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OSCILLATIONS OF A LIQUID IN A ROTATING CYLINDER: PART II. SPIN-UP

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The unsteady motion of a fluid which fills a cylindrical container in a spinning projectile is considered. The spin is imparted impulsively to the cylinder and spin-up of the fluid is the basic flow which is perturbed to study the waves in the rotating fluid. The core flow is perturbed, not the Ekman layer flow. This is called the spin-up eigenvalue problem which is solved with modal solutions. Viscous perturbations are needed because of the boundary layer and critical layer. A numerical method of solution is given and several results

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shown. There are two times at which projectile instability might occur. The results of the theory and method are validated by comparison with experimental results.

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I. INTRODUCTION

For many years it has been observed that liquid-filled projectiles have a proclivity for unusual flight behavior, often being unstable even though the same projectile with a solid payload is stable. The basic states of motion of the liquid payload in a cylinder can be rationally separated into solid-body rotation and spin-up, the transient state that ends with the fluid in solid-body rotation. The determination, according to linear theory, of the frequencies and decay rates of the waves in the rotating fluid is called the "eigenvalue problem." It was solved for the basic state of solid-body rotation in Reference 1. The purpose of this report is to present the theory for the basic state of spin-up and some representative results.

Some attempts to solve the eigenvalue, abbreviated e.v., problem have been made in the past. Reddi[2] attempted to solve the inviscid perturbation equations but had difficulty with a singularity in these equations, related to the critical layer to be discussed in Chapter IV. A critique of this work is given by Lynn.[3] An ad hoc method to solve the e.v. problem was considered in Reference 4. The actual spin-up velocity profile was replaced by distributions in which the angular velocity was constant in an annular layer next to the cylinder and zero elsewhere. These distributions were required to have either the same angular momentum or the same volume of rotating fluid as the correct spin-up profile. In this way the spin-up problem could be fitted into the formalism of the Stewartson theory.[5] This ad hoc approach was not successful, and the e.v.'s disagreed with results from the present theory. A rational approach to the e.v. problem was given by Lynn[3] for a basic flow designated by V_w in Section II. The effect of the critical layer singularity was discussed and the real part of the e.v. calculated for axisymmetric inviscid disturbances, for which no critical layer exists.

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1. C. W. Kitchens, Jr., N. Gerber, and R. Sedney, "Oscillations of a Liquid in a Rotating Cylinder: Part I. Solid Body Rotation," ARBRL-TR-02081, June 1978 (AD A057759).

2. M. M. Reddi, "On the Eigenvalues of Couette Flow in a Fully-Filled Cylindrical Container," Franklin Institute Research Laboratory, Philadelphia, PA, Report No. F-B2294, 1967.

3. Y. M. Lynn, "Free Oscillations of a Liquid During Spin-Up," BRL Report No. 1663, August 1973 (AD 769710).

4. Engineering Design Handbook, Liquid-Filled Projectile Design, AMC Pamphlet No. 706.165, United States Army Material Command, Washington, D.C., April 1969 (AD 853719).

5. K. Stewartson, "On the Stability of a Spinning Top Containing Liquid," Journal of Fluid Mechanics, Vol. 5, Part 4, 1959.

In this report the linear e.v. problem is solved for viscous perturbations, axisymmetric or not, on a basic flow which is the spin-up core flow including diffusion. The modal analysis used allows slip on the cylinder endwalls. The perturbation equations have a singularity at the axis of the cylinder that is handled analytically. Because the Reynolds number is large, an integration scheme with orthonormalization is used that maintains linearly independent solutions. The determination of the eigenfunctions through the critical layer is accomplished numerically.

The significance of spin-up effects on a projectile flight can be estimated in an order of magnitude sense. Let a and c be the radius and half-height of the cylinder, Ω the spin in rad/sec of the projectile (say at the muzzle), ν the kinematic viscosity of the liquid, and

$$Re = \Omega a^2/\nu, \quad E = \nu/\Omega c^2,$$

the Reynolds number and Ekman number. Note that E is sometimes defined using the height, 2c, rather than c, and that for historical reasons we shall use Re in this report rather than the more conventional E. If the Ekman layers, i.e., endwall boundary layers, are laminar, the characteristic time for spin-up is

$$\bar{t}_s = (2c/a)Re^{1/2}/\Omega$$

$$= 2/E^{1/2}\Omega \text{ (sec)}.$$

We also use a nondimensional spin-up time $t_s = \Omega \bar{t}_s$. For $Re > 10^5$, approximately, the Ekman layer may be turbulent, in which case the characteristic spin-up time can be estimated by $\bar{t}_{st} = (28.6\ c/a)\ Re^{1/5}/\Omega$. Basically, these are time scales for the spin-up process and do not necessarily give a measure of how close the spin-up process is to solid-body rotation. This can be measured in several ways, e.g., time for the angular momentum, e.v., or other property to attain a value within 10%, say, of the solid body value. These measures require detailed calculations and so are not very convenient. A rule of thumb is that solid-body rotation is reached at about 4 $\bar{t}_s$ after an impulsive start of the cylinder.

The characteristic spin-up times can be compared with $\bar{t}_{fl}$, the time of flight of the projectile. If $\bar{t}_s$ or $\bar{t}_{st} \ll \bar{t}_{fl}$, spin-up effects can be disregarded and solid-body rotation can be assumed. If $\bar{t}_s$ or $\bar{t}_{st} \approx \bar{t}_{fl}/10$, or larger, spin-up effects probably have to be considered. To put these estimates in perspective, consider the parameters for two of several recent firings of 155mm projectiles conducted by Dr W. P. D'Amico, BRL. For one of these c/a = 3.120, Re = 4 × 10^4, Ω = 754 rad/sec, giving $\bar{t}_s$ = 1.65 sec; for another case c/a = 5.200, Re = 2 × 10^6, Ω = 754 rad/sec giving, $\bar{t}_{st}$ = 3.58 sec. These are referred to as Cases 1 and 2, respectively. For both of these $\bar{t}_{fl}$ = 40 sec. and, from the e.v. histories, spin-up effects are significant.

A question of more practical concern is: To what extent does the spin-up process affect the moment exerted by the liquid on the projectile and its yaw growth rate? The answer requires solution of the forced oscillation problem for the fluid. The e.v. problem is the free oscillation problem, and its role, vis-a´-vis the forced oscillation problem, is the same here as in any vibrating system. For example, the e.v. and eigenfunctions can be used to expand the solution of the forced oscillation problem. The e.v. can also be used to estimate when the projectile might be unstable. For inviscid perturbations on solid-body rotation, Stewartson[5] showed that if the projectile nutational frequency is within a certain band of the eigenfrequencies, the moment has a resonant response and the projectile will be unstable. If this idea is extended to viscous perturbations on spin-up, the real part of the e.v. would be compared with the nutational frequency. The comparison is, at best, a necessary condition for projectile instability. Confidence in application of this theory must be obtained by other means. This has been provided by experiments[6] and numerical simulations.[7]

II. SPIN-UP BASIC FLOW

Before the spin-up e.v. problem is considered, the basic flow requires some discussion. Only those aspects of it germane to the e.v. problem will be discussed. Our interest is in spin-up from rest which is inherently a non-linear problem. Spin-up from one state of solid-body rotation to another can be treated as a linear problem if the change in rotation rate is small. Both types are discussed by Greenspan[8] and Benton and Clark.[9]

Consider the motion of a fluid which fills a cylinder, initially at rest, which is "rapidly" brought to a constant angular velocity. The fluid motion is axisymmetric and time dependent. There are many variations on this problem which are of practical interest, e.g., the fluid only partially filling the cylinder, but they will not be considered here. For our application the angular acceleration of the cylinder is large during the time when the projectile accelerates in the gun tube; the small deceleration in flight is neglected. For artillery projectiles this acceleration time is typically 0.020 sec. which is small compared to t_{f1} ; comparison of angular acceleration

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6. *S. Stergiopoulos, "An Experimental Study of Inertial Waves in a Fluid Contained in a Rotating Cylindrical Cavity During Spin-Up From Rest," Ph.D. Thesis, York University, Toronto, Ontario, February 1982.*

7. *R. Sedney, N. Gerber, and J.M. Bartos, "Oscillations of a Liquid in a Rotating Cylinder," AIAA Preprint 82-0296, 20th Aerospace Sciences Meeting, Orlando, FL, January 1982.*

8. *H. P. Greenspan, The Theory of Rotating Fluids, Cambridge University Press, London and New York, 1968.*

9. *E. R. Benton and A. Clark, Jr., "Spin-Up," article in Annual Review of Fluid Mechanics, Vol. 6, Annual Reviews, Inc., Palo Alto, CA, 1974.*

time with spin-up time is also required. The assumption of an impulsive start for the cylinder is a good approximation for the projectile problem; it may not be in a laboratory apparatus. For an impulsive start our model requires separate treatment of the initial conditions.

The notation here will be the same as in Reference 1. Lengths, velocities, pressure, and time are made nondimensional by a, $a\Omega$, $\rho\Omega^2 a^2$, and Ω^{-1}, respectively, where ρ is the liquid density and Ω is the constant spin rate of the cylinder. In the inertial frame cylindrical coordinates r, θ, z are used, with the origin of z at the center of the cylinder, and velocities U, V, W, respectively. Dimensionless time is t. Note that in Reference 1 the origin of z is at the bottom endwall. Derivatives are indicated by subscripts.

The basic work on spin-up from rest was done by Wedemeyer.[10] In this remarkable paper he shows that the flow must be divided into two regions: the Ekman layers at the endwalls, which can be treated in a quasi-steady fashion, and the rest of the flow, called the core flow. Wedemeyer did not point out that an additional boundary layer, a Stewartson layer, is required at the cylinder wall. Starting with the Navier-Stokes equations for axisymmetric flow:

$$U^*_t + U^* U^*_r + W^* U^*_z - V^{*2}/r = -P^*_r + Re^{-1} (\nabla_2 U^* - U^*/r^2) \qquad (2.1a)$$

$$V^*_t + U^* V^*_r + W^* V^*_z + U^* V^*/r = Re^{-1} (\nabla_2 V^* - V^*/r^2) \qquad (2.1b)$$

$$W^*_t + U^* W^*_r + W^* W^*_z = -P^*_z + Re^{-1} \nabla_2 W^* \qquad (2.1c)$$

$$(rU^*)_r + rW^*_z = 0 \qquad (2.1d)$$

$$\nabla_2() = ()_{rr} + (1/r)\,()_r + ()_{zz}, \qquad (2.1e)$$

where the asterisk indicates the exact solution, he used physical order-of-magnitude arguments to simplify them in the core flow. The approximations reduce the three momentum equations, (2.1, a, b, c), to

$$V_t + U\,(V_r + V/r) = Re^{-1}\,[V_{rr} + (V/r)_r] \qquad (2.2)$$

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10. *S. H. Wedemeyer, "The Unsteady Flow Within a Spinning Cylinder," USA BRL Report No. 1225, October 1963 (AD 431846). Also, Journal of Fluid Mechanics, Vol. 20, Part 3, 1964, pp. 383-399.*

and

$$U_z = V_z = P_z = 0, \qquad (2.3)$$

where the superscript is dropped for this approximate solution. For Re $\to \infty$ Wedemeyer proposed neglecting the diffusion terms in (2.2) and arrived at

$$V_{w_t} + U_w \left(V_{w_r} + V_w/r\right) = 0, \qquad (2.4)$$

where the sub w indicates this approximation. Wedemeyer used (2.4) rather than (2.2) when he applied his model.

Although Wedemeyer showed the crucial importance of the Ekman layers to the spin-up process, his model did not require a solution for the flow in these layers. He did discuss certain properties of the solution which were required in his model for the core flow; exclusion of this solution has important consequences for the e.v. problem.

A more formal approach to this problem was given by Greenspan.[8] In his treatment, lengths are made dimensionless by 2c rather than a, which is more natural for the spin-up process; time is made dimensionless by $\bar{t}_s$ so that the new time is $t' = t/t_s = t/(2c/a)\ Re^{1/2}$; an expansion in the small parameter

$$(1/t_s) = (a/2c)\ Re^{-1/2} \qquad (2.5)$$

is assumed. The form of this expansion

$$\begin{aligned} U^* &= \qquad (1/t_s)\ U_1 + \ldots \\ V^* &= V_0 + (1/t_s)\ V_1 + \ldots \\ W^* &= \qquad (1/t_s)\ W_1 + \ldots \\ P^* &= P_0 + (1/t_s)\ P_1 + \ldots \end{aligned} \qquad (2.6)$$

follows from the facts that in the core flow U^* and W^* are $O(1/t_s)$ and V^* is $O(1)$, if $1/t_s$ is small. See Reference 8 for details. The independent variables are r, z, t'. Substituting the expansion into (2.1) after transforming to t' yields

$$V_{o_{t'}} + U_1 (V_{o_r} + V_o/r) = 0$$

$$V_o^2/r = P_{o_r} \qquad (2.7)$$

$$P_{o_z} = V_{o_z} = U_{1_z} = 0 .$$

These are the same as (2.3) and (2.4) since $V_0 = V_w$, $U_w = (1/t_s)\, U_1$, and $t' = t/t_s$. Although the formalism of matched asymptotic expansions was not used, the first two terms of the outer expansion are apparently given by (2.6). The Ekman layer solution would be obtained from the inner expansion which was not considered in Reference 8.

To solve (2.2), (2.4), or (2.7) a relationship between U and V is necessary. Wedemeyer used the facts that the Ekman layers are steady after one revolution and that the radial mass flux in the core flow must be balanced by that in the Ekman layers to obtain some conditions on the U, V relationship. At this point he was forced to take a phenomenological approach. The relationship is known at $t \to 0$ and $t \to \infty$ and he proposed a linear interpolation between them to obtain an approximate relationship for any t. He tested this idea in some other problems where the solution was known and decided it was satisfactory. Some confusion has appeared in the literature because this step was misinterpreted; this is discussed in a separate report. The result is

$$U = k_\ell (V - r)$$

$$k_\ell = \kappa (a/c)\, Re^{-1/2} \qquad (2.8)$$

$$= \kappa\, \bar{E}^{1/2}$$

$$= 2\kappa/t_s$$

for laminar Ekman layers. Wedemeyer proposed $\kappa = 0.443$ but Greenspan suggested $\kappa = 0.5$; the latter often gives results in better agreement with numerical solutions to the Navier-Stokes equations. Other relationships have been proposed, but they will not be discussed here. For turbulent Ekman layers

$$U = -k_t \, (r-V)^{8/5}$$

$$k_t = 0.035 \, (a/c) \, Re^{-1/5} \qquad (2.9)$$

$$= 1/t_{st},$$

where t_{st} is the nondimensional, turbulent spin-up time. The core flow is assumed to be laminar so that turbulent stresses are not introduced in the right-hand side of (2.2).

Using (2.8), (2.4) can be solved explicitly:

$$V_w = (re^{2k_\ell t} - 1/r)/(e^{2k_\ell t} - 1) \quad \text{for } r > e^{-k_\ell t}$$

$$= 0 \quad \text{for } r \leq e^{-k_\ell t}, \qquad (2.10)$$

so that $r = e^{-k_\ell t}$ separates rotating and nonrotating fluid. Although V_w is continuous there, a discontinuity in shear, i.e., V_{w_r}, exists there. Using (2.9) a numerical integration is necessary to obtain V_w; but the character of the solution is the same as when (2.8) is used. The radial velocity is obtained from (2.8) or (2.9) and W is obtained from the continuity equation

$$W = -(z/r) \, (rU)_r$$

At $r = 1$, $W \neq 0$; thus, the Stewartson layer should be included at $r = 1$, but this has yet to be done. Also $W \neq 0$ at the endwalls $z = \pm c/a$.

Wedemeyer pointed out that the corner in the solution to (2.4) would be smoothed out if the diffusion terms were retained, as in (2.2). But this is a nonlinear second order equation and must be integrated by finite difference methods. This can be done with some standard techniques for diffusion equations, using (2.8) or (2.9). Special treatment is needed near the point $r = 1$, $t = 0$ because, for an impulsive start, a discontinuity in the boundary conditions exists at that point. In our work a local, analytic solution was derived to resolve the discontinuity. In most of our e.v. calculations we used the V from the numerical solution of (2.2) with either (2.8) or (2.9), but other options are also used. For laminar Ekman layers, the spin-up velocity profile

$$V = f \, (r, k_\ell t, k_\ell \, Re);$$

for the turbulent case k_ℓ is replaced by k_t. These functional forms follow directly from (2.2) and (2.8) or (2.9) and represent the most efficient way to display the results.

Some examples of the solution of (2.2) will be given simply to illustrate the form of V which must be perturbed in the e.v. problem. The parameters of the two cases of 155mm projectiles, given in the introduction, will be used. Case 1, with $Re = 4 \times 10^4$, should have laminar Ekman layers. The solution to (2.2) with (2.8) gives the V vs r shown in Figure 1 for 3 values of t. One of them is the nondimensional spin-up time $t_s = 1245$. For t = 4800, approximately $4t_s$, solid-body rotation is achieved, essentially. Case 2, with Re $= 2 \times 10^6$, should have turbulent Ekman layers. The solution to (2.2) with (2.9) gives the V vs r shown in Figure 2 for 3 values of t including $t_{st} = 2700$. For t = 10,000, approximately 4 t_{st}, solid-body rotation has not been achieved to the same degree as in Figure 1. The profile for t = 1300 is relatively steep; early time profiles such as this require more care in the e.v. calculation.

Some results from the Wedemeyer spin-up model have been compared with experiments; these tend to validate the model. Another way to validate the model is to compare its results with solutions to the Navier-Stokes equations. The finite difference solution presented by Kitchens[11] was used for this purpose.

The V's from both solutions were compared over a significant range of parameters; in general, the differences were small for $\kappa = 0.5$ and slightly larger for $\kappa = 0.443$. Comparisons are given in a separate report. Some limited comparisons are shown in the following discussion which actually has a different purpose, viz., to show the flow in the Ekman layer and its transition to the core flow.

To the authors' knowledge no solution for the flow in the Ekman layers during spin-up has been given other than finite difference solutions. In References 8 and 10 certain properties of these were used to determine the core flow. The finite difference solutions to the Navier-Stokes equations by Kitchens[11] provide the complete flow. In Figure 3 results from this method are shown for c/a = 1, $Re = 10^4$, $t_s = 200$ at t = 62.5 and 250. For three values of r, V* vs z + c/a is shown. According to Wedemeyer's spin-up model V is independent of z; this is verified by the calculation for $z + c/a \geq .05$ at t = 62.5 and for $z + c/a \geq .025$ at t = 250. These values of z + c/a can be taken as one measure of Ekman layer thickness. The values of V from the Wedemeyer model are also indicated; the differences from the results of the Navier-Stokes equation are relatively large at t = 62.5 and essentially zero at t = 250. Significant undershoots exist in the V profile at t = 62.5 but

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11. C. W. Kitchens, Jr., "Navier-Stokes Solutions for Spin-Up in a Filled Cylinder," AIAA Journal, Vol. 18, No. 8, August 1980, pp. 929-934. Also Technical Report ARBRL-TR-02193, September 1979 (AD A077115).

only mild ones at t = 250. An example of U vs z + c/a is given in Reference 11. The azimuthal component of the basic flow which must be perturbed to obtain the e.v. is represented in Figure 3.

If only the core flow were perturbed, it would not be possible to satisfy the boundary conditions on the endwalls. Figure 3 also shows that a large error in azimuthal velocity is made in the Ekman layers if the core flow is extended to the endwalls. This is, in fact, what is done in the e.v. problem formulated in this report. Nevertheless, the wave frequency calculated in this way agrees fairly well with some experimental results; however, the calculated wave damping has a substantial error. This is expected since the dissipation in the Ekman layers should affect the wave damping.

III. THE PERTURBED FLOW

The procedure for obtaining the equations governing the perturbed flow is a standard one. It starts from the Navier-Stokes equations for 3-D flow:

$$\bar U_t + \bar U \bar U_r + (\bar V/r)\,\bar U_\theta + \bar W \bar U_z - \bar V^2/r = -\bar P_r + Re^{-1}(\nabla^2 \bar U - \bar U/r^2 - 2\bar V_\theta/r^2)$$

$$\bar V_t + \bar U \bar V_r + (\bar V/r)\,\bar V_\theta + \bar W \bar V_z + \bar U \bar V/r = -\bar P_\theta/r + Re^{-1}(\nabla^2 \bar V - \bar V/r^2 + 2\bar U_\theta/r^2)$$

$$\bar W_t + \bar U \bar W_r + (\bar V/r)\,\bar W_\theta + \bar W \bar W_z = -\bar P_z + Re^{-1}\nabla^2 \bar W \qquad (3.1)$$

$$(r\,\bar U)_r + \bar V_\theta + r\,\bar W_z = 0$$

$$\nabla^2(\ ) = \nabla_2(\ ) + (1/r^2)(\ )_{\theta\theta}$$

The velocity components and pressure are then expressed as the sum of the basic and the perturbation:

$$\bar U = U^*(r,z,t) + u'(r,\theta,z,t) \qquad (3.2)$$

and similarly for $\bar V$, $\bar W$, and $\bar P$. For our application nonaxisymmetric disturbances must be considered, hence the dependence of u' on θ.

The details of the derivation will be given only for the radial momentum equation, the first of (3.1). The representation of the velocity components and pressure as a basic flow plus a perturbation, i.e., (3.2) and the similar expressions for V, W, and P, are substituted into (3.1). The terms are separated into zeroth order terms which are independent of the u', v', w', p', first order terms which are linear in u', v', w', p', etc. The zeroth order terms give exactly (2.1), by construction. The first order terms for the radial momentum equation, the first of (3.1), give

$$u'_t + U^* u'_r + u' U^*_r + (V^*/r)\, u'_\theta + W^* u'_z + w' U^*_z - (2V^* v')/r = \quad (3.3)$$

$$-p'_r + Re^{-1} [\nabla^2 u' - u'/r^2 - 2v'_\theta/r^2].$$

One term does not appear because U^* is independent of θ.

Up to this point the basic flow is an exact solution of the Navier-Stokes equations, (2.1), which would have to be a finite difference solution. It is impractical to use such a solution for the coefficients in (3.3). Therefore, an approximation to it, the Wedemeyer model for the core flow, is used. This leaves open the matter of treating the Ekman layer flow and the attendant difficulties discussed at the end of the last section. A correction is required to account for the Ekman layers; this will not be done here.

The next step is to estimate the order of magnitude of the terms in (3.3). They are not all the same order of magnitude, and this step leads to a significant simplification. The expansion (2.6) is convenient for this purpose although the approach taken in deriving (2.2) was actually used. It is assumed that all perturbations, the primed quantities, and their derivatives are O(1). In particular $u'_t = O(1)$ which means that the dimensional time scale for the perturbations is $O(\Omega^{-1})$ and is therefore small compared to $\tilde{t}_s$; the nondimensional time scale for the perturbations is O(1) which is small compared to t_s. This estimation is used below to make another important simplification. It follows from (2.6), or the equivalent using Wedemeyer's approach, that all coefficients in (3.3) containing U^* and W^* and their derivatives with respect to r and z are $O(1/t_s)$; all those containing V^* and V^*_r are O(1). Neglecting the $O(1/t_s)$ terms on the left-hand side of (3.3) deletes all terms that contain U^* and W^*. Only the terms containing V^* and those not involving the basic flow remain. For V^* we must substitute V of (2.2). Applying this same process to the remaining equations of (3.1) yields the following set of perturbation equations:

$$u'_t + (V/r)\, u'_\theta - 2Vv'/r = -p'_r + Re^{-1} (\nabla^2 u' - u'/r^2 - 2v'_\theta/r^2) \quad (3.4a)$$

$$v'_t + [V_r + (V/r)]\, u' + (V/r)\, v'_\theta = -p'_\theta/r + Re^{-1} (\nabla^2 v' - v'/r^2 + 2u'_\theta/r^2) \quad (3.4b)$$

$$w'_t + (V/r)\, w'_\theta = -p'_z + Re^{-1}\, \nabla^2 w' \qquad (3.4c)$$

$$(ru')_r + v'_\theta + r\, w'_z = 0. \qquad (3.4d)$$

These are the equations that govern viscous perturbations of the core flow but not the Ekman layers.

The presence of the viscous terms on the right-hand side of (3.4a, b, c) requires discussion. It is helpful to refer back to the spin-up basic flow model. A formal, rational expansion such as (2.6) yields (2.7) in which no viscous diffusion terms appear, but Wedemeyer's approach did have these terms, as in (2.2). Without them the predictions of the model would not agree nearly as well with experimental or other numerical results for a range of r and t (See Reference 12 for a discussion of this.) As mentioned before, at the discontinuity in V_{w_r} the diffusion terms are required. They could be included in a "local" sense as Venezian[13] did or in a "global" sense as Wedemeyer did. The situation with (3.4) is analogous. A formal, rational expansion would yield perturbation equations of the same form as (3.4) but with $Re^{-1} = 0$, i.e., the inviscid equations. These are adequate except for two regions where viscous effects are important. These are the boundary layer at $r = 1$ and the critical layer, to be discussed below. Here also the viscous terms could be included in a local sense in both of these regions or in a global sense as we have done by retaining the Re^{-1} terms in (3.4).

There is another restriction on these equations arising from the fact that Wedemeyer's model for the core flow is not correct at $r = 1$, as discussed in Section II. Specifically, $W \neq 0$ at $r = 1$. One can either say that the model is valid only outside a Stewartson layer at $r = 1$ or such a layer could be incorporated. In the second alternative an additional term would appear in (3.4c) arising from the term $U\, W_r$ in the z-momentum equation of (3.1). The first alternative will be adopted here, but the presence of the Stewartson layer is neglected just as the presence of the Ekman layer is. The fact that $W \neq 0$ at $z = \pm c/a$ is part of the overall problem of neglecting the Ekman layer in the basic flow.

The final step is to consider the applicability of the quasi-steady assumption to (3.4). If V(r,t) varies significantly with t over the time scale O(1) appropriate to (3.4), then several coefficients depend on both r

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12. *W. B. Watkins and R. G. Hussey, "Spin-Up From Rest: Limitations of the Wedemeyer Model," The Physics of Fluids, Vol. 16, No. 9, September 1973, pp. 1530-1531.*

13. *G. Venezian, Topics in Ocean Engineering, Vol. 2, pp. 87-96, Gulf Publishing Co., Houston, Texas, 1970.*

and t and separation of variables could not be used to solve (3.4). As pointed out above the time scale for V, the spin-up time, is $O(t_s)$ which is large compared to O(1). Therefore, over the time scale for the perturbations V does not change appreciably. Thus, t can be regarded as a parameter in the solution of (3.4); i.e., we can use the quasi-steady approximation. A possible exception to this might exist for $t \to 0$ and an impulsive start.

In Lynn's work[3] the same equations as (3.4) appeared except that his equations have V_W in place of V. The basic flow used by Lynn was $(0, V_W, 0, P_W)$ which is an exact solution to the steady Navier-Stokes equations; this basic flow would not apply in the two wall layers. The philosophy behind his approach is different from ours even though the equations are the same if $V = V_W$. Lynn solved only the inviscid equations, $Re^{-1} = 0$, for axisymmetric perturbations.

From the discussion in the introduction, it is clear that wave-type solutions to (3.4) are required. Although there may be several approaches to obtaining these, only one besides that actually employed will be mentioned. If the sinusoidal variation in θ and t of (u',v',w',p') is factored out, a set of linear partial differential equations in r and z is obtained. These might have to be solved numerically. This approach would be especially useful if the Ekman layer effects were included. No serious attempt has been made to use this approach. Here a separation of variables is employed to obtain modal type solutions.

IV. THE MODAL SOLUTION AND EIGENVALUE PROBELM

In this section it is convenient to change the origin of the z-coordinate to the bottom endwall which was the convention in Reference 1. Let that coordinate be z' so that $z' = z + (c/a)$; z' will be used instead of z.

It is assumed that the perturbation can be expressed as a superposition of modes or a triple Fourier expansion in θ, z', and t with coefficients functions of r. It is convenient to use complex notation and express the perturbation as

$$u' = \text{Real}\left\{u\ (r) \cos K z' \exp\left[i\ (Ct - m\theta)\right]\right\} \qquad (4.1a)$$

$$v' = \text{Real}\left\{v\ (r) \cos Kz' \exp\left[i\ (Ct - m\theta)\right]\right\} \qquad (4.1b)$$

$$w' = \text{Real}\left\{w\ (r) \sin Kz' \exp\left[i\ (Ct - m\theta)\right]\right\} \qquad (4.1c)$$

$$p' = \text{Real} \left\{ p\ (r) \cos Kz' \exp\ [i\ (Ct - m\theta)] \right\} \tag{4.1d}$$

$$K = k\pi a/2c.$$

The complex quantities u, v, w, and p are solutions of the ordinary differential equations obtained by substituting these forms into (3.4). The integers $m = 0, \pm 1, \ldots$ and $k = 1, 2, \ldots$ are the azimuthal and axial wave numbers, respectively. For axisymmetric perturbations $m = 0$; $m = 1$ is the value relevant to the projectile problem, but we shall not specify m at this point. The nondimensional complex constant $C = C_R + i\ C_I$ is the eigenvalue of the system. The dimensional wave frequency is $C_R\ \Omega$ and the decay rate or damping is $C_I\ \Omega$.

For the free oscillation problem the boundary conditions at the endwalls, $z' = 0, 2c/a$, are $u' = v' = w' = 0$ if the complete flow is being perturbed. Since we are perturbing only the core flow, the boundary conditions are not the same. From (4.1) $w' = 0$ at the endwalls but not u' and v'. Evidently, the form (4.1) represents the first term of an outer expansion of the perturbation. The inner expansion must be introduced to satisfy the no-slip boundary conditions, but for this the inner expansion of the basic flow, the Ekman layers, must be perturbed. In lieu of this, the form (4.1) is used without endwall correction; i.e., there is slip at the endwalls.

The no-slip boundary conditions at $r = 1$ are satisfied by requiring

$$u = v = w = 0 \qquad \text{at } r = 1. \tag{4.2}$$

Three more boundary conditions are necessary. For a filled cylinder the boundary conditions at $r = 0$ depend on m; they are presented in Reference 1 and are simply restated here:

$$\begin{array}{ll} m = 0 & u = v = dw/dr = 0 \\ m = 1 & u - iv = w = p = 0 \\ m > 1 & u = v = w = 0 \end{array} \tag{4.3}$$

Applying the two differential operators common to (3.4a,b,c) to the generic form

$$F = f\ (r) \begin{Bmatrix} \sin Kz \\ \cos Kz \end{Bmatrix} \exp\ [i\ (Ct - m\theta)]$$

yields

$$[\partial/\partial t + (V/r)\ \partial/\partial\theta]\ F = i\ M\ (r)\ F$$

and

$$\nabla^2 F = \Delta_1 f \begin{Bmatrix} \sin Kz \\ \cos Kz \end{Bmatrix} \exp [i (Ct - m\theta)]$$

where

$$M (r) = C - m V/r \tag{4.4}$$

$$\Delta_1 f = f_{rr} + f_r/r - [(m^2/r^2) + K^2] f.$$

Using these the following ordinary differential equations are obtained:

$$[Re^{-1} (\Delta_1 - r^{-2}) - i M] u + (2/r)(V + i m r^{-1} Re^{-1}) v - p_r = 0 \tag{4.5a}$$

$$[Re^{-1} (\Delta_1 - r^{-2}) - i M] v - (\partial V/\partial r + V/r + 2i m Re^{-1}/r^2)u + i m p/r = 0 \tag{4.5b}$$

$$[Re^{-1} \Delta_1 - i M] w + K p = 0 \tag{4.5c}$$

$$(r u)_r - i m v + K r w = 0. \tag{4.5d}$$

Since V is obtained from the finite difference solution of (2.2), numerical integration of these equations is required. Equations (4.5) are a homogeneous, sixth order system with homogeneous, two-point boundary conditions (4.2) at $r = 1$ and (4.3) at $r = 0$. The system is not self-adjoint. Although there is no proof, we assume the eigenvalues form a denumerable, discrete spectrum. With n used to order the spectrum, the e.v. are C_n; for brevity the index n will sometimes be omitted. Eqs. (4.5) must be solved for C_n and the eigenfunctions u_n, v_n, w_n, p_n given V, c/a, Re, m and k; t enters only through V.

If the system were self-adjoint, the index n would indicate the radial mode number, where the mode can be defined in terms of the number of zeros of the eigenfunctions. For a system which is not self-adjoint, it is unclear how to define a mode in a general and unambiguous manner. However, the concept of mode is a very useful one and will be employed here. The mode can be identified by calculating the e.v. at large t, where the radial mode number is known from the solid-body rotation solution, and then tracked as t decreases.

Consider the inviscid limit of Eqs. (4.5), $Re^{-1} = 0$. The order of the system is reduced to two, and, in fact, a single 2nd order equation is obtained (Reference 3); it is analogous to the Rayleigh equation in hydrodynamic stability theory. The coefficient of the highest order derivative contains $M = C - m V/r$ and the C is then real. If $M = 0$ has a root in $0 < r < 1$, the equation has a singular point and the theory breaks down in its neighborhood. Let r_c be the root of $C_R - m V/r$. In analogy with the Orr-

Sommerfeld equation and its inviscid limit, r_c is called the critical level and its neighborhood the critical layer. A review of critical layer dynamics for shear flows was given by Stewartson.[14] The physical interpretation of

$$C_R = m\ V/r$$

at $r = r_c$ is that the wave frequency is an integral multiple of the angular frequency of the basic flow, indicating a resonance. If m = 0 there is no critical layer. If m = 1 there is a critical layer up to a time for which $r_c = 0$; after that a real r_c does not exist.

For the viscous case, (4.5) do not have a singularity when M = 0. However, the critical layer still plays an important role just as it does in the Orr-Sommerfeld equation. The r for which M = 0 is called a turning point. Lynn[3] showed that the thickness of the critical layer is $0(Re^{-1/3})$ in direct analogy with the result for the Orr-Sommerfeld equation. The practical consequence of the existence of a critical layer is that the eigenfunctions develop high frequency, large relative amplitude oscillations there. If the integration scheme is not capable of resolving these, a solution to (4.5) will not be obtained.

There are two other properties of (4.5) which make their numerical integration not straightforward: (1) They have a singularity at r = 0; (2) The coefficient of the highest order derivative, Re^{-1}, is small, which classifies them as stiff equations. Both of these properties exist when the basic flow is solid body rotation. They were discussed in detail in Reference 1, so only a summary of these properties and the methods of treating them will be given.

According to the theory of differential equations (Reference 15), r = 0 is a singularity of the second kind. A singular transformation was found that transforms the system into one with a singularity of the first kind. This is easier to treat, but the transformed system has characteristic values that are not all distinct and differ by integers; in that case there is no simple, systematic method of generating all solutions. (See page 136 of Reference 15.) For the sixth order system (4.5) there should be 6 independent solutions near r = 0 and because of the boundary conditions, 3 of them should be singular and the other 3 regular. It was more efficient to assume power series expansions near r = 0 and determine the 3 regular, linearly independent solutions. The only significant difference, compared to the solid-body rotation case in Reference 1, is that in the present case V must be expressed as a

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14. K. Stewartson, "Marginally Stable Inviscid Flows with Critical Layers," Journal of Applied Mathematics, Vol. 27, 1981, pp. 133-175.

15. E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill Book Co., New York, 1955.

power series in r. Numerical values of these power series solutions at $r = \varepsilon$ are used as initial conditions for the numerical integration of (4.5) over the interval $\varepsilon \leq r \leq 1$. The number of terms in the series and ε are parameters of the integration. A convergence test is applied for each integration. Formulae and recurrence relations for the power series coefficients are given in Appendix A.

The stiff nature of (4.5) was discussed in Reference 1, and orthonormalization was applied to insure that the initially linearly independent solutions remain so during the integration. No additional discussion is needed for the spin-up case. General rules cannot be given for the number of subintervals, N, and the integration step size which are required for a given accuracy in the solution for C; they depend on a large number of parameters, in an interactive manner. What can be said is that spin-up and larger Re require larger N and smaller step size than solid-body rotation and smaller Re.

V. SOLUTION OF THE EIGENVALUE PROBLEM

The procedure for solving (4.5) for C, and the eigenfunctions u, v, w, p is the same as that given in Reference 1 for solid-body rotation. The major steps in the procedure are merely listed here except for the "first guess" strategy; this will be described in detail since it has no analog in the solid-body rotation case.

For numerical integration, (4.5) are put into canonical form, i.e., six first order equations in the manner of Reference 1. A value of C must be specified, the first guess. Next, the power series solutions are evaluated at $r = \varepsilon$. With these as initial conditions, (4.5) is integrated for a specified N and integration step size over $\varepsilon \leq r \leq 1$. Orthonormalization insures that 3 linearly independent solutions are generated. These are tested to see if the boundary conditions are satisfied at $r = 1$; this test requires that the characteristic determinant, $Z = 0$; Z is defined in Appendix B. If $Z \neq 0$ to within a certain tolerance, a new value of C is obtained using, essentially, Newton's method. The iteration then proceeds until $Z = 0$ and two successive C values differ to within specified tolerances. The maximum number of iterations allowed is typically 15. For a convergent iteration process, the first guess must be reasonably close to the converged C.

Obtaining a first guess for a spin-up e.v. at an arbitrary t would be quite difficult. Usually, a spin-up history is needed, i.e., C vs t. The computation is then started at large t, where a first guess can be obtained from the C for solid-body rotation or an approximation to it. (See Reference 1, page 18.) For smaller t, the previously computed C, or the two previous ones, can be used to obtain the first guess. For some cases, at early times, the required time interval between successive calculations of C becomes small, and additional searching for a first guess may be necessary for the iteration process to converge.

The time to calculate one C depends mainly on the first guess for C, t, N, integration step size, tolerances, and severity of convergence test. Typical CPU times for one C calculation are one minute on the CDC 7600 and 15 minutes on the VAX. The output of the standard program is only the e.v., C.

A separate program is used to print out the eigenfunctions with an option to plot real and imaginary parts of w and p vs r.

Some results for eigenfunctions and e.v. will now be shown. In presenting these, it is convenient to designate the 3 wave numbers by the triplet (k, n, m) corresponding to the wave numbers in the (z, r, θ) directions.

The eigenfunctions, of course, are determined only to within a multiplicative constant. The eigenfunctions have not been normalized in this program. In Figure 4 the eigenfunction Real (w) = w_R vs r is shown at t = 7000, 1000, and 400 for Re = 5×10^5, c/a = 2.679 (so that t_s = 3788) and mode (5,1,1). In Figure 4a t/t_s = 1.85, $C_R = 8.354 \times 10^{-2}$, $C_I = 8.363 \times 10^{-4}$ and there is no critical layer; the variation of w_R through the boundary layer can be barely discerned on this scale. In Figure 4b t/t_s = .26, $C_R = 7.060 \times 10^{-2}$, $C_I = 2.269 \times 10^{-2}$; r_c = 0.44 as indicated by the arrow in the figure. The boundary layer is less discernible here than in Figure 4a. In Figure 4c t/t_s = .11, $C_R = 3.015 \times 10^{-2}$, $C_I = 8.066 \times 10^{-2}$, and r_c = 0.66; away from the critical layer w_R is not equal to zero although it appears to be on this scale. For the conditions of Figure 4c, if Ω = 628 rad/sec (100 Hz), $\bar{t}$ = 0.64 sec. For both Figures 4b and 4c, the critical layer oscillations are centered at $r = r_c$.

Some results for C_R are presented next for the two sample cases discussed in the introduction. In Figure 5 C_{R1} vs t is shown for Case 1 with c/a = 3.120, Re = 4×10^4, mode (3,1,1) and Case 2 with c/a = 5.200, Re = 2×10^6, mode (5,1,1). Cases 1 and 2 are calculated with the laminar and turbulent Ekman layer compatibility conditions, (2.8) and (2.9), respectively. For Case 2 there is a maximum in C_{R1} at t = 1500, a phenomenon first determined by this investigation. If the calculation were continued for smaller times, the curve would go through the origin because C_R = 0 for t = 0. For Case 1 a maximum in C_{R1} is not found for $t \geq 180$, the last point computed. The maximum occurs for t < 180 since C_R = 0 for t = 0. For both cases, of course, C_{R1} approaches the result for solid-body rotation as $t \to \infty$.

The significance of a maximum in the C_{R1} vs t curve is related to the necessary condition for projectile instability discussed at the end of the introduction. If the projectile nutational frequency is less than the maximum of C_R, there are two times at which instability might develop. Whether or not it does, at either time, is not within the scope of this report.

Finally, some results for another set of parameters are presented to show the spin-up e.v. histories for n = 1 and 2 and to illustrate the sensitivity of the iteration process convergence to the first guess of C that sometimes occurs at small t. These parameters were chosen by Dr. William P. D'Amico for a firing program, proposed a few years ago, to be conducted in the aerodynamics range for the purpose of obtaining data on spin-up instability of projectiles. Unfortunately, the program was not fired, so that no such data exist to compare with the theory. However, the e.v. calculations made for this case are of interest.

In Figure 6, C_1 and C_2 are presented as time histories of C_R and C_I with c/a = 3.297, Re = 4974, modes (3,1,1) and (3,2,1). For this case the Ekman layers would be laminar, so that (2.8) is used in the spin-up basic flow calculation. The C_{R1} curve has an unusual feature. For 180 < t < 220 it has a shallow minimum and maximum; on the scale of this figure C_{R1} appears to be constant over this range. The variation of C_{I1} is rather typical. Obtaining a first guess for C, for which the iteration process would converge, was made more difficult because C_{R1} was not monotonic. However, the calculation of C_2 was, relative to C_1, considerably more difficult starting at about t = 250. The range of values of C_{R2} , C_{I2} in which a first guess for C_2 had to lie, in order for the iteration process to converge to the n = 2 root of Z = 0, steadily decreased. This is shown in the C_I vs C_R plot in Figure 7. The curve is the e.v. trajectory for 180 < t < 250; the areas in which a first guess must lie are shown for four values of t. As indicated for t = 200 and 250, the areas are off-scale. For t = 180 the rectangle has dimensions (.0008, .0007). The search for a first guess must be done with differences less than .001. The difficulty of obtaining a first guess is compounded by the fact that C_{I2} is nonmonotonic near t = 200. When the iteration process did not converge to the n = 2 root of Z = 0, it almost always converged to the n = 1 root; sometimes it converged to a root for larger n, and a few times it did not converge. There was a very predominant preference for the n = 1 root.

To visualize Z (C), |Z| (C_R, C_I) is plotted in Figure 8. Only the shaded portion is the |Z| surface; the planar regions from the surface to |Z| = 0 are artifacts of the plotting program. The surface is plotted in the neighborhood of the n = 2 root for t = 180, viz., C_{R2} = .101673, C_{I2} = .034867. Only some of the values on the coordinate lines are labeled. The root lies at the bottom of the steep valley (not visible in the plot). To the right of it is a steep peak, and this helps to explain why most guesses did not converge to the n = 2 result. In a sense the valley is shaded by the peak. This case illustrates the fact that for early times the solution to the e.v. problem may become more laborious than for larger times.

VI. DISCUSSION

In this report the theory and a method for the solution of the spin-up e.v. problem are given. The theory for the perturbation flow is a linear one but not for the basic flow. Because viscous effects are important in the e.v. problem at the cylinder wall and critical layer, viscous perturbations are used. Several assumptions are made, and some limitations are contained in the theory. The theory and methods are successful in the sense that they provide results, as shown in the previous section, that are physically meaningful and do not violate intuition or the "physics of the problem." Other investigators have worked on this problem without success in that sense. The more important gauge of success is validation by comparison with either numerical simulation or experimental results.

For the solid-body rotation case and $m = 0$, a detailed validation was given in Reference 7 using finite difference solutions to Navier-Stokes equations. This provides confidence in the treatment of (4.5), and the solid-body rotation endwall correction, for that case. Spin-up and $m = 0$ could be validated in the same way; it would require, relatively, more analysis to reduce the numerical data. At the present time there is no numerical simulation to validate the $m = 1$ case.

Although the importance of the spin-up problem for liquid-filled projectiles has been recognized since the early 1960's, very few experiments relevant to spin-up have been conducted at BRL, and none of these were directed to the modes of oscillation. The only published results seem to be those quoted in References 4 and 10. Experiments in a spin generator determined particle paths during spin-up, and range tests provided spin decay histories; these were used by Wedemeyer[10] to validate his theory for the spin-up basic flow, described in section II. Further validation of that flow was obtained with measurements of velocity using the LDV, e.g., Reference 16.

The only attempts to measure the e.v. were those of Aldridge[17] and the further development of that work reported in Reference 6. In Reference 17, measurements of C_R (not C_I) are given for modes (2,1,0), (2,2,0), and (4,2,0) over the range $1.4 < t/t_c < \infty$ and compared with Lynn's calculation for inviscid perturbations and $m = 0$. For larger times the difference was small; but the trend, as t decreases, was for the differences to grow. Later, Aldridge found that a systematic error existed in the analysis of the data. Some of these experiments were redone using a different data analysis technique and reported in Reference 6; they are presented below. For $m = 0$ there is no critical layer and $t/t_s = 1.4$ is larger than one would want for a spin-up test. In Reference 6 measurements of C_R and C_I are given for modes (2,1,0), (1,2,1) and (1,1,1). The first of these replaces those of Reference 17. The latter two were excited using a rotating cylinder with a precessing top. In Table 1, these experimental results are presented together with calculated results using the method described here. The smallest t/t_s =

16. *W. B. Watkins and G. R. Hussey, "Spin-Up from Rest in a Cylinder,"* *__The Physics of Fluids__, Vol. 20, No. 10, Part 1, 1977, pp. 1596-1604.*

17. *K. D. Aldridge, "Experimental Verification of the Inertial Oscillations of a Fluid in a Cylinder During Spin-Up," BRL Contract Report No 273, Aberdeen Proving Ground, Maryland, September 1975 (AD A018797). Also* *__Geophys. Astrophys. Fluid Dynamics__, Vol. 8, 1977, pp. 279-301.*

1.51. The experimental C_R has an estimated error of less than 1%; the error in C_I can be as much as 10%. The differences between the experimental and calculated frequencies are less than 2%. For the damping the difference is a factor of 3 for t/t_s = 1.51; this could be expected because we have neglected the Ekman layers, as discussed in Sections III and IV. Solid-body rotation is achieved, essentially, for t/t_s = 4.75; thus, the method of Reference 1 can be used, including the solid-body rotation endwall correction (not Ekman layer correction). This gives the C_I in the last column of Table 1, and the difference is now about 10%.

TABLE 1. EXPERIMENTAL AND CALCULATED EIGENVALUES

(2,1,0) MODE • SPIN-UP

Re = 43,000 c/a = 0.995

	Frequency		Damping		
t/t_s	Exp.	Calc.	Exp.	Calc. Uncorr.	Calc. Corr
4.75	1.269	1.268	.0061	.00297	.007
3.04	1.266	1.265	.0062	.00295	
2.02	1.236	1.242	.0068	.00311	
1.68	1.202	1.219	.0101	.00334	
1.51	1.184	1.202	.0105	.00342	

The comparisons of C_R for the m = 1 modes are given in Table 2. For each t, Re varies slightly; the values given are representative. Again, the differences between the experimental C_R and our calculated results are, at most, 2%. Further validation is needed for smaller t/t_s, for both smaller and larger Re and for larger c/a.

For projectile applications m = 1 is the pertinent azimuthal wave number and, to apply the necessary condition for projectile instability, C_R is required. The comparison shown in Table 2 gives the significant result that the method presented here is validated within the limitations noted.

The lack of projectile tests in ballistic ranges has already been noted. Field tests wherein data is obtained from yawsondes can provide some information, but not as detailed as range tests. For the field tests on 155mm projectiles mentioned in the introduction, some of the parameters were chosen on the basis of our calculations to obtain information on instabilities during spin-up. The results are in publication by D'Amico.[18]

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18. *W. P. D'Amico, "Flight Data on Liquid-Filled Shell for Spin-Up Instabilities," BRL Report in preparation, 1983.*

TABLE 2. EXPERIMENTAL AND CALCULATED EIGENFREQUENCIES

c/a = 0.600

Mode (1,2,1)	Re ≅ 50,000		
t/t_s	Calculated $(1-C_R)$	Experimental $(1-C_R)$	Difference
1.473	.8474	.8658	2.1%
1.804	.8352	.8488	1.6%
2.800	.8276	.8305	.35%
4.947	.8255	.8295	.49%
Mode (1,1,1)	Re ≅ 30,000		
1.357	1.3359	1.3487	.95%
1.634	1.3707	1.3736	.2%
3.780	1.4275	1.426	.1%
4.110	1.4278	1.4290	.1%

Free oscillation of the fluid during spin-up was considered in this report. For projectile motion the forced oscillation problem must be solved. The modal solutions to (4.5) would be needed there also, but the boundary conditions are nonhomogeneous. As shown in Reference 19 for the solid-body rotation case, the same techniques discussed in this report are needed in the forced oscillations.

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19. *N. Gerber, R. Sedney, J. M. Bartos, "Pressure Moment on a Liquid-Filled Projectile: Solid Body Rotation," BRL Technical Report ARBRL-TR-02422, October 1982 (AD A120567).*

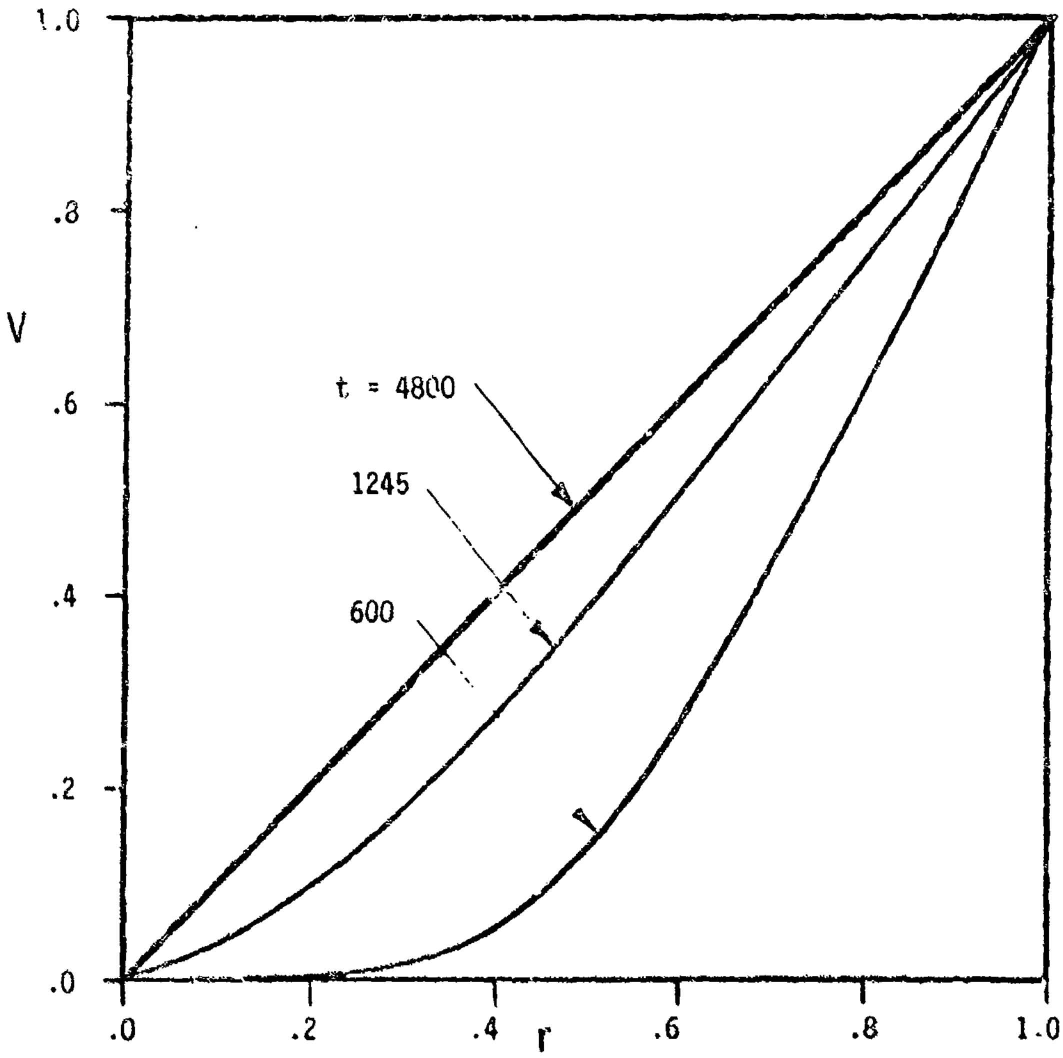


Figure 1. V vs r for Case 1 with c/a = 3.120, Re = 4 x 10^4 at Three Values of t; t_s = 1245. Laminar Ekman Layer.

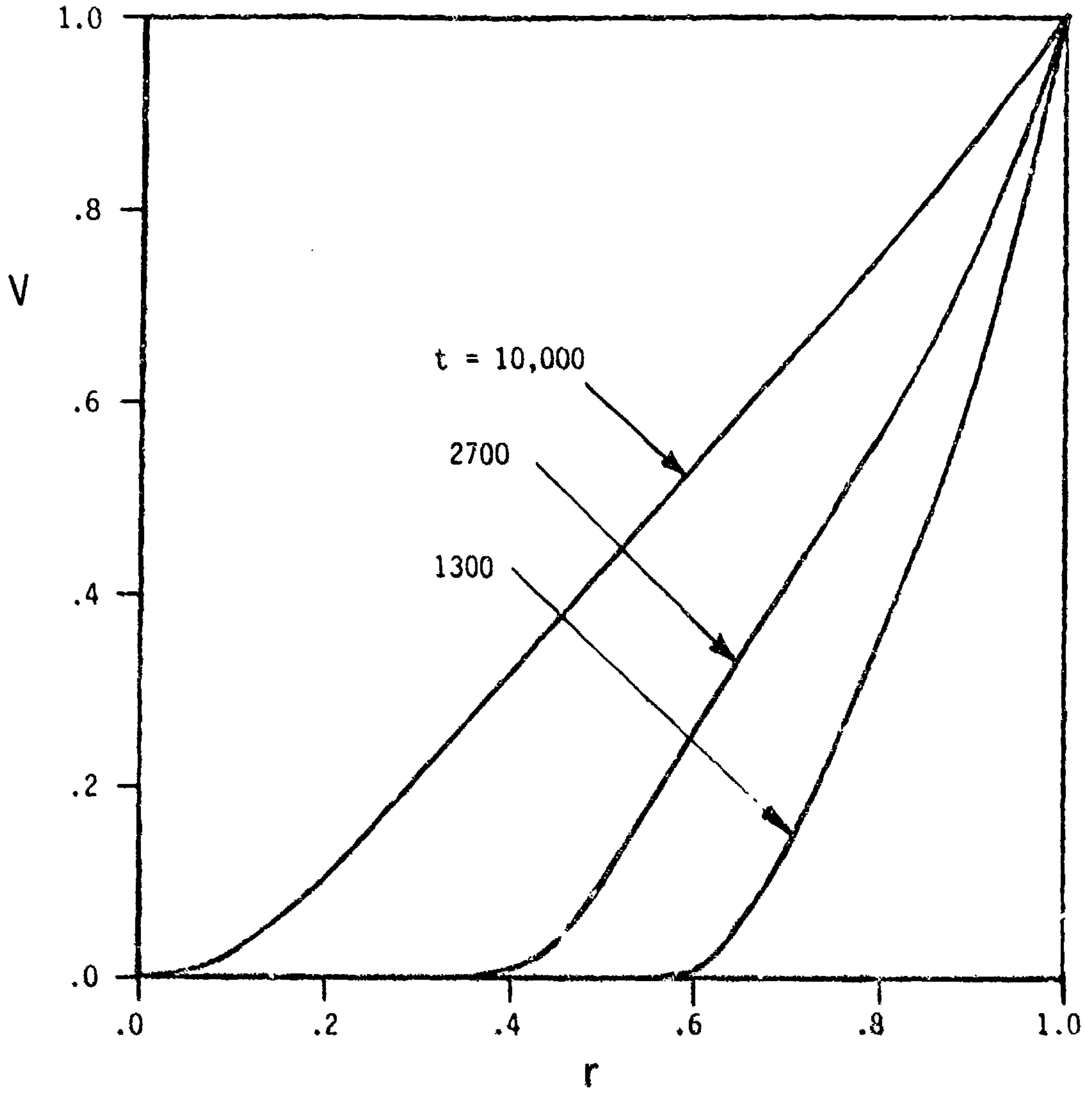


Figure 2. V vs r for Case 2 with c/a = 5.200, Re = 2×10^6 at Three Values of t; $\hat{t}_{st}$ = 2700. Turbulent Ekman Layer.

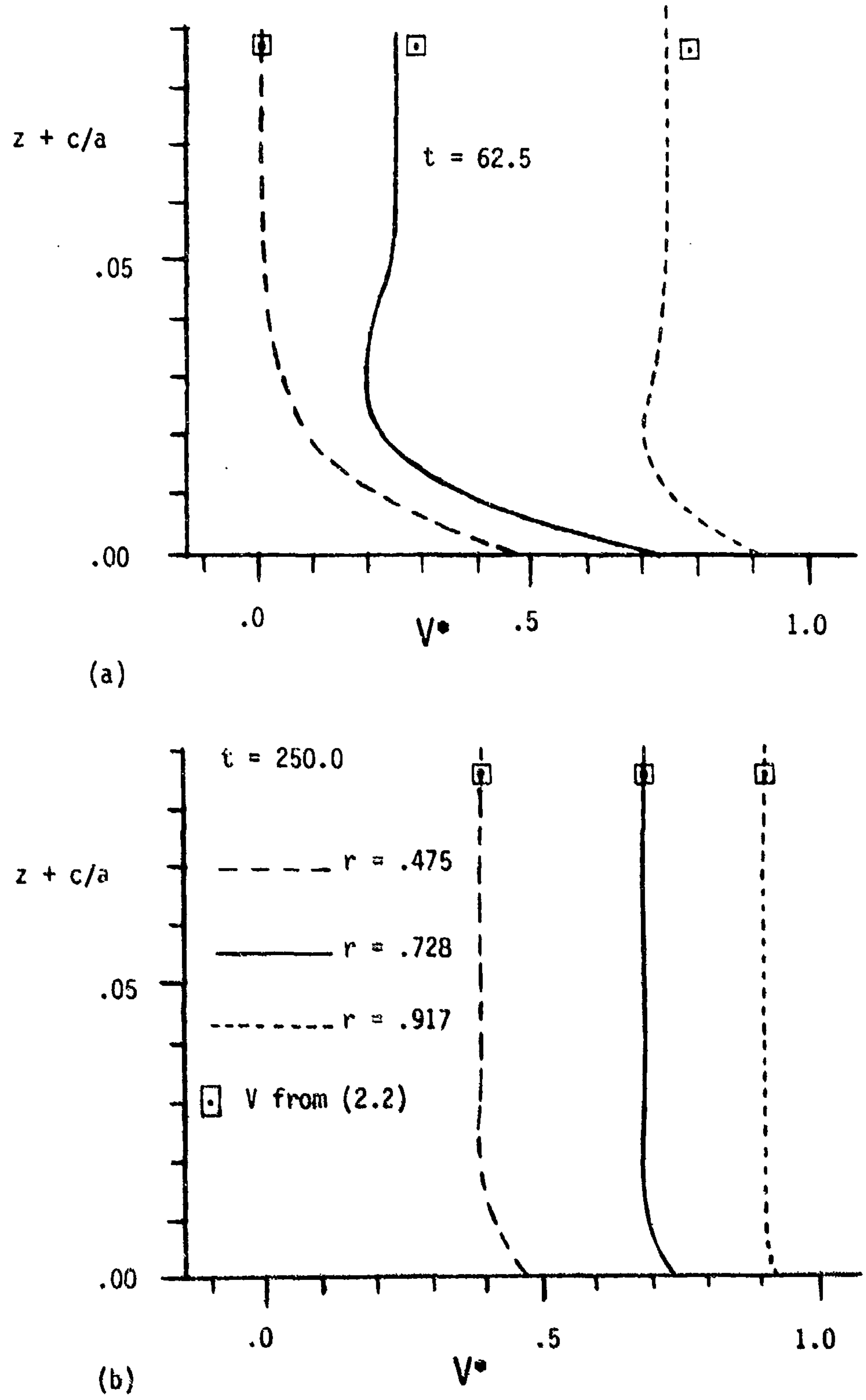
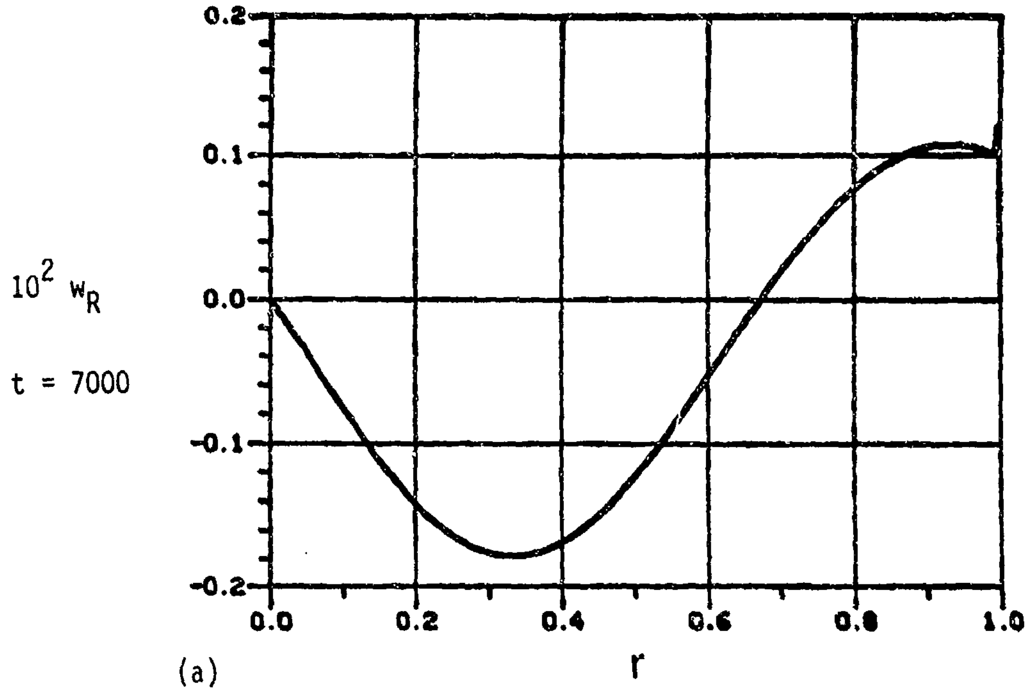
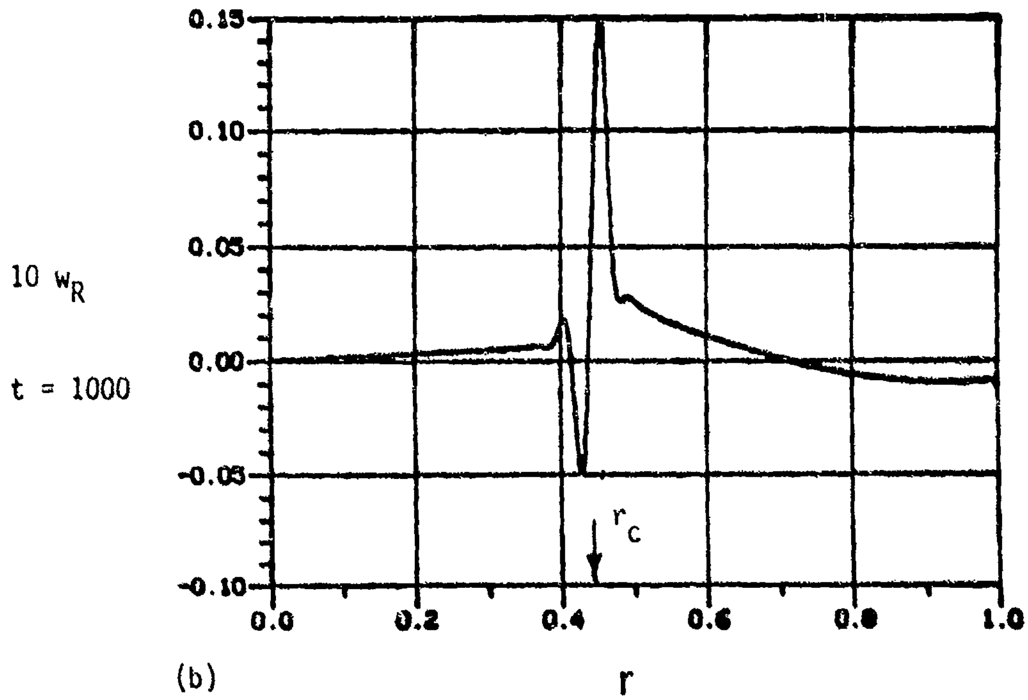


Figure 3. V* vs z + c/a = z' for c/a = 1, Re = 10^4, t_s = 200 at (a) t = 62.5, (b) t = 250. The Solution is Obtained From the Program in Reference 11.



(a)



(b)

Figure 4. The Eigenfunction Real (w) vs r for Re = 5 x 10^5, c/a = 2.679, Mode (5.1,1). (a) t = 7000, (b) t = 1000, (c) t = 400.

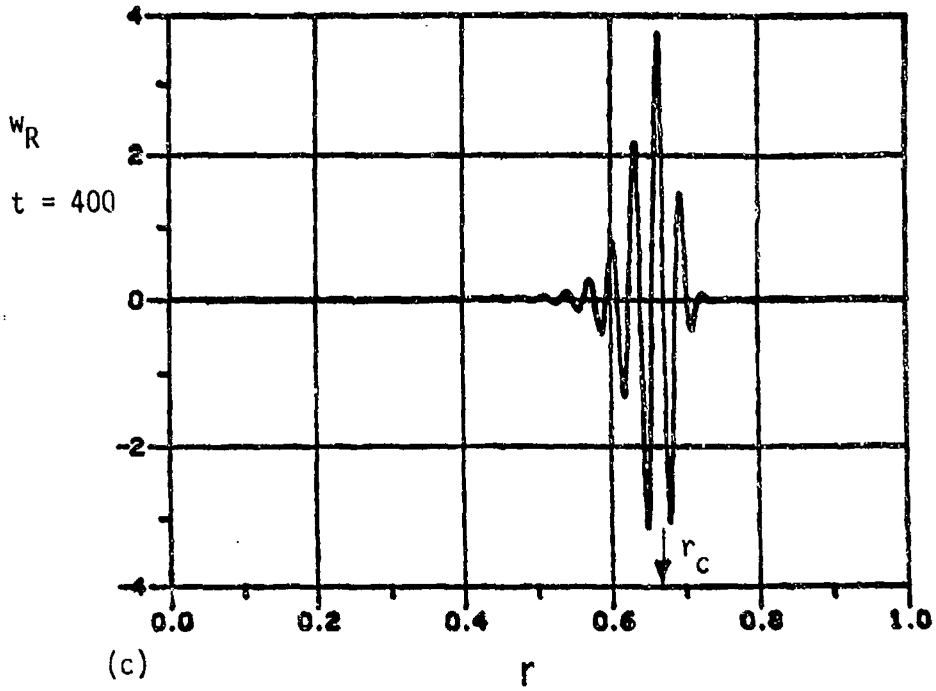


Figure 4. The Eigenfunction Real (w) vs r for Re = 5 x 10^5, c/a = 2.679, Mode (5,1,1). (a) t = 7000, (b) t = 1000, (c) t = 400 (continued).

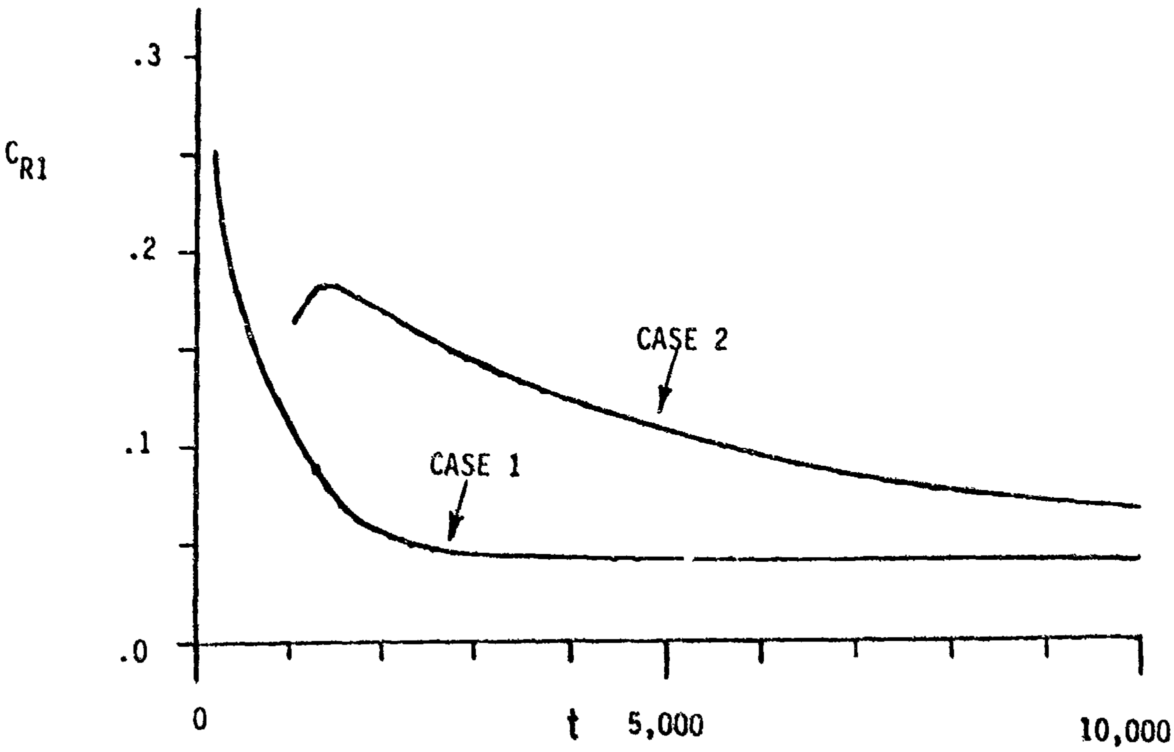


Figure 5. C_{R1} vs t for Case 1 With c/a = 3.120, Re = 4 x 10^4, Mode (3,1,1) and Case 2 With c/a = 5.200, Re = 2 x 10^6, Mode (5,1,1).

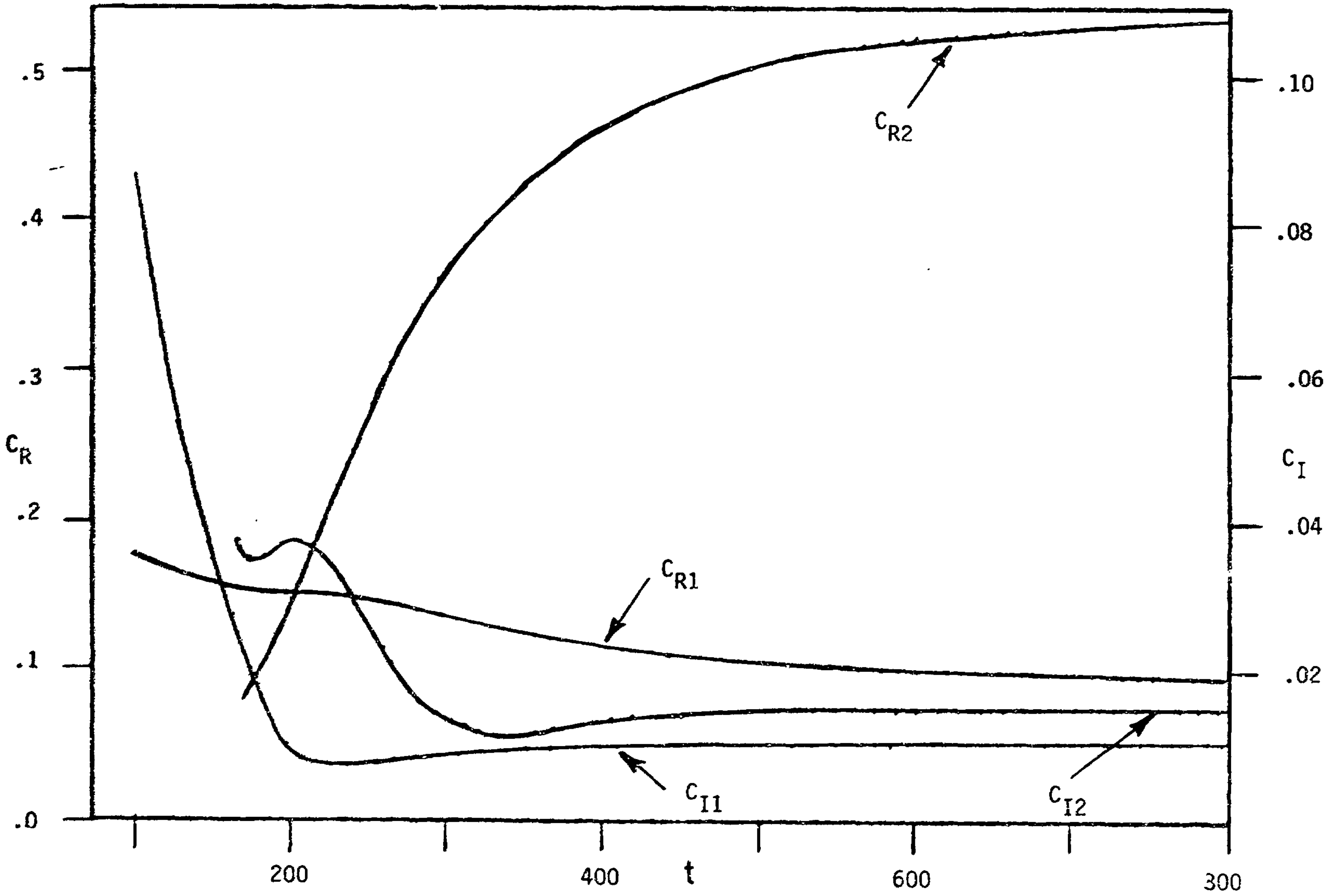


Figure 6. Time Histories of C_R and C_I for c/a = 3.297, Re = 4,974, Modes (3,1,1) and (3,2,1).

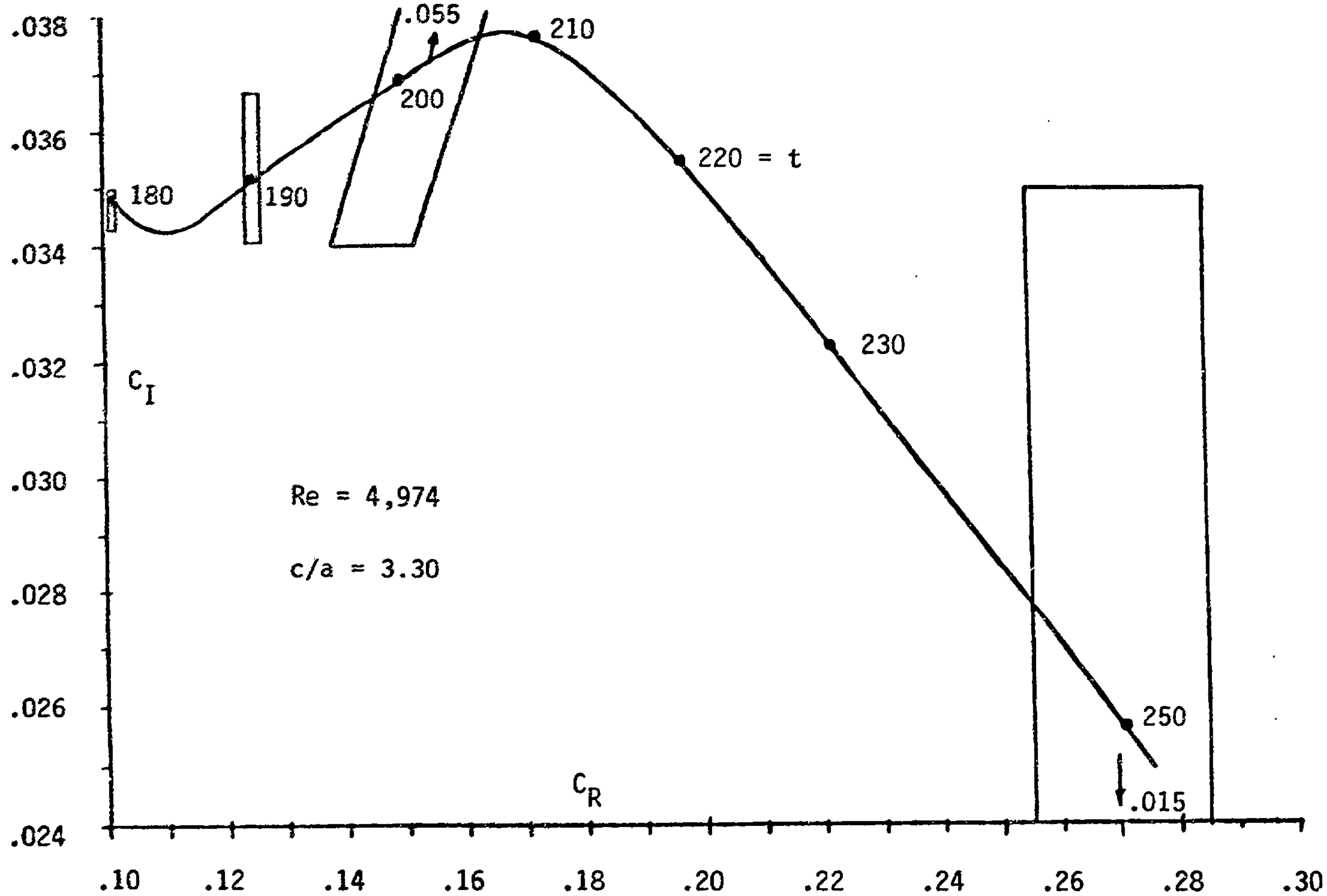


Figure 7. Trajectory of C_2 With t as Parameter; c/a = 3.297, Re = 4,974, Mode (3,2,1). Approximate Regions in Which First Guess for C_2 Must Lie to Have Convergence to n = 2 Mode Are Shown for Four Times.

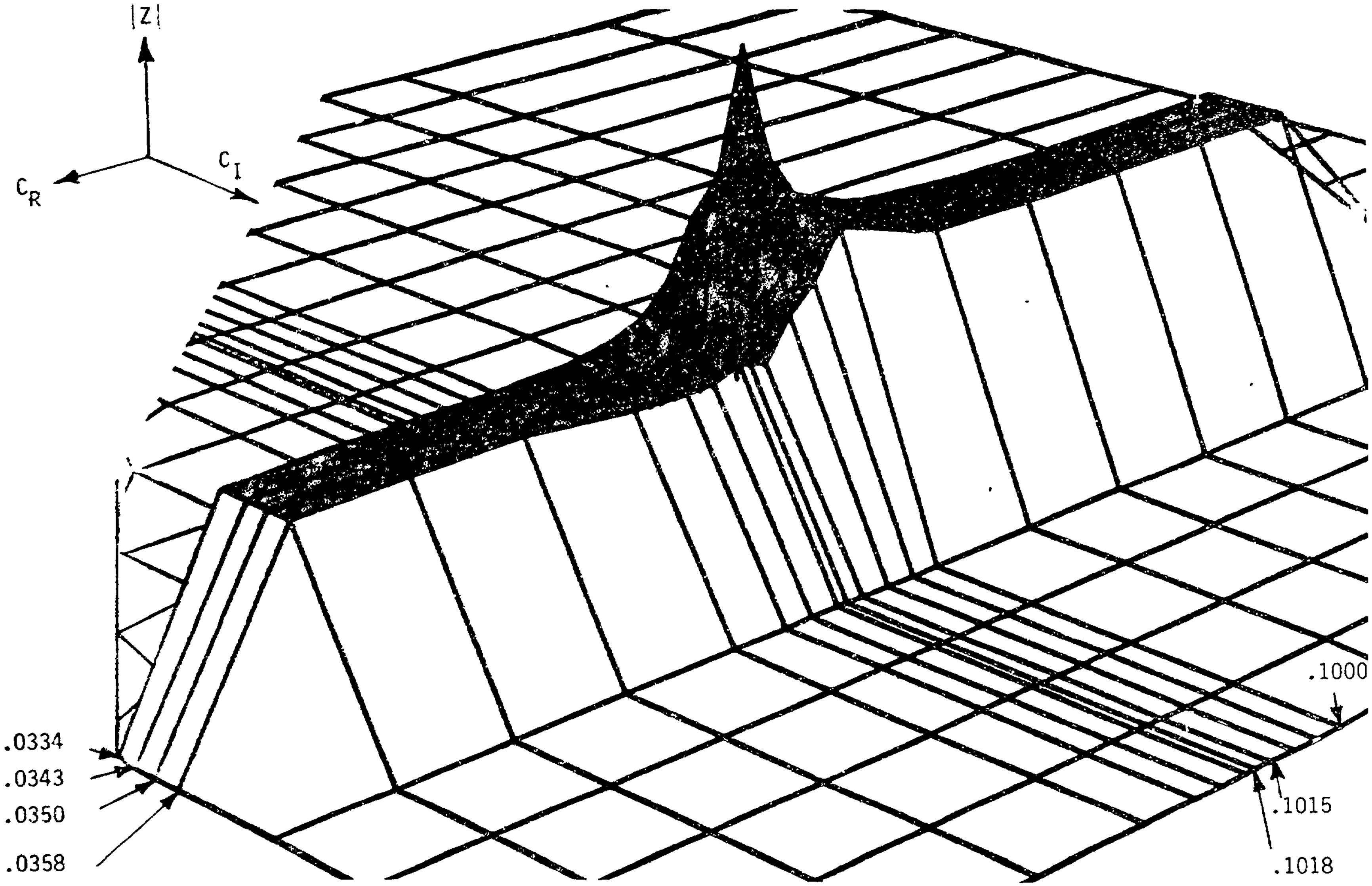


Figure 8. The Surface $|Z|$ (C_R, C_I) for $c/a = 3.297$ and $Re = 4{,}974$ at $t = 180$, in the Neighborhood of the Root $C_{R2} = .101673$, $C_{I2} = .034867$.

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APPENDIX A

Power Series Coefficients

APPENDIX A. POWER SERIES COEFFICIENTS

The need for the power series expansions near $r = 0$ of u, v, w, and p is described in Section IV. The coefficients of these series are recorded here; they are obtained by substituting the series into (4.5). V must also be expressed in a power series

$$V = \sum_{j=1} V_j r^j$$

which is permissible since V is an analytic function of r, near $r = 0$, except that it may not be uniformly so for $t \to 0$. The coefficients V_j could be determined from (2.2) in terms of $\partial V/\partial r$, $\partial(\partial V/\partial t)/\partial r$, etc., at $r = 0$. These would have to be found numerically from the solution V (r,t). Instead we evaluate the V_j directly, fitting a quartic to V (r,t).

Because the boundary conditions (4.3) are different for $m = 0$ and $m = 1$, the coefficients must be given separately for these two cases.

$m = 0$:

The forms of the power series are

$$u = \sum_{j=1} u_j r^j \qquad v = \sum_{j=1} v_j r^j$$

$$w = w_0 + \sum_{j=1} w_j r^j \qquad p = p_0 + \sum_{j=1} p_j r^j$$

The coefficients are given by the following sequence of formulas, where w_0, p_0, and v_1 are arbitrary complex constants:

$$H \equiv K^2 Re^{-1} + i C$$

$$u_1 = -(K/2)w_0, \qquad w_1 = 0$$

$$v_1 = v_1, \qquad p_1 = 0$$

$$u_2 = 0, \qquad w_2 = (Re/4)(Hw_0 - Kp_0)$$

$$v_2 = 0, \qquad p_2 = (1/2)Hu_1 + (K/Re)w_2 - V_1 v_1$$

For $j = 1, 2, 3 \ldots$

$$w_{j+2} = Re\,(j+2)^{-2}\,[Hw_j - Kp_j]$$

$$u_{j+2} = -K(j+3)^{-1} w_{j+1}$$

$$v_{j+2} = \text{Re}\ [(j^2 + 4j + 3)]^{-1} [Hv_j + \sum_{\ell = 1}^{j+1} (\ell+1)\ V_\ell\ u_{j+1-\ell}]$$

$$p_{j+2} = (j+2)^{-1} [-K(j+2)^{-1} (Hw_j - Kp_j) - Hu_{j+1} + 2 \sum_{\ell = 1}^{j+2} V_\ell\ v_{j+2-\ell}]$$

m = 1:

The solution for the azimuthal mode m = 1 is given by

$$u = \sum_{j = 0} a_j r^j, \qquad v = \sum_{j = 0} b_j r^j$$

$$w = \sum_{j = 0} d_j r^j, \qquad p = \sum_{j = 0} e_j r^j$$

Summations are taken from j = 0 for convenience.

The coefficients are given by the following sequence of formulas, where b_0, d_1, and e_1 are arbitrary complex constants:

$$a_0 = ib_0, \quad b_0 = b_0, \quad d_0 = 0, \quad e_0 = 0$$

$$a_1 = 0, \quad b_1 = 0, \quad d_1 = d_1, \quad e_1 = e_1$$

$$H \equiv Re^{-1} K^2 + i\ C$$

$$a_2 = -(1/4)Kd_1 + (1/8)\ \text{Re}\ [b_0\ (iH - V_1) + e_1]$$

$$b_2 = -i(1/4)Kd_1 - (3i/8)\ \text{Re}\ [b_0\ (iH - V_1) + e_1]$$

For j = 0, 1, 2, ...

$$N_{1j} = \text{Re}\ [Ha_{j+1} - \sum_{\ell = 0}^{j+1} V_{\ell+1}\ (2b_{j+1-\ell} + ia_{j+1-\ell})]$$

$$N_{2j} = \text{Re}\ [Hb_{j+1} + \sum_{\ell = 0}^{j+1} V_{\ell+1}\ ((\ell+2)a_{j+1-\ell} - ib_{j+1-\ell})].$$

$$N_j = [N_{1j} - i(j+2)N_{2j}]/[(j+1)\ (j+3)]$$

$$d_{j+2} = Re\ [Hd_j - Ke_j - i \sum_{\ell=0}^{j} V_{\ell+1} d_{j-\ell}]/[(j+1)\ (j+3)]$$

$$a_{j+3} = -[N_j + (j+4)\ Kd_{j+2}]/(j^2 + 8j + 15)$$

$$b_{j+3} = i[Kd_{j+2} + (j+4)N_j]/(j^2 + 8j + 15)$$

$$e_{j+2} = [2a_{j+3} + i\ (j^2 + 6j + 7)\ b_{j+3} - iN_{2j}]/Re$$

APPENDIX B

Characteristic Determinant, Z

APPENDIX B. CHARACTERISTIC DETERMINANT, Z

The characteristic determinant, Z (C), denoted by D (C) in (25) of Reference 1, is defined here for convenience. In the integration procedure we calculate three linearly independent solutions that satisfy three boundary conditions at $r = 0$. These solutions are denoted by $(u_j, u_j - i\, v_j, w_j)$, $j = 1, 2, 3$; p_j is not needed here. The no-slip conditions at the sidewall require the velocity components to vanish at $r = 1$. These components are linear combinations of the three solutions above. Thus

$$\gamma_1\, u_1(1) + \gamma_2\, [u_1(1) - i\, v_1(1)] + \gamma_3\, w_1(1) = 0$$

$$\gamma_1\, u_2(1) + \gamma_2\, [u_2(1) - i\, v_2(1)] + \gamma_3\, w_2\, (1) = 0 \qquad \text{(B1)}$$

$$\gamma_1\, u_3(1) + \gamma_2\, [u_3(1) - i\, v_3(1)] + \gamma_3\, w_3(1) = 0,$$

where γ_1, γ_2, and γ_3 are constants. In order that a non-trivial solution for γ_1, γ_2, and γ_3 be found, the determinant of the coefficients in (B1), which is designated Z, must equal zero.

LIST OF SYMBOLS

a	cross-sectional radius of cylinder
c	half-height of cylinder
C	$\equiv C_R + i\,C_I$, complex eigenvalue
C_I	disturbance decay rate/Ω
C_R	disturbance frequency/Ω
C_{R1}, C_{R2}	C_R's for $n = 1$ and $n = 2$ modes, respectively
E	$= \nu/\Omega\, c^2$, Ekman number
k	axial mode number (see (4.1))
k_ℓ	$\equiv \kappa\,(a/c)\, Re^{-1/2}$ (see (2.8))
k_t	$\equiv 0.035\,(a/c)\, Re^{-1/5}$ (see (2.9))
K	$= k\, \pi a/(2c)$ (see (4.1))
m	azimuthal mode number (see (4.1))
M	$\equiv C - mV/r$
n	radial mode number
N	number of subintervals used in orthonormalization
p	function describing radial variation of p' (see (4.1))
p'	3-D pressure perturbation/$(\rho a^2 \Omega^2)$
P	pressure/$(\rho a^2 \Omega^2)$ of approximate basic flow (see (2.3))
$\bar{P}$	pressure/$(\rho a^2 \Omega^2)$ in 3-D Navier-Stokes equations (see (3.1))
P_0	zeroth order approximation, in $1/t_s$, to P^* (see (2.6))
P_1	coefficient in first order approximation to P^* in (2.6)
P^*	pressure/$(\rho a^2 \Omega^2)$ of exact basic flow (see (2.1))
r	radial coordinate / a
r_c	critical level radius / a
Re	$\equiv a^2 \Omega/\nu$, Reynolds number
t	time $\times\ \Omega$

$\bar{t}_{fl}$	characteristic flight time of projectile
t_s	$\equiv (2c/a)\ Re^{1/2}$, $\bar{t}_s = t_s/\Omega$ = spin-up time (laminar)
t_{st}	$\equiv (28.6c/a)\ Re^{1/5}$, $\bar{t}_{st} = t_{st}/\Omega$ = spin-up time (turbulent)
t'	$\equiv t/t_s$
u, v, w	functions describing radial variation of u', v', w' (see (4.1))
u', v', w'	radial, azimuthal, axial velocity components × 1/(a Ω) of 3-D perturbed flow (see (3.2))
U, V, W	radial, azimuthal, axial velocity components × 1/(a Ω) of approximate basic flow (see (2.2))
U_1, V_1, W_1	coefficients in first order approximations, in $1/t_s$, to U*, V*, W* (see (2.6))
U_w, V_w	radial and azimuthal velocity components × 1/(a Ω) of basic flow without diffusion given by (2.4)
$\bar{U}$, $\bar{V}$, $\bar{W}$	radial, azimuthal, axial velocity components × 1/(a Ω) of the 3-D Navier-Stokes equations (see (3.1))
U*, V*, W*	radial, azimuthal, axial velocity components × 1/(a Ω) of exact basic flow (see (2.1))
V_0	zeroth order approximation, in $1/t_s$, to V* (see (2.6))
z	axial coordinate/a (z = 0 at cylinder midplane)
z'	= z + c/a
Z	characteristic determinant whose roots are eigenvalues
ϵ	value of r where numerical integration is initiated
θ	azimuthal angle
κ	constant in expression for radial velocity in approximate basic flow with laminar Ekman layer (see (2.8))
ν	kinematic viscosity of fluid
ρ	density of fluid
Ω	spin rate of cylinder

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