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TRANSITION FROM TRAVELING TO STATIONARY LOADS IN A HOLLOW CYLINDER

Alexander S. Elder

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The equations for elastic strains due to internal loads moving with constant velocity are formulated in terms of Green-Lame′ potentials, using sin(s)(z-ct) and cos(s)(z-ct) as separation factors in the Fourier integrals for the strains. The remaining factors involve Bessel functions and trigonometric functions of the polar angle. Solenoidal vector potentials are also derived in accord with the general theory. The dynamic stresses for a traveling load approach the equilibrium stresses as the velocity of travel approaches zero.

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I. INTRODUCTION

In this paper we derive formulas for calculating the response of a hollow cylinder to internal loads moving with uniform velocity. This analysis is based on the historic Green-Lamé theory of wave motion[1] and recent work on free vibrations of a hollow cylinder.[2] Only steady state conditions are considered; transients which may arise at the beginning of motion are ignored. Formulas for moving sinusoidal loads are discussed in detail. The response to a moving step function can be obtained by using Cauchy's discontinuous factor.

The transition from traveling to stationary loads is obtained by a limiting process. The biharmonic functions required for the stationary load arise from the confluence of the vector and scalar wave functions at zero velocity. This solution is also derived independently from appropriate elastic potentials. This analysis shows that the calculated displacements for moving loads approach the calculated values for stationary loads as the velocity of travel approaches zero. Physically, this requirement is obvious. Hence, computations for low velocities will serve as a critical test of the analysis and computations.

The results are required to interpret strain measurements obtained with an instrumented gun tube in which arrays of strain gages and pressure gages were mounted in close proximity along the length of the tube. Tube strains based on the measured pressures were calculated according to thick walled cylinder theory,[3] using appropriate mechanical properties for the gun steel. At low velocities the measured and calculated strains agreed reasonably well, but at high projectile velocities the calculated strain distribution profiles were significantly different from the measured strain histories.

This effect is well known in the theory of moving loads.[4] Prof. Ian Sneddon of Glasgow University, a pioneer in the field, suggested this problem to the author during a recent visit.

The analysis is given in considerable detail to facilitate verification of formulas required for programming. Formulation is limited to subsonic velocities which occur in practice; analysis of the moving load problem for supersonic velocities is quite difficult and will not be considered at the present time.

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[1] *M.E. Gurtin, Elasticity, Enc. of Phys., Volume V1 a/2, Mechanics II, Springer Verlag, New York, 1972. See pages 212-214.*

[2] *A.E. Armenakas, G. Herrmann and D.C. Gazis, Free Vibrations of Circular Cylindrical Shells, Pergamon Press, New York, 1969.*

[3] *A.S. Elder and K.L. Zimmerman, "Stresses in a Gun Tube Produced by Internal Pressure and Shear," BRL Memorandum Report No. 2495, June 1975 (AD A012765).*

[4] *L. Taylor, Vibrations of Solids and Structures under Moving Loads, Noordhoff International Publishing, Gronigen, The Netherlands, 1972. See Chapter 17 and Reference 203.*

II. FORMULAS FOR WAVE MOTION IN CYLINDRICAL COORDINATES

In the absence of body forces the equation of motion may be written in the form*

$$(\lambda+2\mu)\nabla\nabla\cdot\hat{u} - \mu \nabla \times (\nabla\times\hat{u}) = \rho \frac{\partial^2\hat{u}}{\partial t^2} \tag{1}$$

where λ and μ are the Lamé constants. The speed of the dilatational wave is

$$c_1^{\ 2} = (\lambda+2\mu)/\rho \tag{2}$$

and the velocity of the shear wave is

$$c_2^{\ 2} = \mu/\rho \quad , \tag{3}$$

where ρ is the density of the elastic material. The equation of motion becomes

$$c_1^{\ 2} \nabla\nabla\cdot\hat{u} - c_2^{\ 2} \nabla \times(\nabla\times\hat{u}) = \frac{\partial^2\hat{u}}{\partial t^2} \tag{4}$$

The Green-Lamé solution is

$$\hat{u} = \nabla \phi + \nabla \times \hat{\psi} \tag{5}$$

in which the potentials ϕ and $\hat{\psi}$ satisfy the equations

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*the notation $\nabla\nabla\cdot\hat{u}$ is to be interpreted as $\nabla(\nabla\cdot\hat{u})$.

$$c_1{}^2 \; \nabla^2 \phi \;=\; \frac{\partial^2 \phi}{\partial t^2} \tag{6}$$

and

$$c_2{}^2 \; \nabla^2 \hat{\psi} \;=\; \frac{\partial^2 \hat{\psi}}{\partial t^2} \tag{7}$$

In cylindrical coordinates we have

$$\nabla^2 \phi \;=\; \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\,\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} \tag{8}$$

The vector potential is given in terms of three components

$$\hat{\psi} \;=\; \hat{e}_r \, \psi_r + \hat{e}_\theta \psi_\theta + \hat{e}_z \, \psi_z \;, \tag{9}$$

where $\hat{e}_r$, $\hat{e}_\theta$, and $\hat{e}_z$ are unit vectors in cylindrical coordinates.[5]

$$\begin{aligned} \nabla^2 \hat{\psi} = \hat{e}_r \quad & \nabla^2 \psi_r - \frac{\psi_r}{r^2} - \frac{2}{r^2}\,\frac{\partial}{\partial \theta}\,\psi_\theta \\ + \hat{e}_\theta \quad & \nabla^2 \psi_\theta - \frac{\psi_\theta}{r^2} + \frac{2}{r^2}\,\frac{\partial}{\partial \theta}\,\psi_r \\ + \hat{e}_z \; & \nabla^2 r_z \end{aligned} \tag{10}$$

[5] *P.C. Chou and N.J. Pagano, Elasticity: Tensor, Dyadic, and Engineering Approaches, D. Van Nostrand Company, Inc., Princeton, 1967, pages 245-265.*

The following scalar partial differential equations are finally obtained:

$$c_1^2 \nabla^2 \phi = \frac{\partial^2 \phi}{\partial t^2} \tag{11}$$

$$c_2^2 \nabla^2 \psi_r - \frac{\psi_r}{r^2} + \frac{2}{r^2} \frac{\partial}{\partial \theta} \psi_\theta = \frac{\partial^2}{\partial t^2} \psi_r \tag{12}$$

$$c_2^2 \nabla^2 \psi_\theta - \frac{\psi_\theta}{r^2} - \frac{2}{r^2} \frac{\partial}{\partial \theta} \psi_r = \frac{\partial^2}{\partial t^2} \psi_\theta \tag{13}$$

$$c_2^2 \nabla^2 \psi_z = \frac{\partial^2}{\partial t^2} \psi_z \tag{14}$$

Four components of the displacement vector are obtained from solutions of these equations. Since only three components are required, we can specify one additional condition. We choose a relation between ψ_r and ψ_θ which will facilitate separation of variables, following the ideas outlined in Reference 2.

III. SEPARATION OF VARIABLES FOR SINUSOIDAL LOADS MOVING WITH CONSTANT VELOCITY

The scalar potential can be obtained directly by separation of variables. For brevity let

$$H_1 = \cos(n\theta)\sin[s(z-ct)] \tag{15}$$

$$H_2 = \cos(n\theta)\cos[s(z-ct)] \tag{16}$$

$$H_3 = \sin(n\theta)\sin[s(z-ct)] \tag{17}$$

$$H_4 = \sin(n\theta)\cos[s(z-ct)] \tag{18}$$

and

$$\phi_i = H_i\, f(r),\ i = 1,2,3,4. \tag{19}$$

In these equations n is an integer since the stresses must be periodic in a complete cylinder. The constant c is the velocity of travel. We assume

$$0 < \varepsilon < c < c_2 - \varepsilon \quad , \tag{20}$$

where

$$0 < \varepsilon << 1 \quad . \tag{21}$$

Let

$$\alpha_1^{\ 2} = c^2/c_1^{\ 2} \tag{22}$$

$$\beta_1^{\ 2} = 1 - \alpha_1^{\ 2} \tag{23}$$

Then

$$f'' + \frac{1}{r} f' - (\frac{n^2}{r^2} + \beta_1^{\ 2} s^2)\ f = 0 \tag{24}$$

and

$$f = A_1 I_n(\beta_1 sr) + A_2 K_n(\beta_1 sr) \tag{25}$$

The axial component of the vector potential can be treated in a similar manner. We let

$$\psi_z = H_i g_z(r), \ i = 1,2,3,4 \tag{26}$$

$$\alpha_2^{\ 2} = c_2^{\ 2}/c^2 \tag{27}$$

$$\beta_2^{\ 2} = 1 - \alpha_2^{\ 2} \tag{28}$$

then

$$g_z = A_3 I_n(\beta_2 sr) + A_4 K_n(\beta_2 sr) \tag{29}$$

The radial and tangential components of the vector potential are coupled. The following four combinations lead to separable solutions:

$$\psi_r = H_1 g_r, \qquad \psi_\theta = - H_3 g_\theta \tag{30}$$

$$\psi_r = H_2 g_r, \qquad \psi_\theta = - H_4 g_\theta \tag{31}$$

$$\psi_r = H_3 g_r, \qquad \psi_\theta = H_1 g_\theta \tag{32}$$

$$\psi_r = H_4 g_r, \qquad \psi_\theta = H_2 g_\theta \tag{33}$$

The combination given in Eq. (32) is discussed in Reference 2; the remaining combinations are introduced to satisfy a variety of boundary conditions on the inner radius. We note that

$$\frac{\partial}{\partial\theta} H_1 = - nH_3 \tag{34}$$

$$\frac{\partial}{\partial\theta} H_2 = - nH_4 \tag{35}$$

$$\frac{\partial}{\partial\theta} H_3 = nH_1 \tag{36}$$

$$\frac{\partial}{\partial\theta} H_4 = nH_2 \tag{37}$$

On referring to Eqs. (12) and (13) we find

$$r^2 g_r'' + r g_r' - (n^2 + 1 + \beta_2^2 s^2 r^2) g_r + 2n g_\theta = 0 \tag{38}$$

$$r^2 g_\theta'' + r g_\theta' - (n^2 + 1 + \beta_2^2 s^2 r^2) g_\theta - 2n g_r = 0 \tag{39}$$

If we let

$$g_\theta = - g_r \tag{40}$$

we find

$$r^2 g_r'' + r g_r' - [(n + 1)^2 + \beta_2^2 s^2 r^2] g_r = 0 \tag{41}$$

and

$$g_r = A_5 I_{n+1}(\beta_2 sr) + A_2 K_{n+1}(\beta_2 sr) \tag{42}$$

If on the other hand we let

$$g_\theta = g_r \quad , \tag{43}$$

then

$$r^2 g_r'' + r g_r' - [(n - 1)^2 + \beta_2^2 s^2 r^2] g_r = 0 \tag{44}$$

and

$$g_r = A_7 I_{n-1}(\beta_2 sr) + A_8 K_{n-1}(\beta_2 sr) \tag{45}$$

In this report the radial and tangential components are based on Eqs. (40), (41) and (42).

As mentioned previously, a complete solution to a specified boundary value problem may be obtained if one of the components of the vector potential is set equal to zero. Hence, although unique solutions for the displacements are expected, the choice of potentials leading to a solution is not unique.

Pao and Mow[6] use vector potentials $\hat{L}$, $\hat{M}$, and $\hat{N}$, which are derived from scalar functions α, ψ, and x in their analysis of diffraction waves in an elastic solid. The relation between this solution and the solution of Herrmann and Gazis may be obtained by equating formulas for the displacements.

IV. COMPONENTS OF THE VECTOR DISPLACEMENT

Each component of the vector displacement consists of one term from the scalar potential and two terms from the vector displacement, according to Eq. (5). These terms must have the same trigonometric factor. In addition, the signs of g_r and g_θ are chosen so these functions involve Bessel functions of order n+1 when $g_\theta = - g_r$. The sign of g_z is chosen so the divergence can be written in the form

$$\nabla \cdot \hat{\psi} = H_i h(r), \ i = 1,2,3,4 \tag{46}$$

where

$$h(r) = \frac{1}{r} \frac{\partial}{\partial r} (r\psi_r) - \frac{n}{r} g_\theta + s g_z \tag{47}$$

We consider four cases, as shown in the table below.

[6]Y. Pao and C. Mow, *Diffraction of Elastic Waves and Dynamic Stress Concentrations*, Crane Russak, Publishers, 1971. See pages 217-239.

TABLE 1. TRIGONOMETRIC FACTORS FOR SCALAR AND VECTOR POTENTIALS

	Function	$f(r)$	$g_r(r)$	$g_\theta(r)$	$g_z(r)$	$h(r)$
Case	I	H_1	H_4	H_2	H_3	H_4
	II	H_2	H_3	H_1	$-H_4$	H_3
	III	H_3	H_2	$-H_4$	H_1	H_2
	IV	H_4	H_1	$-H_3$	$-H_2$	H_1

The displacement components for each case are obtained from the potentials.

Case I

$$ru = [rf' + srg_\theta + ng_z]\ H_1 \tag{48}$$

$$rv = [-nf - srg_r - rg_z']\ H_3 \tag{49}$$

$$rw = [srf + \frac{\partial}{\partial r} rg_\theta - ng_z]\ H_2 \tag{50}$$

Case II

$$ru = [rf' + srg_\theta - ng_z]\ H_2 \tag{51}$$

$$rv = [nf + srg_r + rg_z']\ H_4 \tag{52}$$

$$rw = [- srf + \frac{\partial}{\partial r} rg_\theta - ng_r] H_1 \tag{53}$$

Case III

$$ru = [rf' + srg_\theta - ng_z] H_3 \tag{54}$$

$$rv = [nf - srg_r - rg_z'] H_1 \tag{55}$$

$$rw = [srf - \frac{\partial}{\partial r} rg_\theta + ng_r] H_4 \tag{56}$$

Case IV

$$ru = [rf' + srg_\theta + ng_z] H_4 \tag{57}$$

$$rv = [nf + srg_r + rg_z'] H_2 \tag{58}$$

$$rw = [-srf + \frac{\partial}{\partial r} rg_\theta + ng_r] H_3 \tag{59}$$

Next, we express the displacement and divergence for Case I in terms of Bessel functions of order n and n+1. The following identities are used to eliminate derivatives:

$$XI_n'(X) = nI_n(X) + XI_{n+1}(X) \tag{60}$$

$$XK_n'(X) = nK_n(X) - XK_{n+1}(X) \tag{61}$$

$$XI'_{n+1}(X) = -(n+1)I_{n+1}(X) + XI_n(X) \qquad (62)$$

$$XK'_{n+1}(X) = -(n+1)K_{n+1}(X) - XK_n(X) \qquad (63)$$

On carrying out details of the analysis we find

$$ru = \{A_1[nI_n(\beta_1 sr) + \beta_1 srI_{n+1}(\beta_1 sr)]$$

$$+ A_2[nK_n(\beta_1 sr) - \beta_1 srK_{n+1}(\beta_1 sr)]$$

$$+ A_3 nI_n(\beta_2 sr) + A_4 nK_n(\beta_2 sr)$$

$$- A_5 srI_{n+1}(\beta_2 sr) - A_6\beta_2 srK_{n+1}(\beta_2 sr)\} \cos(n\theta)\sin[s(z-ct)] \quad (64)$$

$$rv = \{- A_1 nI_n(\beta_1 sr) - A_2 nK_n(\beta_1 sr)$$

$$+ A_3[- nI_n(\beta_2 sr) + \beta_2 srI_{n+1}(\beta_2 sr)]$$

$$+ A_4[- nK_n(\beta_2 sr) + \beta_2 srK_{n+1}(\beta_2 sr)]$$

$$- A_5 srI_{n+1}(\beta_2 sr) - A_6 srK_{n+1}(\beta_2 sr)\} \sin(n\theta)\sin[s(z-ct)] , \quad (65)$$

$$rw = \{A_1 srI_n(\beta_1 sr) + A_2 srK_n(\beta_1 sr)$$

$$- A_5 s r I_n(\beta_2 s r) + A_6 \beta_2 s r K_n(\beta_2 s r)\} \cos(n\theta)\cos[s(z-ct)] \tag{66}$$

The divergence of the vector potential is

$$\nabla \cdot \hat{\psi} = [(A_3 + A_5\beta_2) s I_n(\beta_2 s r) + (A_4 - A_5\beta_2) s K_n(\beta_2 s r)]\sin(n\theta)\cos[s(z-ct)] \tag{67}$$

Formulas for the displacement and divergence corresponding to Cases II, III, and IV can be written down by inspection.

The divergence of the vector potential enters into calculations of the rotation vector. The analysis prior to Part IX of this report follows Reference 2 in using a vector potential with nonsolenoidal divergence, as this approach considerably simplifies the analysis. In Part IX a solenoidal vector potential, for which the divergence is zero, is derived by separation of variables. Formulas for the vector displacement and its derivative remain unchanged, but a simpler formula for the rotation vector is obtained. The usual approach by means of the Newtonian potential is not appropriate since the hollow cylinder is a multiply connected region.

V. STRAINS, ROTATION VECTOR, AND STRESSES

The strains are obtained from the displacements by means of the formulas

$$\varepsilon_r = \frac{\partial u}{\partial r} \tag{68}$$

$$\varepsilon_\theta = \frac{u}{r} + \frac{\partial v}{r\partial\theta} \tag{69}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \tag{70}$$

$$\gamma_{r\theta} = \frac{\partial u}{r\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \tag{71}$$

$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \tag{72}$$

$$\gamma_{z\theta} = \frac{\partial v}{\partial z} + \frac{\partial w}{r\partial\theta} \tag{73}$$

We calculate the strains in terms of Bessel functions for Case I. The remaining three cases can be treated in a similar manner. Let

$$r^2\varepsilon_r = H_1 \sum_1^6 A_i R_{1,i} \tag{74}$$

$$r^2\varepsilon_\theta = H_1 \sum_1^6 A_i R_{2,i} \tag{75}$$

$$r^2\varepsilon_z = H_1 \sum_1^6 A_i R_{3,i} \tag{76}$$

$$r^2\gamma_{r\theta} = H_3 \sum_1^6 A_i R_{4,i} \tag{77}$$

$$r^2\gamma_{rz} = H_2 \sum_1^6 A_i R_{5,i} \tag{78}$$

$$r^2\gamma_{\theta z} = H_4 \sum_1^6 A_i R_{6,i} \tag{79}$$

The $R_{i,j}$ functions depend on the radius, and can be expressed in terms of Bessel functions of orders n and n+1.

$$R_{1,1} = [\beta_1{}^2 s^2 r^2 + (n^2-n)]I_n(\beta_1 sr) - \beta_1 srI_{n+1}(\beta_1 sr) \tag{80}$$

$$R_{1,2} = [\beta_1{}^2 s^2 r^2 + (n^2-n)]K_n(\beta_1(sr) + \beta_1 sr K_{n+1}(\beta_1 sr) \tag{81}$$

$$R_{1,3} = (n^2-n)I_n(\beta_2 sr) + \beta_2 sr I_{n+1}(\beta_2 sr) \tag{82}$$

$$R_{1,4} = (n^2-n)K_n(\beta_2 sr) - \beta_2 sr K_{n+1}(\beta_2 sr) \tag{83}$$

$$R_{1,5} = -\ \beta_2 s^2 r^2 I_n(\beta_2 sr) + (n+1) sr I_{n+1}(\beta_2 sr) \tag{84}$$

$$R_{1,6} = \beta_2 s^2 r^2 K_n(\beta_2 sr) + (n+1) sr K_{n+1}(\beta_2 sr) \tag{85}$$

$$R_{2,1} = -\ (n^2-n)I_n(\beta_1 sr) + \beta_1 sr I_{n+1}(\beta_1 sr) \tag{86}$$

$$R_{2,2} = -\ (n^2-n)K_n(\beta_1 sr) - \beta_1 sr K_{n+1}(\beta_1 sr) \tag{87}$$

$$R_{2,3} = -\ (n^2-n)I_n(\beta_2 sr) - \beta_2 nsr I_{n+1}(\beta_2 sr) \tag{88}$$

$$R_{2,4} = -\ (n^2-n)K_n(\beta_2 sr) + \beta_2 nsr K_{n+1}(\beta_2 sr) \tag{89}$$

$$R_{2,5} = -\ (n+1) sr I_{n+1}(\beta_2 sr) \tag{90}$$

$$R_{2,6} = -\ (n+1) sr K_{n+1}(\beta_2 sr) \tag{91}$$

$$R_{3,1} = -\ S^2 r^2 I_n(\beta_1 sr) \tag{92}$$

$$R_{3,2} = - s^2r^2K_n(\beta_1 sr) \tag{93}$$

$$R_{3,3} = 0 \tag{94}$$

$$R_{3,4} = 0 \tag{95}$$

$$R_{3,5} = \beta_2 s^2r^2I_n(\beta_2 sr) \tag{96}$$

$$R_{3,6} = - \beta_2 s^2r^2K_n(\beta_1 nsr) \tag{97}$$

$$R_{4,1} = - 2(n^2 - n)I_n(\beta_1 sr) - \beta_1 nsrI_{n+1}(\beta_1 sr) \tag{98}$$

$$R_{4,2} = - 2(n^2 - n)K_n(\beta_1 sr) + \beta_1 nsrK_{n+1}(\beta_1 sr) \tag{99}$$

$$R_{4,3} = - [2(n^2-n) + \beta_2 s^2r^2]I_n(\beta_2 sr) + 2\beta_2 srI_{n+1}(\beta_2 sr) \tag{100}$$

$$R_{4,4} = - [2(n^2-n) + \beta_2 s^2r^2]K_n(\beta_2 sr) - 2\beta_2 srI_{n+1}(\beta_2 sr) \tag{101}$$

$$R_{4,5} = - \beta_2 s^2r^2I_n(\beta_2 sr) + 2(n + 1)srI_{n+1}(\beta_2 sr) \tag{102}$$

$$R_{4,6} = \beta_2 s^2r^2K_n(\beta_2 sr) + 2(n + 1)srK_{n+1}(\beta_2 sr) \tag{103}$$

$$R_{5,1} = nsrI_n(\beta_1 sr) + \beta_1 s^2r^2I_{n+1}(\beta_1 sr) \tag{104}$$

$$R_{5,2} = nsrK_n(\beta_1 sr) - \beta_1 s^2 r^2 K_{n+1}(\beta_1 sr) \quad (105)$$

$$R_{5,3} = nsrI_n(\beta_2 sr) \quad (106)$$

$$R_{5,4} = nsrK_n(\beta_2 sr) \quad (107)$$

$$R_{5,5} = -\beta_2 nsrI_n(\beta_2 sr) + (\beta_2^2 + 1)s^2 r^2 I_{n+1}(\beta_2 sr) \quad (108)$$

$$R_{5,6} = \beta_2 nsrK_n(\beta_2 sr) + (\beta_2^2 + 1)s^2 r^2 K_{n+1}(\beta_2 sr) \quad (109)$$

$$R_{6,1} = -2nsrI_n(\beta_1 sr) \quad (110)$$

$$R_{6,2} = -2nsrK_n(\beta_1 sr) \quad (111)$$

$$R_{6,3} = -nsrI_n(\beta_2 sr) - \beta_2 s^2 r^2 I_{n+1}(\beta_2 sr) \quad (112)$$

$$R_{6,4} = -nsrI_n(\beta_2 sr) + \beta_2 s^2 r^2 K_{n+1}(\beta_2 sr) \quad 113)$$

$$R_{6,5} = \beta_2 nsrI_n(\beta_2 sr) - s^2 r^2 I_{n+1}(\beta_2 sr) \quad (114)$$

$$R_{6,6} = -\beta_2 nsrK_n(\beta_2 sr) - s^2 r^2 K_{n+1}(\beta_2 sr) \quad (115)$$

The dilatation e is the sum of the principle strains

$$e = \varepsilon_r + \varepsilon_\theta + \varepsilon_z \tag{116}$$

We write

$$r^2 e = \sum_1^2 R_{7,i} \tag{117}$$

where

$$R_{7,1} = - \alpha_1^2 s^2 r^2 I_n(\beta_1 sr) \tag{118}$$

$$R_{7,2} = - \alpha_1 s^2 r^2 K_n(\beta_1 sr) \tag{119}$$

We note that

$$\alpha_1 \rightarrow 0 \text{ as } c \rightarrow 0$$

Hence

$$e \rightarrow 0 \text{ as } c \rightarrow 0$$

In general, therefore, the solution in terms of wave functions does not yield a valid solution in the limit as the velocity of travel approaches zero. Pure torsion is an exception, as only shear strains are involved and the dilatation is zero under both static and dynamic loading conditions.

The rotations are given by the formulas

$$2w_z = \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{\partial v}{\partial r} \tag{120}$$

$$2w_\theta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \tag{121}$$

$$2w_z = \frac{1}{r}\frac{\partial rv}{\partial r} - \frac{1}{r}\frac{\partial u}{\partial \theta} \tag{122}$$

We set

$$2r^2 w_z = H_3 \sum_8^{10} A_i R_{8,i} \tag{123}$$

$$2r^2 w_\theta = H_2 \sum_8^{10} A_i R_{9,i} \tag{124}$$

$$2r^2 w_r = H_4 \sum_8^{10} A_i R_{10,i} \tag{125}$$

for Case I.

$$R_{8,3} = nsrI_n(\beta_2 sr) + \beta_2 s^2 r^2 I_{n+1}(\beta_2 sr) \tag{126}$$

$$R_{8,4} = nsrK_n(\beta_2 sr) - \beta_2 s^2 r^2 K_{n+1}(\beta_2 sr) \tag{127}$$

$$R_{8,5} = \beta_2 nsrI_n(\beta_2 sr) + s^2 r^2 I_{n+1}(\beta_2 sr) \tag{128}$$

$$R_{8,6} = - \beta_2 nsrK_n(\beta_2 sr) + s^2 r^2 K_{n+1}(\beta_2 sr) \quad (129)$$

$$R_{9,3} = nsrI_n(\beta_2 sr) \quad (130)$$

$$R_{9,4} = nsrK_n(\beta_2 sr) \quad (131)$$

$$R_{9,5} = \beta_2 nsrI_n(\beta_2 sr) - \alpha_2 s^2 I_{n+1}(\beta_2 sr) \quad (132)$$

$$R_{9,6} = - \beta_2 nsrK_n(\beta_2 sr) - \alpha_2 s^2 K_{n+1}(\beta_2 sr) \quad (133)$$

$$R_{10,3} = - \beta_2 s^2 r^2 I_n(\beta_2 sr) \quad (134)$$

$$R_{10,4} = - \beta_2 s^2 r^2 K_n(\beta_2 sr) \quad (135)$$

$$R_{10,5} = \beta_2 s^2 r^2 I_n(\beta_2 sr) \quad (136)$$

$$R_{10,6} = - \beta_2 s^2 r^2 K_n(\beta_2 sr) \quad (137)$$

The stresses are derived from the strains by means of generalized Hooke's law.

$$\sigma_r = \lambda e + 2\mu\varepsilon_r \quad (138)$$

$$\sigma_\theta = \lambda e + 2\mu\varepsilon_\theta \quad (139)$$

$$\sigma_z = \lambda e + 2\mu\varepsilon_z \tag{140}$$

$$\tau_{r\theta} = \mu\gamma_{r\theta} \tag{141}$$

$$\tau_{rz} = \mu\gamma_{rz} \tag{142}$$

$$\gamma_{\theta z} = \mu\gamma_{\theta z} \tag{143}$$

For case I we write

$$r^2\sigma_r = H_1 \sum_1^6 A_i S_{1,i} \tag{144}$$

$$r^2\sigma_\theta = H_1 \sum_1^6 A_i S_{2,i} \tag{145}$$

$$r^2\sigma_z = H_1 \sum_1^6 A_i S_{3,i} \tag{146}$$

$$r^2\tau_{r\theta} = H_3 \sum_1^6 A_i S_{4,i} \tag{147}$$

$$r^2\tau_{rz} = H_2 \sum_1^6 A_i S_{5,i} \tag{148}$$

$$r^2\tau_{\theta z} = H_4 \sum_1^6 A_i S_{6,i} \tag{149}$$

Then

$$S_{j,i} = \lambda R_{7,1} + 2\mu R_{j,i}, \; j = 1,2,3; \; i = 1,3,5 \tag{150a}$$

$$S_{j,i} = \lambda R_{7,2} + 2\mu R_{j,i}, \; j = 1,2,3; \; i - 2,4,6 \tag{150b}$$

$$S_{j,i} = \mu R_{j,i}, \; j = 4,5,6; \; i = 1,2,3,4,5,6 \tag{151}$$

Stresses and strains for Cases II, III, and IV can be obtained in a similar manner. The $R_{i,j}$ functions must be calculated for each case to determine the correct algebraic signs in Eqs. (80) - (115) and Eqs. (118) - (119). A similar remark applies to the formulas for the rotations.

VI. RESPONSE TO AN INTERNAL TRAVELING PRESSURE PULSE

First, we consider a sinusoidal pressure pulse traveling with velocity c. The boundary conditions are

$$\sigma_r = \sigma_o \cos(n\theta)\sin[s(z-ct)], \; r = a \tag{152}$$

$$\tau_{r\theta} = 0, \; r = a \tag{153}$$

$$\tau_{rz} = 0, \; r = a \tag{154}$$

$$\tau_r = 0, \; r = b \tag{155}$$

$$\tau_{r\varepsilon} = 0, \; r = b \tag{156}$$

$$\tau_{rz} = 0, \; r = b \tag{157}$$

These boundary conditions lead to a set of six linear equations for the A_i coefficients.

$$\sum_1^6 A_i S_{1,i}(a) = a^2\sigma_o \tag{158}$$

$$\sum_1^6 A_i S_{4,i}(a) = 0 \tag{159}$$

$$\sum_1^6 A_i S_{5,i}(a) = 0 \tag{160}$$

$$\sum_1^6 A_i S_{1,i}(b) = 0 \tag{161}$$

$$\sum_1^6 A_i S_{4,i}(b) = 0 \tag{162}$$

$$\sum_1^6 A_i S_{5,i}(b) = 0 \tag{163}$$

These equations can be solved for the A_i provided c is greater than zero and less than the velocity of the shear wave, according to Eq. (20). It is convenient to write the final results in terms of determinants, using Cramer's rule and then combining the separate terms. Let D(s) be the determinant for Eqs. (158) - (163):

$$D(s) = \begin{vmatrix} S_{1,1}(a) & S_{1,2}(a) & S_{1,3}(a) & S_{1,4}(a) & S_{1,5}(a) & S_{1,6}(a) \\ S_{4,1}(a) & S_{4,2}(a) & S_{4,3}(a) & S_{4,4}(a) & S_{4,5}(a) & S_{4,6}(a) \\ S_{5,1}(a) & S_{5,2}(a) & S_{5,3}(a) & S_{5,4}(a) & S_{5,5}(a) & S_{5,6}(a) \\ S_{1,1}(b) & S_{1,2}(b) & S_{1,3}(b) & S_{1,4}(b) & S_{1,5}(b) & S_{1,6}(b) \\ S_{4,1}(b) & S_{4,2}(b) & S_{4,3}(b) & S_{4,4}(b) & S_{4,5}(b) & S_{4,6}(b) \\ S_{5,1}(b) & S_{5,2}(b) & S_{5,3}(b) & S_{5,4}(b) & S_{5,5}(b) & S_{5,6}(b) \end{vmatrix} \qquad (164)$$

The remaining determinates use the appropriate $R_{j,i}$ or $S_{j,i}$ for the first row; the remaining five rows are identical with the corresponding rows of D(s). We have, for instance,

$$Sr = \begin{vmatrix} S_{1,1}(r) & S_{1,2}(r) & S_{1,3}(r) & S_{1,4}(r) & S_{1,5}(r) & S_{1,6}(r) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \qquad (165)$$

$$S_{r\theta} = \begin{vmatrix} S_{4,1}(r) & S_{4,2}(r) & S_{4,3}(r) & S_{4,4}(r) & S_{4,5}(r) & S_{4,6}(r) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \qquad (166)$$

$$S_{rz} = \begin{vmatrix} R_{1,1}(r) & R_{1.2}(r) & R_{1,3}(r) & R_{1,4}(r) & R_{1,5}(r) & R_{1,6}(r) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \qquad (167)$$

Determinates for the remaining stresses and strains can be written down by inspection. The stresses for Case I are given by

$$r^2\sigma_r = H_1 S_r/D(s) \qquad (168)$$

$$r^2\sigma_\theta = H_1 S_\theta/D(s) \qquad (169)$$

$$r^2\sigma_z = H_1 S_z/D(s) \tag{170}$$

$$r^2\tau_{r\theta} = H_3 S_{r\theta}/D(s) \tag{171}$$

$$r^2\tau_{rz} = H_2 S_{rz}/D(s) \tag{172}$$

$$r^2\tau_{\theta z} = H_4 S_{\theta z}/D(s) \tag{173}$$

Formulas for the strains are obtained in a similar manner.

$$r^2\varepsilon_r = H_1 R_r/D(s) \tag{174}$$

$$r^2\varepsilon_\theta = H_1 R_\theta/D(s) \tag{175}$$

$$r^2\varepsilon_z = H_1 R_z/D(s) \tag{176}$$

$$r^2\gamma_{r\theta} = H_3 R_{r\theta}/D(s) \tag{177}$$

$$r^2\gamma_{rz} = H_2 R_{rz}/D(s) \tag{178}$$

$$r^2\gamma_{\theta z} = H_4 R_{\theta z}/D(s) \tag{179}$$

We can also solve problems specified by the boundary conditions

$$\sigma_r = 0 , \quad r = a \tag{180}$$

$$\tau_{r\theta} = \tau_o \cos(n\theta)\sin[s(z-ct)], \; r = a \tag{181}$$

$$t_{rz} = 0 , \quad r = a \tag{182}$$

and

$$\sigma_r = 0 , \quad r = a \tag{183}$$

$$\tau_{r\theta} = 0 , \quad r = a \tag{184}$$

$$\tau_{rz} = \tau_o \cos(n\theta)\sin[s(z-ct)z] \tag{185}$$

It is assumed the outside surface is stress-free according to Eqs. (155)-(157). The R_{ij} and S_{ij} functions must be derived for each case.

Formulas for the response to a traveling step function of pressure are obtained by writing the stresses and strains in terms of Fourier integrals. In place of Eq. (152) we write

$$\sigma_r = \sigma_o \cos(n\theta) \left[\tfrac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\sin[s(z-ct)]}{sD(s)} ds \right] , \; r = a \; , \tag{186}$$

Then

$$\sigma_r = \sigma_o \cos(n\theta) \; , \quad z>ct \tag{187}$$

$$\sigma_r = \tfrac{1}{2}\, \sigma_o \cos(n\theta) \quad , \qquad z = ct \tag{188}$$

$$\sigma_r = 0 \, , \qquad z<ct \tag{189}$$

The term

$$\sigma_r = \tfrac{1}{2}\, \sigma_o \cos(n\theta) \, , \qquad r = a \tag{190}$$

leads to elementary solutions obtained by evaluating the corresponding Fourier integral at the origin. The number of terms depends in part on the multiplicity of the zero at the origin of the complex s plane.*

$$r^2\sigma_r = \frac{\sigma_o}{\pi} \cos(n\theta) \int_0^\infty \frac{\sin[s(z-ct)]}{sD(s)} \, ds \tag{191}$$

leads to Fourier integrals involving functions occurring on the right-hand side of Eqs. (173) - (184). We have

$$\sigma_r = \frac{1}{\pi} \cos(n\sigma) \int_0^\infty \frac{\sin[s(z-ct)]}{sD(s)} \, S_r \, ds \tag{192}$$

$$\sigma_\theta = \frac{1}{\pi} \cos(n\theta) \int_0^\infty \frac{\sin[s(z-ct)]}{sD(s)} \, S_\theta \, ds \tag{193}$$

$$\sigma_z = \frac{1}{\pi} \cos(n\theta) \int_0^\infty \frac{\sin[s(z-ct)]}{sD(s)} \, S_z \, ds \tag{194}$$

Appendix A. These elementary solutions can be expressed in terms of zonal harmonics.

$$\tau_{r\theta} = \frac{1}{\pi} \sin(n\theta) \int_0^\infty \frac{\sin[s(z-ct)]}{sD(s)} S_{r\theta} \, ds \tag{195}$$

$$\tau_{rz} = \frac{1}{\pi} \cos(n\theta) \int_0^\infty \frac{\sin[s(z-ct)]}{sD(s)} S_{rz} \, ds \tag{196}$$

$$\tau_{\theta z} = \frac{1}{\pi} \sin(n\theta) \int_0^\infty \frac{\cos[s(z-ct)]}{sD(s)} S_{\theta z} \, ds \tag{197}$$

The corresponding formulas for the strains are

$$\varepsilon_r = \frac{1}{\pi} \cos(n\theta) \int_0^\infty \frac{\sin[s(z-ct)]}{sD(s)} R_r \, ds \tag{198}$$

$$\varepsilon_\theta = \frac{1}{\pi} \cos(n\theta) \int_0^\infty \frac{\sin[s(z-ct)]}{sD(s)} R_\theta \, ds \tag{199}$$

$$\varepsilon_z = \frac{1}{\pi} \cos(n\theta) \int_0^\infty \frac{\sin[s(z-ct)]}{sD(s)} R_z \, ds \tag{200}$$

$$\gamma_{r\theta} = \frac{1}{\pi} \sin(n\theta) \int_0^\infty \frac{\sin[s(z-c)]}{sD(s)} R_{r\theta} \, ds \tag{201}$$

$$\tau_{rz} = \frac{2}{\pi} \cos(n\theta) \int_0^\infty \frac{\sin[s(z-ct)}{sD(s)} R_{rz} \, ds \tag{202}$$

$$\tau_{\theta z} = \frac{2}{\pi} \sin(n\theta) \int_0^\infty \frac{\cos[s(z-ct)]}{sD(s)} R_{\theta z} \, ds \tag{203}$$

The total value of the stresses and strains is obtained by adding solutions to the problem defined by Eq. (174).

Solutions to problems involving other boundary conditions may be obtained in a similar manner. In certain cases the Fourier integrals will be divergent. In these cases the limits of integration should be taken between $-\infty$ and $+\infty$, and the Cauchy principal value calculated. The factor

$2/\pi$ must be replaced by $1/\pi$ in these cases.

VII. HARMONIC AND BIHARMONIC FUNCTIONS AS LIMITS OF WAVE FUNCTIONS

As the velocity of travel approaches zero, the wave functions tend to harmonic functions in the limit, as shown below. However, in general, a static problem requires biharmonic functions as well as harmonic functions for a complete solution.* A second limiting process involving an indeterminate quotient is required to extract the biharmonic functions from the wave functions.

We recall the basic equations of Part II.

$$\hat{u} = \nabla\phi + \nabla \times \hat{\psi} \tag{5}$$

$$c_1^2 \nabla^2\phi = \frac{\partial^2\phi}{\partial t^2} \tag{6}$$

$$c_2^2 \nabla^2 \hat{\psi} = \frac{\partial^2\hat{\psi}}{\partial t^2} \tag{7}$$

We find from Eqs. (19), (29), (30), (31), (32), and (23) that

$$\frac{\partial\psi}{\partial t^2} = - c^2 s^2 \phi \tag{204}$$

$$\frac{\partial^2\hat{\psi}}{\partial z^2} = - c^2 s^2 \hat{\psi} \tag{205}$$

On letting $c \to 0$ we find

**Problems involving pure shear, such as torsion of an axisymmetric solid, are exceptions. A scalar biharmonic function is required in problems involving a change in volume, since dilatation derived from a scalar harmonic function is zero.*

$$\nabla^2 \phi = 0 \tag{206}$$

$$\nabla^2 \hat{\psi} = 0 \tag{207}$$

where

$$\phi = [A_1 I_n(sr) + A_2 K_n(sr)]\cos(n\theta)\sin(sz) \tag{208}$$

$$\hat{\psi} = \hat{e}_r [A_5 I_{n+1}(sr) + A_6 K_{n+1}(sr)]\cos(n\theta)\cos(sz) \tag{209}$$

$$- \hat{e}_\theta \; A_5 I_{n+1}(sr) + A_6 K_{n+1}(sr) \; \cos(n\theta) \; \sin(sz)$$

$$+ \hat{e}_z [A_3 I_n(sr) + A_4 K_n(sr)]\sin(n\theta)\sin(sz)$$

The corresponding displacements are found by setting $c = 0$, $\beta_1 = 1$, $\beta_2 = 1$ in Eqs. (64) - (66). We can eliminate two of the A_i coefficients in the result by writing

$$A_7 = A_1 + A_3 \tag{210}$$

$$A_8 = A_2 + A_6 \tag{211}$$

$$A_9 = A_3 - A_5 \tag{212}$$

$$A_{10} = A_4 - A_6 \tag{213}$$

Then

$$ru = [A_7 nI_n(sr) + A_8 nk_n(sr) + A_9 srI_{n+1}(sr) - A_{10} srK_{n+1}(sr)]\cos(n\theta)\cos(sz) \tag{214}$$

$$rv = [-A_7 nI_n(sr) - A_8 nK_n(sr) - A_9 srI_{n+1}(sr) + A_{10} srK_{n+1}(sr)]\sin(n\theta)\sin(sz) \tag{215}$$

$$rw = [A_7 - A_9)srI_n(sr) - (A_8 + A_{10})srK_n(sr)]\cos(n\theta)\cos(sz) \tag{216}$$

We note that the six independent solutions for the displacements for the wave equation reduce to four when the velocity of travel becomes zero. Moreover, the displacements vanish if

$$A_i = 0, \quad i = 7,8,9,10 \tag{217}$$

To obtain additional solutions, let

$$A_i = 1/(\beta_1 - \beta_2), \quad i = 1,2 \tag{218}$$

in Eqs. (217), (218), and (219). We define the functions

$$F_1 = (nI_n(\beta_1 sr) - nI_n(\beta_2 sr)]/(\beta_1 - \beta_2) \tag{219}$$

$$F_2 = [nK_n(\beta_1 sr) - nK_n(\beta_2 sr)]/(\beta_1 - \beta_2) \tag{220}$$

$$F_3 = [\beta_1 srI_{n+1}(sr) - srI_{n+1}(sr)(\beta_2 - sr)]/(\beta_1 - \beta_2) \tag{221}$$

$$F_4 = [\beta_1 srK_{n+1}(\beta_1 sr) - srK_{n+1}(\beta_2 sr)]/(\beta_1 - \beta_2) \tag{222}$$

$$F_5 = [\beta_2 srI_{n+1}(\beta_2 sr) - srI_{n+1}(\beta_2 sr)]/(\beta_1 - \beta_2) \tag{223}$$

$$F_6 = [\beta_2 srK_{n+1}(\beta_2 sr) - srK_{n+1}(\beta_2 sr)]/(\beta_1 - \beta_2) \tag{224}$$

$$F_7 = [srI_n(\beta_1 sr) - \beta_2 srI_n(\beta_2 sr)]/(\beta_1 - \beta_2) \tag{225}$$

$$F_8 = [srK_n(\beta_1 sr) - \beta_2 srK_n(\beta_2 sr)]/(\beta_1 - \beta_2) \tag{226}$$

On referring to Eqs. (64), (65), and (66) we find

$$ru = [\beta_1(F_1 + F_3) + \beta_2(F_2 - F_4)]\cos(n\theta)\sin[s(z-ct)] \tag{227}$$

$$rv = [- \beta_1(F_1 + F_3) + \beta_2(F_2 + F_4)]\sin(n\theta)\sin[s(z-ct)] \tag{228}$$

$$rw = [\beta_1(F_6 - \beta_2 F_8]\cos(n\theta)\cos[s(z-ct)] \tag{229}$$

Each of the F_i functions has the indeterminate form 0/0 when $c = 0$, $\beta_1 = 1$, $\beta_2 = 1$; this is obviously also true for the displacements. The limits of these indeterminate forms can be found by the ordinary rules of calculus. When the velocity c is small we have

$$1 - \beta_1 \sim \tfrac{1}{2}\, \alpha_1^{\,2} \ , \quad 1 - \beta_2 \sim \tfrac{1}{2}\, \alpha_2^{\,2} \qquad (230)$$

and

$$\lim_{c \to 0} \frac{1 - \beta_1}{1 - \beta_2} = \frac{\mu}{\lambda + 2\mu} \qquad (231)$$

on referring to Eqs. (2), (3), (22), (23), (27), and (28). The limits of the F_i functions can be found by using two terms of Taylor's series:

$$I_n(\beta_1 sr) = I_n(sr) + (\beta_1 - 1) sr I_n'(sr) + \delta(\beta_1 - 1)^2 \qquad (232)$$

$$K_n(\beta_1 sr) = K_n(sr) + (\beta_1 - 1) sr K_n'(sr) + \delta(\beta_1 - 1)^2 \qquad (233)$$

$$I_{n+1}(\beta_1 sr) = I_{n+1}(sr) + (\beta_1 - 1) sr I_{n+1}'(sr) + \delta(\beta_1 - 1)^2 \qquad (234)$$

$$K_{n+1}(\beta_1 sr) = K_{n+1}(sr) + (\beta_1 - 1) sr K_{n+1}'(sr) + \delta(\beta_1 - 1)^2 \qquad (235)$$

Additional formulas are obtained by substituting β_2 for β_1.

As c approaches zero we find

$$\lim F_1 = nsr I_n'(sr) \qquad (236)$$

$$\lim F_2 = nsr K_n'(sr) \qquad (237)$$

$$\lim F_3 = -\frac{\mu}{\lambda + \mu} srI_{n+1}(sr) + srI'_{n+1}(sr) \tag{238}$$

$$\lim F_4 = -\frac{\mu}{\lambda + \mu} srK_{n+1}(sr) + srK'_{n+1}(sr) \tag{239}$$

$$\lim F_5 = -\frac{\lambda + 2\mu}{\lambda + \mu} srI_n I_{n+1}(sr) \tag{240}$$

$$\lim F_6 = -\frac{\lambda + 2\mu}{\lambda + \mu} srK_{n+1}(sr) \tag{241}$$

$$\lim F_7 = \frac{\lambda + 2\mu}{\lambda + \mu} srI_n(sr) + srI'_n(sr) \tag{242}$$

$$\lim F_8 = \frac{\lambda + 2\mu}{\lambda + \mu} srK_n(sr) + srK'_n(sr) \tag{243}$$

The derivatives may be eliminated by means of Eqs. (60) - (63). The displacements corresponding to these limits are

$$ru = \lim_{c \to 0} [\beta_1(F_1 + F_3) + \beta_2(F_2 - f_4)]\cos(n\theta)\sin[s(z-ct)] \tag{244}$$

$$rv = \lim_{c \to 0} [-\beta_1(F_1 + F_3) + \beta_2(F_2 + f_6)]\sin(n\theta)\sin[s(z-ct)] \tag{245}$$

$$rw = \lim_{c \to 0} [\beta_1 F_7 - \beta_2 F_8]\cos(n\theta)\cos[s(z-ct)] \tag{246}$$

The total displacements due to static loading are obtained by adding these displacements to the displacements obtained from the harmonic potentials, Eqs. (217) - (219). We now have a total of six linearly independent solutions for the displacements. The strains can be derived from the displacements and the stresses follow immediately from Eqs. (156) and (151). Thus, six independent stress formulas are obtained for meeting six boundary conditions of the type given by Eqs. (152) - (157) with c = 0.

VIII. BIHARMONIC SCALAR AND VECTOR POTENTIALS FOR STATIONARY LOADS

It is possible to derive the displacements given by Eqs. (250) - (252) from vector and scalar potentials according to Eq. (5). Harmonic scalar and vector potentials are given by Eqs. (211) and (212). We now derive the additional biharmonic potentials required for Eqs. (250) - (252).

For the scalar potential, assume

$$\chi = r \frac{\partial \phi}{\partial r} \tag{247}$$

Then

$$\frac{\partial \chi}{\partial r} = r \frac{\partial^2 \phi}{\partial r^2} + \frac{\partial \phi}{\partial r} \tag{247a}$$

or

$$\frac{\partial \chi}{\partial r} = - \frac{1}{r} \frac{\partial^2 \phi}{\partial \theta^2} - r \frac{\partial^2 \phi}{\partial z^2} \tag{247b}$$

since ϕ is a scalar harmonic function it follows, after some routine analysis, that[7]

$$\nabla^2 \chi = - 2 \frac{\partial^2 \phi}{\partial z^2} \tag{248}$$

[7] *A.E.H. Love, A Treatise on the Mathematical Theory of Elasticity, Dover Publications, New York, 1944. See pages 274-277, especially Eqs. (66) and (67).*

and

$$\nabla^4 X = 0 \tag{249}$$

Other scalar biharmonic functions are given by References 8, 9, and 10.

$$X = r^2 \phi \tag{250}$$

$$X = (r \sin\theta)\phi \tag{251}$$

$$X = (r \cos\theta)\phi \tag{252}$$

$$X = z\phi \tag{253}$$

However, the biharmonic functions given by Eqs. (250) - (251) seem to be most closely related to scalar potentials for wave motion, as they can also be derived from a scalar wave function by a limiting process. We have, for instance,

$$\beta_1 = \sqrt{1-c^2/c_1^{\,2}} \tag{254}$$

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[8] *G.C. Fung, Foundations of Solid Mechanics, Prentice Hall, Inc., Englewood Cliffs, NJ, page 208.*

[9] *H. Neuber, Theory of Notch Stresses: Principles for Exact Stress Calculation, David Taylor Model Basin, Washington, DC, Translation 74, November 1945. See pages 25 and 128.*

[10] *A.S. Elder, "Traveling and Stationary Loads on the Half Space," BRL report to be published. See section titled "Biharmonic Functions as Limits of Wave Functions."*

$$\phi_1 = \tfrac{1}{2}\,[A_1 I_n(\beta_1 sr) + A_2 K_n(\beta_1 sr)] \tag{255a}$$

$$\times\ [\sin s(z + ct) + \sin s(z-ct)]\cos(n\theta)$$

or

$$\phi_1 = [A_1 I_n(\beta_1 sr) + A_2 K_n(\beta_1 sr)]\ \sin(sz)\cos(sct)\cos(n\theta) \tag{255b}$$

This scalar potential represents two equal loads traveling with the same speed in opposite directions.

$$\frac{\partial\phi_1}{\partial s} = \tfrac{1}{2}\ [A_1\beta_1 rI_n'(\beta_1 sr) + A\ \beta_2 r_1 K_n'(\beta\ s r_1)][\sin[s(z+ct)] + \sin[s(z-ct)]\cos(n\theta) \tag{256a}$$

$$+\ \tfrac{1}{2}\ [A_1 I_n(\beta_1 sr) + A_2 K_n(\beta_1 sr)]$$

$$\times\ [(z+ct)\cos s(z+ct) + (z-ct)\cos(sz)-ct)]\cos(n\theta)$$

$$\frac{\partial\phi_1}{\partial s} = [A_1\beta_1 rI_n'\ (\beta_1 sr) + A_2\beta_1 rK_n'\ (\beta_1 sr)]\ \sin(sz)\cos(sct)\cos(n\theta)$$

$$+\ [A_1 I_n(\beta_1 sr) + A_2 K_n(\beta_1 sr)][z\cos(sz)]\ \cos(sct)\cos(n\theta) \tag{256b}$$

$$-\ [A_1 I_n(\beta_1 sr) + A_2 K_n(\beta_1 sr)][ct\sin(sz)]\ \sin(sct)\cos(n\theta)$$

On allowing c to approach zero, we find

$$\frac{\partial\phi_1}{\partial s} = [A_1 rI_n'(sr) + A_2 rK_n'(sr)]\sin(sz)\cos(n\theta)$$

$$+ \quad [A_1 z I_n(sr) \quad + \quad A_2 z K_n(sr)] z\cos(sz)\cos(n\theta)$$

We also have

$$\chi \quad = \quad r \frac{\partial}{\partial r} [A_1 I_n(sr) \quad + \quad A_2 K_n(sr)]\sin(sz)\cos(n\theta) \tag{257a}$$

or

$$\chi \quad = \quad [A_1 sr I_n'(sr) \quad + \quad A_2 sr K_n'(sr)]\sin(sz)\cos(n\theta) \tag{257b}$$

which yields two terms in Eq. (256b) except for a factor s. The remaining two terms of (256b) correspond to Eq. (253).

The biharmonic vector potential can be derived in a similar manner. We assume

$$\hat{\omega} \quad = \quad \frac{\partial \hat{\psi}}{\partial r} \tag{258}$$

where

$$\hat{\omega} \quad = \quad \hat{e}_r \, \omega_r \quad + \quad \hat{e}_\theta \omega_\theta \quad + \quad \hat{e}_z \, \omega_z \tag{259}$$

Then

$$\omega_\theta \quad = \quad r \, \frac{\partial}{\partial r} \quad \psi_\theta \tag{260}$$

$$\omega_\theta = r \frac{\partial}{\partial r} \psi_\theta \tag{261}$$

$$\omega_z = r \frac{\partial}{\partial r} \psi_z \tag{262}$$

We find as before

$$\frac{\partial}{\partial r} \omega_r = r \frac{\partial^2}{\partial r^2} \psi_r + \frac{\partial}{\partial r} \psi_r \tag{263}$$

with similar equations for ω_θ and ω_z. The second derivatives are eliminated by using the three equations which result when the derivatives with respect to time are set equal to zero in Eqs. (11), (12), and (13). On referring to Eq. (8) we find

$$\frac{\partial}{\partial r} \omega_r = \frac{1}{r} \psi_r + \frac{2}{r} \frac{\partial}{\partial r} \psi_r - \frac{1}{r} \frac{\partial^2}{\partial \theta^2} \psi_r - r \frac{\partial^2}{\partial z^2} \psi_r \tag{264}$$

$$\frac{\partial}{\partial r} \omega_\theta = \frac{1}{r} \psi_\theta - \frac{2}{r} \frac{\partial}{\partial r} \psi_r - \frac{1}{r} \frac{\partial^2}{\partial \theta^2} \psi_\theta - r \frac{\partial^2}{\partial z^2} \psi_\theta \tag{265}$$

and

$$\frac{\partial}{\partial z} \omega_z = - \frac{1}{r} \frac{\partial^2}{\partial \theta^2} \psi_z - r \frac{\partial^2}{\partial z^2} \psi_z \tag{266}$$

We finally obtain

$$\nabla^2 \omega_r - \frac{1}{r^2} \omega_r + \frac{1}{r^2} \frac{\partial}{\partial \theta} \omega_\theta = - 2 \frac{\partial^2}{\partial z^2} \psi_r \tag{267}$$

$$\nabla^2 \psi_\theta - \frac{1}{r^2} \omega_\theta + \frac{1}{r^2} \frac{\partial}{\partial\theta} \omega_\theta = - 2 \frac{\partial^2}{\partial z^2} \psi_\theta \tag{268}$$

and

$$\nabla^2 \omega_z = - 2 \frac{\partial^2}{\partial z^2} \psi_z \tag{269}$$

Consequently,

$$\nabla^2 \hat{\omega} = - 2 \frac{\partial^2 \hat{\psi}}{\partial z^2} \tag{270}$$

and

$$\nabla^4 \hat{\omega} = 0 \tag{271}$$

The biharmonic scalar and vector potentials are not independent, but are coupled through Navier's equation.

$$(\lambda + 2\mu)\nabla\phi + \mu\nabla\times\hat{\omega} = 0 \tag{272}$$

or

$$\frac{\partial^2}{\partial z^2} [(\lambda + 2\mu)\nabla\phi + \mu\nabla\times\hat{\psi}] = 0 \tag{273}$$

This equation is satisfied if

$$(\lambda + 2\mu)\ \nabla\phi \quad + \quad \mu\nabla\times\hat{\psi} \quad = \quad 0 \tag{274}$$

or, in scalar form,

$$(\lambda + 2\mu)\ \frac{\partial\phi}{\partial r} \quad + \quad \mu\ r\,\frac{\partial}{\partial\theta}\,\psi_z \quad - \quad \frac{\partial}{\partial z}\,\psi_\theta \quad = \quad 0 \tag{275}$$

$$(\lambda + 2\mu)\quad r\ \frac{\partial\phi}{\partial\theta} \quad + \quad \mu\quad \frac{\partial}{\partial z}\,\psi_r \quad - \quad \frac{\partial}{\partial r}\,\psi_z \quad = \quad 0 \tag{276}$$

$$(\lambda + 2\mu)\ \frac{\partial\theta}{\partial z} \quad + \quad \mu\ \frac{1}{r}\ \frac{\partial r}{\partial r}\,\psi_\theta \quad - \quad \frac{1}{r}\ \frac{\partial}{\partial\theta}\,\psi_r \quad = \quad 0 \tag{277}$$

We use Case I as an example.

$$\phi \quad = \quad [A_1 I_n(sr) \quad + \quad A_2 K_n(sr)]\cos(n\theta)\sin(sz) \tag{278}$$

$$\psi_r \quad = \quad [A_5 I_{n+1}(sr) \quad + \quad A_6 K_{n+1}(sr)]\sin(n\theta)\cos(sz) \tag{279}$$

$$\psi_\theta \quad = - \quad [A_5 I_{n+1}(sr) \quad + \quad A_6 K_{n+1}(sr)]\cos(n\theta)\cos(sz) \tag{280}$$

$$\psi_z \quad = \quad [A_3 I_n(sr) \quad + \quad A_4 K_n(sr)]\sin(n\theta)\sin(sz) \tag{281}$$

Relations among the A_i components are obtained by substituting these scalar components into Eqs. (11) - (14) and carrying out the indicated analysis.

We finally obtain

$$\chi = \frac{1}{\lambda+2\mu}[\beta_{11}srI_n'(sr) + \beta_{12}(sr)]\cos(n\theta)\sin(sz) \tag{282}$$

$$\psi_r = \frac{1}{\mu}[\beta_{11}srI_n'(sr) - \beta_{12}srK_{n+1}'(sr)]\sin(n\theta)\cos(sz) \tag{283}$$

$$\psi_\theta = \frac{1}{\mu}[\beta_{11}srI_{n+1}'(sr) + \beta_{12}srK_{n+1}'(sr)]\cos(n\theta)\cos(sz) \tag{284}$$

$$\psi_z = \frac{1}{\mu}[-\beta_{11}srI_n'(sr) + \beta_{12}srK_n'(sr)]\sin(n\theta)\sin(sz) \tag{285}$$

where β_{11} and β_{12} are related linearly to the A_i coefficients. The derivatives may be eliminated by using standard formulas, so that the final result can be expressed in terms of trigonometric factors and Bessel functions of order n and n+1.

Thus, we have a complete solution for static loading in terms of scalar and vector potentials. By contrast, Love's strain function uses a single biharmonic scalar potential based on the Galerkin vector. The displacements in our analysis are given in terms of first partial derivatives of the potential functions, consistent with the Green-Lamé formulation of wave motion, whereas Love's strain function expresses displacements as second derivatives of a scalar potential. The present analysis illustrates a logical connection between static and dynamic problems for the elastic deformation of a hollow cylinder.

IX. HELMHOLTZ THEORY

Helmholtz's theorem concerning resolution of a vector consists of two parts:

$$\hat{u} = \nabla\phi + \nabla\times\hat{\chi} \tag{286}$$

$$\nabla \cdot \hat{\chi} = 0 \tag{287}$$

The vector potential $\hat{\psi}$ of Eq. (5), as derived in the previous analysis, is generally not solenoidal, as required by Eq. (287). A solenoidal vector is obtained by a gauge transformation, as outlined by McQuistan.[11] In the course of McQuistan's analysis a Newtonian potential is used to solve Poisson's partial differential equation, and the proof is restricted to simply connected regions. Unfortunately, a hollow cylinder is multiply connected, and this proof does not apply. We solve Poisson's equation by separation of variables. We have three cases to consider:

1) $\hat{\psi}$ is a harmonic vector potential

2) $\hat{\psi}$ is a biharmonic vector potential

3) $\hat{\psi}$ is a vector wave function.

In each case we assume

$$\hat{\chi} = \hat{\psi} + \nabla\eta \tag{288}$$

where η is a scalar. Then

$$\nabla \cdot \hat{\chi} = \nabla \cdot \hat{\psi} = \nabla \cdot \nabla \eta \tag{289}$$

which, in view of Eq. (287), reduces to

[11] *R.B. McQuistan, Scalar and Vector Fields, A Physical Interpretation, John Wiley and Sons, Inc., New York, 1965. See pages 256-264.*

$$\nabla^2 \eta = - \nabla \cdot \hat{\psi} \tag{290}$$

Since the vector potential $\hat{\psi}$ has been calculated previously, Eq. (290) is Poisson's equation for the scalar η . In each case we have

$$\nabla \cdot \hat{\psi} = \frac{1}{r} \frac{\partial}{\partial r} (r\psi_r) + \frac{1}{r} \frac{\partial}{\partial \theta} \psi_\theta + \frac{\partial}{\partial z} \psi_z \tag{291}$$

where

$$\hat{\psi} = \hat{e}_r \psi_r + \hat{e}_\theta \psi_\theta + \hat{e}_z \psi_z \tag{292}$$

and $\hat{e}_r$, $\hat{e}_\theta$, and $\hat{e}_z$ are the unit vectors in cylindrical coordinates. For Case 1 we have

$$\psi_r = [A_5 I_{n+1}(sr) + A_6 K_{n+1}(sr)] \sin(n\theta) \cos(sz) \tag{293}$$

$$\psi_\theta = - [A_5 I_{n+1}(sr) + A_6 K_{n+1}(sr)] \cos(n\theta) \sin(sz) \tag{294}$$

$$\psi_z = [A_5 I_n(sr) + A_4 K_n(sr)] \sin(n\theta) \sin(sz) \tag{295}$$

After some routine analysis we find

$$\nabla \cdot \hat{\psi} = [A_e + A_5) s I_n(sr) + (A_4 - A_6) s K_n(sr)] \sin(n\theta) \cos(sz) \tag{296}$$

We use the semi-inverse method to solve Eq. (290). This was a lucky guess which saved considerable analysis. Assume

$$\eta = C\,[(A_3 + A_5)I_n'(sr) + (A_4 - A_6)K_n'(sr)]\sin(n\theta)\cos(sz) \tag{297}$$

where C is an unknown constant.

Then

$$\nabla^2\eta = 2\,C\,s^2[(A_3 + A_5)I_n(sr) + (A_4 - A_6)K_n(sr)]\sin(n\theta)\cos(sz) \tag{298}$$

On referring to Eqs. (290) and (291) we see that

$$C = -\tfrac{1}{2}s \tag{299}$$

and, consequently,

$$\eta = -\tfrac{1}{2}\,[(A_3 + A_5)I_n'(sr) + (A_4 - A_6)K_n(sr)]\sin(n\theta)\cos(sz) \tag{300}$$

The biharmonic vector, Case 2, may be treated in a similar manner. We have

$$\psi_r = [\beta_1 sr I_{n+1}'(sr) - \beta_2 sr K_{n+1}'(sr)]\sin(n\theta)\cos(sz) \tag{301}$$

$$\psi_\theta = -[\beta_1 sr I_{n+1}'(sr) - \beta_2 sr K_{n+1}'(sr)]\sin(n\theta)\cos(sz) \tag{302}$$

$$\psi_z = [- \beta_1 s r'_n(sr) + \beta_2 s r K'_n(sr)]\sin(n\theta)\sin(sz) \tag{303}$$

We find

$$\nabla \cdot \hat{\psi} = [\beta_1 s I_n(sr) + \beta_2 s K_n(sr)]\sin(n\theta)\cos(sz) \tag{304}$$

Hence,

$$\nabla^2 \eta = - [\beta_1 s I_n(sr) + \beta_2 s K_n(sr)]\sin(n\theta)\cos(sz) \tag{305}$$

On proceding as in the previous case, we find

$$\eta = - \frac{1}{2} [\beta_1 r I'_n(sr) + \beta_2 r K'_n(sr)]\sin(n\theta)\cos(sz) \tag{306}$$

This result is well known. If the right-hand side of Poisson's equation is a harmonic function, biharmonic functions can be used to solve Poisson's Equation. For instance, if

$$\nabla^2 \eta = \phi, \tag{307}$$

and

$$\eta = - \frac{1}{2} r \frac{\partial \phi}{\partial r} \tag{308}$$

then

$$\nabla^2 \eta = \frac{\partial^2 \phi}{\partial z^2} \tag{309}$$

and we can readily prove that η is biharmonic.

For the wave function, Case 3, we have

$$\psi_r = [A_5 I_{n+1}(\beta_2 sr) + A_6 K_{n+1}(\beta_2 sr)]\sin(n\theta)[\cos s(z-ct)] \tag{310}$$

$$\psi_\theta = - [A_5 I_{n+1}(\beta_2 sr) + A_6 K_n(\beta_2 sr)]\cos(n\theta)\cos[s(z-ct)] \tag{311}$$

$$\psi_z = [A_3 I_n(\beta_2 sr) + A_4 K_n(\beta_2 sr)]\sin(n\theta)\sin[s(z-ct)] \tag{312}$$

where ψ_r, ψ_θ, and ψ_z are components of the vector potential. We have shown that

$$\nabla \cdot \hat{\psi} = [(A_3 + \beta_2 A_5)sI_n(\beta_2 sr) + (A_4 - \beta_2 A_6)sK_n(\beta_2 sr)]\sin(n\theta)\cos[s(z-ct)] \tag{67}$$

Assume

$$\eta = C\, [(A_3 + \beta_2 A_5)sI_n(\beta_2 sr) + (A_4 - \beta_2 A_6)sK_n(\beta_2 sr)]\sin(n\theta)\cos[s(z-ct)] \tag{313}$$

where the constant C is unknown. Then

$$\nabla^2\eta = -\left\{C\ (\beta_2 - 1)s^2\ (A_3 + \beta_2 A_5)sI_n(\beta_2 sr) + (A_4 - \beta_2 A_6)K_n(\beta_2 sr)]\right\} \quad (314)$$

$$\times \left\{\sin(n\theta)\cos[s(z-ct)z]\right\}$$

But

$$\alpha_2^{\ 2} = 1 - \beta_2^{\ 2} \quad \text{so that} \quad (315)$$

$$C = -\ 1/s^2\alpha_2^{\ 2}$$

and

$$\eta = -\frac{1}{s\alpha_2^{\ 2}}[(A_3 + \beta_2 A_5)I_n(\beta_2 sr) + (A_4 - \beta_2 A_6)K_n(\beta_2 sr)]\sin(n\theta)\cos][s(z-ct)] \quad (316)$$

This result is not valid for $\alpha_2 = 0$, corresponding to zero velocity of travel.

Next we recompute the vector displacement by substituting $\hat{\chi}$ from Eq. (288) into Eq. (286). We find

$$\hat{u} = \nabla\phi + \nabla\times\hat{\psi} + \nabla\cdot\nabla\eta \quad (317)$$

The third term on the right-hand side of this equation is zero, so we recover Eq. (5) in the formula for the vector displacement.

$$\hat{u} = \nabla\phi + \nabla\times\hat{\psi} \quad (5)$$

Formulas for the stresses and strain, which involve partial derivatives of $\hat{u}$, remain unchanged.

The vector potential is given by

$$\hat{\omega} = {}^1\!/_2\, \nabla \times \hat{u} \tag{318}$$

or

$$\hat{\omega} = {}^1\!/_2\ \nabla \times \nabla\phi + {}^1\!/_2\ \nabla \times \nabla \times \hat{\psi} \tag{319}$$

But

$$\nabla \times \nabla\phi = 0 \tag{320}$$

and

$$\nabla \times \nabla \times \hat{\psi} = \nabla\,(\nabla \cdot \hat{\psi}) - \nabla^2\, \hat{\chi} \tag{321}$$

Hence,

$$\hat{\omega} = -\frac{1}{2}\ \nabla^2 \hat{\psi} \tag{322}$$

An analogous formula is found for the dilatation

$$e = \nabla \cdot \hat{u} \tag{323}$$

$$e = \nabla \cdot \nabla \phi + \nabla \cdot \nabla \times \hat{\psi} \tag{324}$$

But

$$\nabla \cdot \nabla \times \hat{\psi} = 0 \tag{325}$$

So that

$$e = \nabla^2 \phi \tag{326}$$

The results obtained by using $\hat{\chi}$ in place of $\hat{\psi}$ conform to all requirements of Helmholtz theory provided the z axis is excluded from the region under consideration and the Bessel functions are defined to be single valued. The scalar potential η is determined so that Eq. (287) is satisfied and $\hat{\chi}$ is solenoidal. The potential $\hat{\chi}$ is a single valued function of θ since $\sin(n\,\theta)$and $\cos(n\,\theta)$ are periodic, and are obviously single valued functions of z and t if s is real and positive.

If s is complex, the complex s plane must be cut along the negative real axis to make the modified Bessel functions of the second kind single valued. This cut makes the logarithms occurring in the definition of these functions single valued. In analysis of equilibrium problems, the final equilibrium solution does not involve logarithms of s and becomes single valued; this is also true of dynamic problems involving Bessel functions of the second kind. The proof requires additional analysis which is given in Appendix A.

X. DISCUSSION AND CONCLUSIONS

In this paper we have formulated the equations governing elastic strains in a hollow cylinder due to stationary loads and loads moving with constant subsonic velocity. In addition, scalar and vector potentials are derived for stationary loads, showing the connection between static loading and the Green-Lamé formulation of wave motion. Six linearly independent solutions are obtained for the moving solutions, corresponding to the six boundary conditions on the cylindrical surfaces. However, when the velocity of travel is set equal to zero, two solutions are lost by confluence of the solutions, and only four linearly independent solutions remain. Two additional solutions are obtained by a limiting method, so a total of six linearly independent solutions is available. Biharmonic scalar and vector potentials are also derived from which these two additional solutions can be calculated.

The distribution of characteristic roots is not yet determined. We speculate the roots in the first quadrant of the complex s plane lie between the imaginary axis* and a smooth curve passing through the complex eigenvalues

*Free vibrations of a hollow cylinder lead to real values of ω the circular frequency, and real eigenvalues when the eigenvalues are expressed in Bessel functions of the first kind, provided ω is not too large. Pure imaginary eigenvalues arise when modified Bessel functions are used.

for static loading. To verify this, the characteristic roots for a traveling load on a solid rod will be calculated first, as the calculations should be relatively simple, before investigating the eigenvalues for a hollow cylinder.

The planned programming will follow the general pattern developed for static loading of a hollow cylinder, but simplified and streamlined to expedite the calculations. Each type of loading will be considered separately.

Torsional loading will be considered first due to the simplicity of the analysis. Axisymmetric loading will be considered next, as an extension of our thick-walled cylinder analysis.

The analysis in Parts I -VIII is based on the work of Herrmann and his associates, in which the vector potential $\hat{\psi}$ is not required to be solenoidal. In order to conform to the classical Helmholtz theory, a new vector solenoidal potential $\hat{\chi}$ is derived. The previously derived formulas for the displacements, strains, and stresses are not changed. The formula for the vector rotation is simplified and becomes the vector counterpart of the scalar formula for the dilatation.

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APPENDIX A

QUASI-STATIC SOLUTIONS OF THE WAVE EQUATIONS

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QUASI-STATIC SOLUTIONS OF THE WAVE EQUATIONS

Stresses given in Part V of this report have the generic form

$$\sigma = G(\theta, r, z) + F(\theta, r, z) \tag{A1}$$

where $G(\theta, r, z)$ is the limiting form of the stress at a great distance from the discontinuity in loading and $F(\theta, r, z)$ is a Fourier integral giving the local effects of this discontinuity. Both solutions can be obtained from scalar and vector wave functions. However, $G(\theta, r, z)$ is of a simpler type, and can be obtained from a scalar harmonic function in the independent variables θ, r, $(z-ct)/\beta_1$ and a vector harmonic function in the variables θ, r, $(z-ct)/\beta_2$. These solutions can be expanded in power series in the variable $(z-ct)$; logarithmic solutions may also occur. Exponentially decaying terms do not occur, so the stresses given by $G(\theta, r, z)$ persist at considerable distances.

The scalar wave function is a solution of

$$c_1^2 \nabla^2 \phi = \frac{\partial^2 \phi}{\partial t^2} \tag{A2a}$$

where c_1 is the dilatational wave speed. If $\phi[\theta, r, (z-ct)/\beta_1]$ is a scalar wave function, then $\phi[\theta, r, z/\beta_1]$ is a scalar harmonic function. We have in expanded form

$$c_1^2 \left[\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} \right] = \frac{\partial^2 \phi}{\partial t^2} \tag{A2b}$$

We note that

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial z^2} \tag{A3}$$

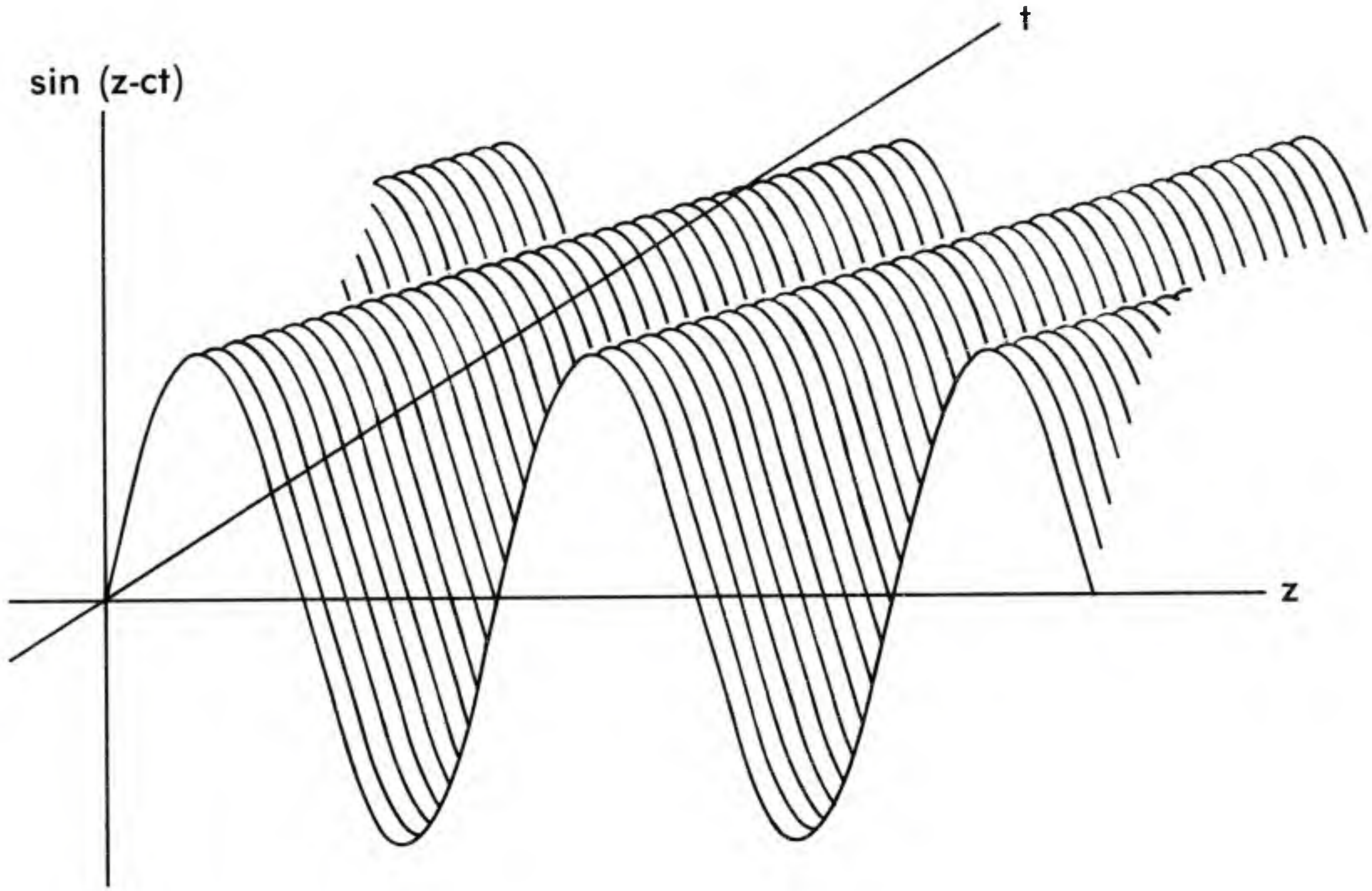

Figure A1. Sinusoidal Pressure Pulse in a Hollow Cylinder so that

$$c_1^2\left[\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right] + [c_1^2 - c^2]\frac{\partial^2\phi}{\partial z^2} = 0 \qquad \text{(A4)}$$

But

$\beta_1^2 = (c_1^2 - c^2)/c_1^2$ We define

$$\zeta_1 = z/\beta_1 \qquad \text{(A5)}$$

Then

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial\phi}{\partial\zeta_1^2} = 0 \qquad \text{(A6)}$$

Analysis of the vector potential yields similar results. If $\hat{\psi}[\theta, r, (z-ct)/\beta_2]$ is a solution of the vector wave equation

$$c_2^2\nabla^2\hat{\psi} = \frac{\partial^2\hat{\psi}}{\partial t^2} \qquad \text{(A7)}$$

then $\hat{\psi}[\theta, r, z/\beta_2]$ is a solution of the vector harmonic equation

$$\nabla^2\hat{\psi} = 0 \qquad \text{(A8)}$$

where c_2 is the velocity of the shear wave and $\beta_2^2 = (c_2^2 - c^2)/c_2^2$

We assume that

$$0 < c < c_2 \tag{A9}$$

in the preceding analysis.

The finite series for G(θ, r, z-ct) can be derived from harmonic scalar potentials in the variables θ, r, z/β_1 and the vector harmonic potentials in the variables in the θ, r, z/β_2. It is sufficient to consider scalar and vector harmonic functions in the variables θ, r, z, as the required potentials in terms of θ, r,(z-ct) can be obtained by appropriate changes in the independent variable z.

We consider the scalar form of Laplace's equation in detail. We have

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{A10}$$

We assume

$$\phi = F \sin(n\theta) \tag{A11}$$

then

$$\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} - \frac{n^2 F}{\partial r^2 F} + \frac{\partial^2 F}{\partial z^2} = 0 \tag{A12}$$

The function

$$F_o = (A_o + A_1 \log r)(1 + \beta_1 z) \tag{A13}$$

is an obvious solution when $n = 0$. For $n > 0$ we have solutions of the type

$$F = (A_o r^{-n} + A_1 r^n)(1+\beta_1 z) \tag{A14}$$

The general, separable solutions involve Bessel and trigonometric functions. We require nonseparable solutions in ascending powers of z. Both positive and negative powers of r are required in order to satisfy boundary conditions at the inner and outer cylindrical surfaces. The required harmonic function can be obtained by expressing spherical harmonics in terms of cylindrical coordinates and rearranging terms in ascending powers of z, References A1 and A2.*

These potentials can be obtained in an elementary manner by assuming

$$F_k = G_o + z^2 G_2 + z^4 G_4 + \dots \tag{A14a}$$

or

$$F_{k+1} = zG_1 + z^3 G_3 + z^5 G_5 \dots \tag{A14b}$$

where

[A1] *E.W. Hobson, The Theory of Spherical and Ellipsoidal Harmonics, University Press, Cambridge, 1931. See Chapter IV.*

[A2] *J. Dougall, "An Analytical Theory of the Equilibrium of an Isotropic Elastic Rod of Circular Cross Section," Transactions of the Royal Society of Edinburg, 1913, Vol. XLIX, Part IV (No. 17), pages 895-978.*

**The fourth degree harmonic $_4$ given on page 348, Reference 12, occurs in the torsion problem. Details will be described in a forthcoming BRL report.*

$$G_i = Ar^k + Br^{-k}, \; i = 0, 1 \tag{A15}$$

The remaining G_i functions, which are also functions of r alone, can be obtained recursively from the formula

$$\frac{\partial^2 G_i}{\partial r^2} + \frac{1}{r}\frac{G_i}{r} - \frac{n^2}{r^2} G_i = (i+2)(i+1)\, G_{i+2} \tag{A16}$$

which is obtained by substituting the preceding expressions for F_k into Eq. (A10), carrying out the indicated operations, and equating the successive powers of z to zero.

As an example, set

$$n = 0,\ k = 4,\ A = 3, B = 3 \tag{A17}$$

We find

$$F_k = 3r^4 - 24r^2 z^2 + 8z^4 + 3r^{-4} - 24r^{-6} z^2 + 8r^{-8} z^4 \tag{A18}$$

The potential given by the first line can be used in the analysis of torsional loads and differs only by a constant from the potential ϕ_4 given in Reference 12, page 348.

Next, we consider the contribution of the Fourier integral to the total solution. We have either

$$F_s(\theta, r, z) = \sin(n\theta) \int_{-\infty}^{\infty} \frac{N_\sigma(r,s)}{D(s)} \frac{\sin[s(z-ct)]}{s}\, ds \tag{A19a}$$

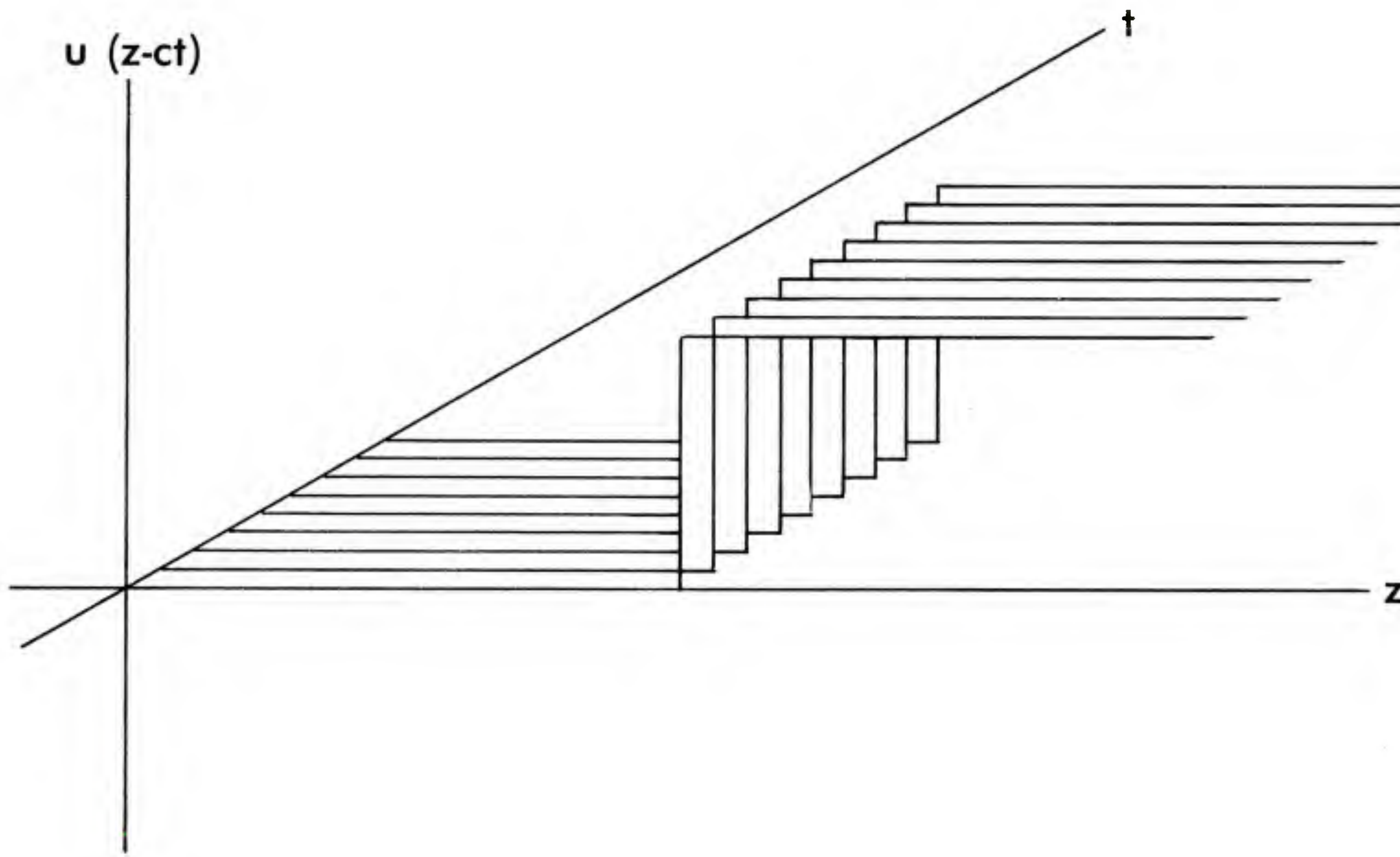

Figure A2. Stepped Pressure Pulse in a Hollow Cylinder or

$$F_c(\theta, r, z) = \cos(n\theta) \int_{-\infty}^{\infty} \frac{N_\sigma(r,s)}{D(s)} \frac{\text{sins } s(z-ct)}{s} ds \qquad \text{(A19b)}$$

for stresses which are odd in z. The cosine replaces the sine under the integral sign if the stress is even in z. For values of z which are not too small, we calculate F_s or F_c by the theory of residues, which is validated in cases of interest by Jordan's lemma. If $z>ct$ we use a contour in the upper half plane, indented at the origin of the complex s plane. If $z<ct$ we use the corresponding contour in the lower half plane. The complex variable theory follows suggestions in a landmark paper by Dougall (A2). Dougall uses an entirely different set of potentials to calculate the displacements and strain; however, his remarks on the nature and distribution of the characteristic roots are still valid.

It is obvious the characteristic functions for the solid rod do not have a branch point at the origin of the complex s plane, since only ordinary or modified Bessel functions of the first kind are involved. These functions are analytic in the entire complex plane with the exception of essential singularities of exponential type at infinity. A detailed analysis is required for the hollow cylinder, since functions of the first and second kind are both required in order to satisfy boundary conditions at the inner and outer cylindrical surfaces. Bessel functions of the second kind have a logarithmic singularity at the origin, plus a finite Laurent series in the case of functions of integral orders greater than zero. We must show the characteristic functions do not have a branch point at the origin; multiple poles will, in general, occur due to the reciprocal powers in the Laurent series.

To this end, we write D(s) in extended form. We have from Eq. (164)

$$D(s) = \begin{vmatrix} S_{1,1}(a) & S_{1,2}(a) & S_{1,3}(a) & S_{1,4}(a) & S_{1,5}(a) & S_{1,6}(a) \\ S_{4,1}(a) & S_{4,2}(a) & S_{4,3}(a) & S_{4,4}(a) & S_{4,5}(a) & S_{4,6}(a) \\ S_{5,1}(a) & S_{5,2}(a) & S_{5,3}(a) & S_{5,4}(a) & S_{5,5}(a) & S_{5,6}(a) \\ S_{1,1}(b) & S_{1,2}(b) & S_{1,3}(b) & S_{1,4}(b) & S_{1,5}(b) & S_{1,6}(b) \\ S_{4,1}(b) & S_{4,2}(b) & S_{4,3}(b) & S_{4,4}(b) & S_{4,5}(b) & S_{4,6}(b) \\ S_{5,1}(b) & S_{5,2}(b) & S_{5,3}(b) & S_{5,4}(b) & S_{5,5}(b) & S_{5,6}(b) \end{vmatrix} \qquad (164)$$

To eliminate the apparent branch point from the determinant D(s) we manipulate the columns in pairs. Columns 1, 3, and 5 are derived from the even-numbered equations in the sequence of Eqs. (80-115). These columns therefore contain only modified Bessel functions of the first kind, and are analytic functions of s. Columns 2, 4, and 6 are derived from odd-numbered equations of the above sequence, and contain only modified Bessel functions of the second kind. The logarithmic term of the modified Bessel functions contain the logarithm of s, which can be eliminated algebraically.[A3] The logarithms of a and b which also occur can not be eliminated. For convenience suppose n is even. The variable r in the following equation can represent either a or b.

$$K_n(sr) = L_n(s)I_n(sr) + Ln(r)I_n(sr) + L_n(sr) + P_n(sr) \qquad (20)$$

where $L_n(sr)$ is a Laurent series having a pole of order n at the origin in the complex s plane and $P_n(sr)$ is a power series in ascending powers of (sr), with zero and positive powers only. Similarly,

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[A3] *M. Abramowitz and S.A. Stegun, Editors. Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables. U.S. Department ofCommerce, Applied Mathematic Series No. 55, 1964. See Eq. (9.6.11), page 375.*

$$K_{n+1}(sr) = - L_n(s)I_n(sr) - Ln(r)I_n(sr) - L_{n+1}(sr) - P_{n+1}(sr) \tag{A21}$$

Now multiply columns 1, 3, and 5 by $L_n(s)$ and subtract the results from columns 2, 4, and 6, respectively. We find the logarithm of s is eliminated. If n is an odd integer we add instead of subtracting. In either case, we find D(s) is an analytic function of s except for poles. A multiple pole generally occurs at the origin.

It is believed the characteristic roots are simple except at the origin. Asymptotic methods may be used for characteristic roots of large modulus. The smallest nonzero root can be examined by expanding D(s) in a Laurent series. Extensive calculations have shown the characteristic equations arising from axisymmetric loading are simple for a large range of wall ratios (A4). Moreover, a double zero leads to functions of the type $R(r) \times [s(z-ct) \sin s(z-ct) + \cos s(z-ct)]$, which do not satisfy the wave equations. Classical methods of determining the multiplicity of characteristic roots are forbiddingly difficult when applied to the determinant D(s); hence, detailed analysis will be reserved for specific types of loading under consideration.

[A4] *A.S. Elder and K.L. Zimmerman, "Stresses in a Gun Tube Produced by Internal Pressure and Shear," BRL Memorandum Report No. 2495, June 1975 (AD A012765).*

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