \longrightarrow -\infty \quad \text{as} \quad \|z\| \to \infty . \tag{7.15}$$

In the following $c_1,\ldots,c_6$ will denot positive constants.

Let $z = \binom{p}{q} \in W_j^-$ and set

$$z = z^* + z_0 + \hat{z}$$

where

$$z^* = \binom{p^*}{q^*} \in M_{\lambda_{-j}} \oplus M_{\lambda_{-j+1}} \oplus \cdots \oplus M_{\lambda_{-1}} \oplus M_{\lambda_1} \oplus \cdots \oplus M_{\lambda_j}$$

$$z_0 = \binom{p_0}{q_0} \in \operatorname{Ker} L, \qquad \hat{z} = \binom{\hat{p}}{\hat{q}} \in W_{-j-1}^- = \overline{\bigoplus_{k<-j-1} M_{\lambda_k}} .$$

Then, by using Lemma (7.2), it is easy to see that

$$(7.16)\qquad \beta f(z) + \langle f'(z),z\rangle \le (\tfrac{\beta}{2}+1)(\langle Lz^*,z^*\rangle + \langle L\hat{z},\hat{z}\rangle) -$$

$$- \mu(\|p^*\|^2_{L^2} + \|\hat{p}\|^2_{L^2} + \|p_0\|^2_{L^2}) -$$

$$- c_1(\|q^*\|^\alpha_{L^2} + \|\hat{q}\|^\alpha_{L^2} + \|q_0\|^\alpha_{L^2}) + C_2 \le$$

$$\le (\tfrac{\beta}{2}+1)(\langle Lz^*,z^*\rangle) - \frac{1+j}{2+j}\|\hat{z}\|^2) -$$

$$- \mu\|p^*\|^2_{L^2} - c_1\|q^*\|^\alpha_{L^2} - c_3(\|\hat{z}\|^2_{L^2} + \|z_0\|^2_{L^2}) + c_2$$

$$\le h(z^*) - c_4(\|\hat{z}\|^2 + \|z_0\|^2_{L^2}) + c_2$$

where

$$h(z^*) = (\tfrac{\beta}{2}+1)\langle Lz^*,z^*\rangle - \mu\|p^*\|^2_{L^2} - c_1\|q^*\|^\alpha_{L^2}\ .$$

The above formula shows that (7.15) is verified once we prove that

$$(7.17)\qquad h(z^*) \to -\infty \quad \text{as} \quad \|z^*\|_{L^2} \to +\infty\ .$$

In order to prove (7.17) we need to find a more "explicit" form of $\langle Lz^*,z^*\rangle$, $\|p^*\|_{L^2}$, $\|q^*\|_{L^2}$. We set

$$z^* = \sum_{\ell=1}^{1} (z_\ell + z_{-\ell}) \qquad z_\ell = \binom{p_\ell}{q_\ell} \in M_{\lambda_\ell}\ .$$

It is not difficult to verify that for any ℓ we have

$$p_\ell = \sum_{k=1}^{n} a_{\ell k}\cos\ell t e_k - b_{\ell k}\sin\ell t e_k$$

$$q_\ell = \sum_{k=1}^{n} a_{\ell k}\sin\ell t e_k + b_{\ell k}\cos\ell t e_k$$

where $e_k (k = 1,\dots,n)$ is the standard basis in $\mathbb{R}^n$ and $a_{\ell k}, b_{\ell k}$ are real coefficients.

By straight computations we obtain

$$(7.18)\qquad \langle Lz^*, z^*\rangle \le \sum_{\ell=1}^{j} \ell\left(\|z_\ell\|^2_{L^2} - \|z_{-\ell}\|^2_{L^2}\right) = \sum_{\ell=1}^{j}\sum_{k=1}^{n} 2\ell\left(a^2_{\ell k} + b^2_{\ell k} - a^2_{-\ell k} - b^2_{-\ell k}\right) .$$

Moreover

$$(7.19)\qquad \|p^*\|^2_{L^2} = \sum_{\ell=1}^{j}\sum_{k=1}^{n} (a_{\ell k} + a_{-\ell k})^2 + (b_{\ell k} - b_{-\ell k})^2$$

and

$$(7.20)\qquad \|q^*\|^2_{L^2} = \sum_{\ell=1}^{j}\sum_{k=1}^{n} (a_{\ell k} - a_{-\ell k})^2 + (b_{\ell k} + b_{-\ell k})^2 .$$

Then

$$h(z^*) \le q(z^*) \quad \text{where}$$

$$q(z^*) = \sum_{\ell=1}^{j}\sum_{k=1}^{n} \left(\frac{\beta}{2} + 1\right)2\ell\left(a^2_{\ell k} - a^2_{-\ell k}\right) - \mu(a_{\ell k} + a_{-\ell k})^2 - c_5|a_{\ell k} - a_{-\ell k}|^\alpha$$

$$+ \left(\frac{\beta}{2} + 1\right)\left(b^2_{\ell k} - b^2_{-\ell k}\right) - \mu(b_{\ell k} - b_{-\ell k})^2 - c_5|b_{\ell k} + b_{-\ell k}|^\alpha .$$

Since $\alpha > 2$ *it can be verified that*

$$q(z^*) \to -\infty \quad \text{as} \quad \|z^*\|^2_{L^2} = \sum_{\ell=1}^{j}\sum_{k=1}^{n} a^2_{\ell k} + a^2_{-\ell k} + b^2_{\ell k} + b^2_{-\ell k} \to \infty .$$

Then (7.17) easily follows. □

Proof of Theorem 0.1. We will apply Theorem 1.4.

By Lemma 6.6, (f_1), (f_2) and (f_3) follow. Since the Hamiltonian H does not depend on t, also (f_4) is satisfied. It remains to verify the geometrical assumptions (f_5).

We set

$$c_0 = \max\{1, -2\pi \cdot \inf_{z \in \mathbb{R}^{2n}} H(z)\} + 1 .$$

The constant c_0 is well defined because by Lemma 7.2, H is bounded from below.

By virtue of Lemma 7.1, it is possible to choose $R > 0$ and $j \in \mathbb{Z}$ such that

$$f(z) > c_0 \quad \text{for every} \quad z \in W_j^+; \ \|z\| = R .$$

Now set

$$V = W_j^+$$

and, chosen n arbitrarily, set

$$W = W_{j+n}^- = (W_{j+n}^+)^{\perp} .$$

With such a choice of V and W, the assumptions (f_5)(i), (ii), (iii) and (iv) are trivially satisfied. Moreover (f_5)(v) is satisfied by virtue of Lemma 7.3 and (f_5)(vi) is satisfied by our choice of c_0.

Then the conclusion of Theorem 1.5 applies and we get the existence of at least

$$\frac{1}{2}\,(\dim(V \cap W) - \operatorname{codim}(V + W)) = n$$

critical values with critical points $z_1, \ldots, z_n$ such that

$$f(z_k) > c_0 . \tag{7.21}$$

It remains to show that the corresponding critical points are not constants.

Suppose that one of them is a constant function $\bar{z}$. Then we have

$$f(\bar{z}) = -2\pi H(z) < c_0 .$$

This contradicts (7.21).

By the arbitrariness of n the conclusion follows. □

<u>Proof of Theorem 0.2</u>. It follows the same argument of the proof of Theorem 0.1 except that we use Theorem 1.5 instead of Theorem 1.4. □

<u>Proof of Theorem 0.3</u>. We shall apply Theorem 1.9.

We can assume without loss of generality that

$$H(t,0) = 0 \quad \text{for every} \quad t \in \mathbb{R} .$$

It is not difficult to prove that f is twice Frechet differentiable for $z = 0$. Then by

(H_4), we have:

$$f(z) = f(0) + \langle f'(0),z\rangle + \frac{1}{2} f''(0)[z,z] + o(\|z\|^2) = \tag{7.22}$$

$$= \frac{1}{2} \langle Lz,z\rangle - \frac{\omega}{2}\int_0^{2\pi} (H_{zz}(\omega t,0)z|z)dt + o(\|z\|^2)$$

where $\omega = \frac{T}{2\pi}$. and $z \in W^{1/2}$. By (H_6), it follows that

$$\omega \int_0^{2\pi} (H_{zz}(\omega t,0)z,z)dt < \gamma \int_0^{2\pi} |z|^2 dt .$$

Then by the above inequality and (7.22)

$$f(z) > \frac{1}{2} \langle Lz,z\rangle - \frac{\gamma}{2} \|z\|^2_{L^2} + o(\|z\|^2) . \tag{7.23}$$

By the definition of $\langle Lz,z\rangle$, we have that

$$\langle Lz,z\rangle > \|z\|^2_{L^2} \quad \text{for every} \quad z \in W^+ .$$

Then by the above inequality, (7.23) and (6.8)(b) we get

$$f(z) > \frac{1}{2} (1 - \gamma)\langle Lz,z\rangle + \frac{\gamma}{2} \langle Lz,z\rangle - \frac{\gamma}{2} \|z\|^2_{L^2} + o(\|z\|^2) >$$

$$> \frac{1}{4} (1 - \gamma)\|z\|^2 + o(\|z\|^2) \quad \text{for every} \quad z \in W^+ .$$

So there exist ρ, $c_0 > 0$ such that

$$f(z) > c_0 \quad \text{for every} \quad z \in W^+, \quad \|z\| = \rho . \tag{7.24}$$

Now let $e \in W^+$ be the eigenfunction corresponding to the first positive eigenvalue λ_1 of L and let R_1,R_2 be two positive constants. We set

$$T = \{se : s \in [0,R_1]\}, \quad Q = \{u + v \mid u \in W^- \oplus \ker L, \|u\| < R_2 \text{ and } v \in T\} .$$

Observe that $Q \subset W_1^-$. Then by Lemma 7.3

$$\sup_{z \in Q} f(z) < +\infty .$$

Moreover, by Lemma 7.3, if R_1 and R_2 are large enough we get that

$$f(z) < 0 \quad \text{for every} \quad z \in \partial Q .$$

Thus all the assumptions of Lemma 1.9 are satisfied with $V = W^+$. Then f has a critical value c

$$c > c_0 > 0 \,. \tag{7.25}$$

The corresponding critical point $\bar{z} \in W^{1/2}$ cannot be constant because in this case we would have

$$c = f(\bar{z}) = -\int_0^{2\pi} H(\omega t, z) < 0$$

and this inequality contradicts (7.25). □

We end this section considering Hamiltonians $H(z)$ which do not depend on t and grow more then quadratically in both the variables.

More precisely we suppose that there exist positive constants c_1, c_2, c_3, c_4, α, β with $\alpha > \beta$ and $\beta > 0$ such that

$$\text{(7.26)} \quad \begin{aligned} &\text{(a)} \quad |H_z(z)| < c_1 + c_2|z|^{\beta} \quad \text{for every} \quad z \in \mathbb{R}^{2n}\,, \\ &\text{(b)} \quad \frac{1}{2}\,(H_z(z)|z) - H(z) > c_3|z|^{\alpha} - c_4 \quad \text{for every} \quad z \in \mathbb{R}^{2n}\,. \end{aligned}$$

Observe that this "superquadraticity" condition (7.26)(b) covers cases which are not covered by (0.3). For example the function

$$H(z) = |z|^2 \log(1 + |z|^2)$$

satisfy the (7.26) but not (0.3). For Hamiltonians of this type the following theorem holds.

Theorem 7.4. If $H \in C^1(\mathbb{R}^{2n},\mathbb{R})$ satisfies (7.26), then for every $T > 0$, the Hamiltonian system (0.2) has infinitely many *nonconstant* T-periodic solutions for any period $T > 0$.

Sketch of the Proof. We apply Theorem 1.4. (f_1) and (f_2) are verified as in the proof of Theorem 0.1. (f_3) follows from Lemma 6.2. (f_4) follows by the fact that H is time independent. Since H satisfies (7.12)(a), Lemma 7.1 holds, and by (7.26)(b) it is easy to show that the analogous of Lemma 7.3 is true. Then reasoning as in the proof of Theorem 0.1, the conclusion follows. □

8. Asymptotically Quadratic Hamiltonians.

Proof of Theorem 0.5. Let L_0 and $L_\infty : W^{1/2} \to W^{-1/2}$ be the operator defined as follows

$$L_\infty z = -J\dot{z} - \omega H_{zz}(\infty)z$$

$$L_0 z = -J\dot{z} - \omega H_{zz}(0)z \ .$$

Then if we set

$$(L_\infty z|v)_{W^{1/2}} = {}_{W^{-1/2}}\langle L_\infty z, v\rangle_{W^{1/2}} \qquad \forall v \in W^{1/2}$$

$$(L_0 z, v)_{W^{1/2}} = {}_{W^{-1/2}}\langle L_0 z, v\rangle_{W^{1/2}}$$

it follows that L_0 and L_∞ are two self-adjoint operators in $W^{1/2}$. It is easy to see that the spectrum of L_0 and L_∞ consists of eigenvalues of finite multiplicity having $+1$ and -1 as accumulation points.

Let M_μ^0 (resp. M_μ^∞) denote the eigenspace of L_0 (resp. L_∞) corresponding to the eigenvalue μ. We set

$$W_0^+ = \overline{\bigoplus_{\mu>0} M_\mu^0}, \quad W_0^- = \overline{\bigoplus_{\mu<0} M_\mu^0}, \quad W_\infty^+ = \overline{\bigoplus_{\mu>0} M_\mu^\infty}, \quad W_\infty^- = \overline{\bigoplus_{\mu<0} M_\mu^\infty}$$

where the closures are taken in $W^{1/2}$. We initially suppose that the Hamiltonian H satisfies (0.8), (0.9), (0.10), (0.12) and (0.13). We can write the action functional as follows:

$$f(z) = +\frac{1}{2}(L_\infty z|z) - \omega\int_0^{2\pi}\left(H(z) - \frac{1}{2}(H_{zz}(\infty)z|z)\right)dt \ .$$

We shall show that f satisfies the assumptions of Theorem 1.5 with:

$$L = L_\infty, \quad \psi(z) = \omega\int_0^{2\pi}\left(H(z) - \frac{1}{2}(H_{zz}(\infty)z|z)\right)dt \ ,$$

$$V = W_0^+ \ \text{ and } \ W = W_\infty^- \ .$$

It is easy to see that (f_1), (f_2), (f_4) are satisfied. Moreover, by virtue of the nonresonance assumption (0.10), standard argument show that also (f_3) is satisfied (cf. the proof of Theorem 6.1 an Remark 4.10 in [B2]). Let us now prove that also (f_5) is satisfied.

$(f_5)(i)$ is obviously satisfied, moreover, since $L_\infty - L_0$ is compact, also $(f_5)(ii)$ holds.

Because $H_{zz}(\infty)$ is positive definite, we have

$$\{\text{constant functions}\} = \mathrm{Fix}(S^1) \subset W_\infty^- .$$

Then also $(f_5)(iii)$ is satisfied. Let $z \in W_0^+$ then,

$$f(z) = f(0) + \langle f'(0),z\rangle + \frac{f''(0)}{2}[z,z] + o(\|z\|^2) = \frac{1}{2}(L_0 z|z) + o(\|z\|^2)$$

$$\geq \frac{\mu_0}{2}\|z\|^2 + o(\|z\|^2) \quad \text{as} \quad \|z\| \to 0$$

where $\mu_0 = \min\{\mu \in \sigma(L_0) | \mu > 0\}$.

So also assumption $(f_5)(iv)$ holds. Moreover, by (0.13), assumption $(f_5)(vi)$ holds.

Let us finally verify that $(f_5)(v)$ is satisfied. Let $z \in W_\infty^- = W$ then

$$f(z) \leq \mu_1\|z\|^2 - \omega\int_0^{2\pi}\left(H(z) - \frac{1}{2}(H_{zz}(\infty)z|z)\right)dt \tag{8.1}$$

where $\mu_1 = \max\{\mu \in \sigma(L_\infty) | \mu < 0\}$. If we set

$$g(z) = H_z(z) - H_{zz}(\infty)z$$

then, by (0.8),

$$\frac{g(z)}{z} \to 0 \quad \text{as} \quad |z| \to +\infty . \tag{8.2}$$

With this notation we have

$$\int_0^1 (H_z(sz) - H_{zz}(\infty)(sz)|z)ds = \int_0^1 (g(sz)|z)ds .$$

So

$$H(z) - \frac{1}{2}(H_{zz}(\infty)z|z) = \int_0^1 (g(sz)|z)ds .$$

From the above formula, we have

(8.3) $$\forall z \in \mathbb{R}^{2n} \quad |H(z) - \frac{1}{2}(H_{zz}(\infty)z|z)| < |z| \int_0^1 |g(sz)|ds .$$

By (8.2), for every $\varepsilon > 0$, there exists $M > 0$ such that

(8.4) $$|g(z)| < \varepsilon|z| \quad \text{for} \quad |z| > M .$$

Let $|z| > M$ and set

$$A_1(z) = \{t \in [0,1] \,|\, |tz| < M\}$$

$$A_2(z) = \{t \in [0,1] \,|\, |tz| > M\} .$$

Then, by (8.4), we have

(8.5) $$\int_0^1 |g(sz)|ds = \int_{A_1(z)} |g(sz)|ds + \int_{A_2(z)} |g(sz)|ds < c_1 + \frac{\varepsilon}{2}|z|$$

where $c_1 = \sup\{|g(z)| \,|\, |z| < M\}$.

Using (8.4) and (8.5),

(8.6) $$\forall z \in \mathbb{R}^{2n}, \ |z| > M \qquad |H(z) - \frac{1}{2}(H_{zz}(\infty)z|z)| < c_1|z| + \frac{\varepsilon}{2}|z|^2 .$$

Then, by (8.1) and (8.6), we easily deduce that

$$\forall z \in W_\infty^- \qquad f(z) < \mu_1 \|z\|^2 + \omega(\|z\|_{L^1} + \frac{\varepsilon}{2}\|z\|^2_{L^2}) + c_2$$

where c_2 is a positive constant depending on ε.

So if we choose ε sufficiently small, by the above formula f is bounded from above on $W_\infty^- = W$, i.e. (f_5) holds. Thus all the assumptions of Theorem 1.5 are satisfied. Therefore it follows that f has at least

(8.7) $$\frac{1}{2}[\dim(W_0^+ \cap W_\infty^-) - \operatorname{cod}(W_0^+ + W_\infty^-)]$$

nontrivial periodic solution.

In Lemma 6.6 of [B2], it has been proved that the number (8.7) is just equal to $\frac{1}{2}(\omega H_{zz}(\infty), \omega H_{zz}(0))$. Then the first part of Theorem 0.5 is proved.

In order to prove the second part we set

$$\tilde{f}(z) = -f(z) = \frac{1}{2}\int_0^{2\pi} [(J\dot{z}|z) + \omega H(z)]dt .$$

The functional $\tilde{f}$ satisfies the assumptions of Theorem 1.5 with

$$L = -L_\infty$$

$$\psi(z) = \omega \int_0^{2\pi} [-H(z) + \frac{1}{2} (H_{zz}(0)z|z)]dt$$

$$V = W_0^- \text{ and } W = W_\infty^+ .$$

At this point we argue exactly in the same way as in the proof of the first part of the theorem *in order* to verify (f_5). We observe that in this case we have

$$\{\text{constant function}\} = \mathrm{Fix}(S^1) \subset W_0^- = V .$$

Then when we verify (f_5)(iii) the first alternative holds. This is the reason why in [B2], a similar result has not been proved.

Since all the assumptions of Theorem 1.5 are verified it follows that there exist at least

(8.8) $$\frac{1}{2} [\dim(W_0^- \cap W_\infty^+) - \mathrm{cod}(W_0^- + W_\infty^+)]$$

nonconstant $2\pi\omega$-periodic solutions.

By Lemma 6.6 of [B2], the number (8.8) is equal to

$$\frac{1}{2} \theta(\omega H_{zz}(0), \omega H_{zz}(\infty)) . \qquad \square$$

Remark 8.1. If the nonresonance condition (0.10) is replaced by assumptions (0.14) and (0.15), by virtue of Lemma 6.2 (f_3) is satisfied. Then the assertion of Remark 0.7 holds.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper is divided in two parts. In the first part some abstract critical point theorems are proved using minimax arguments. The second part is devoted to applications. We study the existence of periodic solutions of the Hamiltonian systems.

20. ABSTRACT - cont'd.

$$\dot{p} = -\frac{\partial H}{\partial q}(p,q)$$

(1)

$$\dot{q} = \frac{\partial H}{\partial p}(p,q)$$

where $p,q \in \mathbf{R}^n$ and $H \in C^1(\mathbf{R}^{2n},\mathbf{R})$. First we consider Hamiltonian function having the following form:

(2) $$H(p,q) = \sum_{ij} a_{ij}(q)p_i p_j + \sum_i b_i(q)p_i + V(q)$$

where the matrix $a_{ij}(q)$ is positive definite and $V(q)$ grows more rapidly than quadratically as $|q| \to +\infty$. We prove that (1) has infinitely many periodic solutions of any period $T > 0$ under suitable assumptions on the Hamiltonian (2). Then we consider asymptotically linear Hamiltonians:

(3) $$H_z(z) = H_{zz}(\infty)\, z + o(|z|) \quad \text{for} \quad |z| \to +\infty$$

where $z = (p,q)$ and $H_{zz}(\infty)$ is a symmetric operator in $\mathbf{R}^n$. We also give an estimate for the periodic solutions of (1) when the Hamiltonian satisfies (3). Time-dependent Hamiltonians also are considered.

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