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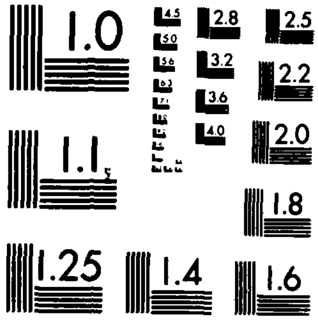

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MULTIVARIATE DEPENDENT RENEWAL PROCESSES

by

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A new class of reliability point-process models for dependent components in introduced. The dependence is expressed through a regression, following a form suggested by Cox (1972) for survival data analysis involving the current life-length of the components. After formulating the current-life process as a Markov process with stationary transitions and stating some general results on asymptotic behavior, we describe the stationary distributions in come bivariate examples. Finally, we discuss statistical inference for the new models, exhibiting and justifying full and partial-likelihood methods for their analysis.

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Multivariate Dependent Renewal Processes

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Abstract. A new class of reliability point-process models for dependent components is introduced. The dependence is expressed through a regression, following a form suggested by Cox (1972) for survival data analysis involving the current life-length of the components. After formulating the current-life process as a Markov process with stationary transitions and stating some general results on asymptotic behavior, we describe the stationary distributions in some bivariate examples. Finally, we discuss statistical inference for the new models, exhibiting and justifying full- and partial-likelihood methods for their analysis.

Key words: multivariate renewal process, dependent components, Markov process, reliability theory, partial likelihood.

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Multivariate Dependent Renewal Processes

1. Introduction. The standard place of renewal point-processes in Reliability theory is in the analysis of repair-vs-replace decisions and of the costs associated with various maintenance policies (cf. Barlow and Proschan, 1975, Chapter 6). However, even when attention centers on a single device, a realistic model of reliability would require description of dependent components which affect each other's failure rates but fail and are replaced separately. The simplest case of such a model would be a system of independent components in which failure of one component places an extra load or shock on the others, with the effect of permanently multiplying their hazard rates (by a fixed function of time). Freund (1961) introduced this model for two component lifetimes T_1, T_2, defining conditional hazard functions (cf. the definitions in Cox and Lewis, 1967)

$$\lim_{\Delta \to 0} \Delta^{-1} P\{T_i \le t+\Delta \mid \min(T_1, T_2) \ge t\} = \begin{cases} \alpha & \text{for } i=1 \\ \beta & \text{for } i=2 \end{cases}$$

$$\lim_{\Delta \to 0} \Delta^{-1} P\{T_i \le t+\Delta \mid T_j = s < t \le T_i\} = \begin{cases} \alpha' & \text{for } i=1,\ j=2 \\ \beta' & \text{for } i=2,\ j=1 \end{cases}$$

In the context of dependence between death and censoring times in survival data analysis, Slud and Rubinstein (1983) generalized the case $\beta = \beta'$ of the Freund model to allow α, β, and α' to be arbitrary hazard functions depending on t. In the present paper, we define a general model of dependence between a finite number k of component lifetimes, including the Freund and generalized Freund models as special cases. The form of dependence - involving the current life-lengths of the components in a regression model like Cox's (1972) for the conditional hazards - allows the instantaneous replacement of failed components by independent new ones, leading to a class of dependent failure point-processes generalizing the renewal point-processes.

In Section 2 we define our model, formulate the vector process of current component lifetimes as a Markov process and study the behavior of stationary distributions. Some explicit calculations for asymptotic joint distributions of component current-lives are given in Section 3 for a special model with one-way dependence. Also for two generalizations of Freund's (1961) shock model, we discuss in Section 3 the qualitative behavior of stationary distributions. Finally, statistical inference procedures for the regression coefficients in our model are derived both from full and partial likelihood score statistics, and we compute asymptotic relative efficiency of the two approaches in a bivariate example.

2. Model assumptions. Analysis of Markov process. We suppose the right-continuous counting processes $N_j(t)$, $j = 1,\ldots,k$ are defined on $[0,\infty)$ and have unit jumps with probability 1. It is natural to interpret $N_j(t)$ as the number of failures (with instantaneous replacement by an independent component) up to

time t of the j^{th} component of an assembly. Our main model assumption, given in terms of conditional hazard functions as defined by Cox and Lewis (1972), is

$$(*) \quad \lim_{h\to 0+} h^{-1} P(N_j(s+h)-N_j(s) = 1 \mid \{N_i(u):\ 0 \le u \le s,\ 1 \le i \le k\})$$

$$= \lambda_j(\tau_j(s)) \cdot \exp\Big(\sum_{i=1}^{k} \beta_{ij}\varphi_{ij}(\underline{\tau}(s))\Big) \equiv g_j(\underline{\tau}(s))$$

where for $s > 0$

$$\tau_j(s) \equiv s - \min\{u \le s:\ N_j(u) = N_j(s)\}$$

$$\underline{\tau}(s) = (\tau_1(s), \ldots, \tau_k(s)),$$

the $\lambda_j(\cdot)$ are hazard functions and lim means limit in the mean. The random variable $\tau_j(s)$ is the current life at time s of component j, and has been a principal focus of Reliability theory. The model assumption (*), like the regression models of Cox (1972) which inspired it, can be given a more general form - the log-linear form of regression is especially tractable for statistical inference, but by no means necessary for the Markov process formulations of the present Section. The important feature of (*) is that the dependence between components involves only their current ages and no past history. (For a model where past cumulative lifetimes of components, from the time the assembly was placed in service, affect current hazards, see Slud and Winnicki (1982).) We remark at once that in the simple case of (*) where all φ_{ij} are constants, it is clear that the $N_j(\cdot)$ are k independent (delayed unless $\tau_j(0) = 0$) renewal counting-processes.

In the present Section, we adopt the simpler notations

$$g_j(\underline{x}) = \lambda_j(x_j)\exp(\sum_{i=1}^{k} \beta_{ij}\varphi_{ij}(\underline{x})), \qquad g(\underline{x}) = \sum_{j=1}^{k} g_j(\underline{x}).$$

The regularity conditions on these functions needed for further work are contained in the following Theorem, which summarizes for general $g_j(\cdot)$ the information available from standard Markov process theory. Some additional notations required for the Theorem are:

$$T_i(s) = \inf\{t > s: N(t) = N(s) + i\} \qquad \text{for } i \geq 1$$

$$\Delta_i(s) = j \quad \text{for which} \quad N_j(T_i(s)) - N_j(T_i(s)-) = 1$$

$$N(\cdot) = \sum_{j=1}^{k} N_j(\cdot), \quad \underline{N}(\cdot) = (N_1(\cdot),\ldots,N_k(\cdot)), \quad \underline{1} = (1,\ldots,1) \in \mathbb{R}^k$$

$$\underline{e}_j = (0,\ldots,0,1,0,\ldots,0) = \text{the } j^{\text{th}} \text{ canonical basis vector in } \mathbb{R}^k$$

$$\underline{x}_j' = \underline{x} - x_j\,\underline{e}_j \qquad \text{for } \underline{x} \in \mathbb{R}^k$$

Theorem 2.1. Assume that $\{N_i(\cdot)\}_{i=1}^{k}$ are right-continuous counting processes on $[0,\infty)$ without fixed discontinuities, defined on some probability space (Ω,F,P) and satisfying (*) with functions $g_j(\cdot)$ which are Borel measurable, non-negative, and such that for all $\underline{x}$

$$\text{(A)} \qquad \int_0^\varepsilon g_j(\underline{x} + u\underline{1})du \begin{cases} \to \infty & \text{as } \varepsilon \to \infty \\ < \infty & \text{for sufficiently small } \varepsilon = \varepsilon(\underline{x}) > 0 \end{cases}$$

for $j = 1,\dots,k$, where $\varepsilon(\underline{x})$ is such that there exist $\delta, \varepsilon_0 > 0$ with $\varepsilon(\underline{x}) \ge \varepsilon_0$ whenever $\min\{x_j : 1 \le j \le k\} \le \delta$.

(i) Then $\{\underline{\tau}(s): s \ge 0\}$ is a stationary-transition Markov process (with piecewise-constant right-continuous paths) essentially determined by the joint conditional density given $F_s \equiv \sigma(\{\underline{N}(u): 0 \le u \le s\})$ of $T_1(s)$ and $\Delta_1(s)$:

$$(2.1) \qquad f(u, j \mid F_s) = g_j(\underline{\tau}(s) + u\underline{1})\exp[-\int_0^u g(\underline{\tau}(s) + v\underline{1})dv].$$

Moreover, all stationary distributions for the process $\{\underline{\tau}(s): s \ge 0\}$ have densities $\pi(\cdot)$ which satisfy for Lebesgue almost all $\underline{x}$

$$(2.2) \quad \sum_{j=1}^{k} \int_{[y_i<x_i, i\ne j]} \pi(\underline{y}_j' + x_j\underline{e}_j)d\underline{y}_j' = -\int_{[y_i<x_i, 1\le i\le k]} \pi(\underline{y})g(\underline{y})d\underline{y} + \sum_{j=1}^{k} \int_{[y_i<x_i, i\ne j]} \pi(\underline{y})g_j(\underline{y})d\underline{y}.$$

(ii) If for each j and almost all $\underline{x}$, $g_j(\underline{x}) \ge h_j(x_j) \ge 0$ for measurable $h_j(\cdot)$ with $\int_0^T g(\underline{x} + u\underline{1})du < \infty$ for all $T < \infty$,

$\mu_j = \int_0^\infty \exp(-H_j(u))du < \infty$, where $H_j(u) = \int_0^u h_j(z)dz$, $H_j(+\infty) = \infty$, and

$$g(\underline{x} + u\underline{1}) - g(\underline{x}_j' + u\underline{1}) \ge h_j(x_j + u) - h_j(u) \quad \text{for } \underline{x} \in [0,\infty)^k, \; u \ge 0,$$

then the laws of $\underline{\tau}(s)$ converge exponentially fast in total variation to a probability law with density $\pi(\cdot)$ uniquely characterized by (2.2).

Remark. The assumptions (A) turn out to guarantee that the random variables $T_i(s)$ for all $i \geq 1$, $s > 0$, and arbitrary $\underline{\tau}(0)$, are almost surely strictly positive and finite.

Proof of Theorem. For each $M \geq 0$, $s < u_1 < u_2 < \dots < u_M \leq t$, and $\underline{j}_M = (j_1, \dots, j_M) \in \{1, \dots, k\}^M$, we define $\underline{u}_M = (u_1, \dots, u_M)$ and

$$(2.3) \quad c_M(t, \underline{u}_M, \underline{j}_M \mid F_s) \equiv \lim_{\delta_1, \dots, \delta_M \to 0+} (2^M \delta_1 \cdots \delta_M)^{-1} \cdot P\{\underline{N}(s) = \underline{N}(u_1 - \delta_1), \underline{N}(u_i + \delta_i) - \underline{N}(u_i - \delta_i) = \underline{e}_{j_i},\ 1 \leq i \leq M,\ \underline{N}(t) = \underline{N}(u_M + \delta_M) \mid F_s\}$$

where existence in the mean of the limits will be established inductively in what follows. Here $\underline{N}(\cdot)$ is almost surely right continuous without fixed discontinuities on $[s, \infty)$ given F_s, so that if the limit in (2.3) exists then the random function $c_M(\cdot, \underline{u}_M, \underline{j}_M) \mid F_s)$ is a.s. continuous on $[u_M, \infty)$. In particular, $c_0(t \mid F_s) = P\{\underline{N}(t) = \underline{N}(s) \mid F_s\}$ exists and is a.s. continuous in t for $t \geq s$. Moreover, if we denote differentiation from the right in t by D_+, then for $M \geq 0$ and $t > s$, existence of (2.3) together with (*) implies

$$(2.4) \quad D_+ c_M(t, \underline{u}_M, \underline{j}_M \mid F_s) = -g(\underline{\tau}(t)) c_M(t, \underline{u}_M, \underline{j}_M \mid F_s)$$

where $\underline{\tau}(t)$ is uniquely determined from the conditions

$$T_i(s) = u_i, \quad \Delta_i(s) = j_i \quad \text{for } 1 \leq i \leq M; \quad T_{M+1}(s) > t$$

by the formulas

$$(2.5)\quad \underline{\tau}(T_i(s)) = \underline{\tau}(T_{i-1}(s)) + (T_i(s) - T_{i-1}(s))\underline{1}$$

$$- (\tau_{j_i}(T_{i-1}(s)) + T_i(s) - T_{i-1}(s))\underline{e}_{j_i}, \qquad 2 \le i \le M$$

$$\underline{\tau}(t) = \underline{\tau}(T_M(s)) + (t - T_M(s))\underline{1}.$$

From (2.4) it follows (Yosida, 1980, p. 239) that a.s. for $t > s$

$$(2.6)\quad \begin{aligned} c_0(t|F_s) &= \exp[-\int_0^{t-s} g(\underline{\tau}(s) + v\underline{1})dv] \\ c_M(t,\underline{u}_M,\underline{j}_M|F_s) &= \exp[-\int_0^{t-u_M} g(\underline{\tau}(u_M) + v\underline{1})dv]c_M(u_M,\underline{u}_M,\underline{j}_M|F_s) \end{aligned}$$

while a direct application of (*) to (2.4) implies, assuming the existence of (2.3) for given M,

$$(2.7)\quad c_{M+1}(t,(\underline{u}_M,t),(\underline{j}_M,\ell)|F_s) = g_\ell(\underline{\tau}(t))c_M(t,\underline{u}_M,\underline{j}_M|F_s).$$

Now (2.6) and (2.7) show, by a simple induction on M, that all the limits (2.3) exist and a.s. for $M \ge 0$ (taking $u_0 = s$)

$$(2.8)\quad c_M(t,\underline{u}_M,\underline{j}_M|F_s) = \exp[-\int_s^t g(\underline{\tau}(v))dv]\cdot \prod_{i=1}^M g_{j_i}(\underline{\tau}(u_i)) =$$

$$\exp[-\int_0^{t-u_M} g(\underline{\tau}(u_M)+v\underline{1})dv]\cdot \prod_{i=1}^M \{\exp[-\int_0^{u_i-u_{i-1}} g(\underline{\tau}(u_{i-1})+v\underline{1})dv]g_{j_i}(\underline{\tau}(u_i))\}.$$

The factored form of the functions c_M together with the formulas (2.5) determining $\underline{\tau}(\cdot)$ on $[s,t]$ readily implies that $\{\underline{\tau}(s), s \ge 0\}$ is a stationary-transition (Strong-) Markov process on $[0,\infty)^k$.

Now we need to make some technical comments regarding the transition functions

$$P(t,\underline{x},A) = P\{\underline{\tau}(t) \in A \mid \underline{\tau}(0) = \underline{x}\}$$

$$P_{\underline{n}}(t,\underline{x},A) = P\{\underline{\tau}(t) \in A\,,\ \underline{N}(t) = \underline{n} \mid \underline{\tau}(0) = \underline{x}\}$$

where $A \in \mathcal{B}([0,\infty)^k)$ is a Borel set and $\underline{n} = (n_1,n_2,\dots,n_k)$ is a vector of nonnegative integers. First, since the existence of limits in (2.3) implies with $s = 0$, $\underline{\tau}(0) =$ [illegible] $M = \sum_{j=1}^{k} n_j$,

$$\text{(2.9)} \quad P_{\underline{n}}(t,\underline{x},A) = \sum_{\underline{j}_M : n_j = \#\{i : 1 \le i \le M,\ j_i = j\}} \int_{[u_1 < \cdots < u_M < t,\ [illegible] \in A]} c_M(t,\underline{u}_{[illegible]} [illegible]) d\underline{u}_M$$

(where # denotes cardinality of a set), it is easy to check by (2.8) that for each $\underline{n}$, $\underline{x}$, T

$$\sum_{\underline{n}: n_j \ge 0,\, 1 \le j \le k} \int_0^T \int P_{\underline{n}}(s,\underline{x},d\underline{y})g(\underline{y})ds = P\{N(T) \ge 1 \mid \underline{\tau}(0) = \underline{x}\}\,;$$

hence

$$\sum_{\underline{n}: n_j \ge 0,\ 1 \le j \le k} \int P_{\underline{n}}(t,\underline{x},d\underline{y})g(\underline{y}) < \infty \quad \text{for a.e.} \quad t \in [0,\infty).$$

Secondly, from formula (2.9) it is clear that if $n_j \ge 1$ for $j = 1,\dots,k$, then $P_{\underline{n}}(t,\underline{x},\cdot)$ is absolutely continuous with respect to Lebesgue measure on $[0,\infty)^k$. Next, if $n_j \ge 2$ for $j = 1,\dots,k$, (2.9) implies

$$\text{(2.10)} \quad \frac{\partial}{\partial t} P_{\underline{n}}(t,\underline{x},[0,z_1)\times\cdots\times[0,z_k)) = -\sum_{j=1}^{k} \frac{\partial}{\partial z_j} P_{\underline{n}}(t,\underline{x},[0,z_1)\times\cdots\times[0,z_k))$$
$$- \int_{[0,z_1)\times\cdots\times[0,z_k)} P_{\underline{n}}(t,\underline{x},d\underline{y})g(\underline{y}) + \sum_{j=1}^{k} \int_{[y_i < z_i,\, i \ne j]} P_{\underline{n}-\underline{e}_j}(t,\underline{x},d\underline{y})g_j(\underline{y})$$

and the same formula holds with $P_{\underline{n}}$ replaced by $\sum_{\underline{n}:n_j \geq 2} P_{\underline{n}}$.
Since

$$\int_0^\infty g(\underline{x} + u\underline{1})du = \infty \quad \text{for all } \underline{x} \text{ implies}$$

$$\sum_{\underline{n}:\, n_j<2 \text{ for some } j} P_{\underline{n}}(t,\underline{x},[0,\infty)^k) \to 0 \quad \text{as} \quad t \to \infty$$

the last statement of (i) follows easily.

Finally, in part (ii) of the Theorem, if we define random functions $\bar{c}_M$ as in (2.8) with all $g_j(\underline{x})$ replaced by $h_j(x_j)$, and define $Q_{\underline{n}}$ from $\bar{c}_M$ just as $P_{\underline{n}}$ was defined from c_M in (2.9), it follows inductively from assumptions in (ii), for fixed M, $0 < u_1 < u_2 < \ldots < u_M < t$, and $\underline{j}_M$, if $\underline{\tau}(\cdot)$ is given on $[0,t]$ by (2.5) with $s = 0$, that

$$\int_0^t (g(\underline{x} + v\underline{1}) - g(\underline{\tau}(v)))dv \geq \int_0^t (h(\underline{x} + v\underline{1}) - h(\underline{\tau}(v)))dv$$

where $h(\underline{x}) \equiv \sum_{j=1}^k h_j(x_j)$. Therefore (putting $s \equiv 0$ and writing $c_M(\cdot)$ for $c_M(\cdot\,|\,F_0)$)

$$\exp\left[\int_0^t g(\underline{x} + v\underline{1})dv\right]c_M(t,\underline{u}_M,\underline{j}_M) \geq \exp\left[\int_0^t h(\underline{x} + v\underline{1})dv\right]\bar{c}_M(t,\underline{u}_M,\underline{j}_M)$$

and by (2.9) we have

$$(2.11)\quad P_{\underline{n}}(t,\underline{x},A)\exp\left[\int_0^t g(\underline{x} + u\underline{1})du\right] \geq Q_{\underline{n}}(t,\underline{x},A)\exp\left[\sum_{j=1}^k (H_j(t + x_j) - H_j(x_j))\right]$$

However, $Q_{\underline{n}}(t,\underline{x},A)$ are precisely the transition functions $P\{\underline{\tau}_H(t) \in A,\ \underline{N}_H(t) = \underline{n} \mid \underline{\tau}(0) = \underline{x}\}$ for a process $(\underline{\tau}_H(s),\underline{N}_H(s):\ s \geq 0)$

satisfying (*) with $g_j(\underline{x})$ replaced by $h_j(x_j)$, that is, for the current-life and counting processes associated with k independent renewal processes with renewal hazards $h_j(\cdot)$. The assumption $\mu_j < \infty$ implies these renewal current-life processes are positive-recurrent with (Karlin and Taylor, 1975, p. 193)

$$(2.12)\quad Q(t,\underline{x},A) \equiv \sum_{\underline{n}\geq\underline{0}} Q_{\underline{n}}(t,\underline{x},A) \to \frac{\int_A \exp[-\sum_{j=1}^{k} H_j(y_j)]dy}{\mu_1\cdot\mu_2\cdots\mu_k} \equiv Q(A)$$

as $t\to\infty$, uniformly in $\underline{x}\in[0,\infty)^k$, $A\in B([0,\infty)^k)$. But

$$(2.13)\quad P(t,\underline{x},A^c) \geq Q(t,\underline{x},A^c)\exp[\sum_{j=1}^{k}(H_j(t+x_j)-H_j(x_j))-\int_0^t g(\underline{x}+u\underline{1})du]$$

implies, since $P(t,\underline{x},\cdot)$ and $Q(t,\underline{x},\cdot)$ are probability measures,

$$(2.14)\quad P(t,\underline{x},A) \leq$$
$$1-\exp[\sum_{j=1}^{k}(H_j(t+x_j)-H_j(x_j))-\int_0^t g(\underline{x}+u\underline{1})du]\cdot(1-Q(t,\underline{x},A)).$$

Fix $1>\delta>0$. By comparison with the independent renewal processes with hazards h_j, one can find t_0 so large that

$$(2.15)\quad P\{N_j(t_0)\geq 1 \text{ for } j=1,\ldots,k \mid \underline{\tau}(0)=\underline{x}\} \geq 1-\delta$$

uniformly in $\underline{x}\in[0,\infty)^k$. Then (2.12) says that for t_1 large enough, for all $\underline{x}$ and A,

$$(2.16)\quad |Q(t_1-t_0,\underline{x},A)-Q(A)| \leq \delta.$$

On the event $[N_j(t_0) \geq 1$ for $j = 1,\ldots,k]$, by local integrability of g, there exis+= $\delta_1 > 0$ such that, for the fixed t_1 in (2.16)

$$(2.17) \qquad \inf_{\underline{\tau}(0)=\underline{x}\in[0,\infty)} \exp\Big[\sum_{j=1}^{k} \big(H_j(t_1-t_0+\tau_j(t_0)) - H_j(\tau_j(t_0))\big) - \int_0^{t_1-t_0} g(\underline{\tau}(t_0) + u\underline{1})du\Big] \geq \delta_1 .$$

Therefore by (2.14) - (2.17), for all $\underline{x} \in [0,\infty)^k$ and all A

$$(2.18) \qquad \begin{aligned} P(t_1,\underline{x},A) &= \int P(t_0,\underline{x},d\underline{y})P(t_1-t_0,\underline{y},A) \\ &\leq \delta + (1-\delta)[1 - \delta_1 + \delta_1(\delta + Q(A))] \\ &= 1 - \delta_1(1-\delta)[1 - \delta - Q(A)] \end{aligned}$$

so that, with $\varepsilon \equiv \min((1-\delta)/2 , \delta_1(1-\delta)^2/2))$

$$(2.19) \qquad \text{if} \quad Q(A) \leq \varepsilon, \quad \text{then} \quad P(t_1,\underline{x},A) \leq 1 - \varepsilon .$$

This is Doeblin's condition (D) (Doob, 1953, p. 256) and by a Theorem proved in Doob (1953) our Theorem is established.

By the method of characteristics (Courant, Hilbert, 1962, pp. 170 ff.) it follows immediately from (2.2) in Theorem 2.1 (i) that for $\underline{x} \in [0,\infty)^k$

$$(2.20) \qquad \left\{ \begin{aligned} \pi(\underline{x}) &= \pi(\underline{x} - x_j\underline{1})\cdot\exp\Big[-\int_0^{x_j} g(\underline{x} + (u-x_j)\underline{1})du\Big] \\ \pi(\underline{x} - x_j\underline{1}) &= \int_0^{\infty} \pi(\underline{x}-x_j\underline{1}+y\underline{e}_j)g_j(\underline{x}-x_j\underline{1}+y\underline{e}_j)dy \end{aligned} \right. \qquad \text{when } x_j = \min_i x_i .$$

The system (2.20) of integral equations is already a non-trivial generalization of the renewal equations which determine the current-life asymptotic density $\pi(\cdot)$ in case all $\beta_{ij} = 0$ (see Karlin and Taylor, 1975, pp. 192-3). Theorem 2.1 contains useful information on existence and uniqueness of solutions of systems of the general form (2.20).

To apply these asymptotic results in reliability theory, one would want also to know the asymptotic distributions of the times between successive jumps for each of the counting processes $N_j(\cdot)$ and for the superposition counting-process $\sum_{j=1}^{k} N_j(\cdot) \equiv N(\cdot)$. These are easy to derive by ergodicity of the process $\{\underline{\tau}(s)\}$ with initial density $\pi(\cdot)$, using a well-known generalization (e.g., see Rolski, 1981, Ch. 1) of a Strong Law technique given by Karlin and Taylor (1975). We state only the result for densities of single times between replacements, although corresponding asymptotic joint densities of successive times between replacements can also be readily expressed in terms of $\pi(\cdot)$.

<u>Proposition 2.2</u>. Under the hypotheses of Theorem 2.1 ensuring the validity of (2.2), if (2.2) has a unique solution and we define

$$W_j^{(m)} \equiv \min\{t \geq 0: N_j(t) = m\} - \min\{t \geq 0: N_j(t) = m-1\}$$

$$W^{(m)} \equiv \min\{t \geq 0: N(t) = m\} - \min\{t \geq 0: N(t) = m-1\}$$

then as $m \to \infty$ the asymptotic densities $f(\cdot)$ of $W^{(m)}$ and $f_j(\cdot)$ of $W_j^{(m)}$ on $[0,\infty)$ are

$$
(2.21)\quad
\begin{aligned}
f_j(x) &= -\frac{d}{dx}\int_{[0,\infty)^{k-1}} \pi(\underline{y}_j'+x\underline{e}_j)d\underline{y}_j' \Big/ \int_{[0,\infty)^{k-1}} \pi(\underline{y}_j')d\underline{y}_j' \\
f(x) &= -\frac{d}{dx}\int_{[x,\infty)^{k-1}} \sum_{j=1}^{k} \pi(\underline{y}_j'+x\underline{e}_j)d\underline{y}_j' \Big/ \int_{[0,\infty)^{k-1}} \sum_{j=1}^{k} \pi(\underline{y}_j')d\underline{y}_j'
\end{aligned}
$$

Corollary 2.3. The asymptotic marginal densities (under (ii) of Theorem 2.1) as $s \to \infty$

$$\int_{[0,\infty)^{k-1}} \pi(\underline{y}_j'+x\underline{e}_j)d\underline{y}_j' \quad \text{for} \quad \tau_j(s)$$

and

$$\sum_{j=1}^{k}\int_{[0,\infty)^{k-1}} \pi(\underline{y}_j'+x\underline{e}_j)d\underline{y}_j' \quad \text{for} \quad \min\{\tau_1(s),\dots,\tau_k(s)\}$$

are non-increasing functions of x.

3. Examples. Consider the model of the previous Section in the case $k = 2$, $\varphi_{21}(\cdot) \equiv 0$, $\beta_{11} = \beta_{22} = 0$, $\beta_{12} = 1$, $\lambda_2(\cdot) \equiv \lambda_2$ constant, and $\varphi_{12}(x,y) \equiv \ln h(x)$. Then $g_1(x,y) = \lambda_1(x)$, $g_2(x,y) = h(x)\lambda_2$, and we define $H(x) = \int_0^x h(z)dz$, $\Lambda_1(x) = \int_0^x \lambda_1(z)dz$. Since $g_1(\cdot)$ depends on x alone, the counting process $N_1(\cdot)$ will marginally be an ordinary (delayed) renewal counting process with interoccurrence distribution function $F_1(x) = 1 - \exp(-\int_0^x \lambda_1(z)dz)$, while $N_2(\cdot)$ will be a "doubly stochastic Poisson" process. We will calculate for this example an explicit form for the asymptotic joint density $\pi(\cdot)$ of current lives $\tau_1(s),\tau_2(s)$.

First, we observe that the renewal counting process $N_1(\cdot)$ has corresponding stationary distribution if and only if

$$\mu_1 = \int_0^\infty \exp(-\Lambda_1(t))dt < \infty. \tag{3.1}$$

Then the condition $P\{N(t) < \infty\} = 1$ for all t is equivalent to

$$P\{N_1(t) = 0 , N_2(t) < \infty\} = \exp(-\Lambda_1(t)) \quad \text{for all} \quad t \geq 0$$

which says that with probability 1 there are finitely many jumps in $N_2(\cdot)$ between successive zeroes of $\tau_1(\cdot)$. But between zeroes of $\tau_1(\cdot)$, $N_2(\cdot)$ is the counting process for a nonhomogeneous Poisson process with rate $\lambda_2 h(\cdot)$. Therefore

$$\begin{aligned} &P\{N(t) < \infty\} = 1 \quad \text{for all } t \text{ if and only if} \\ &H(t) < \infty \quad \text{for all finite } t \text{ with } \Lambda_1(t) < \infty. \end{aligned} \tag{3.2}$$

Under assumptions (3.1) and (3.2), it is easy to check that the hypothesis of Theorem 2.1 (ii) is equivalent to $\alpha \equiv \text{ess.inf.}h(\cdot) > 0$, together with $\Lambda_1(\infty) = \infty$, $\Lambda_1(x) + H(x) < \infty$ when $x < \infty$. Rather than impose these last hypotheses, we calculate directly the unique solution $\pi(x,y)$ of (2.20). Theorem 2.1 (i) then implies, by standard arguments, that $\{\underline{\tau}(s): s \geq 0\}$ is ergodic and for all initial distributions, as $s \to \infty$ the distributions of $\underline{\tau}(s)$ converge weakly to the measure with density π on $[0,\infty)^2$.

Equations (2.20) can be rewritten in our example to give

$$(3.3)\quad \pi(x,y) = \begin{cases} \pi(x-y\,,0)\exp[-\Lambda_1(x)+\Lambda_1(x-y)-\lambda_2(H(x)-H(x-y))] & \text{if } x>y \\ \pi(0,y-x)\exp[-\Lambda_1(x)-\lambda_2 H(x)] & \text{if } x<y \end{cases}$$

$$\pi(x,0) = \int_0^\infty \lambda_2 h(x)\pi(x,y)dy, \qquad \pi(0,y) = \int_0^\infty \lambda_1(x)\pi(x,y)dx.$$

Since $N_1(\cdot)$ is a renewal counting process with interoccurrence time distribution function $1-\exp(-\Lambda_1(x))$, standard renewal theory implies

$$\int_0^\infty \pi(x,y)dy = \mu_1^{-1}\exp(-\Lambda_1(x))$$

and by (3.3),

$$(3.4)\quad \begin{aligned} \pi(x,0) &= \mu_1^{-1}\lambda_2 h(x)\exp(-\Lambda_1(x)) \\ \pi(0,y) &= \int_0^y \lambda_1(x)\pi(0,y-x)\exp[-\Lambda_1(x)-\lambda_2 H(x)]dx \\ &\quad + \int_y^\infty \lambda_2\mu_1^{-1}\lambda_1(x)h(x-y)\exp[-\Lambda_1(x)-\lambda_2(H(x)-H(x-y))]dx. \end{aligned}$$

From the second of these equations follows

$$(3.5)\quad q(t) \equiv \int_0^\infty \exp(-ty)\pi(0,y)dy$$

$$= \int_0^\infty \frac{\lambda_1(x)}{\mu_1} e^{-\Lambda_1(x)-\lambda_2 H(x)-tx}\left[\int_0^x \lambda_2 h(z)e^{\lambda_2 H(z)+tz}dz\right]dx \Big/$$

$$\left[1-\int_0^\infty \lambda_1(x)e^{-tx-\Lambda_1(x)-\lambda_2 H(x)}dx\right].$$

Then the known Laplace transform $q(t)$ determines $\pi(0,y)$, and (3.3), (3.4) give the exact asymptotic joint density $\pi(x,y)$. In particular, the form of (3.3) immediately implies that under the model of this Section as $s \to \infty$, for any initial distribution $\pi_0(x,y)$,

> $\tau_1(s)$ and $\tau_2(s) - \tau_1(s)$ are asymptotically conditionally independent given $\operatorname{sgn}(\tau_2(s) - \tau_1(s))$.

There are two generalizations of Freund's (1961) bivariate shock model which may be of interest in Reliability. Let $k = 2$, $\beta_{11} = \beta_{22} = 0$, and $\beta_{12} = \beta_{21} = 1$ in (*). Then the two general models are given by

$$(F_1) \qquad \begin{cases} \varphi_{12}(x,y) = I_{[x<y]} \ln h_1(y) \\ \varphi_{21}(x,y) = I_{[y<x]} \ln h_2(x) \end{cases}$$

and

$$(F_2) \qquad \begin{cases} \varphi_{12}(x,y) = I_{[x<y]} \ln h_1(y-x) \\ \varphi_{21}(x,y) = I_{[y<x]} \ln h_2(x-y) \end{cases}$$

where $I_{[\ ]}$ denotes indicator function. In these models, component 1 acquires the multiplicative factor $h_2(\cdot)$ in its hazard the instant that component 2 fails; however, further failures of the replacements of component 2 during the life of the same component 1 do not further change the hazard. Model (F_1) says essentially that failure of one component instantaneously shifts the other component to a new survival curve, while (F_2) says that the proportionality factor for hazard due to shock can vary with time only as measured from the instant of shock. The bivariate model of Slud and Rubinstein (1983) is a special case of (F_1).

Under model (F_1), (2.20) easily implies the asymptotic density for $\underline{\tau}$ has the form

$$(3.6)\quad \pi(x,y) = \begin{cases} \pi(0,y-x)\exp[-\Lambda_1(x) - \int_0^x h_1(y-x+u)\lambda_2(y-x+u)du] & \text{if } x < y \\ \pi(x-y,0)\exp[-\int_0^y h_2(x-y+u)\lambda_1(x-y+u)du - \Lambda_2(y)] & \text{if } x > y \end{cases}$$

so that under (F_1)

(3.7) $\min(\tau_1(s),\tau_2(s))$ and $\max(\tau_1(s),\tau_2(s)) - \min(\tau_1(s),\tau_2(s))$ are asymptotically conditionally independent given $\operatorname{sgn}(\tau_1(s) - \tau_2(s))$ as $s \to \infty$.

Although no such appealing property can hold for $\pi(\cdot)$ under (F_2), except in Freund's case with λ_i and h_i constant, it seems to be physically the more reasonable of the two models and should, together with the statistical procedures of the following Section, find applications in Reliability. It is interesting also to remark that if in (F_2) one has $h_i(\cdot)$ sharply peaked at 0 with values rapidly decaying to 1, then model (F_2) mimics the behavior of the well-known model of Marshall and Olkin for dependent failure times (see Barlow and Proschan, 1981, Chapter 5).

4. Statistical inference for dependent renewal processes.

Until now, we have made no use of the special regression form for $g_j(\cdot)$ in (*). We recall

$$g_j(\underline{x}) = \lambda_j(x_j)\exp\left(\sum_{i=1}^{k} \beta_{ij}\varphi_{ij}(\underline{x})\right)$$

which for convenience (and identifiability of parameters β_{ij}) we rewrite

$$g_j(\underline{x}) = \lambda_j(\underline{x})\exp\left(\sum_{i\neq j} \beta_{ij}\varphi_{ij}(\underline{x})\right).$$

Now suppose that data on the process $\{\underline{\tau}(s), 0 \le s \le t\}$ is available in the following form:

$$\underline{\tau}(0) \quad \text{and} \quad \{T_\ell, \Delta_\ell : \ell = 1,\ldots,N(t)\} \quad \text{are observed}$$

where

$$T_\ell = \inf\{s > 0 : N(s) = \ell\}, \qquad \Delta_\ell = j \quad \text{iff} \quad \tau_j(T_\ell) = 0.$$

The full likelihood $L = L(\underline{\tau}(0), (T_\ell, \Delta_\ell); (\beta_{ij}))$ conditional on $\underline{\tau}(0)$ for this data is

$$L = \prod_{\ell=1}^{N(t)} \prod_{j=1}^{k} [g_j(\underline{\tau}(T_\ell -))]^{I[\Delta_\ell = j]} \cdot \exp\left(-\int_0^t g(\underline{\tau}(u))du\right)$$

which makes sense because for all $\ell > 0$ (with $T_0 \equiv 0$)

$$\underline{\tau}(T_{\ell+1}^-) \equiv \underline{\tau}(T_\ell) + (T_{\ell+1} - T_\ell)\underline{1}.$$

The score statistic S_{ij} for β_{ij} is given by

$$(4.1)\quad \frac{\partial}{\partial\beta_{ij}} \log L = \sum_{\ell=1}^{N(t)} I_{[\Delta_\ell = j]}\varphi_{ij}(\underline{\tau}(T_\ell -)) - \int_0^t \varphi_{ij}(\underline{\tau}(u))g_j(\underline{\tau}(u))du$$

from which the identifiability of β_{ij} for each j and $i \neq j$ is seen to be assured if the known measurable functions φ_{ij} for $i \in \{1,\ldots,k\}\setminus\{j\}$ are linearly independent on a subset with positive Lebesgue measure of $\{\underline{x}: \lambda_j(\underline{x}) > 0\}$. The log-concavity of L follows from

$$(4.2)\quad \frac{\partial^2}{\partial\beta_{ij}\partial\beta_{i'j'}} \log L = -I_{[j'=j]} \int_0^t \varphi_{ij}(\underline{\tau}(u))\varphi_{i'j}(\underline{\tau}(u))g_j(\underline{\tau}(u))du.$$

If we assume that (2.2) has a unique solution π, so that $\{\underline{\tau}(s),\ s \geq 0\}$ is an ergodic Markov process by Theorem 2.1(i), then for $(\beta_{ij}: i,j = 1,\ldots,k,\ i \neq j)$ in an open region $D \subset \mathbb{R}^{k(k-1)}$, as $t \to \infty$

$$(4.3)\quad -\frac{\partial^2}{\partial\beta_{ij}\partial\beta_{i'j'}} \log L \sim tI_{[j'=j]} \int_{[0,\infty)^k} \varphi_{ij}(\underline{x})\varphi_{i'j}(\underline{x})g_j(\underline{x})\pi(\underline{x})d\underline{x}\,.$$

Moreover, for large t the maximum-likelihood estimates $(\hat{\beta}_{ij})$ for which $S_{ij} = 0$ are unique, strongly consistent, and asymptotically jointly normal with asymptotic variance-covariance matrix V_β^F the inverse of the matrix tI_β^F with ij,i'j' component equal to the right-hand side of (4.3) (see for example Basawa and Prakasa Rao, 1980).

Another framework for the consistent estimation of (β_{ij}) is given by the recent generalization by Andersen and Gill (1982) of the partial likelihood methods of inference due to Cox (1972, 1975). Following Cox and Andersen and Gill, we define the partial likelihood for (β_{ij}) based on the data $\{T_\ell, \Delta_\ell, \ \ell = 1,\ldots,N(t)\}$

$$(4.4) \quad L^{(P)} = \prod_{\ell=1}^{N(t)} \left(\prod_{j=1}^{k} g_j(\underline{\tau}(T_\ell-))^{I[\Delta_\ell=j]} \right) \Big/ g(\underline{\tau}(T_\ell-))$$

leading to the "partial-likelihood score statistic for β_{ij}"

$$(4.5) \quad \frac{\partial}{\partial\beta_{ij}} \log L^{(P)} = \sum_{\ell=1}^{N(t)} \left\{ I_{[\Delta_\ell=j]}\varphi_{ij}(\underline{\tau}(T_\ell-)) - \varphi_{ij}(\underline{\tau}(T_\ell-)) \frac{g_j(\underline{\tau}(T_\ell-))}{g(\underline{\tau}(T_\ell-))} \right\}$$

all linear combinations of which are martingales in the continuous parameter t. As in the maximum-likelihood theory, an argument exactly analogous to that in Andersen and Gill (1982) shows that for large t the maximum partial likelihood estimates $(\hat{\beta}_{ij,P})$ where (4.5) equals 0 are unique, strongly consistent, and asymptotically jointly normally distributed with asymptotic variance the inverse of the matrix with $ij,i'j'$ entry

$$(4.6) \quad -\frac{\partial^2}{\partial\beta_{ij}\partial\beta_{i'j'}} \log L^{(P)} =$$

$$I_{[j=j']} \sum_{\ell=1}^{N(t)} \varphi_{ij}(\underline{\tau}(T_\ell-))\varphi_{i'j}(\underline{\tau}(T_\ell-)) \frac{g_j(\underline{\tau}(T_\ell-))}{g(\underline{\tau}(T_\ell-))} \left[1 - \frac{g_j(\underline{\tau}(T_\ell-))}{g(\underline{\tau}(T_\ell-))} \right].$$

The right-hand side of (4.6) is asymptotic in probability as $t \to \infty$ to a quantity $t \cdot (I_\beta^P)_{ij,i'j'}$ which can be explicitly calculated in terms of π in special cases.

We mention the partial likelihood methods of estimation so prominently for several reasons. First, with their most recent justification by Andersen and Gill (1982), they provide easily constructed consistent estimates in many cases where ordinary maximum likelihood methods are not available: for example, where all $g_j(\underline{x})$ contain a common but unknown "nuisance" factor $\lambda(\underline{x})$, or where data-collection is subject to independent random censoring with unknown distribution. (We give no detailed discussion of either of these situations.) Another reason for introducing partial likelihood methods here is the broad exposure they have been given in the recent literature on statistical analysis of failure time data (see for example Kalbfleisch and Prentice, 1980, Miller, 1981). That literature showed for independent failure times the considerable theoretical and practical efficiency of partial likelihood methods relative to maximum likelihood. In the present context of dependent failure times, we show by calculations along the line of Slud (1982) in an example that partial likelihood estimation may typically be quite inefficient in ignoring the lengths of intervals between events.

Consider the model (*) with $k = 2$, $\lambda_1(x)$ general, $\lambda_2(x) = \lambda_2$ constant, $\beta_{11} = \beta_{22} = \beta_{21} = 0$, $\beta_{12} = \beta$, and $\varphi_{12}(x,y) = I_{[x<y]}$. Again the condition

$$\mu_1 = \int_0^\infty \exp(-\Lambda_1(x))dx < \infty$$

guarantees that $N_1(\cdot)$ is the counting process for positive-recurrent renewals. By (2.5) we find the unique solution of (2.3) satisfies

$$\pi(x,y;\beta) = \pi(0,y-x;\beta)\exp[-\int_0^x (\lambda_1(u) + e^{\beta}\lambda_2)du] \quad \text{for} \quad x<y.$$

By the known asymptotic marginal behavior of $\tau_1(s)$,

$$\int_0^{\infty} \pi(x,z;\beta)dz = \mu_1^{-1}\exp(-\Lambda_1(x)), \quad x \geq 0.$$

Therefore the quantity I_{β}^F from (4.3) is given in the present example by

$$(4.7) \quad \begin{aligned} I_{\beta}^F &= \iint_{[x<y]} e^{\beta}\lambda_2\pi(x,y;\beta)dx\,dy \\ &= \iint_{[x<y]} e^{\beta}\lambda_2\pi(0,y-x;\beta)\cdot\exp(-\Lambda_1(x) - e^{\beta}\lambda_2 x)dy\,dx \\ &= e^{\beta}\lambda_2 \int_0^{\infty} \exp[-\Lambda_1(x) - e^{\beta}\lambda_2 x]dx \int_0^{\infty} \pi(0,z;\beta)dz \\ &= e^{\beta}\lambda_2\mu_1^{-1} \int_0^{\infty} \exp[-\Lambda_1(x) - e^{\beta}\lambda_2 x]dx. \end{aligned}$$

From (4.6) we find that I_{β}^P is given by

$$I_{\beta}^P = \underset{t\to\infty}{\text{p.lim}}\ t^{-1} \sum_{\ell=1}^{N(t)} I_{[\tau_1(T_\ell-)<\tau_2(T_\ell-)]} \frac{\lambda_1(\tau_2(T_\ell-))\lambda_2 e^{\beta}}{(\lambda_1(\tau_2(T_\ell-)) + \lambda_2 e^{\beta})^2}.$$

Now a failure with $\ell \geq 2$ can have $\tau_1(T_\ell-) < \tau_2(T_\ell-)$ only if it follows a failure with $\Lambda_{\ell-1} = 1$. Moreover, immediately following the failure at $T_{\ell-1}$ with $\Delta_{\ell-1} = 1$ the conditional hazard for $N_2(\cdot)$ is the constant $\lambda_2\exp(\beta)$, so that the time from $T_{\ell-1}$ until the next jump in $N_2(\cdot)$ is exponential with parameter $\lambda_2 \exp(\beta)$, and the time $T_\ell - T_{\ell-1}$ has conditional survival function $\exp(-\lambda_2 e^\beta - \Lambda_1(x))$. Since the number $N_1(t)$ of failures with $\Delta_{\ell-1} = 1$ is asymptotic to t/μ_1 by the elementary renewal theorem, we conclude

$$(4.8) \qquad I_\beta^P = \lambda_2 e^\beta \mu_1^{-1} \int_0^\infty \frac{\lambda_1(y)}{\lambda_1(y) + \lambda_2 e^\beta} \exp(-\lambda_2 e^\beta - \Lambda_1(y))dy.$$

From (4.7) and (4.8), it is apparent that $I_\beta^P < I_\beta^F$ for all β, λ_2, $\lambda_1(\cdot)$, with the interpretation for $\lambda_1(\cdot)$ uniformly bounded by a constant λ_1^* that the asymptotic relative efficiency of maximum partial likelihood to maximum likelihood estimation of β in this example is at best $\lambda_1^*/(\lambda_1^* + \lambda_2 e^\beta)$, and I_β^P / I_β^F can be nearly 1 only when $\lambda_1(\cdot)$ is very large. The special forms of the estimators may also be of interest here:

$$\exp(\hat{\beta}) = \lambda_2^{-1} \sum_{\ell=1}^{N(t)} I_{[\Delta_\ell = 2\, ,\, \Delta_{\ell-1}=1]} \Big/ \sum_{\ell=1}^{N(t)} I_{[\Delta_{\ell-1}=1]}(T_\ell - T_{\ell-1})$$

where $\Delta_0 = 1$ iff $\tau_1(0) < \tau_2(0)$, while $\hat{\beta}_P$ is the unique solution β of

$$0 = \sum_{\ell=1}^{N(t)} \left\{ I_{[\Delta_\ell = 2\, ,\, \Delta_{\ell-1}=1]} - I_{[\Delta_{\ell-1}=1]} \frac{\lambda_2 e^\beta}{\lambda_2 e^\beta + \lambda_1(\tau_1(T_\ell -))} \right\}.$$

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