NRL Memorandum Report 5080

Stabilizing Effect of Gas Conductivity Evolution on the Resistive Sausage Mode of a Propagating Beam

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June 8, 1983

This work was supported by Defense Advanced Research Projects Agency under ARPA Order 4395, Amendment No. 1, monitored by Naval Surface Weapons Center under Order N60921-82-WR-W0066.

NAVAL RESEARCH LABORATORY
Washington, D.C.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NRL Memorandum Report 5080	2. GOVT ACCESSION NO. AD-A128922	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) STABILIZING EFFECT OF GAS CONDUCTIVITY EVOLUTION ON THE RESISTIVE SAUSAGE MODE OF A PROPAGATING BEAM		5. TYPE OF REPORT & PERIOD COVERED Interim report on a continuing NRL problem. 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) M. Lampe and G. Joyce		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Research Laboratory Washington, D.C. 20375		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 47-0900-0-2
11. CONTROLLING OFFICE NAME AND ADDRESS Defense Advanced Research Projects Agency Arlington, VA 22209		12. REPORT DATE June 8, 1983 13. NUMBER OF PAGES 26
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) Naval Surface Weapons Center Detachment Silver Spring, MD 20910		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

This work was supported by Defense Advanced Research Projects Agency under ARPA Order 4395, Amendment No. 1, monitored by Naval Surface Weapons Center under Order N60921-82-WR-W0066.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Beam propagation
Beam instabilities
Relativistic electron beams
Charged particle beams
Sausage instability
Axisymmetric instability of beams

The authors

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Previous theoretical work has shown that a highly current neutralized charged particle beam propagating in a pre-ionized plasma channel of fixed conductivity is subject to a resistive sausage instability. We show that the instability if stabilized, for the case of beam propagation into initially unionized gas, when the effect of beam-collisional ionization on the gas conductivity is modeled fully self-consistently.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

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STABILIZING EFFECT OF GAS CONDUCTIVITY EVOLUTION ON THE RESISTIVE SAUSAGE MODE OF A PROPAGATING BEAM

1. Introduction

A self-pinched electron or ion beam propagating in gas will excite a substantial return current if the gas is pre-ionized, if it is rapidly ionized collisionally by the beam head[1], or if avalanche breakdown is driven by inductive electric fields at the beam head. One may well expect highly current neutralized beams to be subject to a variety of instabilities excited by the repulsive force between beam and return current, and several model calculations[2-5] have reached this conclusion with regard to beams propagating in resistive plasma. The situation differs sharply from that of a non-current-neutralized beam in a resistive plasma, for which the hose mode is the only unstable mode.

Under conditions where they are unstable, the axisymmetric beam modes appear to be particularly dangerous to propagation, because they are almost inevitably excited at large amplitude. Unless the beam emittance is perfectly matched at injection, the beam will oscillate in radius, and in particular, the violent pinchdown associated with the process of nose expansion and erosion[1,6] can be expected to excite some radial oscillations. Non-axisymmetric modes, such as hose and filamentation, must grow out of initially low-level noise if the accelerator produces a high-quality beam, and thus must e-fold many more times before they pose a threat to beam integrity.

We classify the linearized normal modes of a beam by an azimuthal mode number m and a radial mode number n. For a mode(m,n), all perturbed quantities ψ are of the form

$$\psi(r,\theta,z,\zeta) = \exp(im\theta - i\Omega z)\, \hat{\psi}(r,\zeta),$$

where $\zeta = ct - z$ is the position in the beam measured back from the beam head. Roughly speaking, n is the number of oscillations of $\hat{\psi}(r,\zeta)$ as r varies

Manuscript approved March 8, 1983.

from zero to the outside of the beam, for fixed ζ. The $m = 0$ (axisymmetric) modes are sometimes referred to collectively as sausage modes, but we shall reserve this term for the $m = 0$, $n = 1$ mode, which corresponds roughly to self-similar radial oscillation or "breathing" of the beam, and we shall refer to the next axisymmetric mode, $m = 0$, $n = 2$, as the axisymmetric hollowing mode. A recent theoretical study by Uhm and Lampe[4], based on the simplifying assumptions of a fixed flat-topped radial profile of plasma conductivity and a flat-topped beam radial profile, predicted sausage instability when the ratio of plasma return current to beam current[7] $I_p/I_b > 0.50$ and axisymmetric hollowing instability when $I_p/I_b > 0.38$. Lee[5] extended the theory of the sausage mode to arbitrary beam profiles and arbitrary (but fixed) conductivity profiles, and also predicted sausage instability when I_p/I_b exceeds a threshold whose exact value depends on the profile. For similar Bennett profiles of beam current and conductivity, the threshold is 0.69; when $I_p/I_b > 0.75$, instability at $\Omega = 0$ is predicted. However the sausage instability has not been observed, as far as we are aware, for beams propagating in neutral gas, although I_p/I_b often exceeds any of these instability thresholds.

In this paper, we present a more complete linearized theory of the sausage mode of a relativistic electron beam which begins from Lee's formulation[5] but includes a self-consistent treatment of the plasma conductivity, including collisional ionization of the gas by the perturbed beam. We find that the conductivity channel perturbs in such a way as to follow the distortions of the beam. This inhibits the separation of beam and plasma current, which is the cause of the instability, and consequently leads to a more stringent instability condition, which is never satisfied for beams propagating in high density initially unionised gas. (The situation at low gas density, $\lesssim$ 50 torr depending on conditions, is discussed later in this introduction.) This instability condition is [see Eq. (41) in Sec. 2]

$$\frac{I_p}{I_{eff}} > \frac{(2\lambda + \sqrt{3})^2}{\lambda + 1} H, \tag{1}$$

where I_{eff} is a radially-averaged net current which determines the mean pinch force,

$$\lambda \equiv \frac{\kappa I_b}{2c} \equiv \frac{d}{d\zeta} \left(\frac{\pi\sigma(r=0,\zeta)a^2}{2c}\right)$$

is a measure of the rate of change of conductivity σ due to beam-molecule collisions, κ is a coefficient depending on the ionization coefficient and mobility of the particular gas, and H is a factor of order unity.

The inequality (1) can easily be satisfied for beams propagating into pre-ionized gas. The instability predicted under those conditions typically has a smaller growth rate than the hose mode, but could dominate if the sausage mode is initiated at larger amplitude, as it normally will be in a well-prepared beam.

For beams injected into neutral gas, a very brief burst of avalanche ionization at the beam head typically has a strong influence on the degree of current neutralization. This effect is tacitly included in our theory, since I_{eff}/I_b is treated as a free parameter. However our analytic theory treats only beam-collisional ionization, and not avalanche, in the beam body where the instability grows. This is usually a good approximation, except in low density gas (5 - 50 torr depending on beam current density and gas type). We have found some cases in low density gas where avalanche at the beam head is so strong that the instability condition (1) is satisfied, but in all of these cases noted to date avalanche should also be included in modeling in the beam body. Heuristic considerations indicate that the neglected avalanche term in the beam body would enhance stability, by further perturbing the conductivity channel so as to follow the beam distortions. We have tested this idea by performing a few simulations with the axisymmetric beam envelope code VIPER-0, which includes avalanche everywhere and permits sausage-like oscillations to develop (but does not permit any higher axisymmetric perturbations). Sausage instability did not occur in these cases, even though condition (1) was satisfied. Further study of this low-density regime is needed, however.

Particle simulations at several laboratories[8-10] have recently observed strong axisymmetric instabilities under a variety of conditions that are compatible with the instability condition

$$I_p/I_b \gtrsim 0.50 \qquad (2)$$

but are incompatible with condition (1). We have shown by means of a simulation analysis that these instabilities involve the hollowing mode, not the sausage mode, and are triggered by a complex set of circumstances with other requirements in addition to (2). These results are reported in a separate paper.[11]

The outline of this paper is as follows. We introduce our model and list its assumptions in Sec. 2. Our analytic calculation of the sausage mode dispersion relation is presented in Sec. 3. Our conclusions, with regard to beam propagation in initially neutral gas, pre-ionized gas, or in a channel of fixed conductivity profile, are discussed in Sec. 4.

2. Formalism and Assumptions

In this section we develop a fully analytic theory of the sausage mode of a relativistic electron beam that includes the modifications of the channel conductivity that result from beam-collisional ionization treated self-consistently with the sausaging of the beam. We find that the conductivity channel tends to follow the sausage distortion of the beam; as a result, spatial separation of the plasma return current density J_p from the beam current density J_b is reduced, and the mode is found to be much more stable than it would be in a fixed conductivity channel.[5]

In order to carry out the analysis in simple form, we make a number of simplifying assumptions, most of them having wide validity. Four of these assumptions have been widely employed in beam propagation theory. They are that the background gas can be regarded as an immobile medium with a scalar conductivity $\sigma(\underset{\sim}{x},t)$, that the beam is highly relativistic ($\gamma \gg 1$) and paraxial ($v_\perp \ll v_z$ for all electrons), and that therefore $v_z = c$.

We consider only instability growth in the region of the beam which we shall call the beam "body", behind the pinch point[1,6] but forward of the beam tail where recombination limits the conductivity. This is the region where violent axisymmetric instability has been observed in simulations,[8-10] and where theory indicates that instabilities should grow most rapidly. Here the conductivity σ is large enough to insure space charge neutrality in the vicinity of the beam, and Maxwell's equations reduce to Ampere's law,

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} A(r,\zeta,z) - \frac{4\pi\sigma(r,\zeta,z)}{c}\frac{\partial A}{\partial \zeta} = -\frac{4\pi}{c} J_b(r,\zeta,z) \qquad (3)$$

for an axisymmetric beam, where $A(r,\zeta,z)$ is the axial component of the vector potential. We can expect the beam body in equilibrium to have a Bennett radial profile,

$$J_{bo}(r,\zeta,z) = \frac{\tilde{J}_{bo}}{(1 + r^2/a_o^2)^2} . \qquad (4)$$

(Other profiles could be treated.) In the beam body, the Bennett radius a_o typically decreases slowly with ζ and increases very slowly (erosion and Nordsieck expansion[1,6]) with z; we use the approximation that a_o is completely independent of both z and ζ in equilibrium. Since avalanche is normally unimportant in the beam body except for beams propagating in low density gas, e.g. $P \lesssim 50$ torr for a typical induction linac beam with $I_b = 10$ kA, $a_o = 0.5$ cm, and since recombination is by definition unimportant in the beam body, we assume that the conductivity evolution is determined by direct collisional ionization of the gas by the beam,

$$\frac{\partial}{\partial \zeta} \sigma(r,\zeta,z) = \kappa J_b . \qquad (5a)$$

(However the effect of avalanche in the beam head is tacitly included as an initial condition, i.e. the values of σ and I_p/I_b specified at the front of the beam body segment included in the calculation, $\zeta = \zeta_o$, depend implicitly on avalanche in the beam head, $\zeta < \zeta_o$.) We neglect the fairly weak dependence of σ on plasma temperature (through the plasma electron collision frequency ν_m) and thus treat κ as a constant. A reasonable estimate of κ, based on the typical value[12] $\nu_m = 2 \times 10^{12}$ sec^{-1} for air at standard density and 0.5 eV $\leq T_e \leq$ 1 eV, is

$$\kappa = 8.8 \times 10^{-4} \text{ cm/statcoul.} \qquad (5b)$$

To facilitate the analysis, we also make the following additional assumptions. The equilibrium electric field $E_{zo}(\zeta,z)$, which has a weak radial dependence[1,6]

$$E_{zo}(r,\zeta,z) \propto \ln\left(\frac{1 + b^2/a_o^2}{1 + r^2/a_o^2}\right), \tag{6a}$$

is regarded as independent of r,

$$E_{zo}(r,\zeta,z) = E_{zo}(0,\zeta,z). \tag{6b}$$

In (6a) b is a large radius where charge neutrality fails. Approximation (6b) is common to all previous stability analyses. We note elsewhere that the breakdown of this approximation plays a key role in the destabilization of the hollowing mode,[11] but we believe that the approximation is acceptable in analysis of the sausage mode.

As a result of Eqs. (4) - (6), the equilibrium conductivity and plasma current density, $\sigma_o(r,\zeta)$ and $J_{po}(r,\zeta)$, both have Bennett profiles of radius a_o. This turns out to be very helpful to the analysis: it allows us to reduce a problem in r,ζ and z to the form of ordinary differential equations in ζ and z only. For related problems of interest, the r-dependence cannot be eliminated. For example, in the beam "tail" where beam-collisional ionization is balanced by recombination, $\sigma(r) \propto [J_b(r)]^{1/2}$, a broader profile than $J_b(r)$. As a result, much of the plasma current flows outside the beam and has no destabilizing effect on the beam, which essentially guarantees that the sausage mode will be stable. On the other hand, if the channel is fully ionized, as is the case typically for applications to ion-beam inertial fusion, $\sigma(r) \propto T_e^{3/2} \propto [J_b(r)]^{3/2}$ for Spitzer conductivity and no effective heat loss mechanism. Thus $\sigma(r)$ is narrower than $J_b(r)$, causing the plasma current to peak on axis, further destabilizing all beam modes.

We assume for convenience that J_{bo} and J_{po} are also independent of ζ. (The analysis could be carried out without these assumptions, but not in closed form.) The justification for the latter is that the decay length τ_o of J_{bo} is long compared to the decay length τ_1 of current perturbations,

$$\tau_o = \frac{2\pi\sigma a^2}{c} \ln \frac{b}{a_o} \tag{7a}$$

$$\tau_1 = \frac{\pi\sigma a^2}{2c} ; \tag{7b}$$

τ_1 also characterizes the instability growth length.

We consider only beam perturbations in the form of self-similar radial expansion and contraction, i.e. the perturbed beam is of the form

$$J_b(r,\zeta,z) = \frac{I_b}{\pi a^2(\zeta,z)} \frac{1}{[1 + r^2/a^2(\zeta,z)]^2} \tag{8}$$

which is a reasonable but not exact representation of the sausage mode. The variation of the root-mean-square beam radius[13] $\bar{a}$ (ξ,z) is calculated from the Lee-Cooper[14] envelope equation,

$$\frac{\partial^2 \bar{a}}{\partial z^2} = \frac{\varepsilon^2}{\bar{a}^3} - \frac{U^2}{\bar{a}} , \tag{9}$$

where U^2 is a measure of the average pinch force,

$$U^2 \equiv \frac{2I_b}{I_A} \int_0^\infty dr \frac{2\pi r J_b(r)}{I_b} \int_0^r dr' \frac{2\pi r' [J_b(r') + J_p(r')]}{I_b} , \tag{10}$$

$I_A = 17\gamma$ kA is the Alfvén current, J_p is the plasma current density[7], and ε is the emittance, defined as $\bar{a}\ \langle\theta\rangle$, where $\langle\theta\rangle$ is the root-mean-square beam electron velocity angle.

The principal effect omitted by the model (8) - (10) is sausage oscillation damping due to phase mixing among beam electrons of different betatron frequency.[15,16] This effect is included phenomenologically by adding a damping term in the form derived by Lee and Yu[17],

$$\frac{\partial \varepsilon^2}{\partial z} = -\left(\frac{2\ \alpha\ \bar{a}^{-2}}{\frac{I_A}{I_{eff}}\ \frac{\varepsilon}{\bar{a}} + \frac{\bar{a}}{\varepsilon}}\right)\frac{\partial^2 \bar{a}}{\partial z^2}\ . \qquad (11)$$

The damping constant α is sensitive to the beam profile.[13] Lee[5,17] estimates $\alpha = 0.7$ for use with a Bennett profile.

We perform a linearized perturbation calculation in which small perturbations are added to the equilibrium profiles of J_b, J_p, σ and A_z described above. Perturbed quantities are calculated from the linearized forms of Eqs. (3), (5), (9), (11), and Ohm's law,

$$J_p = \sigma\ E_z. \qquad (12)$$

Because of the assumed radial dependences of the equilibrium and perturbations, the equations reduce to coupled ordinary differential equations.

We note particularly that both the equilibrium and the perturbed conductivity are calculated self-consistently from Eq. (5); this is the new feature of the present calculation and leads to enhanced mode stability.

3. Calculation

The equilibrium described in Sec. 2 is specified by[7]

$$J_{bo} = \tilde{J}_{bo}(1 + r^2/a_o^2)^{-2}, \tag{13a}$$

$$J_{po} = \tilde{J}_{po}(1 + r^2/a_o^2)^{-2}, \tag{13b}$$

$$\sigma_o = \kappa \tilde{J}_{bo}\, \zeta(1 + r^2/a_o^2)^{-2}, \tag{13c}$$

$$A_o = \tilde{A}_o(\zeta)\ell n \frac{1 + r^2/a_o^2}{1 + b^2/a_o^2} \approx - \tilde{A}_o(\zeta)\ \ell n\ \frac{b^2}{a_o^2}\ , \tag{13d}$$

$$E_{oz} = - \frac{dA_o}{d\zeta} \approx \frac{d\tilde{A}_o}{d\zeta}\ \ell n\ \frac{b^2}{a_o^2}\ . \tag{13e}$$

Equations (3), (9) and (10) reduce to the equilibrium relations

$$\tilde{A}_o + \tau_o \frac{d\tilde{A}_o}{d\zeta} = - \frac{\pi a_o^2}{c} J_{bo}, \tag{14}$$

$$\varepsilon_o^2/\ \bar{a}_o^{2} = U_o^2 = I_{eff}/I_A, \tag{15}$$

where

$$I_{eff} \equiv I_b - I_p \tag{16}$$

for the present case of beam and plasma currents with the same profile[7]
The front of the beam segment of interest is taken to be at $\zeta = \zeta_o$; the value of ζ_o may be chosen so that Eq. (13c) gives the conductivity desired as an initial condition in ζ.

The perturbed parts of J_b, J_p, σ and A, designated δJ_b, δJ_p, $\delta\sigma$ and δA, are treated as small quanities. To first order Eqs. (3), (5), (9), (11) and (12) become

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\delta A - \frac{4\pi\sigma_o}{c}\frac{\partial \delta A}{\partial \zeta} = \frac{4\pi}{c}\frac{\partial A_o}{\partial \zeta}\delta\sigma - \frac{4\pi}{c}\delta J_b, \tag{17a}$$

$$\frac{\partial \delta\sigma}{\partial \zeta} = \kappa\, \delta J_b, \tag{17b}$$

$$\delta J_p = -\sigma_o \frac{\partial \delta A}{\partial \zeta} - \frac{\partial A_o}{\partial \zeta}\delta\sigma, \tag{17c}$$

and

$$\frac{\partial^2 \delta\bar{a}}{\partial z^2} = -\frac{2U_o^2}{\bar{a}_o^2}\delta\bar{a} - \alpha\, U_o \bar{a}_o^{-2}\frac{\partial \delta\bar{a}}{\partial z} - \frac{\delta U^2}{\bar{a}_o}, \tag{18}$$

where we have used (11) and (15) to simplify (9). These equations admit the solution

$$\delta J_b = -\tilde{J}_b(\zeta,z)\frac{1 - r^2/a_o^2}{(1 + r^2/a_o^2)^3}, \tag{19a}$$

$$\delta J_p = -\tilde{J}_p(\zeta,z)\frac{1 - r^2/a_o^2}{(1 + r^2/a_o^2)^3}, \tag{19b}$$

$$\delta\sigma = -\tilde{\sigma}(\zeta,z)\frac{1 - r^2/a^2}{(1 + r^2/a_o^2)^3}, \tag{19c}$$

$$\delta A = -\tilde{A}\,(\xi,z)\frac{1 - r^2/a_o^2}{1 + r^2/a_o^2}, \tag{19d}$$

which corresponds to self-similar oscillations, since

$$\frac{1 - r^2/a_o^2}{(1 + r^2/a_o^2)^3} = -\frac{a^3}{2}\frac{d}{da}\left[\frac{1}{a^2}\frac{1}{(1 + r^2/a^2)^2}\right]_{a=a_o}. \tag{20}$$

Equations (17) and (18) then reduce to the ordinary differential equations

$$\left(1 + \frac{\pi a_o^2 \kappa \tilde{J}_{bo}}{2c}\zeta\frac{\partial}{\partial\zeta}\right)\tilde{A} = \frac{\pi a_o^2}{2c}\tilde{J}_b + \frac{\pi a_o^2}{2c}\frac{d\tilde{A}_o}{d\zeta}\ln\left(\frac{b^2}{a_o^2}\right)\tilde{\sigma}, \tag{21a}$$

$$\frac{\partial\tilde{\sigma}}{\partial\zeta} = \kappa\tilde{J}_b, \tag{21b}$$

$$\tilde{J}_p = -\tilde{\sigma}_o\frac{\partial\tilde{A}}{\partial\zeta} - \frac{dA_o}{d\zeta}\tilde{\sigma}, \tag{21c}$$

$$\frac{\partial^2\tilde{J}_b}{\partial z^2} = -\frac{2U_o^2}{\bar{a}_o^2}\tilde{J}_b - \alpha U_o \bar{a}_o^{-2}\frac{\partial\tilde{J}_b}{\partial z} + \frac{\delta U^2}{\bar{a}_o}. \tag{21d}$$

We address the linearized envelope equation (21d) first. Since I_b and I_p are unchanged by self-similar expansion, only the cross terms between J_p and J_b in Eq. (10) contribute to the perturbed pinch force δU^2. Recalling our convention that I_p and I_b are both positive[7], we find

$$\frac{\delta U^2}{U_o^2} = \frac{1}{3\tilde{J}_{bo}}\left(-\frac{I_p}{I_{eff}}\tilde{J}_b + \frac{I_b}{I_{eff}}\tilde{J}_p\right), \tag{22}$$

and the perturbed envelope equation (21d) then reduces to

$$\frac{\partial^2\tilde{J}_b}{\partial(z/\lambda_\beta)^2} = -\alpha\frac{\partial\tilde{J}_b}{\partial(z/\lambda_\beta)} - \left(2 - \frac{2}{3}\frac{I_p}{I_{eff}}\right)\tilde{J}_b + \frac{2}{3}\frac{I_b}{I_{eff}}\tilde{J}_p, \tag{23}$$

where we have defined an average betatron wavelength

$$\lambda_\beta = \bar{a}_o/U_o. \tag{24}$$

The quantity[13] $\bar{a}_o$ appears only through λ_β.

In order to close the analysis, we must express $\tilde{J}_p$ in terms of $\tilde{J}_b$ though Eqs. (21a-c). First we simplify Eq. (21a) by using the approximation that I_p as well as I_b are independent of ζ, so that[7]

$$\frac{\tilde{J}_{po}}{\tilde{J}_{bo}} = -\frac{I_p}{I_b} = \text{const} \tag{25}$$

and from Eqs. (12) and (13e)

$$\tilde{J}_{po} = \kappa\, \tilde{J}_{bo} \zeta \frac{d\tilde{A}_o}{d\zeta} \ln \frac{b^2}{a_o^2} . \tag{26}$$

(The analysis could be carried out without these approximations, but would then require numerical solution of a complicated ordinary differential equation, rather than yielding the solutions we shall find in closed form.) Thus Eq. (21a) can be written as

$$\left(\frac{2c}{\pi a_o^2} + \kappa\, \tilde{J}_{bo} \zeta \frac{\partial}{\partial \zeta}\right) \tilde{A} = \tilde{J}_b - \frac{I_p}{\kappa I_b} \frac{\tilde{\sigma}}{\zeta} . \tag{27}$$

Next we rewrite Eq. (21b) as

$$\left(1 + \zeta \frac{\partial}{\partial \zeta}\right) \frac{\tilde{\sigma}}{\zeta} = \kappa\, \tilde{J}_b, \tag{28}$$

and use (28) to elimate $\tilde{\sigma}$ from (27), which yields an expression for $\tilde{A}$ in terms of $\tilde{J}_b$,

$$\left(1 + \zeta \frac{\partial}{\partial \zeta}\right)\left(\frac{2c}{\pi a_o^2} + \kappa\, \tilde{J}_{bo} \zeta \frac{\partial}{\partial \zeta}\right) \tilde{A} = \left(1 + \zeta \frac{\partial}{\partial \zeta} - \frac{I_p}{I_b}\right) \tilde{J}_b . \tag{29}$$

We also use Eqs. (13), (19) and (26) in (17c), to obtain

$$\tilde{J}_p = \frac{I_p}{\kappa I_b} \frac{\tilde{\sigma}}{\zeta} - \kappa\, \tilde{J}_{bo}\, \zeta \frac{\partial \tilde{A}}{\partial \zeta}\,, \tag{30}$$

and using (28) again in (30) gives

$$\left(1 + \zeta \frac{\partial}{\partial \zeta}\right) \tilde{J}_p = \frac{I_p}{I_b} \tilde{J}_b - \kappa\, \tilde{J}_{bo}\left(1 + \zeta \frac{\partial}{\partial \zeta}\right) \zeta \frac{\partial \tilde{A}}{\partial \zeta}\,. \tag{31}$$

Equations (29) and (31) together specify $\tilde{J}_p$ as a linear function of $\tilde{J}_b$.

We observe that (23), (29) and (31) constitute a closed system of linear equations with constant coefficients if $(\partial/\partial z)$ and $\zeta(\partial/\partial\zeta) \equiv (\partial/\partial \ln \zeta)$ are regarded as the basic operators. Thus it is natural to use z and $\ln \zeta$ as the independent variables. A complete solution of the problem with perturbation initial conditions at $\zeta = \zeta_o$ would require a Laplace transform analysis, which is beyond the scope of this paper, but the equations admit a free mode solution in the form

$$f(\zeta, z) = \hat{f} \exp\left[-i\Omega z/\lambda_\beta - i\bar{\omega} \ln (\zeta/\zeta_o)\right]$$

$$= \hat{f}\, e^{-i\Omega z/\lambda_\beta} \left(\frac{\zeta}{\zeta_o}\right)^{\bar{\omega}_i}\left[\cos\left(\bar{\omega}_r \ln \frac{\zeta}{\zeta_o}\right) - i \sin\left(\bar{\omega}_r \ln \frac{\zeta}{\zeta_o}\right)\right], \tag{32}$$

where $f(\zeta, z)$ is any perturbed quantity. The z-dependence of the mode is in the usual exponential form, but in ζ the mode shows power law growth or decay and oscillation with steadily increasing wavelength, due to the non-uniformity of the equilibrium in ζ, i.e. the linear increase of $\sigma_o(r,\zeta)$ with ζ.

Using (32) in (29) and (31) yields the required relation between $\tilde{J}_p$ and $\tilde{J}_b$,

$$\tilde{J}_p = \left[-\frac{I_p/I_b}{1 - i\bar{\omega}} + \frac{i\bar{\omega}(1 - i\bar{\omega} - I_p/I_b)}{(1 - i\bar{\omega})\left(\frac{2c}{\pi\kappa\, \tilde{J}_{bo}\, a_o^2} - i\bar{\omega}\right)}\right] \tilde{J}_b. \tag{33}$$

We also use (32) to reduce the envelope equation (23) to algebraic form, and use (33) to eliminate $\tilde{J}_p$. After some algebra, but no approximations, the envelope equation reduces to the dispersion relation

$$- \Omega^2 - i\alpha\Omega = - 2$$

$$+ \frac{2}{3} \frac{1}{1 + \lambda^2 \bar{\omega}^2} \left[\bar{\omega}^2 \left(\frac{1 + \lambda}{1 + \bar{\omega}^2} \frac{I_p}{I_{eff}} - \lambda^2 \right) + i\bar{\omega} \left(\lambda - \frac{1 - \lambda \bar{\omega}^2}{1 + \bar{\omega}^2} \frac{I_p}{I_{eff}} \right) \right], \quad (34)$$

where

$$\lambda \equiv \frac{d\tau_1}{d\zeta} \equiv \frac{\kappa I_b}{2c} , \quad (35a)$$

τ_1 from Eq. (7b) is the characteristic decay time of the perturbed plasma current, and α is the phenomenological damping constant from Eq. (11). If we use Eq. (5b) for κ, we can write Eq. (35a) in the convenient form

$$\lambda = 0.044 \left(\frac{I_b}{1 \text{ kA}} \right). \quad (35b)$$

4. Results

The dispersion relation (34) can be solved for either Ω or $\bar{\omega}$ as a function of the other. The mode is unstable if $\Omega_1 > 0$ for real $\bar{\omega}$ or if $\bar{\omega}_1 > 0$ for real Ω. We consider the instability condition $\Omega_1 > 0$ for real $\bar{\omega}$. Rewriting (34) in the schematic form

$$-\Omega^2 - i\alpha\Omega = F_1(\lambda, I_p/I_{eff}, \bar{\omega}) + iF_2(\lambda, I_p/I_{eff}, \bar{\omega}), \tag{36}$$

where F_1 and F_2 are real, this condition is

$$F_1 > -F_2{}^2/\alpha^2. \tag{37}$$

For $\alpha \to 0$ (no phase mix damping included in the formalism) all modes are unstable; as might be expected the conditions for instability become more restrictive as α increases, but instability can occur even in the limit $\alpha \to \infty$ if $F_1 > 0$, i.e.

$$\frac{1+\lambda}{1+\bar{\omega}^2}\,\frac{I_p}{I_{eff}} > 4\lambda^2 + \frac{3}{\bar{\omega}^2}\,. \tag{38}$$

There are modes $\bar{\omega}$ that satisfy condition (38) if and only if

$$\frac{I_p}{I_{eff}} > \frac{(2\lambda + \sqrt{3})^2}{\lambda + 1}\,, \tag{39}$$

and the range of unstable modes is given by

$$C_1 - C_2 < \bar{\omega}^2 < C_1 + C_2, \tag{40a}$$

where

$$C_1 \equiv \frac{1}{8\lambda^2} \; [(1+\lambda)\frac{I_p}{I_{eff}} - 4\lambda^2 - 3], \tag{40b}$$

$$C_2 \equiv \frac{1}{8\lambda^2} \{[(1+\lambda)\frac{I_p}{I_{eff}} - 4\lambda^2 - 3]^2 - 48\lambda^2\}^{1/2}. \tag{40c}$$

If condition (39) is satisfied, there are very firm model-independent grounds for expecting strong instability, independent of the damping coefficient α. In fact, if one solves for $\bar{\omega}(\Omega)$ it is seen that instability can occur ($\bar{\omega}_i > 0$) in this case even with $\Omega = 0$, i.e. mere time-independent beam non-uniformities can grow unstably as one moves back in the beam. If the weaker condition (37) is satisfied, instability is still predicted, but of an oscillatory and somewhat weaker form, somewhat dependent on the model and the value of α. Lee[5] uses $\alpha = 0.7$, corresponding to a Bennett profile truncated at three to four Bennett radii[13]. We can write the critical value of I_p/I_{eff} for instability, from Eq. (37), in the form of a correction to (39),

$$\frac{I_p}{I_{eff}} > \frac{(2\lambda + \sqrt{3})^2}{\lambda + 1} H\,(\alpha = 0.7, \lambda). \tag{41}$$

Numerical evaluation shows that $H(\alpha = 0.7, \lambda)$ is quite close to unity, varying only from 0.75 for $\lambda = 0$ to 0.81 for $\lambda \to \infty$, so (39) is a reasonably accurate instability criterion even for finite α.

Under conditions where avalanche is unimportant even at the pinch point, e.g. for a typical induction linac beam[18,19] with $I_b \lesssim 10$ kA, $a_e \gtrsim 0.5$ cm, in air at density close to or above ambient, the instability condition (41) is never satisfied. In these cases, Sharp and Lampe[1] have shown that I_p/I_{eff} reaches a peak value

$$I_p/I_{eff} = 3\lambda \text{ to } 4\lambda \tag{42a}$$

at the pinch point, and then falls off to a fairly constant value

$$I_p/I_{eff} \sim (1 \pm 0.3)\lambda \tag{42b}$$

a few centimeters further back in ζ. Equation (42b) is incompatible with condition (41).

If avalanche is important near the pinch point, I_p/I_{eff} is increased. Nevertheless a survey performed with the simulation code SIMMO[11], which does include avalanche, indicates that over a very wide range of beam parameters in air the instability condition (41) is not satisfied over a long enough stretch of beam to permit effective mode growth. [In some cases (41) is satisfied for a region of only a few centimeters about the pinch point.] For example, beams with I_b = 10 kA and radius $\gtrsim$ 0.5 cm are predicted to be sausage-stable in air at densities above 50 torr, and beams with I_b up to at least 100 kA are predicted to be sausage-stable in air at ambient density. We have found, however, that the instability condition (41) is satisfied in some low-density regimes where avalanche is so strong near the pinch point that I_{eff}/I_b is small, e.g. in air at 10 torr $I_{eff}/I_b = 0.08$ for a 10 kA beam with radius 0.5 cm and current risetime 0.33 nsec. This low density regime around 10 torr is of interest, since considerable experimental effort has been devoted to it in past[20] and present experiments.[18]

However in this regime avalanche often dominates the conductivity physics even well behind the pinch point. Thus the conductivity model (5) used in the present theory is seriously incomplete. Inclusion of avalanche self-consistently in the model of the beam body probably would be stabilizing, since it further reduces the spatial separation of the plasma current from the beam current. To date, we have been unable to treat avalanche self-consistently in an analytic theory, but we are presently using the axisymmetric beam envelope code VIPER-0 to simulate sausage evolution in this regime. In limited studies to date we have found no case for which the sausage mode is unstable for beam injection into neutral gas.[21]

Recent particle simulations[8-10] have shown very strong axisymmetric instabilities in some regimes where (41) predicts stability. We present a detailed simulation analysis of these instabilities in a companion paper[11]. We find that the unstable mode is the ($m = 0$, $n = 2$) hollowing mode, not the ($m = 0$, $n = 1$) sausage mode.

For beams propagating into a pre-ionized channel, I_p/I_{eff} can be arbitrarily large, and the instability conditions (41) or even (39) can be satisfied. The limit of a <u>fixed</u> pre-ionized channel, i.e. the case in which beam-induced conductivity augmentation is negligible compared to the pre-existing conductivity, has been considered previously by Lee[5]. In our formalism, this is the limit $\lambda \to 0$, Ω fixed, $\lambda\bar{\omega}(\Omega)$ fixed, $\bar{\omega}(\Omega)\to\infty$, ζ restricted to a range $\zeta_o \leq \zeta \leq \zeta_o + \Delta\zeta$, where $\bar{\omega}\Delta\zeta \ll 1$ and $\tilde{\sigma}_o(\zeta)$ has the essentially constant value $\kappa\tilde{J}_{bo}\,\zeta_o$. Since κ is proportional to λ, we note also that $\zeta_o \to \infty$ is the limit. The dependence of any perturbed quantity f on ζ, from Eq. (32), then reduces to

$$f(\zeta) = \hat{f}\,\exp\left(-i\bar{\omega}\,\frac{\zeta - \zeta_o}{\zeta_o}\right) \equiv \hat{f}\,\exp\left[-i\omega(\zeta - \zeta_o)\right], \qquad (43)$$

i.e. exponential growth and sinusoidal oscillation with complex frequency

$$\omega \equiv \bar{\omega}/\zeta_o. \qquad (44)$$

Our dispersion relation (34) reduces to one previously derived by Lee[5],

$$-\Omega^2 - i\alpha\Omega = -2 + \frac{2}{3}\left(\frac{I_p}{I_{eff}} + \frac{I_b}{I_{eff}}\frac{i\omega}{1 - i\omega}\right). \qquad (45)$$

Our condition (39) for instability to occur and not be stabilized by any value of the damping coefficient α reduces to

$$I_p/I_{eff} > 3, \qquad (46)$$

in agreement with Ref. 5.

In conclusion, then, we have determined the mode structure, Eq. (32), for instability growth in the beam body, and have calculated a dispersion relation, Eq. (34), and instability conditions, (39) and (41), for the sausage mode. We find that the sausage mode instability condition is not satisfied for beams injected into neutral gas under conditions that satisfy our assumptions. However the instability conditions are usually satisfied for beams injected into a pre-ionized gas channel with the same profile as the beam (and with $4\pi\sigma_o a_o \gtrsim c$), because of the substantial current neutralization under those conditions. There is a region of low neutral gas density where our theory would predict instability but where the theory is itself inapplicable because avalanche is strong and persistent. Although sausage stability properties are not well understood in this regime, early indications are that sausage instability does not occur.

To our knowledge, sausage instability has not been observed experimentally for beams injected into neutral gas, in agreement with our conclusions.

Acknowledgments

We are grateful to Drs. Richard F. Hubbard and William M. Sharp for many contributions, and to Dr. Hubbard in particular for performing the VIPER-O simulations mentioned in this paper. We also wish to thank Dr. Edward P. Lee for discussing his formulation of the sausage problem[5] with us in advance of publication, and to acknowledge that unpublished calculations of hose instability by Drs. Lee, R. J. Briggs, J. M. Leary, and L. D. Pearlstein were of use to us.

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Supported by the Defense Advanced Research Projects Agency under ARPA Order 4395, Amendment No. 1.

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